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Tetravalent edge-transitive graphs of order p^2q

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Abstract A graph is called edge-transitive if its full automorphism group acts transitively on its edge set. In this paper, by using classification of finite simple groups, we classify tetravalent edge-transitive graphs of order p^2q with p, q distinct odd primes. The result generalizes certain previous results. In particular, it shows that such graphs are normal Cayley graphs with only a few exceptions of small orders.

Keywords edge-transitive graph, automorphism group, normal Cayley graph

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1 Introduction

Graphs considered in this paper are assumed to be finite, simple, connected and undirected. Given a graph Γ, denote by $V\Gamma$, $E\Gamma$ and $A\Gamma$ the vertex set, edge set and arc set of Γ, respectively. The size $|V\Gamma|$ is called the order of Γ. Let $\mathrm{Aut}\Gamma$ be the full automorphism group of Γ and X a subgroup of $\mathrm{Aut}\Gamma$. Then Γ is called *X-vertex-transitive*, *X-edge-transitive*, or *X-arc-transitive*, if X is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$, respectively. Moreover, Γ is called *half-transitive* if $\mathrm{Aut}\Gamma$ is transitive on $V\Gamma$ and $E\Gamma$ but not on $A\Gamma$. Let $\mathrm{val}(\Gamma)$ denote the valency of Γ. If $\mathrm{val}(\Gamma) = 4$, then Γ is called a tetravalent graph.

Let G be a group and S a subset of $G \setminus \{1\}$ such that $S = S^{-1} := \{g^{-1} \mid g \in S\}$. The Cayley graph on G with respect to S is defined with vertex set G and two vertices x and y are adjacent if and only if $yx^{-1} \in S$. We denote this Cayley graph by $\mathrm{Cay}(G, S)$. It is well known that a graph Γ can be viewed as a Cayley graph of a group G if and only if $\mathrm{Aut}\Gamma$ contains a subgroup which is regular on $V\Gamma$ and isomorphic to G, see [2, Proposition 16.3]. For an X-edge-transitive graph with $X \leqslant \mathrm{Aut}\Gamma$, if X contains a normal subgroup G that is regular on $V\Gamma$, then Γ is called an *X-normal edge-transitive Cayley graph* of G. Normal edge-transitive Cayley graphs have some nice properties, see [15, 18, 21, 23].

Tetravalent graphs have received much attention in the literature. For example, a classification of tetravalent arc-transitive abelian Cayley graphs is given in [33], and some characterizations of tetravalent edge-transitive nonabelian Cayley graphs are obtained in [8, 16, 17]. Let p, q be distinct primes. Some characterizations of tetravalent graphs of order $4p$ are obtained in [11, 36], and tetravalent half-transitive graphs of orders p^3 and p^4 are classified in [10, 34], respectively. More recently, a classification of tetravalent half-transitive graphs of order $2pq$ is presented in [9]. In this paper, we shall classify tetravalent edge-transitive graphs of order p^2q with p, q distinct odd primes.

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Throughout the paper, we always use the following notation. For a positive integer n, denote by $\mathbb{Z}_n$, D_{2n}, A_n and S_n the cyclic group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively. For two groups N and H, denote by $N.F$ an extension of N by H, by $N{:}H$ a semi-direct product of N by H, and by $N \times H$ the direct product of N and H. Moreover, for a group X and a subgroup $H \subseteq X$, we denote by $C_X(H)$ and $N_X(H)$ the centralizer and normalizer of H in X, respectively.

This paper is organized as follows. After this introduction, we give some preliminary results on both graph theory and group theory in Section 2, and then construct some examples in Section 3. By proving some technical lemmas in Section 4, we finally present the classification in Section 5.

2 Preliminaries

In this section, we quote some preliminary results which will be used in the subsequent sections.

2.1 Some graph-theoretic results

Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph of a group G. Let $\hat{G} = \{\hat{g} \mid \hat{g} : x \mapsto xg, \text{for all } g, x \in G\}$, $\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid S^\sigma = S\}$. Then both $\hat{G}$ and $\mathrm{Aut}(G, S)$ are subgroups of $\mathrm{Aut}\Gamma$. For convenience, we denote the regular subgroup $\hat{G}$ still by G.

Lemma 2.1 (See [12, Lemma 2.1]). *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph. Then the normalizer $N_{\mathrm{Aut}\Gamma}(G) = G{:}\mathrm{Aut}(G, S)$.*

Let X be a group, H a core-free subgroup of X (i.e., H contains no nontrivial normal subgroup of X), and S a subset of $X \setminus \{1\}$. Then the *coset graph*, denoted by $\mathrm{Cos}(X, H, HSH)$, is defined with vertex set $[X : H] = \{Hx \mid x \in X\}$ such that Hx is adjacent to Hy if and only if $yx^{-1} \in HSH$. The following lemma is known, refer to [26].

Lemma 2.2. *Let Γ be an X-vertex-transitive and X-edge-transitive graph with $X \leqslant \mathrm{Aut}\Gamma$. Then $\Gamma \cong \mathrm{Cos}(X, X_\alpha, X_\alpha\{g, g^{-1}\}X_\alpha)$ for some $g \in X$ and $\alpha \in V\Gamma$, and Γ is connected if and only if $\langle X_\alpha, g\rangle = X$. Furthermore, if Γ is X-arc-transitive, then $\Gamma \cong \mathrm{Cos}(X, X_\alpha, X_\alpha f X_\alpha)$ with f a 2-element of X such that $f^2 \in X_\alpha$.*

For a graph Γ and a positive integer s, an *s-arc* of Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices such that v_{i-1}, v_i are adjacent for $1 \leqslant i \leqslant s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$. A graph Γ is called *(X, s)-arc-transitive*, where $X \leqslant \mathrm{Aut}\Gamma$, if X is transitive on the set of s-arcs of Γ. If Γ is (X, s)-arc-transitive but not $(X, s+1)$-arc-transitive, then Γ is called *(X, s)-transitive*. In particular, an $(\mathrm{Aut}\Gamma, s)$-transitive graph is simply called *s-transitive*.

The following result characterizes the vertex stabilizers of tetravalent edge-transitive graphs of odd order, refer to [30] or [17, Lemma 2.5], which will play an important role in our later discussion.

Lemma 2.3. *Let Γ be a tetravalent X-edge-transitive graph of odd order, where $X \leqslant \mathrm{Aut}\Gamma$. Let $\alpha \in V\Gamma$. Then either*

(1) X_α is a 2-group, and Γ is X-half-transitive or $(X, 1)$-transitive; or

(2) Γ is (X, s)-transitive with $2 \leqslant s \leqslant 3$, and $|X_\alpha| \mid 144$. Furthermore, the pair (s, X_α) satisfies the following table:

s	2	3
X_α	$\mathrm{A}_4 \leqslant X_\alpha \leqslant \mathrm{S}_4$	$\mathrm{A}_4 \times \mathbb{Z}_3 \leqslant X_\alpha \leqslant \mathrm{S}_4 \times \mathrm{S}_3$

Lemma 2.3 has the following corollary.

Corollary 2.4. *Let Γ be a tetravalent X-edge-transitive graph of odd order, where $X \leqslant \mathrm{Aut}\Gamma$. If X is insoluble, then Γ is not an X-normal edge-transitive Cayley graph.*

Proof. In fact, if $\Gamma := \mathrm{Cay}(G, S)$ is an X-normal edge-transitive Cayley graph, then $G \lhd X$ is soluble as $|G| = |V\Gamma|$ is odd. Now, because $X = G{:}X_\alpha$ is insoluble, where $\alpha \in V\Gamma$, we conclude that X_α is insoluble, which is not possible by Lemma 2.3. □

A typical method for studying vertex-transitive graphs is taking certain quotients. Let Γ be an X-vertex-transitive graph with $X \leqslant \mathrm{Aut}\Gamma$. Suppose that X has a normal subgroup N which is intransitive on $V\Gamma$. Let $V\Gamma_N$ denote the set of N-orbits in $V\Gamma$. Then the *normal quotient graph* of Γ induced by N, denoted by Γ_N, is defined with vertex set $V\Gamma_N$ such that two vertices $B, C \in V\Gamma_N$ are adjacent if and only if some $\alpha \in B$ is adjacent in Γ to some $\beta \in C$. If Γ and Γ_N have the same valency, then Γ is called a *normal cover* of Γ_N.

The following theorem provides a basic method for studying 2-arc-transitive graphs [22, Theorem 4.1].

Theorem 2.5. *Let Γ be an $(X, 2)$-arc-transitive graph, and let $N \lhd X$ have at least three orbits on $V\Gamma$, where $X \leqslant \mathrm{Aut}\Gamma$. Then N is semiregular on $V\Gamma$, $X/N \leqslant \mathrm{Aut}(\Gamma_N)$, Γ_N is $(X/N, 2)$-arc-transitive and Γ is a normal cover of Γ_N.*

For a reduction, we need some information of certain tetravalent edge-transitive graphs of order a product of two distinct odd primes.

Lemma 2.6. *Let Γ be a tetravalent edge-transitive graph of order pq, where $p < q$ are odd primes. Suppose $\mathrm{Aut}\Gamma$ is insoluble. Then the triple $(pq, \mathrm{Aut}\Gamma, (\mathrm{Aut}\Gamma)_\alpha)$ lies in the following Table 1, where $\alpha \in V\Gamma$.*

Table 1 Tetravalent edge-transitive graphs of order a product of two distinct primes

pq	transitivity	$\mathrm{Aut}\Gamma$	$(\mathrm{Aut}\Gamma)_\alpha$
15	1-transitive	S_5	$\mathbb{Z}_2^2$
21	1-transitive	$\mathrm{PGL}(2,7)$	D_{16}
35	3-transitive	S_7	$\mathrm{S}_4 \times \mathrm{S}_3$
55	2-transitive	$\mathrm{PGL}(2,11)$	S_4
253	2-transitive	$\mathrm{PSL}(2,23)$	S_4

Remark on Lemma 2.6. Since $|V\Gamma| = pq$ is odd, Γ is vertex-transitive. By [1,28], no tetravalent edge-transitive graph of order pq is half-transitive, so Γ is arc-transitive. Furthermore, if $p = 3$, by [29, p. 215, Theorem], all tetravalent arc-transitive graphs of order $3q$ are either as in rows 1, 2 of Table 1, or isomorphic to the graph $\mathrm{G}(3q, 2)$ (see [29, p. 204] for the definition of the graph). However, by [29, Example 3.4], $\mathrm{Aut}(\mathrm{G}(3q, 2)) \cong \mathbb{Z}_q{:}\mathbb{Z}_2.\mathrm{S}_3$ is soluble, a contradiction. For the case $p \geqslant 5$, Γ lies in Rows 3–5 of Table 1 by [24, Section 4] and [25, p. 248, Table 1].

2.2 Some group-theoretic results

Lemma 2.7 (See [14, Chapter I, Theorem 4]). *The quotient group $N_X(H)/C_X(H)$ is isomorphic to a subgroup of the automorphism group of H.*

For a given group X, its *Fitting subgroup* is the largest nilpotent normal subgroup of X. Obviously, the Fitting subgroup of X is a characteristic subgroup of X.

Lemma 2.8 (See [27, p. 30, Corollary]). *Let F be the Fitting subgroup of a group X. If X is soluble, then $F \neq 1$ and $C_X(F) \leqslant F$.*

From a classification of transitive permutation groups of prime degree [6, p. 99], we have the following lemma.

Lemma 2.9. *Let $X \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group of prime degree p. Then either $X \leqslant \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ is affine, or X is almost simple and 2-transitive on Ω.*

The next result slightly generalizes [31, Theorem 3.4], and the proof is similar and thus omitted.

Lemma 2.10. *Let $X \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group on Ω, and let p^m be a divisor of $|\alpha^X|$, where $\alpha \in \Omega$ and p is a prime. If X has a subgroup H such that $(p, |X : H|) = 1$, then p^m divides $|\alpha^H|$. In particular, if $(|\Omega|, |X : H|) = 1$, then H is transitive on Ω.*

Let $X = N.H$ be an extension of N by H. If $N \leqslant Z(X)$, the center of X, then $X = N.H$ is called a *central extension*. A group X is called *perfect* if $X = X'$, the commutator subgroup of X. For a given group H, if N is the largest abelian group such that $X{:}=N.H$ is perfect and the extension is a central extension, then N is called the *Schur Multiplier* of H, denoted by $\mathrm{Multi}(H)$. The Schur multipliers of all finite simple groups are known, see [13, p. 302].

The following lemma is probably known. For the completeness of the paper, a proof is given.

Lemma 2.11. *Let N be a group of order a prime or a prime square. Let T be a nonabelian simple group. Then $X{:}=N.T$ is a central extension. Moreover, $X=NX'$ and $X'=H.T$, where $H\leqslant N$ and $H\leqslant \mathrm{Multi}(T)$.*

Proof. We first prove that $X=N.T$ is a central extension. If N is a cyclic group, the result is known. Suppose that N is not cyclic. Then $N\cong \mathbb{Z}_p^2$ for some prime p. Let $C=C_X(N)$. Then $N\leqslant C$ and $C\lhd X$. Since $C/N\lhd X/N\cong T$, we have that either $C/N=1$ or $C/N=X/N$. For the former, $C=N$, then Lemma 2.7 implies $T\cong X/C\leqslant \mathrm{Aut}(N)\cong \mathrm{GL}(2,p)$. However, by [7, Lemma 2.7], $\mathrm{GL}(2,p)$ has no nonabelian simple subgroup, a contradiction. Thus, $C=X$, i.e., $N\subseteq Z(X)$, as required.

Now, since X is insoluble, we have $X'\not\subseteq N$. Then as $1\neq NX'/N\lhd X/N\cong T$, it follows that $X=NX'$ and $X'=(NX')'=X''$. Let $H=X'\cap N$. Then $H\leqslant N$, $X'/H=X'/(X'\cap N)\cong X'N/N=X/N\cong T$ and $H\subseteq X'\cap Z(X)=Z(X')$. Hence $H\leqslant \mathrm{Multi}(T)$ by the definition of the Schur multiplier. □

3 Constructions

In this section, we construct some examples which will appear in Theorem 5.3 in Section 5. First, with the use of [3], one may check up the following example.

Example 3.1. Let $G=\mathrm{PSL}(2,17)$. Then G has a maximal subgroup $H\cong D_{16}$ and an involution g such that $|H{:}H\cap H^g|=4$ and $\langle H,g\rangle=G$. The coset graph $\mathrm{Cos}(G,H,HgH)$, denoted by $\mathcal{G}_{153}$, is a tetravalent arc-transitive graph of order 153. Furthermore, each tetravalent edge-transitive graph of order 153 admitting G as an edge-transitive automorphism group is isomorphic to $\mathcal{G}_{153}$.

The following lemma gives a general construction of normal edge-transitive tetravalent Cayley graph of order p^2q, where p,q are distinct odd primes.

Lemma 3.2. *Let $\Gamma=\mathrm{Cay}(G,S)$ be an X-normal edge-transitive tetravalent Cayley graph of order p^2q, where p,q are distinct odd primes. Let $\mathbf{1}$ denote the vertex of Γ corresponding to the identity element of G. Then either*

(i) *Γ is $(X,1)$-transitive, and $S=\{a,a^\sigma,a^{\sigma^2},a^{\sigma^3}\}$, where $\sigma\in\mathrm{Aut}(G)$ is of order 4; or*

(ii) *$X_{\mathbf{1}}\leqslant \mathbb{Z}_2^2$ and $S=\{a,a^\tau,a^{-1},(a^{-1})^\tau\}$, where $\tau\in\mathrm{Aut}(G)$ is an involution.*

Proof. By Lemma 2.1, $X_{\mathbf{1}}\leqslant \mathrm{Aut}(G,S)$. Since Γ is connected, $\langle S\rangle=G$ and then $X_{\mathbf{1}}$ acts faithfully on $\Gamma(\mathbf{1})=S$, which implies $X_{\mathbf{1}}\leqslant \mathrm{S}_4$. If $3\mid |X_{\mathbf{1}}|$, then Γ is $(X,2)$-arc-transitive by Lemma 2.3, so $X_{\mathbf{1}}$ is 2-transitive on S. It follows that elements in S are involutions for otherwise an element in S with the order bigger then 2 and its inverse would form a nontrivial block of $X_{\mathbf{1}}$ on S. However, as $|G|=p^2q$ is odd, G has no involution, which is a contradiction.

Thus, $X_{\mathbf{1}}$ is a 2-group and hence $X_{\mathbf{1}}\leqslant D_8$. Let $a\in S$. If $X_{\mathbf{1}}\geqslant\langle\sigma\rangle\cong\mathbb{Z}_4$, then $\langle\sigma\rangle$ is regular on S, hence $S=\{a,a^\sigma,a^{\sigma^2},a^{\sigma^3}\}$, part (i) holds. If $X_{\mathbf{1}}\leqslant\mathbb{Z}_2^2$, then there exists an involution $\tau\in X_{\mathbf{1}}$ such that $a^\tau\neq a$ or a^{-1}, it follows that $S=\{a,a^\tau,a^{-1},(a^{-1})^\tau\}$, part (ii) holds. □

By Lemma 3.2, more specific constructions of the graphs depend on the automorphism group of group G. Since there are dozens of isomorphic classes of groups with order p^2q (refer to [35, p. 317] for $p<q$ and [4] for $p>q$), it is inconvenient to give specific constructions case by case. Here, we determine the graphs where G is a Frobenius group.

Example 3.3. Let $\Gamma=\mathrm{Cay}(G,S)$ be an X-normal edge-transitive tetravalent Cayley graph of a group G, and let $G=\langle a\rangle{:}\langle b\rangle\cong\mathbb{Z}_m{:}\mathbb{Z}_n$ be a Frobenius group, where $(m,n)=(p^2,q)$ or (q,p^2) with p,q distinct odd primes. Then Γ is X-half-transitive, and $S=\{b^k,(b^k)^\tau,b^{-k},(b^{-k})^\tau\}$, where $\tau\in\mathrm{Aut}(G)$ is an involution, $(k,n)=1$, and $1\leqslant k\leqslant(n-1)/2$.

Proof. Since G is a Frobenius group, G is center-free. Then, as $\langle a\rangle \cong \mathbb{Z}_m$ is a characteristic subgroup of G, it is easy to show that $\mathrm{Aut}(G) \cong \mathbb{Z}_m{:}\mathbb{Z}_{\phi(m)}$, where $\phi(m)$ is the Euler phi-function, and each $\sigma \in \mathrm{Aut}(G)$ has a presentation $\sigma : a \to a^i,\ b \to a^j b$, where $(i,m)=1,\ 1 \leqslant j \leqslant m$. In particular, there is no automorphism of G which maps b to b^{-1}.

Now, by the normality of Γ, $G \lhd X$ and so $X_{\mathbf{1}} \leqslant \mathrm{Aut}(G,S)$ by Lemma 2.1. As X is transitive on $E\Gamma$, $X_{\mathbf{1}}$ has at most two orbits on $\Gamma(\mathbf{1})$, and $S = \{s_1, s_1^{-1}\}^{X_{\mathbf{1}}}$ for some $s_1 \in S$, where $\mathbf{1}$ denotes the vertex of Γ corresponding to the identity element of G. It follows that each element in S has the same order n. Since $n = q$ or p^2, and Sylow p- or Sylow q-subgroups of G are conjugate, s_1 is conjugate to b^k for some k which is coprime to n. Then as $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,S^\pi)$ for each $\pi \in \mathrm{Aut}(G)$, we may assume that $b^k \in S$. Then as $b^{-k} \in S$, we may also assume that $1 \leqslant k \leqslant (n-1)/2$. Furthermore, as $G = \langle a\rangle{:}\langle b^k\rangle$, there is no $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(b^k) = b^{-k}$, which follows that $X_{\mathbf{1}} \leqslant \mathrm{Aut}(G)$ is not transitive on S. Thus, Γ is X-half-transitive, and by Lemma 3.2, $S = \{b^k, (b^k)^\tau, b^{-k}, (b^{-k})^\tau\}$ for some involution $\tau \in \mathrm{Aut}(G)$, as required. □

4 A few technical lemmas

For later discussion, we prove several technical lemmas in this section.

Lemma 4.1. *Let $X = D_{2m}$ be a dihedral group. Then X has a 2-transitive permutation representation on a set Ω with $|\Omega| \geqslant 3$ if and only if $3 \mid m$ and $|\Omega| = 3$.*

In particular, a dihedral group has no 2-transitive permutation representation of degree 4.

Proof. Suppose that X acts 2-transitively on Ω. Then the induced permutation group X^Ω is 2-transitive. Since $X = D_{2m}$, $X^\Omega \cong \mathbb{Z}_n{:}\mathbb{Z}_2$ for some $n \mid m$, it then follows easily that $|\Omega| = n = 3$.

Conversely, if $3 \mid m$, let H be a subgroup of X such that $H \cong D_{2m/3}$ and let $\Delta = [X : H]$. Then $|\Delta| = 3$ and X acts 2-transitively on Δ by the coset action. □

The next two lemmas give some properties of tetravalent graphs.

Lemma 4.2. *Suppose that X is an insoluble group and $p \geqslant 5$ is a prime. Then there is no tetravalent X-edge-transitive graph of order p^2.*

Proof. Suppose, by contradiction, that Γ is a tetravalent X-edge-transitive graph of order p^2. Let $\alpha \in V\Gamma$. Since $\mathrm{val}(\Gamma) = 4$, X_α is a $\{2,3\}$-group. Then as $|X| = p^2|X_\alpha|$ and X is insoluble, we have 3 divides $|X_\alpha|$, it then follows from Lemma 2.3 that Γ is $(X,2)$-arc-transitive.

Now, by [19, Corollary 1.2(ii)], Γ is a normal cover of $\Gamma_N = \mathbf{K}_{p^m}^l$, where $N \lhd X$, $lm \leqslant 2$, and $\mathbf{K}_{p^m}^l$ denotes the l-terms direct product of the complete graph $\mathbf{K}_{p^m}$. Then $4 = \mathrm{val}(\Gamma) = \mathrm{val}(\mathbf{K}_{p^m}^l) = (p^m-1)^l$, which implies $p = 5$, $N \cong \mathbb{Z}_5$ and $m = l = 1$, i.e., $\Gamma_N \cong \mathbf{K}_5$. Since $X = N.(X/N) \leqslant \mathbb{Z}_5.\mathrm{S}_5$ is insoluble, $X \cong \mathbb{Z}_5.\mathrm{A}_5$ or $\mathbb{Z}_5.\mathrm{S}_5$. Since $\mathrm{Multi}(\mathrm{A}_5) = \mathbb{Z}_2$, we have $X = \mathbb{Z}_5 \times \mathrm{A}_5$ or $(\mathbb{Z}_5 \times \mathrm{A}_5).\mathbb{Z}_2$. In particular, X always has a normal subgroup M such that $M \cong \mathrm{A}_5$. If M has at least three orbits on $V\Gamma$, then M is semiregular on $V\Gamma$ by Theorem 2.5, which implies $|M| = |\mathrm{A}_5|$ divides $|V\Gamma| = 25$, not possible. Thus, as $|V\Gamma|$ is odd, M is transitive on $V\Gamma$, which is also not possible. □

Lemma 4.3. *Let Γ be a tetravalent X-edge-transitive graph with odd but not a prime power order, where $X \leqslant \mathrm{Aut}\Gamma$. Suppose that N is a nilpotent normal subgroup of X. Then N is semiregular on $V\Gamma$.*

Proof. Since N is nilpotent, we may suppose that $N = N_1 \times \cdots \times N_s$, where N_i is a Sylow p_i-subgroup of N for $1 \leqslant i \leqslant s$. Since $|V\Gamma|$ is odd, $p_i \neq 2$ for each i. Let $\alpha \in V\Gamma$.

It is sufficient to prove that each N_i is semiregular on $V\Gamma$. If there is some N_i which is not semiregular on $V\Gamma$, as $p_i \neq 2$, $(N_i)_\alpha \neq 1$ is not a 2-group, so is X_α. By Lemma 2.3, Γ is $(X,2)$-arc-transitive. Now, as $|V\Gamma|$ is not a prime power, N_i has at least three orbits on $V\Gamma$, it then follows from Theorem 2.5 that N_i is semiregular on $V\Gamma$, a contradiction. □

The following lemma classifies certain simple groups, which will be used later.

Lemma 4.4. *Let T be a nonabelian simple group such that $|T|$ divides $144p^2q$, where p,q are distinct odd primes. Then the triple $(T, |T|, \mathrm{Out}(T))$ lies in the following table:*

Table 2 Nonabelian simple groups of order dividing $144p^2q$

3-Prime factor			4-Prime factor		
T	$\|T\|$	Out(T)	T	$\|T\|$	Out(T)
A_5	$2^2 \cdot 3 \cdot 5$	$\mathbb{Z}_2$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$\mathbb{Z}_2$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathbb{Z}_2^2$	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
PSL$(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathbb{Z}_2$	PSL$(2,16)$	$2^4 \cdot 3 \cdot 5 \cdot 7$	$\mathbb{Z}_4$
PSL$(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathbb{Z}_3$	PSL$(2,25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$\mathbb{Z}_2^2$
PSL$(2,17)$	$2^4 \cdot 3^2 \cdot 17$	$\mathbb{Z}_2$	PSL$(2,r)$ ($r = p$ or q)	$\frac{r(r^2-1)}{2}$	$\mathbb{Z}_2$
PSL$(3,3)$	$2^4 \cdot 3^3 \cdot 13$	$\mathbb{Z}_2$			

Proof. If $|T|$ has exactly three prime factors, by [13, pp. 12–14], T lies in Column 1 of the table.

Suppose that $|T|$ has four prime factors in the following. Then $p, q \geqslant 5$. If T is a sporadic simple group, $T = M_{11}$ by [13, pp. 135–136]. If $T = A_m$ is an alternating group, then $7 \leqslant m \leqslant 10$, it then easily follows that $T = A_7$.

Now, suppose $T = X(t)$ is a Lie group, where X is one type of Lie groups, and $t = r^d$ is a prime power. If $r = 2$, as $2^5 \nmid |T|$, it is easy to show that $T = \mathrm{PSL}(2,16)$ by [13, pp. 135–136].

Assume $r \geqslant 3$. If $T \neq \mathrm{PSL}(2,t)$, by [13, pp. 135–136], we always have t^3 divides $|T| = |X(t)|$, which contradicts that $|T| \mid 144p^2q$. So $T = \mathrm{PSL}(2, r^d)$ and $d \leqslant 2$. If $r = 3$, because PSL$(2,3)$ is not a simple group and $\mathrm{PSL}(2,9) \cong A_6$, which are not the cases. Suppose $r > 3$. Then $r = p$ or q. If $d = 1$, then $T = \mathrm{PSL}(2,r)$ as in Row 5 of Column 4 of the table. If $d = 2$, then $|T| = \frac{1}{2}r^2(r^2-1)(r^2+1)$, so $r = p$ and $\frac{(r^2-1)(r^2+1)}{2} \mid 144q$. Since $(r^2-1, \frac{r^2+1}{2}) = 1$ and $\frac{r^2+1}{2}$ is odd and does not divide 9, we conclude that $\frac{r^2+1}{2} = q$ and hence $(r^2-1) \mid 144$. This implies that $r = 5$ and $T = \mathrm{PSL}(2,25)$.

Finally, the outer automorphism groups of the groups in Table 2 follow directly by [5]. □

Remark on Lemma 4.4. If T is a nonableian simple group such that $|T|$ has exactly three prime factors, then either T lies in the column 1 of Table 2, or $T = \mathrm{PSU}(3,3)$ or PSU$(4,2)$, see [13, pp. 12–14].

5 Classification

From now on, we always use the following convention: Let Γ be a tetravalent X-edge-transitive graph of order p^2q, where $X \leqslant \mathrm{Aut}\Gamma$ and p, q are distinct odd primes.

Since $|V\Gamma|$ is odd, X is transitive on $V\Gamma$. Let $\alpha \in V\Gamma$. Then X_α is a $\{2,3\}$-group, and hence X is a $\{2,3,p,q\}$-group. Obviously, X has no nontrivial normal 2-subgroup, and if $p, q > 3$, then X has no nontrivial normal 3-subgroup.

We first treat the case where X is insoluble. For convenience, for a positive integer $m = p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$ and a subset ϕ of $\{p_1, p_2, \ldots, p_s\}$, where $p_1, p_2, \ldots, p_s$ are distinct primes, we denote $m_\phi = \prod_{p_i \in \phi} p_i^{r_i}$.

Lemma 5.1. *Suppose that X is insoluble. Then one of the following holds*:

(1) Γ *is of order* $45, 63, 75$ *or* 147;

(2) X *is almost simple.*

Proof. Let N be the *socle* of X, denoted by soc(X), the product of all minimal normal subgroups of X. Let M be the *soluble radical* of X, the largest normal soluble subgroup of X. Let $\alpha \in V\Gamma$. Because X_α is a $\{2,3\}$-group, we may suppose that $|X| = 2^i3^jp^2q$ for some integers i, j.

Case 1. Assume $M = 1$. Then each nontrivial normal subgroup of X is insoluble, by [6, Theorem 4.3A], $N = M_1 \times \cdots \times M_s$, where $M_1, \ldots, M_s$ are the all minimal normal subgroups of X. Suppose that $M_k = T_k^{d_k}$, where T_k is a nonabelian simple group and $1 \leqslant k \leqslant s$.

If X_α is a 2-group or $p = 3$, then N is a $\{2,p,q\}$-group with $|N|_q = q$, obviously N is a simple group and X is almost simple, part (2) holds. Thus suppose that 3 divides $|X_\alpha|$ and $p > 3$ in the following. By Lemma 2.3, $|X_\alpha|$ divides 144, and hence $i \leqslant 4$ and $j \leqslant 2$. Furthermore, each T_k lies in Table 2 as $|T_k|$ divides $144p^2q$.

Subcase 1.1. Suppose $q = 3$. Then $|X| = 2^i 3^{j+1} p^2$ with $p \geqslant 5$. So $s \leqslant 2$, and T_i lies in Column 1 of Table 2. In particular, $|T_i|_p = p$ and $(|\mathrm{Out}(T_i)|, p) = 1$. If N is a simple group, then $C_X(N) = 1$ and $X \leqslant \mathrm{Aut}(N) = N.\mathrm{Out}(N)$, which implies that p^2 does not divide $|X|$, a contradiction. Thus N is not a simple group and p^2 divides $|N|$. By Lemma 2.10, p^2 divides $|\alpha^N| = |N{:}N_\alpha|$.

Assume $s = 1$. Then $N = T_1^2$, and T_1 lies in column 1 of Table 2. If $T_1 \neq \mathrm{A}_5$, then $2^3 \mid |T_1|$, implying $2^6 \mid |N|$, which is not possible. Suppose $T_1 = \mathrm{A}_5$. Then $p = 5$ and $|V\Gamma| = 75$. Since N is the unique minimal normal subgroup of X, $C_X(N) = 1$, by [6, Exercise 4.3.9] we have $X{:} = N.O \leqslant \mathrm{Aut}(N) = \mathrm{S}_5 \wr \mathbb{Z}_2$. It then follows from Lemma 2.10 that $N = \mathrm{A}_5^2$ is transitive on $V\Gamma$, and hence $|N_\alpha| = |N|/|V\Gamma| = 48$. Furthermore, as $X_\alpha/N_\alpha = X_\alpha/(X_\alpha \cap N) \cong X_\alpha N/N = X/N \cong O$, $|O| \mid 8$, and by Lemma 2.3, 32 does not divide $|X_\alpha|$, we conclude that $|X_\alpha| = |N_\alpha| = 48$, which is not possible by Lemma 2.3.

Assume next $s = 2$. Then $N = M_1 \times M_2 = T_1 \times T_2$. If $T_1 \cong T_2$, then arguing as above, one may draw a contradiction. Suppose $T_1 \ncong T_2$. Since $N_\alpha \geqslant (T_1)_\alpha \times (T_2)_\alpha$, we have that $|N{:}N_\alpha|$ divides $|T_1{:}(T_1)_\alpha||T_2{:}(T_2)_\alpha|$, which follows that p divides both $|T_1{:}(T_1)_\alpha|$ and $|T_2{:}(T_2)_\alpha|$. Furthermore, as $|T_1{:}(T_1)_\alpha|$ divides $|N{:}N_\alpha|$, and $|N{:}N_\alpha|$ divides $3p^2$, we obtain that $|T_1{:}(T_1)_\alpha| = p$ or $3p$. Similarly, $|T_2{:}(T_2)_\alpha| = p$ or $3p$, i.e., T_1 and T_2 have a subgroup with index p or $3p$. Checking the candidates in Column 1 of Table 2, by [5], we conclude that T_1 and T_2 lie in $\{\mathrm{A}_5, \mathrm{A}_6, \mathrm{PSL}(2,7), \mathrm{PSL}(3,3)\}$. If $T_k = \mathrm{PSL}(3,3)$ with $k = 1$ or 2, then 3^4 divides $|N|$, not possible; if $T_k = \mathrm{PSL}(2,7)$, then $p = 7$, and as $|T_1|_p = |T_2|_p$ we have $T_1 \cong T_2$, not the case. Thus $\{T_1, T_2\} = \{\mathrm{A}_5, \mathrm{A}_6\}$, $p = 5$ and $|V\Gamma| = 75$. Now, noting that $|N|_{\{3,5\}} = |X|_{\{3,5\}} = 3^3 \cdot 5^2$, by Lemma 2.10, $N = \mathrm{A}_5 \times \mathrm{A}_6$ is transitive on $V\Gamma$, so $|N_\alpha| = |N|/|V\Gamma| = 288$, which is impossible by Lemma 2.3.

Subcase 1.2. Suppose $p, q > 3$. Then $|X|_{\{p,q\}} = p^2 q$ and $s \leqslant 3$. Assume $s = 1$. Then $N = T_1^{d_1}$ and $d_1 \leqslant 2$. If $d_1 = 2$, then T_1 is an $\{2,3,p\}$-group, and lies in Column 1 of Table 2. However, as $(|\mathrm{Out}(T_1)|, q) = 1$ and $X \leqslant N.(\mathrm{Out}(T_1) \wr \mathrm{S}_2)$, we conclude that q does not divide $|X|$, yielding a contradiction. So $d_1 = 1$ and X is almost simple.

Assume $s = 2$. Suppose first that both $|M_1|$ and $|M_2|$ have exactly three prime factors. Then T_1, T_2 lie in Column 1 of Table 2 and in particular $(|\mathrm{Out}(T_i)|, pq) = 1$. It is then easy to show that $|N|_{\{p,q\}} = p^2 q$ as $|X|_{\{p,q\}} = p^2 q$. Without lose of generality, we may suppose that T_1 is a $\{2,3,p\}$-group and T_2 is a $\{2,3,q\}$-group. Then $M_1 = T_1^2$ and $M_2 = T_2$. By Lemma 2.3, we have $|T_2{:}(T_2)_\alpha| = q$, and $|M_1{:}(M_1)_\alpha| = p^2$, which implies that $|T_1{:}(T_1)_\alpha| = p$ and $(M_1)_\alpha = (T_1)_\alpha^2$. Now, noting that $|(T_k)_\alpha|$ with $1 \leqslant k \leqslant 2$ has two prime divisors and $N_\alpha \geqslant (T_1)_\alpha^2 \times (T_2)_\alpha$, it is not possible by Lemma 2.3.

Suppose now that $|M_1|$ has four prime factors and $|M_2|$ has exactly three prime factors. Then $M_1 = T_1$, $M_2 = T_2$, and $p^2 q$ divides $|N|$, so N is transitive on $V\Gamma$ by Lemma 2.10. Then, because $p^2 q = |N{:}N_\alpha|$ divides $|T_1{:}(T_1)_\alpha||T_2{:}(T_2)_\alpha|$, and $|(T_k)_\alpha|$ divides $|N{:}N_\alpha| = p^2 q$, we conclude that $|T_1{:}(T_1)_\alpha| = pq$, $|T_2{:}(T_2)_\alpha| = p$ and $N_\alpha = (T_1)_\alpha \times (T_2)_\alpha$. Since T_2 lies in column 1 of Table 2 and $|T_2{:}(T_2)_\alpha| = p$, we have $T_2 = \mathrm{A}_5, \mathrm{PSL}(2,7)$ or $\mathrm{PSL}(3,3)$, and $(T_2)_\alpha = \mathrm{A}_4, \mathrm{S}_4$ or $(3^2{:}2\mathrm{S}_4)^2$, respectively. Now, by Lemma 2.3, $(T_1)_\alpha \times (T_2)_\alpha = N_\alpha \leqslant \mathrm{S}_4 \times \mathrm{S}_3$, we have $(T_1)_\alpha \leqslant \mathrm{S}_3$. It follows that $|T_1| = |(T_1)_\alpha||T_1{:}(T_1)_\alpha|$ divides $6pq$, hence T_1 is soluble, a contradiction.

Finally, assume $s = 3$. Then $N = T_1 \times T_2 \times T_3$, where T_1, T_2 are $\{2,3,p\}$-groups and T_3 is a $\{2,3,q\}$-group. It follows that $|T_1{:}(T_1)_\alpha| = |T_2{:}(T_2)_\alpha| = p$ and $|T_3{:}(T_3)_\alpha| = q$. Noting that $(T_1)_\alpha$, $(T_2)_\alpha$ and $(T_3)_\alpha$ are not 2-groups, and $N_\alpha \geqslant (T_1)_\alpha \times (T_2)_\alpha \times (T_3)_\alpha$, it is not possible by Lemma 2.3.

Case 2. Assume $M \neq 1$. Let F be the Fitting subgroup of M. Then $F \lhd X$ and $F \neq 1$ by Lemma 2.8.

We consider Γ_F. Let K be the kernel of X acting on $V\Gamma_F$. Then $K = FK_\alpha$, and hence K is soluble as K_α is soluble by Lemma 2.3. If $\mathrm{val}(\Gamma_F) = 2$, then Γ_F is a cycle and $X/K \leqslant \mathrm{Aut}(\Gamma_F) = D_{2m}$, where $m = |V\Gamma_F|$. However, as K is soluble, X is soluble, which is not the case. Thus, $\mathrm{val}(\Gamma_F) = 4$. Then $K = F$ and $X/F \leqslant \mathrm{Aut}(\Gamma_F)$. Further, by Lemma 4.3, F is semiregular on $V\Gamma$ and hence $|F|$ divides $p^2 q$.

If $|F| = p^2 q$, then Γ is an X-normal edge-transitive Cayley graph of F, which is not possible by Lemma 2.4.

Suppose $|F| = q$. Then Γ_F is a tetravalent X/F-edge-transitive graph of order p^2. If $p = 3$, then $|V\Gamma_F| = 9$, by [20], $\Gamma_F = DW(3,3)$ is a deleted wreath graph and $\mathrm{Aut}(\Gamma_F) = \mathbb{Z}_3^2{:}D_8$. It follows that Γ is a tetravalent X-normal edge-transitive Cayley graph of a group isomorphic to $\mathbb{Z}_q{:}\mathbb{Z}_3^2$, which is impossible by Lemma 2.4. If $p \geqslant 5$, then as X/F is insoluble, a contradiction occurs by Lemma 4.2.

Suppose next $|F| = pq$. Then $F \cong \mathbb{Z}_{pq}$ and X/F is a permutation group of prime degree p. By Lemma 2.9, either $X/F \leqslant \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ is an affine group or X/F is 2-transitive on $V\Gamma_F$. For the former case, Γ is an X-normal edge-transitive Cayley graph of a group isomorphic to $\mathbb{Z}_{pq}{:}\mathbb{Z}_p$, which is not possible by Lemma 2.4. Thus, X/F is 2-transitive on $V\Gamma_F$ and so $\Gamma_F = \mathbf{K}_p$. Since $\mathrm{val}(\Gamma_F) = 4$, $p = 5$. As $X/F \leqslant \mathrm{Aut}(\mathbf{K}_5) = \mathrm{S}_5$ is insoluble, we have $X \cong \mathbb{Z}_{5q}.\mathrm{A}_5 = \mathbb{Z}_{5q} \times \mathrm{A}_5$ or $\mathbb{Z}_{5q}.\mathrm{S}_5 = (\mathbb{Z}_{5q} \times \mathrm{A}_5).\mathbb{Z}_2$, and $X_\alpha \cong \mathrm{A}_4$ or S_4, respectively. Hence X always has a normal subgroup Y isomorphic to A_5, and Γ is $(X, 2)$-arc-transitive by Lemma 2.3. If Y has at least three orbits on $V\Gamma$, then Theorem 2.5 implies that Y is semiregular on $V\Gamma$, which is not possible as $|V\Gamma|$ is odd. Thus, Y is transitive on $V\Gamma$. However, as $|V\Gamma| = 25q$ does not divide $|Y| = 60$, this is also not possible.

For the case that $|F| = p^2$, arguing as above, we have that $q = 5$, $\Gamma_F \cong \mathbf{K}_5$ and $X/F = \mathrm{A}_5$ or S_5. Let $Z = F.\mathrm{A}_5 \leqslant X$. By Lemma 2.11, $Z = F.\mathrm{A}_5$ is also a central extension. Then, one may easily draw a contradiction with the same discussion as above.

Finally, suppose $|F| = p$. Then Γ_F is a tetravalent X/F-edge-transitive graph of order $|V\Gamma_F| = pq$. So Γ_F satisfies Table 1 in Lemma 2.6. If $pq = 15$ or 21, then $|V\Gamma| = 45, 63, 75$ or 147, (1) of Lemma 5.1 holds.

We consider Rows 3–5 of Table 1. If $pq = 35$ as in Row 3, then $X/F \leqslant \mathrm{Aut}(\Gamma_F) = \mathrm{S}_7$. Noting that X/F is insoluble, and since X/F is edge-transitive on $V\Gamma_F$, $70 \mid |X/F|$, we conclude from [5] that $X/F \geqslant \mathrm{A}_7$. Similarly, for the cases where $pq = 55$ or 253, as in Rows 4 or 5, we have $X/F \geqslant \mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 23)$, respectively.

Now, let $T = \mathrm{soc}(X/F)$ and let Q be a normal subgroup of X such that $Q/F = T$. Then $T = \mathrm{A}_7, \mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 23)$. Since $\mathrm{Out}(T) = \mathbb{Z}_2$ for each T, $X = Q$ or $Q.\mathbb{Z}_2$, it is then easy to show that Γ is Q-edge-transitive. Furthermore, as $|X_\alpha{:}Q_\alpha| \leqslant 2$, Q_α is not a 2-group, then Lemma 2.3 implies that Γ is $(Q, 2)$-arc-transitive. Thus by Lemma 2.2, we may suppose $\Gamma = \mathrm{Cos}(Q, Q_\alpha, Q_\alpha f Q_\alpha)$ for some 2-element f. Because $F \cong \mathbb{Z}_p$, $Q = F.T$ is a central extension. Since $p \geqslant 5$, $\mathrm{Multi}(\mathrm{A}_7) = \mathbb{Z}_6$ and $\mathrm{Multi}(\mathrm{PSL}(2, 11)) = \mathrm{Multi}(\mathrm{PSL}(2, 23)) = \mathbb{Z}_2$, by Lemma 2.11, we have $Q = F \times Q'$ and $Q' \cong T$. This implies that each element of $Q \setminus Q'$ has order a multiple of p. Now, as $p \geqslant 5$, Q_α is a $\{2, 3\}$-group, and f is a 2-element, we conclude that $Q_\alpha \subseteq Q'$ and $f \in Q'$. It follows that $\langle Q_\alpha, f\rangle \subseteq Q' \neq Q$, which contradicts the connectivity of Γ. □

The case where X is soluble is considered by the next lemma.

Lemma 5.2. *Suppose that X is soluble. Then Γ is an X-normal edge-transitive Cayley graph.*

Proof. Since X_α is a $\{2, 3\}$-group, $|X| = 2^i 3^j p^2 q$ for some integers i, j. Let F be the Fitting subgroup of X. By Lemma 2.8, $F \neq 1$, $C_X(F) \leqslant F$, and $F = O_p(X) \times O_q(X)$, where $O_p(X)$ and $O_q(X)$ denote the largest normal p-subgroup and q-subgroup of X, respectively. By Lemma 4.3, F is semiregular on $V\Gamma$ and hence $|F|$ divides $p^2 q$. In particular, F is abelian and $C_X(F) = F$.

If $F \cong \mathbb{Z}_p$, then $X/F \leqslant \mathrm{Aut}(F) \cong \mathbb{Z}_{p-1}$, it follows that p^2 does not divide $|X|$, which is not possible.

Assume $|O_p(X)| = p^2$. Then the quotient graph $\Gamma_{O_p(X)}$ is an X/K-edge-transitive graph of order q, where K is the kernel of X acting on $V\Gamma_{O_p(X)}$. Then $K = O_p(X)K_\alpha$. Suppose $\mathrm{val}(\Gamma_{O_p(X)}) = 4$. Then K_α fixes each neighbour of α in Γ as K fixes each orbit of $O_p(X)$ in Γ. Then connectivity of Γ implies that $K_\alpha = 1$ and so $K = O_p(X)$. As $|V\Gamma_{O_p(X)}| = q$ is a prime, by Lemma 2.9, either $X/O_p(X) \leqslant \mathbb{Z}_q{:}\mathbb{Z}_{q-1}$ or $X/O_p(X)$ is 2-transitive on $V\Gamma_{O_p(X)}$. For the former case, X has a normal subgroup isomorphic to $O_p(X).\mathbb{Z}_q$ which is regular on $V\Gamma$, Γ is an X-normal edge-transitive Cayley graph. For the latter case, we have $\Gamma_{O_p(X)} = \mathbf{K}_q$, then as $\mathrm{val}(\Gamma_{O_p(X)}) = 4$, it follows that $q = 5$ and $X/O_p(X) \leqslant \mathrm{S}_5$. Now since $X/O_p(X)$ is soluble, q divides $|X/O_p(X)|$ and $X/O_p(X)$ is transitive on $E\Gamma_{O_p(X)}$, we conclude that $X/O_p(X) = \mathbb{Z}_5{:}\mathbb{Z}_2$ or $\mathbb{Z}_5{:}\mathbb{Z}_4$. It follows from Lemma 2.10 that X has a normal subgroup which is isomorphic to $O_p(X).\mathbb{Z}_5$ and regular on $V\Gamma$. So Γ is an X-normal edge-transitive Cayley graph. Suppose now $\mathrm{val}(\Gamma_{O_p(X)}) = 2$. Then $\Gamma_{O_p(X)}{:} = \{B_1, B_2, \ldots, B_q\}$ is a cycle of length q, where B_i is adjacent to B_{i+1} in $\Gamma_{O_p(X)}$ for $1 \leqslant i \leqslant q-1$, so the induced subgraph $[B_i, B_{i+1}]$ is a cycle of length $2p^2$. This implies that $K_\alpha \leqslant \mathbb{Z}_2$, $K = O_p(X)$ or $O_p(X).\mathbb{Z}_2$, and $X \leqslant K.\mathrm{Aut}(\Gamma_{O_p(X)}) = K.D_{2q}$. It follows that X has a normal Hall $\{p, q\}$-subgroup which is regular on Γ, hence Γ is also an X-normal edge-transitive Cayley graph. Assume next that $F \cong \mathbb{Z}_{pq}$. Then Γ_F is an edge-transitive graph of prime order p. With the same discussion as above, one may prove that Γ is an X-normal edge-transitive Cayley graph isomorphic to $\mathbb{Z}_{pq}.\mathbb{Z}_p$.

Finally, we treat the remaining case where $F \cong \mathbb{Z}_q$. Then as p^2 divides $|X|$ and $X/F \leqslant \mathrm{Aut}(F) \cong \mathbb{Z}_{q-1}$, we have $p^2 \mid (q-1)$. If $p=3$, Γ_F is an edge-transitive graph of order 9. By [20], $\Gamma_F = DW(3,3)$ is a deleted wreath graph, and $\mathrm{Aut}(\Gamma_F) = \mathbb{Z}_3^2.D_8$. It then easily follows that Γ is an X-normal Cayley graph of a group isomorphic to $\mathbb{Z}_q.\mathbb{Z}_3^2$.

Suppose $p \geqslant 5$. Let H be a Hall $\{p,q\}$-subgroup of X. Then $|H| = p^2q$ and $(|V\Gamma|, |X:H|)=1$, by Lemma 2.10, H is regular on $V\Gamma$, that is, Γ is a Cayley graph of H. Furthermore, since $F < H$ and $X/F \leqslant \mathbb{Z}_{q-1}$ is abelian, we have $H/F \lhd X/F$, and so $H \lhd X$. Hence Γ is an X-normal edge-transitive Cayley graph of H. □

A census (not necessarily complete) of tetravalent edge-transitive graphs of order up to 150 is presented in [32]. By using [3], we confirmed that the items of the order $45, 63, 75$ or 147 in the census are complete.

Now, we are ready to prove the main result of this paper.

Theorem 5.3. *Let Γ be a tetravalent X-edge-transitive graph of order p^2q, where $X \leqslant \mathrm{Aut}\Gamma$ and p,q are distinct odd primes. Then one of the following statements holds*:

(1) Γ *is of order* $45, 63, 75$ *or* 147*, given in* [32]*. In particular, there are exactly* 17 *pairwise non-isomorphic graphs in this case*;

(2) $\Gamma \cong \mathcal{G}_{153}$*, given in Example* 3.1;

(3) $\Gamma = \mathrm{Cay}(G,S)$ *is an X-normal edge-transitive Cayley graph, and either*

(i) Γ *is* $(X,1)$*-transitive, and* $S = \{a, a^{\sigma}, a^{\sigma^2}, a^{\sigma^3}\}$*, where* $\sigma \in \mathrm{Aut}(G)$ *is of order* 4; *or*

(ii) $X_{\mathbf{1}} \leqslant \mathbb{Z}_2^2$ *and* $S = \{a, a^{\tau}, a^{-1}, (a^{-1})^{\tau}\}$*, where* $\tau \in \mathrm{Aut}(G)$ *is an involution.*

Proof. If X is soluble, by Lemma 5.2, Γ is an X-normal edge-transitive Cayley graph, it then follows from Lemma 3.2 that part (3) of Theorem 5.3 holds. Suppose that X is insoluble. Then by Lemma 5.1, either $|V\Gamma| = 45, 63, 75$ or 147, or X is almost simple. For the former, by [32] and the remark before Theorem 5.3, part (1) of Theorem 5.3 holds. Let $\alpha \in V\Gamma$. Then $|X| = |X_\alpha| \cdot p^2q$.

Assume $|V\Gamma| \neq 45, 63, 75$ or 147 in the following. Then $T := \mathrm{soc}(X)$ is a nonabelian simple group, and $|T{:}T_\alpha|$ divides p^2q. If T_α is a 2-group, then T is a $\{2,p,q\}$-group. By the remark on Lemma 4.4, either $T = \mathrm{PSU}(3,3)$ or $\mathrm{PSU}(4,2)$, or T lies in Table 2. If $T = \mathrm{PSU}(3,3)$, then $|T| = 2^5 \cdot 3^3 \cdot 7$, so $3 \mid |T_\alpha|$, a contradiction. Similarly, one may prove that $T \neq \mathrm{PSU}(4,2)$. Thus, T lies in Table 2 in this case. If T_α is not a 2-group, so is X_α and hence $|X_\alpha| \mid 144$ by Lemma 2.3. This implies $|T| \mid 144p^2q$, hence T also lies in Table 2 by Lemma 4.4.

We consider all the candidates in Table 2. If $T = \mathrm{PSL}(2,8)$, then $X = \mathrm{PSL}(2,8)$ or $\mathrm{PSL}(2,8).\mathbb{Z}_3$, and $|X| = 2^3 \cdot 3^2 \cdot 7$ or $2^3 \cdot 3^3 \cdot 7$, respectively. This implies $(p,q) = (3,7)$ and $|V\Gamma| = 63$, not the case. Suppose $T \neq \mathrm{PSL}(2,8)$. By Table 2, we always have $(|\mathrm{Out}(T)|, pq) = 1$, then since $X \leqslant T.\mathrm{Out}(T)$, we conclude $(p^2q, |X{:}T|) = 1$. It then follows from Lemma 2.10 that T is transitive on $V\Gamma$. So $|T:T_\alpha| = p^2q$, i.e., T has a subgroup with index p^2q.

We claim that $T \neq \mathrm{PSL}(2,r)$, in Column 4 of Table 2. In fact, if $T = \mathrm{PSL}(2,r)$, then $r = q$, and p^2 divides $\frac{q+1}{2}$ or $\frac{q-1}{2}$. If p^2 divides $\frac{q+1}{2}$, then $q-1$ divides $|T_\alpha|$, so $T_\alpha = D_{q-1}$ as $\mathrm{PSL}(2,q)$ has no subgroup with order a proper multiple of $(q-1)$. Now, as $\mathrm{Out}(T) = \mathbb{Z}_2$, $X \leqslant T.\mathbb{Z}_2 = \mathrm{PGL}(2,q)$, we have $X_\alpha = D_{q-1}$ or $D_{2(q-1)}$. On the other hand, since T has four prime factors, T_α is not a 2-group, it follows that Γ is $(X,2)$-arc-transitive by Lemma 2.3. Hence X_α acts 2-transitive on $\Gamma(\alpha)$, which is no possible by Lemma 4.1. Similarly, one may also draw a contradiction if p^2 divides $\frac{q-1}{2}$. Thus the claim is true. Furthermore, if T lies in Column 4 of Table 2, as $p^2q \mid |T|$, the only possibility is $T = \mathrm{PSL}(2,25)$.

Hence, either $T = \mathrm{PSL}(2,25)$, or T lies in Column 1 of Table 2 but $T \neq \mathrm{PSL}(2,8)$. Because T has a subgroup T_α with index p^2q, by [5], the following are all the possibilities of couple (T, T_α):

T	$\mathrm{PSL}(2,17)$	$\mathrm{PSL}(2,25)$	$\mathrm{PSL}(3,3)$
T_α	D_{16}	S_4 or D_{24}	$2.\mathrm{S}_4$

If $T = \mathrm{PSL}(3,3)$, then as $\mathrm{Out}(T) = \mathbb{Z}_2$, $2.\mathrm{S}_4 \leqslant X_\alpha \leqslant 2.\mathrm{S}_4.\mathbb{Z}_2$, so $|X_\alpha| = 48$ or 96, which is not possible by Lemma 2.3. If $T = \mathrm{PSL}(2,25)$, then $T_\alpha \cong \mathrm{S}_4$ by Lemma 2.3. Using [3], one may easily check out that no example occurs. Thus, $T = \mathrm{PSL}(2,17)$ and $T_\alpha = D_{16}$. By Example 3.1, $\Gamma = \mathcal{G}_{153}$, part (2) of Theorem 5.3 holds. This completes the proof. □

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