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正曲率齐性Finsler空间的分类:偶数维情形下的一种新方法*

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摘要:本文介绍了正曲率齐性Finsler流形的分类。在偶数维的情形下,给出了一种新方法,证明了偶数维光滑陪集空间上有正曲率齐性Finsler度量,当且仅当其上面有正曲率齐性黎曼度量。

关键词:旗曲率;齐性Finsler空间;不变Finsler度量;正曲率Finsler流形

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Classification of positively curved homogeneous Finsler spaces: a new approach in the even dimensional case*

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Abstract: In this paper, we briefly survey the classification of positively curved homogeneous Finsler manifolds. We propose a new method in the even dimensional case, which proves that an even dimensional smooth coset space admits positively curved homogeneous Finsler metrics if and only if it admits positively curved homogeneous Riemannian metrics.

Keywords: flag curvature; homogeneous Finsler space; invariant Finsler metric; positively curved Finsler manifold

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0 Introduction

0.1 Survey on the classification of positively curved homogeneous Finsler manifolds

The study on compact Riemannian manifolds with positive sectional curvature is one of the hottest projects in geometry and topology. The known examples are relatively rare and most of them are diffeomorphic to homogeneous manifolds. The classification of positively curved homogeneous Riemannian manifolds was accomplished in the 1960's and 1970's[1-4]. See references [5-6] for some minor corrections and [7] for a modern proof.

In Finsler geometry, sectional curvature is naturally generalized to flag curvature. So it is important and natural to classify positively curved homogeneous Finsler manifolds, i.e., homogeneous Finsler manifolds with positive flag curvature. Generally

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speaking, curvatures in Finsler geometry are hard to calculate, so this project has not been touched for many years. After 2014, researchers found many crucial techniques in homogeneous Finsler geometry, including the homogeneous curvature formula[8-10], the submersion technique[11-12], totally geodesic technique[13], etc. Since then, there are many progresses on the classification of positively curved homogeneous Finsler manifolds.

Using the Finsler submersion, normal homogeneity and generalized normal homogeneity (i.e., δ-homogeneity) can be defined in Finsler geometry[11,14]. All (generalized) normal homogeneous Finsler manifolds have non-negative flag curvature and zero S-curvature. In [11, 14], the authors proved that a smooth coset space admits positively curved (generalized) normal homogeneous Finsler metrics if and only if it admits positively curved normal homogeneous Riemannian metrics. So the classification list for positively curved (generalized) normal homogeneous Finsler manifolds coincides with the following Berger's list[1]:

(1) Compact rank-one symmetric spaces, $S^n = SO(n+1)/SO(n)$ with $n>1$, $CP^n = SU(n+1)/S(U(n)U(1))$ with $n>0$, $HP^n = Sp(n+1)/Sp(n)Sp(1)$ with $n>0$, and $OP^2 = F_4/Spin(9)$;

(2) Other homogeneous spheres and complex projective spaces, $S^{2n-1} = SU(n)/SU(n-1)$ with $n>1$, $S^{2n-1} = U(n)/U(n-1)$ with $n>1$, $S^{4n-1} = Sp(n)/Sp(n-1)$ with $n>0$, $S^{4n-1} = Sp(n)U(1)/Sp(n-1)U(1)$ with $n>0$, $S^{4n-1} = Sp(n)Sp(1)/Sp(n-1)Sp(1)$ with $n>0$, $S^6 = G_2/SU(2)$, $S^7 = Spin(7)/G_2$, $S^{15} = Spin(9)/Spin(7)$, and $CP^{2n-1} = Sp(n)/Sp(n-1)U(1)$;

(3) Two Berger spaces $SU(5)/Sp(2)U(1)$ and $Sp(2)/SU(2)$;

(4) $SU(3)\times SO(3)/U(2)$ (in Riemannian geometry, it was missed in reference [1] and added by Wilking[5]).

Notice that this list is only complete up to local isometries. For example, $RP^n = SO(n+1)/O(n)$ does not show up because it is locally isometric to the symmetric S^n.

Using a homogeneous flag curvature formula (see Theorem 0.2 below), Xu et al[12] proved that an even dimensional smooth coset space admits positively curved homogeneous Finsler metrics if and only if it admits positively curved homogeneous Riemannian metrics. So the classification list for even dimensional positively curved homogeneous Finsler manifolds coincides with Wallach's list[2], which consists of those in Berger's list with even dimensions, and three Wallach spaces, $SU(3)/T^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$.

The odd dimensional case is the hardest. Until now, there are only some partial results. If we require the metric to be reversible, i.e., any tangent vector has the same length as its opposite, an odd dimensional smooth coset space admits positively curved invariant reversible Finsler metrics if and only if one of the following possibilities happens[13,15]. Either it admits positively curved homogeneous Riemannian metrics besides those in Berger's list with odd dimensions, it could be an Aloff-Wallach space $SU(3)/S^1$[3], or it belongs to a short list of undetermined candidates[13]. If we require its S-curvature to be vanishing, then all non-Riemannian homogeneous Randers and (α,β) manifolds with positive flag curvature and zero S-curvature can be classified[16-17].

0.2 A new approach in the even dimensional case

In this paper, we will provide a new proof for the following main theorem in reference [12]:

Theorem 0.1 A smooth even dimensional coset space admits positively curved homogeneous Finsler metrics if and only if it admits positively curved homogeneous Riemannian metrics.

When the authors of [12] proved Theorem 0.1, they used the following crucial homogeneous flag curvature formula.

Theorem 0.2[12] Let $(G/H, F)$ be a homogeneous Finsler manifold, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition for G/H. Then for any linearly independent commuting pair $u, v \in \mathfrak{m} = T_{eH}(G/H)$, such that $g_u(u, [u, \mathfrak{m}]_{\mathfrak{m}}) = 0$, the flag curvature

$K(o,u,u\wedge v)$ satisfies

$$K(o,u,u\wedge v)=\frac{g_u(U(u,v),U(u,v))}{g_u(u,u)g_u(v,v)-g_u(u,v)^2},$$

in which g_u is the fundamental tensor for the Minkowski norm $F=F(eH,\cdot):\mathfrak{m}\to\mathbf{R}$, and $U(u,v)$ is determined by $2g_u(U(u,v),w)=g_u([u,w]_{\mathfrak{m}},v)+g_u(u,[v,w]_{\mathfrak{m}})$, $\forall\, w\in\mathfrak{m}$.

Using Theorem 0.2, the authors of [12] proved the following lemma.

Lemma 0.3[12] Suppose $(G/H,F)$ is an even dimensional positively curved homogeneous Finsler space with compact G and H, then there do not exist a pair of linearly independent roots α and β of $\mathfrak{g}$, such that α and β are not roots of $\mathfrak{h}$, and $\alpha\pm\beta$ are not roots of $\mathfrak{g}$.

Wallach called the root pair in Lemma 0.3 strongly orthogonal. All compact G/H without strongly orthogonal root pairs are classified in [2], which produces Wallach's list. To summarize, whenever Lemma 0.3 is proved, Theorem 0.2 follows after it immediately.

It is not easy task to prove Theorem 0.2 directly in [12], or to prove Huang's homogeneous curvature formula in [8]. So we propose a easier shortcut to Lemma 0.3, which involves the following observations. Firstly, the flag curvature for a symmetric Finsler space is easy to calculate[18]. In particular, a symmetric Finsler $S^2\times S^2$ is not positively curved. Secondly, if Lemma 0.3 is not valid, then we can find a homogeneous totally geodesic submanifold G'/H' of $(G/H,F)$. The submanifold $(G'/H',F)$ itself is a positively curved 4-dimensional homogeneous Finsler space. Thirdly, we use the following theorem.

Theorem 0.4 Suppose that two homogeneous Finsler manifolds $(G_1/H_1,F_1)$ and $(G_2/H_2,F_2)$ have the same reductive decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, in which $\mathfrak{g}=\mathrm{Lie}(G_1)=\mathrm{Lie}(G_2)$ and $\mathfrak{h}=\mathrm{Lie}(H_1)=\mathrm{Lie}(H_2)$, such that F_1 and F_2 induce the same Minkowski norm F on $\mathfrak{m}$, then $(G_1/H_1,F_1)$ and $(G_2/H_2,F_2)$ are locally isometric.

Analysis the roots of G'/H', and apply Theorem 0.4, we see that $(G'/H',F)$ is locally isometric to a symmetric Finsler $S^2\times S^2$, which is not positively curved. The contradiction is found.

This paper is scheduled as follows. In Section 1, we summarize some necessary knowledge for later discussion. In Section 2, we prove Theorem 0.4 and then use it to prove Lemma 0.3.

1 Preliminaries

1.1 Minkowski norm and Finsler metric

A Minkowski norm on a real vector space V is a continuous function $F:V\to[0,+\infty)$ which meets the following requirements:

(1) F is positive and smooth on $V\setminus\{0\}$;

(2) $F(\lambda y)=\lambda F(y)$, $\forall\lambda\geq 0$, $y\in V$;

(3) The fundamental tensor $g_y(u,v)=\frac{1}{2}\left[F^2(y+su+tv)\right]_{st}\Big|_{s=t=0}$ is positive definite for any $y\in V\setminus\{0\}$.

A Minkowski norm is Euclidean if and only if its Cartan tensor $C_y(u,v,w)=\frac{1}{2}\left[g_{y+tw}(u,v)\right]_t\Big|_{t=0}$ vanishes identically.

A Finsler metric on a smooth manifold M is a continuous function $F:TM\to[0,+\infty)$, such that the restriction of F to $TM\backslash 0$ is smooth and $F(x,\cdot)$ is a Minkowski norm for each $x\in M$. Riemannian metrics are an important subclass of Finsler metrics, which only involves Euclidean metrics. That means, a Finsler metric is Riemannian if and only if its Cartan tensor vanishes everywhere.

Let (M,F) be a Finsler manifold. Suppose that we have $x\in M$, $y\in T_xM\backslash\{0\}$, and a tangent plane $P=\mathrm{span}\{y,u\}\subset T_xM$. Then the flag curvature $K(x,y,P)$ (or $K(x,y,y\wedge u)$) for the flag triple (x,y,P) is defined as

$$K(x,y,P)=\frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u)-g_y(u,y)^2}.$$

When F is Riemannian, $K(x,y,P)$ is the sectional curvature $K(x,P)$. Here R_y is the Riemann curvature operator. See reference [19] for its formula and more details.

1.2 Homogeneous Finsler manifold and reductive decomposition

A connected Finsler manifold (M, F) is called homogeneous if its isometry group $I(M, F)$ acts transitively on M. Since $I(M, F)$ is a Lie transformation group, we may present M as G/H, in which G is any connected Lie subgroup of $I(M, F)$ which acts transitively on M.

For a homogeneous Finsler manifold $(G/H, F)$, we can always find an $\mathrm{Ad}(H)$-invariant linear decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, in which $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. We call it a reductive decomposition. Usually, we only use the $\mathrm{Ad}(H)$-invariance of a reductive decomposition in the Lie algebra level, i.e., $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The subspace $\mathfrak{m}$ can be naturally identified with the tangent space $T_o(G/H)$ at $o = eH$, such that the $\mathrm{Ad}(H)$-action on $\mathfrak{m}$ coincides with the isotropic H-action on $T_o(G/H)$. The G-invariant Finsler metric F is one-to-one determined by its restriction to $T_o(G/H)$, which is any arbitrary $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$. See reference [18] for more details.

1.3 Symmetric Finsler space and its flag curvature

A connected Finsler manifold (M, F) is called symmetric if for each $x \in M$, there exists an isometry s_x on (M, F), such that $s_x(x) = x$ and the tangent map $(s_x)_*: T_xM \to T_xM$ is $-\mathrm{id}$. It can be presented as a homogeneous Finsler space $M = G/H$ with $G = I(M, F)$, and it has a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ i.e., a reductive decomposition satisfying $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Theorem 5.5 in reference [18] can be reformulated as follows.

Theorem 1.1 Let G/H be a homogeneous manifold with a connected and simply connected G and a connected H. Suppose that it has a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then any $\mathrm{Ad}(H)$-invariant reversible Minkowski norm on $\mathfrak{m}$ induces a symmetric Finsler metric on G/H.

Example 1.2 $S^2 \times S^2 = G/H = (SU(2) \times SU(2))/(U(1) \times U(1))$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ which is orthogonal with respect to the Killing form of $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_1$. Here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{m} = \mathfrak{g}_{\pm\alpha} + \mathfrak{g}_{\pm\beta}$ is the linear direct sum of two commuting root planes. Any $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$ is reversible because it is an (α_1, α_2) norm[19], so by Theorem 1.1, any G-invariant Finsler metric on $S^2 \times S^2$ is symmetric.

The flag curvature for a Finsler symmetric space has a simple formula.

Lemma 1.3[18] Let $(\mathfrak{g}, \sigma, F_0)$ be a Minkowski symmetric Lie algebra and (G, H) a pair associated with $(\mathfrak{g}, \sigma)$. Suppose there exists an invariant Finsler metric F on G/H such that the restriction of F to $\mathfrak{m}$ is F_0. Then the curvature tensor of F is given by

$$R_0(u, v)w = -[[u, v], w], \forall u, v, w \in \mathfrak{m},$$

and the flag curvature of the flag (o, y, P), $y \neq 0$, $y \in P$, is given by

$$K(o, y, P) = g_l([[l, v], l], v),$$

where $l = y/F$ (the distinguished section) and l, u is an orthonormal basis of the plane P with respect to g_l.

As a direct corollary of this theorem, only those which are of compact type and rank one are positively curved. So we see the following lemma.

Lemma 1.4 Any homogeneous Finsler manifold which is locally isometric to a Finsler symmetric $S^2 \times S^2 = (SU(2) \times SU(2))/(U(1) \times U(1))$ is not positively curved.

1.4 Totally geodesic submanifold

A submanifold N in a Finsler manifold (M, F) is called totally geodesic, if for any $x \in N$ and $y \in T_xN \backslash \{0\}$, the geodesic $c(t)$ on (M, F) satisfying $c(0) = x$ and $\dot{c}(0) = y$ is contained in N (at least for t sufficiently close to 0). Denote by $F|_N$ or simply F, the induced Finsler metric on N. The flag curvature K^F for (M, F) and the flag curvature $K^{F|_N}$ for a totally geodesic $(N, F|_N)$ have the following relation (see Proposition 2.2 in reference [11]).

Lemma 1.5 For any $x \in N$, $y \in T_xN \backslash \{0\}$ and tangent plane $P \subset T_xN$ containing y, $K^{F|_N}(x, y, P) = K^F(x, y, P)$.

The common fixed point set $\mathrm{Fix}(S, M)$ of a family isometries S on (M, F) is a closely imbedded totally geodesic submanifold. In particular, when (M, F) is a compact homogeneous Finsler manifold, we have the following consequence of Corollary II.5.7 in reference [19].

Lemma 1.6 Let $(G/H, F)$ be a homogeneous Finsler manifold with compact G and H, K a closed subgroup in H and G' the identity component of the normalizer $N_G(K)$. Then connected component $\mathrm{Fix}_o(K, G/H)$ of the fixed point set $\mathrm{Fix}(K, G/H)$, which contains $o = eH$, is the homogeneous totally geodesic submanifold $G' \cdot o = G'/(G' \cap H)$.

2 Proofs of the main results

2.1 Proof of Theorem 0.4

Without loss of generality, we may assume G_1 and G_2 are connected Lie groups.

Firstly, we prove Theorem 0.4 when G_2/H_2 is a universal cover of G_1/H_1. To be more precise, we assume that G_2 is the universal cover of G_1, and denote by π the covering map from G_2 to G_1, which is a Lie group homomorphism. For G_2/H_2 to be simply connected, H_2 must be the identity component of $\pi^{-1}(H_1)$. Then π induces a smooth covering map $\bar{\pi}$: $G_2/H_2 \to G_1/H_1$, $\bar{\pi}(g_2 H_2) = \pi(g_2) H_1$. We will prove $\bar{\pi}^* F_1 = F_2$, i.e., $\bar{\pi}$ is a local isometry.

Since both tangent spaces at $o_1 = e_1 H_1 \in G_1/H_1$ and $o_2 = e_2 H_2 \in G_2/H_2$ are identified as $\mathfrak{m}$, the tangent map $\bar{\pi}_*$: $\mathfrak{m} = T_{o_2}\left(G_2/H_2\right) \to T_{o_1}\left(G_1/H_1\right) = \mathfrak{m}$ becomes the identity map.

So for any $u \in \mathfrak{m} = T_{o_2}\left(G_2/H_2\right)$, we have $F_1(\bar{\pi}_*(u)) = F(u) = F_2(u)$. That means $\bar{\pi}^* F_1|_{T_{o_1}(G_1/H_1)} = F_2|_{T_{o_2}(G_2/H_2)}$, i.e., the local isometry is valid at the origin.

Since π is a Lie group homomorphism, we have

$$\bar{\pi} \circ g_2 = \pi(g_2) \circ \bar{\pi} \text{ is valid everywhere on } G_2/H_2, \quad (1)$$

for any $g_2 \in G_2$. In equation(1), we have used g_2 and $\pi(g_2)$ to denote their actions on G_2/H_2 and G_1/H_1 respectively. The invariance of F_1 and F_2 provides

$$F_2|_{T_{g_2^{-1}H_2}(G_2/H_2)} = g_2^*(F_2|_{T_{o_2}(G_2/H_2)})$$

and

$$\pi(g_2)^*(F_1|_{T_{o_1}(G_1/H_1)}) = F_1|_{T_{\pi(g_2^{-1})H_1}(G_1/H_1)}.$$

So we may differentiate equation(1) and get

$$F_2|_{T_{g_2^{-1}H_2}(G_2/H_2)} = g_2^*(F_2|_{T_{o_2}(G_2/H_2)}) = g_2^*(\bar{\pi}^*(F_1|_{T_{o_1}(G_1/H_1)})) = \bar{\pi}^*(\pi(g_2)^*(F_1|_{T_{o_1}(G_1/H_1)})) = \bar{\pi}^*(F_1|_{T_{\pi(g_2^{-1})H_1}(G_1/H_1)}),$$

i.e., $\bar{\pi}$ is a local isometry at each $g_2 H_2 \in G_2/H_2$. This proves Theorem 0.4 when G_2/H_2 is the universal cover of G_1/H_1.

Nextly, we prove Theorem 0.4 in the general situation.

We denote by $\tilde{G}_1$ the universal cover of G_1, by π_1: $\tilde{G}_1 \to G_1$ the corresponding covering map, which is a Lie group homomorphism, and by $\tilde{H}_1$ the identity component of $\pi^{-1}(H_1)$. So $\tilde{H}_1$ is a closed connected Lie subgroup generated by the Lie subalgebra $\mathfrak{h}$ in $\tilde{G}_1$. Then $\tilde{G}_1/\tilde{H}_1$ is the universal cover of G_1/H_1.

Obviously, we have $\mathrm{Lie}(\tilde{G}_1) = \mathrm{Lie}(G_1) = g$, $\mathrm{Lie}(\tilde{H}_1) = \mathrm{Lie}(H_1) = \mathfrak{h}$. Since we have $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $\tilde{H}_1$ is connected, the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is $\mathrm{Ad}(\tilde{H}_1)$-invariant, i.e., it is a reductive decomposition for $\tilde{G}_1/\tilde{H}_1$. The Ad $(\tilde{H}_1)$-invariance of the Minkowski norm F on $\mathfrak{m}$ implies

$$g_y([w, u], v) + g_y(u, [w, v]) + 2C_y(u, v, [w, y]) = 0, \quad \forall y \in \mathfrak{m} \backslash \{0\}, u, v \in \mathfrak{m}, w \in \mathfrak{h},$$

in the Lie algebra level, which is equivalent to the Ad $(\tilde{H}_1)$-invariance of F. To summarize, the Minkowski norm F on $\mathfrak{m}$ induces a $\tilde{G}_1$-invariant Finsler metric $\tilde{F}_1$ on $\tilde{G}_1/\tilde{H}_1$. Previous argument proves that $(G_1/H_1, F_1)$ is locally isometric to $(\tilde{G}_1/\tilde{H}_1, \tilde{F}_1)$.

By similar argument, we can find a universal cover $\tilde{G}_2/\tilde{H}_2$ for G_2/H_2, and a $\tilde{G}_2$-invariant Finsler metric $\tilde{F}_2$ on $\tilde{G}_2/\tilde{H}_2$ which is locally isometric to F_2.

Finally, we have $\tilde{G}_1 = \tilde{G}_2$ because they are both connected and simply connected and have the same Lie algebra. We have $\tilde{H}_1 = \tilde{H}_2$ because they are connected Lie subgroups generated by the same Lie subalgebra. We have $\tilde{F}_1 = \tilde{F}_2$ because they are induced by the same Minkowski norm F, with respect to the same reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for $\tilde{G}_1/\tilde{H}_1 = \tilde{G}_2/\tilde{H}_2$. To summarize, $(G_1/H_1, F_1)$ and $(G_2/H_2, F_2)$ are locally isometric through $(\tilde{G}_1/\tilde{H}_1, \tilde{F}_1) = (\tilde{G}_2/\tilde{H}_2, \tilde{F}_2)$.

This ends the proof of Theorem 0.4.

2.2 Proof of Lemma 0.3

Let $(G/H, F)$ be a positively curved homogeneous Finsler manifold, where G and H are compact Lie groups. The dimension of G/H is even implies that rank G = rank H[12]. Then we can find a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, which is contained in $\mathfrak{h}$. We fix a $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle_{\mathrm{bi}}$ on $\mathfrak{g}$. For simplicity, we use $\langle \cdot, \cdot \rangle_{\mathrm{bi}}|_{\mathfrak{t}\times\mathfrak{t}}$ to identify root systems $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{h}}$ of $\mathfrak{g}$ and $\mathfrak{h}$ respectively as subsets of $\mathfrak{t}\backslash\{0\}$. We denote by $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the reductive decomposition for G/H, which is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathrm{bi}}$. It is compatible with the root plane decomposition $\mathfrak{g} = \mathfrak{t} + \sum_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_{\pm\alpha}$ in the sense that each root and root plane of $\mathfrak{h}$ are also root and root plane of $\mathfrak{g}$ respectively, and $\mathfrak{m}$ is the linear direct sum of those root planes $\mathfrak{g}_{\pm\alpha}$ with $\alpha \notin \Delta_{\mathfrak{g}}\backslash\Delta_{\mathfrak{h}}$.

Assume that there exist a pair of linearly independent roots α and β in $\Delta_{\mathfrak{g}}\backslash\Delta_{\mathfrak{h}}$, such that $\alpha \pm \beta$ are not roots of $\mathfrak{g}$. Denote $\mathfrak{t}'$ to be orthogonal complement of $R\alpha + R\beta$ in $\mathfrak{t}$ and denote T', the torus generated by $\mathfrak{t}'$. By Lemma 1.6, $\mathrm{Fix}_o(T', G/H) = G' \cdot o = G'/H' = G'/(G' \cap H)$, where $\mathfrak{g}' = \mathrm{Lie}(G') = \mathfrak{t}' \oplus \mathfrak{g}''$ with $\mathfrak{g}'' = R\alpha + R\beta + \mathfrak{g}_{\pm\alpha} + \mathfrak{g}_{\pm\beta}$, $\mathfrak{h}' = \mathrm{Lie}(H') = \mathfrak{t}' \oplus \mathfrak{h}''$ with $\mathfrak{h}'' = R\alpha + R\beta$. Let G'' be the connected subgroup generated by $\mathfrak{g}''$ in G. It is a compact Lie subgroup because its Lie algebra $\mathfrak{g}'' = \mathfrak{a}_1 \oplus \mathfrak{a}_1$ is compact semi-simple. Since $G' = G''T'$ and T' acts trivially on $\mathrm{Fix}_o(T', G/H)$, $\mathrm{Fix}_o(T', G/H) = G'' \cdot o = G''/H''$ where the Lie algebra of $H'' = G'' \cap H'$ is $\mathfrak{h}''$, and the submanifold metric $F|_{\mathrm{Fix}_o(T', G/H)}$ is G''-invariant.

Because $\mathfrak{g}'' = \mathfrak{a}_1 \oplus \mathfrak{a}_1$ and $\mathfrak{h}''$ is a Cartan subalgebra of $\mathfrak{g}''$, it has the same Cartan decomposition as $S^2 \times S^2 = (SU(2) \times SU(2))/(U(1) \times U(1))$ in Example 1.2. By Theorem 0.4 and Theorem 1.1, $F|_{\mathrm{Fix}_o(T', G/H)}$ is locally isometric to a Finsler symmetric metric on $S^2 \times S^2 = (SU(2) \times SU(2))/(U(1) \times U(1))$. By Lemma 1.3, $F|_{\mathrm{Fix}_o(T', G/H)}$ is not positively curved. This is a contradiction to Lemma 1.4 because $(G/H, F)$ is positively curved.

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