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Spanning 3-ended trees in k-connected $K_{1,4}$-free graphs

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Abstract A tree with at most m leaves is called an m-ended tree. Kyaw proved that every connected $K_{1,4}$-free graph with $\sigma_4(G) \geqslant n-1$ contains a spanning 3-ended tree. In this paper we obtain a result for k-connected $K_{1,4}$-free graphs with $k \geqslant 2$. Let G be a k-connected $K_{1,4}$-free graph of order n with $k \geqslant 2$. If $\sigma_{k+3}(G) \geqslant n+2k-2$, then G contains a spanning 3-ended tree.

Keywords spanning tree, degree sum, insertible vertex, segment insertion

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a set X, the cardinality of X is denoted by $|X|$. Given a vertex v in a graph G, we let $N_G(v)$ denote the set of neighbors of v in G and let $d_G(v)$ denote $|N_G(v)|$, the degree of v in G. For simplicity, we will write $N(v)$ and $d(v)$ instead of $N_G(v)$ and $d_G(v)$. Let H and K be subgraphs of G, we denote the neighourhood of H in G by $N(H)$ and define $N_K(H) = N(H) \cap V(K)$. $G[K]$ is the subgraph induced by $V(K)$ in G. We write $G-K$ instead of $G-V(K)$. Let $\omega(G)$ denote the number of components of G. A subset $U \subseteq V(G)$ is called an independent set of G if no two vertices of U are adjacent in G. We define $\sigma_k(G) = \min \{d(v_1)+\cdots+d(v_k) \mid \{v_1,\ldots,v_k\}$ is an independent set in $G\}$. Clearly, $\sigma_1(G) = \delta(G)$ is the minimum degree of G. A $K_{1,r}$-free graph is a graph without an induced $K_{1,r}$ subgraph.

Let $P[a,b]$ denote a path connecting a and b in G and define the orientation from a to b the positive direction of P. Given a vertex x on P, we let x^+ denote the successor of x and x^- denote the predecessor of x if they are well-defined. We let $x\overrightarrow{P}y$ denote the subpath from x to y and $y\overleftarrow{P}x$ denote the subpath from y to x on P. Let i and j be nonnegative intergers such that $i<j$, and we denote the integers from i to j by $[i,j]$. For further explanation of terminologies and notation, we refer to [3].

A tree with at most m leaves is called an *m-ended tree*. There are several well-known conditions ensuring that a graph G contains a spanning m-ended tree. Win [11] obtained a sufficient condition related to independent number for k-connected graph that confirms a conjecture of Las Vergnas. Broersma and Tuinstra [1] gave a degree sum condition for a spanning m-ended tree.

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Theorem 1.1 (See [11]). *Let $m \geqslant 2$ and let G be a k-connected graph. If $\alpha(G) \leqslant k+m-1$, then G has a spanning m-ended tree.*

Theorem 1.2 (See [1]). *Let $m \geqslant 2$ and let G be a connected graph of order $n \geqslant 2$. If $\sigma_2(G) \geqslant n-m+1$, then G has a spanning m-ended tree.*

On the other hand, Flandrin et al. [4] obtained a neighborhood union condition for a spanning m-ended tree. We denote the minimum order of the neighborhoods of an independent set of order m by $N_m(G)$.

Theorem 1.3 (See [4]). *Let $m \geqslant 2$ and let G be a connected graph of order n. If $N_m(G) > \frac{m}{m+1}(n-m)$, then G has a spanning m-ended tree.*

There are also several results for the claw-free graphs and $K_{1,4}$-free graphs, where a claw-free graph is a $K_{1,3}$-free graph. Matthews and Sumner [8] obtained a degree sum condition for a claw-free graph to have a hamiltonian path. Kano et al. [5] obtained a slightly stronger result than a generalization of it.

Theorem 1.4 (See [5]). *Let G be a connected claw-free graph of order n with $m \geqslant 2$. If $\sigma_{m+1}(G) \geqslant n-m$, then G has a spanning m-ended tree with the maximum degree at most 3.*

Recently, Kyaw [6, 7] presented some sharp sufficient conditions for a connected $K_{1,4}$-free to have a spanning m-ended tree.

Theorem 1.5 (See [6]). *Every connected $K_{1,4}$-free graph with $\sigma_4(G) \geqslant n-1$ contains a spanning 3-ended tree.*

Theorem 1.6 (See [7]). *Let G be a connected $K_{1,4}$-free graph.*

(i) *If $\sigma_3(G) \geqslant |G|$, then G has a hamiltonian path.*

(ii) *If $\sigma_{m+1}(G) \geqslant |G| - \frac{m}{2}$ for an integer $m \geqslant 3$, then G has a spanning m-ended tree.*

As for the spanning tree with certain extremal properties in a k-connected $K_{1,4}$-free graph, Chen and Schelp [2] gave some degree conditions for the hamiltonicity of a k-connected $K_{1,4}$-free graph where a hamiltonian path is just a spanning tree with two leaves.

Theorem 1.7 (See [2]). *Let G be a k-connected $K_{1,4}$-free graph of order $n \geqslant 3$. If $\sigma_{k+1}(G) \geqslant n+k$, then G is hamiltonian.*

Theorem 1.8 (See [2]). *Let G be a k-connected $K_{1,4}$-free graph of order n. If $\sigma_k(G) \geqslant n+k+1$, then G is hamiltonian-connected.*

Here, we give a degree sum condition assuring the existence of a spanning 3-ended tree in a k-connected $K_{1,4}$-free graph with $k \geqslant 2$.

Theorem 1.9. *Let G be a k-connected $K_{1,4}$-free graph of order n with $k \geqslant 2$. If $\sigma_{k+3}(G) \geqslant n+2k-2$, then G contains a spanning 3-ended tree.*

2 Preliminaries

According to the algorithm used by Zhang in [12], Chen and Schelp defined a concept, i.e., *insertible vertex*, in [2] as follows.

Let G be a non-hamiltonian connected graph and C be a maximal cycle of G with an orientation. Assume that H is a connected component of $G - V(C)$ and $\{v_1, v_2, \ldots, v_h\}$ are h vertices in $N_C(H)$ with $x_i v_i \in E(G)$, where $x_i \in V(H)$ for $1 \leqslant i \leqslant h$. We define the counter-clockwise direction the positive direction of C. We also assume that $v_1, v_2, \ldots, v_h$ are labeled in the order of the positive orientation of C, i.e., $v_i \in C(v_{i-1}, v_{i+1})$. The vertices $v_1, v_2, \ldots, v_h$ divide the cycle C into h segments, $Q_i = C(v_i, v_{i+1}]$ for $1 \leqslant i \leqslant h$, where the subscripts are taken modulo h.

A vertex w_i in Q_i is called an *insertible vertex* if there is a pair of consecutive vertices w and w^+ in $C - Q_i$ such that $w_i w$, $w_i w^+ \in E(G)$. If w_i is an insertible vertex, we define $I(w_i)$ as some fixed vertex in $C - Q_i$ such that $w_i I(w_i)$, $w_i(I(w_i))^+ \in E(G)$.

Motivated by the method of *segment insertion* used by Chen and Schelp in [2], we define *segment insertion with respect to* X as follows. Suppose that $X := \{x_1, \ldots, x_{\alpha_i}\}$, where $x_1, \ldots, x_{\alpha_i}$ are insertible vertices of Q_i in order along Q_i. Let β_1 be the largest integer in $[1, \alpha_i]$ such that $I(x_1) = I(x_{\beta_1})$, and let β_2 be the largest integer in $[\beta_1 + 1, \alpha_i]$ such that $I(x_{\beta_1+1}) = I(x_{\beta_2})$. Continuing in the same manner, we will have $\beta_t = \alpha_i$ for some $t \geqslant 1$. Then we insert the segment $C[x_1, x_{\beta_1}]$ between $I(x_1)$ and $(I(x_1))^+$, the segment $C[x_{\beta_1+1}, x_{\beta_2}]$ between $I(x_{\beta_1+1})$ and $(I(x_{\beta_1+1}))^+$,..., the segment $C[x_{\beta_{t-1}+1}, x_{\beta_t}]$ between $I(x_{\beta_{t-1}+1})$ and $(I(x_{\beta_{t-1}+1}))^+$. We call such an insertion a *segment insertion with respect to* X and denote it by $SI[Q_i, X]$. After such an insertion, we obtain a path $P'[x^+_{\alpha_i}, x^-_1]$ in G such that $V(P') \supseteq V(C[x^+_{\alpha_i}, x^-_1]) \cup X$. We denote $SI[Q_i, X]$ by $SI[C[x_1, x_{\alpha_i}]]$ if $X = V(C[x_1, x_{\alpha_i}])$.

In [2], Chen and Schelp gave the following two lemmas.

Lemma 2.1 (See [2]). *For each Q_i, there is a non-insertible vertex in $Q_i - \{v_{i+1}\}$.*

For each Q_i, let w_i be the first noninsertible vertex in $Q_i - \{v_{i+1}\}$. Then the following lemma holds.

Lemma 2.2 (See [2]). *Let $1 \leqslant i < j \leqslant h$. Then for $u_i \in C[v^+_i, w_i]$ and $u_j \in C[v^+_j, w_j]$, the following properties hold*:

(i) *There does not exist a path $Q[u_i, u_j]$ in G such that $Q[u_i, u_j] \cap V(C) = \{u_i, u_j\}$.*

(ii) *For every $v \in C[u^+_i, u^-_j]$, if $vu_i \in E(G)$, then $v^-u_j \notin E(G)$. Similarly, for every $w \in C[u^+_j, u^-_i]$, if $wu_j \in E(G)$, then $w^-u_i \notin E(G)$.*

(iii) *For every $v \in C[u_i, u_j]$, if $vu_i, vu_j \in E(G)$, then $v^+v^- \notin E(G)$.*

3 Proof of Theorem 1.9

Let G be a k-connected $K_{1,4}$-free graph of order n with $k \geqslant 2$ such that $\sigma_{k+3}(G) \geqslant n + 2k - 2$. Suppose, to the contrary, that every spanning tree of G has at least 4 leaves. Let P be a longest path of G. Clearly, $V(P) \subset V(G)$. We additionally assume that the following conditions hold.

(P1) $w(G - P)$ is minimum; and

(P2) subject to (P1), $|P[a, v]|$ achieves the minimum, where a is the first vertex of P and v is the first vertex of P that is adjacent to some vertex in $G - P$.

Let a and b be the first and the last vertexes of P along its orientation respectively, and H be a component of $G - P$. Let $v_1, \ldots, v_t$ be the neighbors of H on P in the order along the orientation of P with $y_iv_i \in E(G)$, where $y_i \in H$ for $i \in [1, t]$.

Let G^* be the graph obtained from G by adding a new vertex v_0 and joining it to every vertex of G. Set $C := v_0a\overrightarrow{P}bv_0$. Then C is a longest cycle of G^*. The vertices $v_0, \ldots, v_t$ divide the cycle C into $t + 1$ segments. Let $Q_i = C(v_i, v_{i+1}]$ for $0 \leqslant i \leqslant t$, where the indices are taken modulo $t + 1$.

From Lemma 2.1, there exists a non-insertible vetex in $Q_i - \{v_{i+1}\}$ for each Q_i. Let w_i be the first non-insertible vertex in $Q_i - \{v_{i+1}\}$ and let $W = \{w_0, w_1, \ldots, w_t\}$.

Claim 1. *We have $a = w_0$ and $b \notin N(w_i)$ for $i \in [0, t - 1]$.*

Proof. Suppose, to the contrary, that $a \neq w_0$. Then a is an insertible vertex in Q_0. The existence of the path $a^+\overrightarrow{P}I(a)a(I(a))^+\overrightarrow{P}b$ contradicts (P2). Thus $a = w_0$. Note that $b \in Q_t$ and $w_ib^+ = w_iv_0 \in E(G)$, we have $w_ib \notin E(G)$ for $i \in [0, t - 1]$. □

Lemma 3.1. *Let I be an independent set of H. Then the following inequality holds,*

$$\sum_{i=0}^{t} d_P(w_i) + \sum_{x \in I} d_P(x) \leqslant \begin{cases} |P| + t - 1, & \textit{if } I \neq \emptyset, \\ |P| - 1, & \textit{if } I = \emptyset. \end{cases}$$

Proof. We define a function τ from $V(G^*)$ to Z^+ such that $\tau(v) = |N(v) \cap W|$. For each $A \subseteq V(G)$, $\tau(A) = \sum_{v \in V(A)} \tau(v)$.

To prove Lemma 3.1, it suffices to show that

$$\tau(P) + \sum_{x \in I} d_P(x) \leqslant \begin{cases} |P| + t - 1, & \text{if } I \neq \emptyset, \\ |P| - 1, & \text{if } I = \emptyset. \end{cases}$$

For $I=\emptyset$, we will show that $\tau(P)\leqslant|P|-1$. From (i) of Lemma 2.2, W is an independent vertex set. Then the cycle C is a disjoint union of intervals $T=C[c,d]$ with c, $d^+\notin N(W)$ and $V(C[c^+,d])\subseteq N(W)$. Notice that $V(C[c^+,d])=\emptyset$ if $c=d$. Such intervals are called *W-segments*.

From Claim 1, the endvertex b has at most one neighbour w_t in W. Since $v_0\in N(W)$ and w_0 is the successor of v_0 on C, v_0 is the last vertex of some W-segment denoted by T_0. Set

$$T_0=\begin{cases}C[b,v_0], & \text{if } b\notin N(w_t),\\ C[u,v_0], & \text{if } b\in N(w_t),\end{cases}$$

where $u\in V(C[w_t,b^-])$ satisfying that $N(W)\cap V(C[u,v_0])=V(C[u^+,v_0])$.

Firstly, we show that $\tau(T_0)=|T_0|+t-1$.

If $b\notin N(w_t)$, we have $|T_0|=|C[b,v_0]|=2$, $\tau(v_0)=t+1$ and $\tau(b)=0$, then $\tau(T_0)=t+1=|T_0|+t-1$.

If $b\in N(w_t)$, from Lemma 2.2(ii) and Claim 1, every vertex in $T_0-\{u,v_0\}$ is only adjacent to w_t in W. For $v\in T_0-\{u,v_0\}$, $\tau(v)=1$. Then we can also have

$$\tau(T_0)=\sum_{v\in T_0-\{u,v_0\}}\tau(v)+\tau(v_0)=(|T_0|-2)+(t+1)=|T_0|+t-1.$$

Next, we show that $\tau(T)\leqslant|T|$ for every W-segment T other than T_0.

Let T be such a W-segment. From Lemma 2.2(i), there is an integer l such that $T\subseteq C[w_l,w_{l+1}^-]$, where the indices are taken modulo $t+1$. Without loss of generality, we assume that $T\subseteq C[w_1,w_2^-]$.

If $|T|=1$, then $\tau(T)=0$. So assume that $|T|\geqslant 2$. From Lemma 2.2(ii),

$$N(w_1)\cap T, N(w_0)\cap T, N(w_t)\cap T,\dots,N(w_2)\cap T$$

form consecutive closed subintervals of T (possibly some of them are empty) which can only have their endvertices in common. We assume that $T=\{c,c_1,\dots,c_s\}$ such that $c\notin N(W)$ and $c_i\in N(W)$. We claim that there exists at most one vertex c_i in T such that $|N(c_i)\cap W|\geqslant 2$.

Suppose that $|N(c_i)\cap W|\geqslant 2$ for some $i\in[1,s]$. Say w_{j_1}, $w_{j_2}\in N(c_i)$ for some $j_1\neq j_2$. From Lemma 2.2(i), $c_i\notin V(C[v_2^+,w_2^-])$. Thus $c_i\in V(C[w_1^+,v_2])$. We claim that $1\in\{j_1,j_2\}$. Suppose not. If $c_i\neq v_2$, then from Lemma 2.2(iii), $G[\{c_i,c_i^-,c_i^+,w_{j_1},w_{j_2}\}]$ is an induced $K_{1,4}$, a contradiction. Hence $c_i=v_2$. But in this case, $G[\{c_i,c_i^-,y_2,w_{j_1},w_{j_2}\}]$ is an induced $K_{1,4}$, also a contradiction.

Thus $w_1\in N(c_i)$, and there exists at most one $j\neq 1$ such that $w_j\in N(c_i)$. Thus $\tau(c_i)\leqslant 2$ and $\tau(c_j)=1$ for $j\in[1,s]-\{i\}$. Hence

$$\tau(T)\leqslant 2+(s-1)=s+1=|T|. \tag{3.1}$$

To sum up,

$$\tau(C)=\sum_{T\neq T_0}\tau(T)+\tau(T_0)\leqslant\sum_{T\neq T_0}|T|+|T_0|+t-1=|C|+t-1.$$

Hence

$$\tau(P)=\tau(C)-\tau(v_0)\leqslant(|C|+t-1)-(t+1)=|C|-2=|P|-1.$$

On the other hand, for $I\neq\emptyset$, we will show that

$$\tau(P)+\sum_{x\in I}d_P(x)\leqslant|P|+t-1.$$

Let I be an independent set in H. Obviously, $N(I)\cap V(P)\subseteq\{v_1,\dots,v_t\}$, where $V(P)=(\bigcup_{i=0}^{t-1}V(P[w_i,w_{i+1}^-]))\cup V(P[w_t,b])$. Then

$$\tau(P)+\sum_{x\in I}d_P(x)=\sum_{i=0}^{t-1}\left(\tau(P[w_i,w_{i+1}^-])+\sum_{x\in I}d_{P[w_i,w_{i+1}^-]}(x)\right)+\tau(P[w_t,b]).$$

For $i\in[0,t-1]$, we take $\tau(P[w_i,w_{i+1}^-])+\sum_{x\in I}d_{P[w_i,w_{i+1}^-]}(x)$ into consideration.

Without loss of generality, we choose $P[w_1, w_2^-]$. Firstly, we consider the W-segment containing v_2 which is denoted by T_{v_2}.

If $\tau(v_2) = 0$, then $\{v_2\}$ is the first vertex of T_{v_2}. From Lemma 2.2(i), every vertex other than v_2 in T_{v_2} is only adjacent to w_2 in W. In this case, $\tau(T_{v_2}) \leqslant |T_{v_2}| - 1$. Since G is $K_{1,4}$-free, v_2 can be adjacent to at most two distinct vertices in I. Otherwise, if $z_1, z_2, z_3 \in N(v_2) \cap I$, then $G[\{v_2, v_2^-, z_1, z_2, z_3\}]$ is an induced $K_{1,4}$, a contradiction.

By inequality (3.1), for any W-segment T in $P[w_1, w_2^-]$ other than T_{v_2}, $\tau(T) \leqslant |T|$. Then

$$\begin{aligned}\tau(P[w_1, w_2^-]) + \sum_{x\in I} d_{P[w_1,w_2^-]}(x) &= \sum_{T\neq T_{v_2}} \tau(T) + \tau(T_{v_2}) + d_H(v_2)\\ &\leqslant \sum_{T\neq T_{v_2}} |T| + |T_{v_2}| - 1 + 2\\ &= |P[w_1, w_2^-]| + 1.\end{aligned}$$

If $\tau(v_2) \neq 0$, by inequality (3.1), $\tau(T) \leqslant |T|$ for any W-segment T in $P[w_1, w_2^-]$. We claim that v_2 can be adjacent to at most one vertex in I. Suppose, to the contrary, that v_2 is adjacent to two distinct vertices z_1 and z_2 in I. Let w_j be a vertex in W that is adjacent to v_2. If $j = 1$, then $G[\{v_2, w_1, v_2^+, z_1, z_2\}]$ is an induced $K_{1,4}$, a contradiction. Otherwise, $G[\{v_2, w_j, v_2^-, z_1, z_2\}]$ is an induced $K_{1,4}$, also a contradiction. Then

$$\begin{aligned}\tau(P[w_1, w_2^-]) + \sum_{x\in I} d_{P[w_1,w_2^-]}(x) &= \sum_{T\neq T_{v_2}} \tau(T) + \tau(T_{v_2}) + d_H(v_2)\\ &\leqslant \sum_{T\neq T_{v_2}} |T| + |T_{v_2}| + 1\\ &= |P[w_1, w_2^-]| + 1.\end{aligned}$$

For $\tau(P[w_t, b])$, from Lemma 2.2(i) and Claim 1, every vertex in $P[w_t^+, b]$ can be only adjacent to at most one vertex w_t in W. Then $\tau(P[w_t, b]) \leqslant |P[w_t, b]| - 1$.

To sum up, we get the conclusion that

$$\begin{aligned}\tau(P) + \sum_{x\in I} d_P(x) &= \sum_{i=0}^{t-1}(\tau(P[w_i, w_{i+1}^-]) + \sum_{x\in I} d_{P[w_i,w_{i+1}^-]}(x)) + \tau(P[w_t, b])\\ &\leqslant \sum_{i=0}^{t-1}(|P[w_i, w_{i+1}^-]| + 1) + |P[w_t, b]| - 1 \leqslant |P| + t - 1.\end{aligned}$$ □

Claim 2. *For every component H in $G - P$, we have $|N_P(H)| = k$ and H is hamiltonian-connected if $|H| \geqslant 2$.*

Proof. From Lemma 2.2(i), for $0 \leqslant i \neq j \leqslant t$, w_i and w_j have no common neighbor in $G - V(P)$. Then $\sum_{i=0}^{t} d_{G-V(P)}(w_i) \leqslant n - |P| - |H|$.

Since G is k-connected, we have $t \geqslant k$. We claim that $t = k$. Suppose, to the contrary, that $t \geqslant k + 1$.

If $t \geqslant k + 2$, then $\{w_0, \ldots, w_t\}$ is an independent set of order at least $k + 3$. By taking $I = \emptyset$ in Lemma 3.1, we have

$$\sum_{i=0}^{t} d(w_i) = \sum_{i=0}^{t} d_P(w_i) + \sum_{i=0}^{t} d_{G-V(P)}(w_i) \leqslant (|P| - 1) + (n - |P| - |H|) = n - |H| - 1,$$

a contradiction to $\sigma_{k+3}(G) \geqslant n + 2k - 2$.

If $t = k + 1$, let z be a vertex in $V(H)$, then $\{z, w_0, \ldots, w_{k+1}\}$ is an independent set of order $k + 3$. By taking $I = \{z\}$ in Lemma 3.1, we have $\sum_{i=0}^{k+1} d_P(w_i) + d_P(z) \leqslant |P| + (k + 1) - 1 = |P| + k$. Then

$$\sum_{i=0}^{k+1} d(w_i) + d(z) = \sum_{i=0}^{k+1} d_P(w_i) + d_P(z) + \sum_{i=0}^{k+1} d_{G-V(P)}(w_i) + d_H(z)$$

$$\leqslant (|P| + k) + (n - |P| - |H|) + (|H| - 1)$$
$$= n + k - 1,$$

also a contradiction to $\sigma_{k+3}(G) \geqslant n + 2k - 2$ since $k \geqslant 2$.

Suppose that $|H| \geqslant 2$ and H is not hamiltonian-connected. From Ore's theorem in [9], there exists a pair of nonadjacent vertices z_1 and z_2 in H such that $d_H(z_1) + d_H(z_2) \leqslant |H|$. Since $\{z_1, z_2, w_0, \ldots, w_k\}$ is an independent set of order $k + 3$, by taking $I = \{z_1, z_2\}$ in Lemma 3.1, we have

$$\sum_{i=0}^{k} d_P(w_i) + d_P(z_1) + d_P(z_2) \leqslant |P| + k - 1.$$

Then

$$\sum_{i=0}^{k} d(w_i) + d(z_1) + d(z_2) = \sum_{i=0}^{k} d_P(w_i) + d_P(z_1) + d_P(z_2) + \sum_{i=0}^{k} d_{G-V(P)}(w_i) + d_H(z_1) + d_H(z_2)$$
$$\leqslant (|P| + k - 1) + (n - |P| - |H|) + |H| = n + k - 1,$$

also a contradiction to $\sigma_{k+3}(G) \geqslant n + 2k - 2$ since $k \geqslant 2$.

Hence $|N_P(H)| = k$ and H is hamiltonian-connected if $|H| \geqslant 2$. □

Claim 3. *Let v_i and v_j be distinct vertices in $N_P(H)$. Then for $0 \leqslant i \neq j \leqslant k$, $G[V(H) \cup \{v_i, v_j\}]$ contains a hamiltonian path from v_i to v_j.*

Proof. Note that $\min\{d_H(v_i), d_H(v_j)\} \geqslant 1$. If $|H| = 1$, there exists only one vertex named h in H, then $v_i h v_j$ is a hamiltonian path from v_i to v_j in $G[V(H) \cup \{v_i, v_j\}]$.If $|H| \geqslant 2$, from Claim 2, H is hamiltonian-connected. It suffices to show that $|N_H(v_i) \cup N_H(v_j)| \geqslant 2$. Otherwise, both v_i and v_j have exactly one neighbor y in H. Then $(N_P(H) - \{v_i, v_j\}) \cup \{y\}$ is a separate set of $k - 1$ vertices which contradicts the k-connectedness of G. □

It follows from Claim 2 that $t = k$. For $1 \leqslant i \neq j \leqslant k$, we denote the hamiltonian path from v_i to v_j in $G[V(H) \cup \{v_i, v_j\}]$ by $v_i H v_j$. If $\omega(G - P) = 1$, then G contains a spanning 3-ended tree. So we have $\omega(G - P) \geqslant 2$. Assume that H' is a component in $G - P - H$.

Claim 4. $N(w_i) \cap V(H') \neq \emptyset$ *for some* $i \in [1, k]$.

Proof. From Claim 1, we have $w_0 = a$. Then $i \neq 0$. Suppose, to the contrary, that $N(w_i) \cap V(H') = \emptyset$ for $i \in [1, k]$. Let y and y' be vertices in H and H', respectively. Then $\{w_0, \ldots, w_k, y, y'\}$ is an independent set of order $k + 3$. By taking $I = \{y\}$ in Lemma 3.1, we have $\sum_{i=0}^{k} d_P(w_i) + d_P(y) \leqslant |P| + k - 1$. In this case, $\sum_{i=0}^{k} d_{G-V(P)}(w_i) \leqslant n - |P| - |H| - |H'|$. Then

$$\sum_{i=0}^{k} d(w_i) + d(y) + d(y') = \sum_{i=0}^{k} d_P(w_i) + + d_P(y) + \sum_{i=0}^{k} d_{G-V(P)}(w_i) + d_H(y) + d_P(y') + d_{H'}(y')$$
$$\leqslant (|P| + k - 1) + (n - |P| - |H| - |H'|) + (|H| - 1) + k + (|H'| - 1)$$
$$\leqslant n + 2k - 3,$$

a contradiction. □

In the following, we assume that $N(w_i) \cap V(H') \neq \emptyset$ for some $i \in [1, k]$. From Lemma 2.2(i), $N(w_j)$ $\cap V(H') = \emptyset$ for $j \in [0, k] - \{i\}$.

Claim 5. *There exists a second non-insertible vertex w_i' in $Q_i - \{v_{i+1}\}$ and $w_i' \notin N(H')$.*

Proof. Suppose, to the contrary, that w_i is the only non-insertible vertex in $Q_i - \{v_{i+1}\}$. Let $X = Q_i - \{w_i\}$ in G^*. Then every vertex of X can be inserted into $C[v_{i+1}, v_i]$. By using the segment insertion $SI[Q_i, X]$, we get a path $P'[v_{i+1}, v_i]$ in G^* such that $V(P'[v_{i+1}, v_i]) \supseteq C[v_{i+1}, v_i] \cup X = V(C) - \{w_i\}$. From Claim 3, the hamiltonian path $v_i H v_{i+1}$ in $G[V(H) \cup \{v_i, v_{i+1}\}]$ together with P' forms a cycle C^*

in G^* with $V(C^*) \supseteq (V(C)-\{w_i\})\cup V(H)$. From Claim 4, $N(w_i)\cap V(H') \neq \emptyset$. If $|H|=1$,then $C^*-\{v_0\}$ is a path in G which contradicts (P1). If $|H| \geqslant 2$, then $C^*-\{v_0\}$ is a path in G which contradicts to the maximality on the length of P. Thus there exists a second non-insertible vertex w_i' in Q_i.

Now, suppose that $w_i' \in N(H')$. By using the segment insertion $SI[C[w_i^+, w_i'^-]]$, we get a path $Q' := Q'[w_i', w_i]$ in G^* with $V(Q') = V(C)$. This together with the non-trivial (w_i, w_i')-path in $G[V(H') \cup\{w_i, w_i'\}]$ forms a cycle in G^* longer than C, a contradiction. □

Lemma 3.2. *Let j be an integer such that $j \in [0,k]-\{i\}$. Then for every $u_i \in C[w_i^+, w_i']$ and $u_j \in C[v_j^+, w_j]$, the following properties hold.*

(i) *There does not exist a path $Q[u_i,u_j]$ in G^* such that $Q[u_i,u_j]\cap V(C)=\{u_i,u_j\}$.*

(ii) *For every $v\in C[u_i^+,u_j^-]$, if $vu_i\in E(G^*)$, then $v^-u_j\notin E(G^*)$. Similarly, for every $w\in C[u_j^+,u_i^-]$, if $wu_j\in E(G^*)$, then $w^-u_i\notin E(G^*)$.*

(iii) *For every $v\in C[u_i^+,u_j^-]$, if vu_i, $vu_j\in E(G^*)$, then $v^-v^+\notin E(G^*)$. Similarly, for every $w\in C[u_j^+,u_i^-]$, if $wu_i, wu_j\in E(G^*)$, then $v^-v^+\notin E(G^*)$.*

Proof. We prove this lemma by induction on $l(u_i,u_j)=|V(C[w_i^+,u_i])\cup V(C[v_j^+,u_j])|$.

For $l=0$, i.e., $u_i=w_i^+$ and $u_j=v_j^+$, by using the segment insertion $SI[C[v_i^+,w_i^-]]$, we obtain a path $S[w_i^+,v_i]$ such that $V(S[w_i^+,v_i])=V(C)-\{w_i\}$. From Lemma 2.2(i), we have $N(v_j^+)\cap V(C[v_i^+,w_i^-])=\emptyset$ which implies that no vertices in $V(C[v_i^+,w_i^-])$ are inserted between v_j and v_j^+. Thus v_j^+ is still the successor of v_j on $S[w_i^+,v_i]$.

To prove (i), assume to the contrary, there is a path $Q[w_i^+,v_j^+]$ in G^* such that $Q[w_i^+,v_j^+]\cap V(C)=\{w_i^+,v_j^+\}$. Let $C_1:=w_i^+\overrightarrow{S}v_jHv_i\overleftarrow{S}v_j^+\overleftarrow{Q}w_i^+$. Then $R_1:=C_1-\{v_0\}$ is a path in G of order at least $|P|$ and with less components in $G-R_1$, a contradiction.

To prove (ii), assume to the contrary, there is a vertex $v\in C[u_i^+,u_j^-]$ such that vu_i and $v^-u_j\in E(G^*)$. Then from Lemma 2.2(ii), no vertex in $C[v_i^+,w_i^-]$ is adjacent to v. Hence no vertices in $C[v_i^+,w_i^-]$ are inserted between v and v^-. Thus v^- and v are two consecutive vertices on the path $S[w_i^+,v_i]$. Let $C_2:=w_i^+v\overrightarrow{S}v_jHv_i\overleftarrow{S}v_j^+v^-\overleftarrow{S}w_i^+$. Then $R_2:=C_2-\{v_0\}$ is a path in G of order at least $|P|$ and with less components in $G-R_2$, a contradiction.

To prove (iii), assume to the contrary, there is a vertex $v\in C[u_i^+,u_j^-]$ such that vu_i, $vu_j\in E(G^*)$ and $v^-v^+\in E(G^*)$. From Lemma 2.2(iii), $N(v)\cap C[v_i^+,w_i^-]=\emptyset$. Thus v^- is still the immediate predecessor of v in $S[w_i^+,v_j]$ and v^+ is still the immediate successor of v in $S[w_i^+,v_j]$. Let $C_3:=w_i^+\overrightarrow{S}v^-v^+\overrightarrow{S}v_jHv_i\overleftarrow{S}v_j^+vw_i^+$. Then $R_3:=C_3-\{v_0\}$ is a path in G of order at least $|P|$ and with less components in $G-R_3$, a contradiction.

Now we see that (i)–(iii) are true for $l=0$. Assume that (i)–(iii) are true for any pair of vertices z_i and z_j with $z_i\in C[w_i^+,u_i]$ and $z_j\in C[v_j^+,u_j]$ such that $l(z_i,z_j)<l(u_i,u_j)$.

Let $X=V(C[v_i^+,u_i^-])-\{w_i\}$, then every vertex of X is an insertible vertex in Q_i. Using the segment insertion $SI[Q_i,X]$, we obtain a path $S'[u_i,v_i]$ in G^* such that $V(S'[u_i,v_i])\supseteq V(C)-\{w_i\}$. From the induction hypothesis on (i) and Lemma 2.2, there are no edge between the vertex sets $V(C[v_i^+,u_i^-])$ and $V(C[v_j^+,u_j])$. Thus no vertices in $V(C[v_i^+,u_i^-])$ are inserted between any pair of vertices in $V(C[v_j^+,u_j])$.

From the induction hypothesis on (ii), $I(z_i)\neq I(z_j)$ for any $z_i\in V(C[v_i^+,u_i^-])$ and $z_j\in V(C[v_j^+,u_j^-])$. Then $(I(z_j))^+$ is also the immediate successor of $I(z_j)$ on the path $S'[u_i,v_i]$. Using the segment insertion $SI[C[v_j^+,u_j^-]]$, we can insert every vertex in $C[v_j^+,u_j^-]$ in $S'[u_i,v_j]$ or $S'[u_j,v_i]$ to obtain two vertex disjoint paths $T_1[u_i,v_j]$ and $T_2[u_j,v_i]$ such that $V(T_1[u_i,v_j])\cup V(T_2[u_j,v_i])\supseteq V(C)-\{w_i\}$. For any pair of two consecutive vertices w^- and w of $V(C)-V(C[v_i^+,u_i^-])\cup V(C[v_j^+,u_j^-])$, only one of the following three properties holds:

1) w^- and w are two consecutive vertices on one of the paths $T_1[u_i,v_j]$ and $T_2[u_j,v_i]$.

2) There is a segment $C[z_i,z_i']$ in $C[v_i^+,u_i^-]$ inserted between w^- and w with $N(z_i)\cap N(z_i')\supseteq\{w^-,w\}$.

3) There is a segment $C[z_j,z_j']$ in $C[v_j^+,u_j^-]$ inserted between w^- and w with $N(z_j)\cap N(z_j')\supseteq\{w^-,w\}$.

To prove (i), suppose, to the contrary, that there is a path $Q[u_i,u_j]$ in G^* such that $Q[u_i,u_j]\cap V(C)=\{u_i,u_j\}$. Let $C_1'=u_i\overrightarrow{T_1}v_jHv_i\overleftarrow{T_2}u_j\overleftarrow{Q}u_i$. Then $R_1':=C_1'-\{v_0\}$ is a path in G of order at least $|P|$ and with less components in $G-R_1'$, a contradiction.

To prove (ii), suppose, to the contrary, that there are two consecutive vertices w and w^- in $C[u_i^+, u_j^-]$ such that $wu_i \in E(G^*)$ and $w^-u_j \in E(G^*)$. Because $w^-u_j \in E(G^*)$, by our induction hypothesis on (ii) and Lemma 2.2, we have $wz_i \notin E(G^*)$ for every $z_i \in V(C[v_i^+, u_i^-])$. Hence no vertices in $V(C[v_i^+, u_i^-])$ are inserted between w and w^-. In the same manner, we can show that no vertices in $V(C[v_j^+, u_j^-])$ are inserted between w^- and w. Thus w^- and w are two consecutive vertices on the path $T_1[u_i, v_j]$. Let $C_2' = u_i w \overrightarrow{T_1} v_j H v_i \overleftarrow{T_2} u_j w^- \overleftarrow{T_1} u_i$. Then $R_2' := C_2' - \{v_0\}$ is a path in G of order at least $|P|$ and with less components in $G - R_2'$, a contradiction.

To prove (iii), suppose, to the contrary, that without loss of generality, there is a vertex $v \in V(C[u_i^+, u_j^-])$ such that vu_i, $vu_j \in E(G^*)$ and $v^-v^+ \in E(G^*)$. By the induction hypothesis on (iii) and Lemma 2.2, $N(v) \cap (V(C[v_i^+, u_i^-]) \cup V(C[v_j^+, v_j^-])) = \emptyset$. Thus v^- is still the immediate predecessor of v in $T_1[u_i, v_j]$ and v^+ is still the immediate successor of v in $T_1[u_i, v_j]$. Let $C_3' = u_i \overrightarrow{T_1} v^- v^+ \overrightarrow{T_1} v_j H v_i \overleftarrow{T_2} u_j v u_i$. Then $R_3' := C_3' - \{v_0\}$ is a path in G of order at least $|P|$ and with less components in $G - R_3'$, a contradiction.

This completes the proof of Lemma 3.2. □

We replace w_i with w_i' in W to obtain $W' = \{w_0, \ldots, w_{i-1}, w_i', w_{i+1}, \ldots, w_k\}$. From Lemmas 2.2 and 3.2, the vertices of W' have the same properties as that of W. Using the same method as in the proof of Lemma 3.1, we also have $\sum_{x \in W'} d_P(x) \leqslant |P| - 1$.

Let y and y' be vertices in H and H', respectively. From Lemma 2.2 and Claim 5, $W' \cup \{y, y'\}$ is an independent set in G of order $k+3$. Then $\sum_{x \in W'} d_{G-V(P)}(x) \leqslant n - |P| - |H| - |H'|$. Now we have

$$\sum_{x \in W'} d(x) \leqslant (|P| - 1) + (n - |P| - |H| - |H'|) = n - 1 - |H| - |H'|.$$

On the other hand, $d(y) \leqslant |H| - 1 + k, d(y') \leqslant |H'| - 1 + k$. Hence

$$\begin{aligned} \sum_{x \in W'} d(x) + d(y) + d(y') &\leqslant (n - 1 - |H| - |H'|) + (|H| - 1 + k) + (|H'| - 1 + k) \\ &= n + 2k - 3, \end{aligned}$$

a contradiction. This completes the proof of Theorem 1.9. □

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