The joint distributions of first hitting and last exit for Brownian motion*

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Let $X=\{x(t,\omega),\ t\geqslant 0\}$ be $d(\geqslant 3)$-dimensional Brownian motion on probability space $(\Omega,\ \mathscr{F},\ P)$ with values in Euclidean space R^d; $\mathscr{B}^d$ be the Borel σ-algebra in R^d. The transition probability density of X is

$$p(t,x,y)=(2\pi t)^{-d/2}\exp(-|x-y|^2/2t).$$

The semigroup of transition operators is $T_t f(x)=\int p(t,x,y)f(y)\mathrm{d}y$, where f is bounded $\mathscr{B}^d$ measurable function, $\int=\int_{R^d}$; the Green function is $g(x,y)=\int_0^\infty p(t,x,y)\,\mathrm{d}t$; the equilibrium measure of a relatively compact set B is denoted by μ_B.

For $B\in\mathscr{B}^d$ we define the first hitting time and last exit time for X by

$$h_B=\inf(t>0,\ x_t\in B),\qquad l_B=\sup(t>0,\ x_t\in B),$$

and by convention $\inf(\phi)=\infty$, $\sup(\phi)=0$, where ϕ is the empty set and $x_t=x(t,\omega)$. Let $B^c=R^d\backslash B$. We call $e_B=h_{B^c}$ the first exit time of B. Evidently, if $l_B>0$, then $l_B\geqslant h_B$ and $l_B\geqslant e_B$; because $\forall\varepsilon>0$, $x(l_B+\varepsilon)\in B^c$, hence $l_B+\varepsilon\geqslant e_B$.

The first hitting location, last exit location and first exit location are denoted by $x(h_B)$, $x(l_B)$ and $x(e_B)$, respectively. Since $d\geqslant 3$, X is transient, i.e.

$$P_x(\lim_{t\to\infty}|x_t|=\infty)=1,\ \forall x\in R^d. \tag{1}$$

Therefore, if B is bounded, then $\forall x\in B$. We have $e_B<\infty$, $l_B>0$, P_x-a.s.

The distributions of $(h_B,\ x(h_B))$ and $(l_B,\ x(l_B))$ are discussed in refs.[1—3] and refs.[1, 3, 4—6], respectively. For the first exit probabilities of Brownian motion on manifolds, see ref. [7]. The purpose of this note is to investigate the joint distribution of h_B, $x(h_B)$, l_B, $x(l_B)$ and some limit distributions. Under certain conditions their exact mathematical formulas

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can be found. In particular, take $x_0=0$, B_r, the ball with center O and radius $r>0$; S_r, its sphere. Then the distribution of the first hitting location and last exit location has spherical symmetry; the joint density and the conditional density of this distribuiton have the same expression; as $d\to\infty$, we meet a new kind of functions (similar to, but not, the Dirac functions) defined on infinite dimensional space.

For $D\in\mathscr{B}^d$, let $p_D(t,x,A)$ be the (sub)transition density on D, i.e.

$$\begin{aligned} p_D(t,x,A) &= P_x(e_D>t,\ x_t\in A)=P_x(x_u\in D,\ u\leqslant t,\ x_t\in A), \quad x\in D;\\ &=0, \qquad x\overline{\in} D. \end{aligned}$$

Let $T_t^D f(x)=\int_D p_D(t,x,\mathrm{d}y)f(y)$.

We fix $B\in\mathscr{B}^d$. Omit B and put $h=h_B$ if there is no ambiguity. Denote

$$H(z,C)=p_z(x(h)\in C), \qquad E(z,C)=p_z(x(e)\in C),$$
$$L(z,C)=P_z(l>0,\ x(l)\in C).$$

Since $x(\infty)$ is undefined, $(x(h)\in C)=(h<\infty,\ x(h)\in C)$ by convention; the same convention is for e, l.

Theorem 1. *Let* $B\in\mathscr{B}^d$, $\forall x\in R^d$, $s>0$, $t>0$. *We have*

$$\begin{aligned} &P_x(h>s,\ x(h)\in A,\ l-h>t,\ x(l)\in C)\\ &=\int_A P_y(l>t,\ x(l)\in C)P_x(h>s,\ x(h)\in \mathrm{d}y) \qquad (2)\\ &=\int_A T_t L(y,\ C)\cdot T_S^{B^c}H(x,\mathrm{d}y). \qquad (3) \end{aligned}$$

Proof. Let $\mathscr{F}_h$ be the pre-σ-algebra of stopping time h; θ_t be the shift operator of X. By strong Markov property the left side of eq. (2) equals

$$\begin{aligned} &P_x(h>s,\ x(h)\in A,\ \theta_h l>t,\ x(l)\in C)\\ &=\int_{(h>s,\ x(h)\in A)} P_x(\theta_h l>t,\ x(l)\in C|\mathscr{F}_h)P_x(\mathrm{d}\omega)\\ &=\int_{(h>s,\ x(h)\in A)} P_{x(h)}(l>t,\ x(l)\in C)P_x(\mathrm{d}\omega)\\ &=\int_A P_y(l>t,\ x(l)\in C)P_x(h>s,\ x(h)\in\mathrm{d}y). \end{aligned}$$

But

$$P_y(l>t,\ x(l)\in C)=\int p(t,y,z)L(z,C)\mathrm{d}z=T_tL(y,C), \qquad (4)$$

$$
\begin{aligned}
&P_x(h>s,\ x(h)\in G)=P_x(x_u\bar{\in} B,\ u\leqslant s,\ x(h)\in G)\\
&=\int_{(x_u\bar{\in} B,\ u\leqslant s)} P_x(x(h)\in G|\mathscr{F}_s)P_x(\mathrm{d}\omega)\\
&=\int_{(x_u\bar{\in} B,\ u\leqslant s)} P_{x(s)}(x(h)\in G)P_x(\mathrm{d}\omega)\\
&=\int_{B^c} P_z(x(h)\in G)P_x(x_u\bar{\in} B,\ u\leqslant s,\ x(s)\in \mathrm{d}z)\\
&=\int_{B^c} H(z,G)p_{B^c}(s,x,\mathrm{d}z)=T_S^{B^c}H(x,G).
\end{aligned}
\tag{5}
$$

Substituting eqs. (4), (5) into eq. (2) we get eq. (3). If $x\in B$, both sides of eqs. (2) and (3) are 0 and the theorem is obviously true.

Since $e_B=h_{B^c}$ and $l_B\geqslant e_B$ if $l_B>0$, we have

Theorem 1′. *Let $B\in\mathscr{B}^d$, $e=e_B$, $l=l_B$, $\forall x\in R^d$, $s>0$, $t>0$. We have*

$$
\begin{aligned}
&P_x(e>s,\ x(e)\in A,\ l-e>t,\ x(l)\in C)\\
&=\int_A P_y(l>t,\ x(l)\in C)P_x(e>s,\ x(e)\in \mathrm{d}y) &(6)\\
&=\int_A T_tL(y,C)\cdot T_S^BE(x,\mathrm{d}y). &(7)
\end{aligned}
$$

Remark 1. Theorems 1 and 1′ are true for general strong Markov processes with continuous path, because the characteristic property of Brownian motion is not used in the proof.

Let $B\in\mathscr{B}^d$ be a bounded non-empty open set. For fixed $s>0$, $P_x(e>s,\ x_s\in\mathrm{d}y)$ has density $p_B(s,x,y)$ with respect to Lebesque measure and

$$p_B(s,\ x,\ y)=\sum_n \mathrm{e}^{-\lambda_n s}\varphi_n(x)\varphi_n(y), \tag{8}$$

where $\varphi_n(x)\,(x\in B)$ is the eigenfunction corresponding to eigenvalue λ_n of $\frac{1}{2}\Delta=\frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$ on B. The series in eq. (8) converges absolutely and uniformly on $B\times B$[3]. Therefore, the following exchange of limits is reasonable.

Let

$$T(t,\ y,\ z)=\int_t^\infty p(u,\ y,\ z)\,\mathrm{d}u.$$

Theorem 2. *Let B be bounded non-empty open set, $\forall x\in B$, $s>0$, $t>0$, we have*

$$
\begin{aligned}
&P_x(e>s,\ x(e)\in A,\ l-e>t,\ x(l)\in C)\\
&=\sum_n \mathrm{e}^{-\lambda_n s}\varphi_n(x)\int_{z\in C}\int_{y\in A}\int_{v\in B}\varphi_n(v)E(v,\mathrm{d}y)\mathrm{d}vT(t,\ y,\ z)\mu_B(\mathrm{d}z).
\end{aligned}
\tag{9}
$$

Proof.

$$P_x(e>s,\ x(e)\in A)=\int_B E(v,A)p_B(s,x,\mathrm{d}v)$$
$$=\int_B E(v,A)\sum_n e^{-\lambda_n s}\varphi_n(x)\varphi_n(v)\,\mathrm{d}v$$
$$=\sum_n \mathrm{e}^{-\lambda_n s}\varphi_n(x)\int_B \varphi_n(v)E(v,A)\,\mathrm{d}v. \tag{10}$$

By ref. [1] we have

$$P_y(l>t,\ x(l)\in C)=\int p(t,y,z)P_z(l>0,\ x(l)\in C)\,\mathrm{d}z$$
$$=\int p(t,y,z)\int_C g(z,a)\mu_B(\mathrm{d}a)\,\mathrm{d}z$$
$$=\int p(t,y,z)\int_C\left(\int_0^\infty p(u,z,a)\,\mathrm{d}u\right)\mu_B(\mathrm{d}a)\,\mathrm{d}z$$
$$=\int_C\int_0^\infty p(t+u,y,a)\,\mathrm{d}u\mu_B(\mathrm{d}a)=\int_C T(t,y,a)\mu_B(\mathrm{d}a). \tag{11}$$

Substituting eqs. (10), (11) into eq. (6), we can obtain equation (9).

Let

$$R(t)=(2\pi)^{d/2}\left(\frac{d}{2}-1\right)t^{d/2-1}.$$

Theorem 3. *For bounded $B\in\mathscr{B}^d$ and compact A, we have*

$$R(s)R(t)P_x(h>s,\ x(h)\in A,\ l-h>t,\ x(l)\in C)$$
$$\to \mu_B(A)\mu_B(C)P_x(h=\infty),\quad (t\to\infty,\ s\to\infty) \tag{12}$$
$$\to \mu_B(A)\mu_B(C),\qquad (|x|\to\infty). \tag{13}$$

Proof. By Theorem 1 the left side of eq. (12) is

$$\int_A R(t)P_y(l>t,\ x(l)\in C)R(s)P_x(h>s,\ x(h)\in \mathrm{d}y). \tag{14}$$

On compact set A we have

$$\lim_{t\to\infty} R(t)P_y(l>t,\ x(l)\in C)=\mu_B(C)$$

uniformly in y. Therefore, when $t\to\infty$, eq. (14) tends to $\mu_B(C)R(s)P_x(h>s,\ x(h)\in A)$, which approaches

$$P_x(h_B=\infty)\mu_B(A)\mu_B(C)\quad \text{if } s\to\infty.$$

Take r large enough such that the ball $B_r\supset B$. When $|x|>r$ we have

$$1-\left|\frac{r}{x}\right|^{d-2}=P_x(h_{B_r}=\infty)\leqslant P_x(h_B=\infty),$$

so that $\lim_{|x|\to\infty} P_x(h_B=\infty)=1$ and eq. (13) is proved.

Some interesting results can be obtained if B is the ball B_r or sphere S_r. The first hitting time and the last exit time are denoted by h_r and l_r, respectively, the first exit time of B by e_r, and the uniform distribution on S_r by U_r. We have[4]

$$H_r(y,D)=P_y(x(h_r)\in D)=\int_D \frac{r^{d-2}\|y|^2-r^2|}{|y-z|^d}\ U_r(\mathrm{d}z), \tag{15}$$

$$L_r(y,D)=P_y(l_r>0,\ x(l_r)\in D)=\int_D\left|\frac{r}{y-z}\right|^{d-2}U_r(\mathrm{d}z), \tag{16}$$

$$T_tL_r(y,D)=P_y(l_r>t,\ x(l_r)\in D)=\frac{1}{(2\pi t)^{d/2}}\int\exp\left(-\frac{|y-u|^2}{2t}\right)\int_D\left|\frac{r}{u-z}\right|^{d-2}U_r(\mathrm{d}z)\,\mathrm{d}u. \tag{17}$$

The equilibrium distributions of B_r and S_r are the same,

$$\mu_r(\mathrm{d}z)=2\pi^{d/2}r^{d-2}U_r(\mathrm{d}z)/\Gamma\left(\frac{d}{2}-1\right). \tag{18}$$

Let

$$K(d,r)=2\pi^{d/2}r^{2d-4}/\Gamma\left(\frac{d}{2}-1\right),$$

$$\Phi_n(y,r)=\int_{B_r}\varphi_n(v)\frac{\|v|^2-r^2|}{|v-y|^d}\ \mathrm{d}v,$$

where φ_n is the eigenfunction corresponding to eigenvalue λ_n of $\frac{1}{2}\Delta$ on open ball B_r. When $x_0\in B_r$, $h_r=e_r$, the distribution $E_r(v,D)$ of e_r coincides with $H_r(v,D)$, $v\in B_r$. Substituting this fact and eqs. (15), (18) in to eq. (9), we get

Theorem 2′. *Let B_r be open ball. Then $\forall x\in B_r$, $s>0$, $t>0$, $A\subset S_r$, $C\subset S_r$. We have*

$$\begin{aligned}&P_x(h_r>s,\ x(h_r)\in A,\ l_r-h_r>t,\ x(l_r)\in C)\\&=K(d,r)\sum_n \mathrm{e}^{-\lambda_n s}\varphi_n(x)\int_{z\in C}\int_{y\in A}\Phi_n(y,r)T(t,y,z)U_r(\mathrm{d}y)U_r(\mathrm{d}z).\end{aligned} \tag{18 }$$

Put

$$Q(d,r,s)=\sum_{i=1}^{\infty}\xi_{di}\exp(-q_{di}^2 s/2r^2),$$

where q_{di} are the positive roots of Bessel function $J_v(z)=0$, $\left(v=\frac{d}{2}-1\right)$, and

$$\xi_{di}=q_{di}^{v-1}/2^{v-1}\Gamma(v+1)J_{v+1}(q_{di}).$$

Under P_0, h_r and $x(h_r)$ are independent; $x(h_r)$ is uniformly distributed on S_r; $P_0(h_r>s)=Q(d,r,s)$[2]. By eq. (2) we have

Corollary 1.

$$P_0(h_r>s,\ x(h_r)\in A,\ l_r-h_r>t,\ x(l_r)\in C)=Q(d,r,s)\int_A T_tL_r(y,C)U_r(\mathrm{d}y). \tag{19}$$

By strong Markov property and eqs. (15) and (16) it is easy to prove

Theorem 4. *$\forall x \in B_r$, $A \subset S_r$, $C \subset S_r$, we have*

$$P_x(x(h_r)\in A,\ x(l_r)\in C)=\int_A P_y(x(l_r)\in C)P_x(x(h_r)\in \mathrm{d}y)=\int_A\int_C \frac{r^{2d-4}||x|^2-r^2|}{|y-x|^d|y-z|^{d-2}}\ U_r(\mathrm{d}y)U_r(\mathrm{d}z). \quad (20)$$

Corollary 2.

$$P_0(x(h_r)\in A,\ x(l_r)\in C)=\int_A\int_C\left|\frac{r}{y-z}\right|^{d-2}U_r(\mathrm{d}y)U_r(\mathrm{d}z)=P_0(x(h_r)\in C,\ x(l_r)\in A).$$

It follows that the distribution of $x(h_r)$ and $x(l_r)$ is symmetric on the sphere, i.e. starting from 0, the events "first hitting A, last exiting from C" and "first hitting C, last exiting from A" on S_r have the same probability. Moreover, the joint distribution of $x(h_r)$ and $x(l_r)$ has the joint density

$$f(y,z)=\left|\frac{r}{y-z}\right|^{d-2},\quad (y\in S_r,\ z\in S_r) \quad (21)$$

with respect to $U_r\times U_r$. Now we are going to find the conditional distribution density $f_l(z|y)$ of $x(l_r)$ with respect to U_r when $x(h_r)=y\in S_r$ is fixed. Using

$$\int_{S_r}\left|\frac{r}{y-z}\right|^{d-2}U_r(\mathrm{d}z)=P_y(h_r<\infty)=\begin{cases}1, & \text{if } |y|\leqslant r;\\ |r/y|^{d-2}, & \text{if } |y|>r,\end{cases}$$

we see that

$$f_l(z|y)=\left|\frac{r}{y-z}\right|^{d-2}\Big/\int_{S_r}\left|\frac{r}{y-z}\right|^{d-2}U_r(\mathrm{d}z)=\left|\frac{r}{y-z}\right|^{d-2}\qquad (z\in S_r). \quad (22)$$

By symmetry, given $x(l_r)=z\in S_r$, the conditional distribution density of $x(h_r)$ with respect to U_r is

$$f_h(y|z)=\left|\frac{r}{y-z}\right|^{d-2}\qquad (y\in S_r). \quad (23)$$

Now the four probabilistic meanings of $\left|\frac{r}{y-z}\right|^{d-2}$ can be seen, namely eqs.(16), (21), (22) and (23). Of course the variables y, z play a different role in each case.

What will appear as the dimension of the space $d\to\infty$? In order to emphasize d one rewrite S_r as S_r^d, and (21) as

$$f_d(y,z)=\left|\frac{r}{y-z}\right|^{d-2}\quad (y\in S_r^d,\ z\in S_r^d). \quad (21)$$

Intuitively, as d increases, $f_d(y,z)$ monotonely increases to ∞ if $|y-z|\equiv c<r$; it means that $x(h_r)$ and $x(l_r)$ approach each other on S_r^∞ with large probability; if $|y-z|\equiv c'>r$, then $f_d(y,z)$ monotonely decreases to 0; it follows that the probability of $|x(h_r)-x(l_r)|>r$ becomes smaller and smaller. Hence we introduce the limit function

$$F(y,z)=\begin{cases}\infty, & \text{if } |y-z|<r;\\ 1, & \text{if } |y-z|=r;\\ 0, & \text{if } |y-z|>r,\end{cases}$$

where $F(y, z)$ is a "function" defined on $S_r^\infty \times S_r^\infty$; S_r^∞ is a sphere with radius r in infinite dimensional space l_2, and

$$l_2 = \{y:\ y = (y_1,\ y_2, \cdots),\ |y|^2 = \sum_i y_i^2 < \infty\},$$

$$S_r^\infty = \{y:\ y \in l_2,\ |y| = r\}.$$

$F(z, y)$ is a new "function", similar to (but not) Dirac function. Perhaps it will interest some researchers.

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