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General inverse of Stokes, Vening-Meinesz and Molodensky formulae

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Abstract The operator operations between the disturbing potential and the geoidal undulation, the gravity anomaly, the deflection of the vertical are defined based on the relations among the gravity potential, the normal gravity potential and the disturbing potential. With the sphere as the boundary surface, based on the solution of the external boundary value problem for the disturbing potential by the spherical harmonics in the physical geodesy, the general inverse Stokes' formula, the general inverse Vening-Meinesz formula and the general Molodensky's formula are derived from the operator operations defined. The general formulae can get rid of the restriction of the classical formulae only used on the geoid. If the boundary surface is defined as the geoid, the general formulas are degenerated into the classic ones.

Keywords: spherical harmonics, boundary value problem, general inverse Stokes' formula, general inverse Vening-Meinesz formula, general Molodensky's formula.

The undulation of the geoid, the gravity anomaly and the deflection of the vertical are the three basic observations describing the shape and the gravity field of the earth. The Stokes' formula that computes the undulation of the geoid using the gravity anomaly on the geoid under spherical approximate conditions was first put forward by Stokes[1]. According to Stokes' theory, The Vening-Meinesz formula that computes the meridian and the prime vertical components of the deflection of the vertical was also derived by Vening-Meinesz utilizing the gravity anomaly on the geoid[2]. The applications of the Stokes' formula and the Vening-Meinesz formula have drawn many important problems[3]. For example, it is necessary that the gravitational gradient or the earth's crust density be supposed and the mass of the earth's crust be moved when processing gravity reduction; on the other hand, the surface for the boundary value problem——the geoid is unknown[4]. Molodensky advanced a new theory to research the shape and the gravity field of the earth in 1945. It is a boundary value problem on the earth's surface[5]. It does not need the information of the earth's crust density in the theory application. Afterwards, Molodensky and his colleagues derived the inverse Vening-Meinesz formula to compute the deflection of the vertical using the gravity anomaly, the inverse Stokes' formula to compute the undulation of the geoid using the gravity anomaly and the Molodensky's formula to compute quasi-geoid height using the deflection of the vertical[5,6]. The above formulae theoretically describe internal relationships between the earth's shape and the gravity field of the earth through converting between the undulation of the geoid, the gravity anomaly and the deflection of the vertical. According to Bruns' formula and utilizing the relationship between the disturbing potential and the gravity anomaly, Rummel similarly derived the inverse Stokes' formula in 1977[3,7,8]. On the basis of the expansions of the spherical harmonics for the gravitational potential and Green's anamorphosis formula, Hwang derived similar Molodensky's formula and inverse Vening-Meinesz formula in 1998[9,10]. In above-mentioned formulae which the undulation of the geoid, the gravity anomaly and the deflection of the vertical are transformed mutually, they have a common characteristic, that is, the undulation of the geoid, the gravity anomaly and the deflection of the vertical share identical boundary surface (*e.g*. geoid or quasi-geoid). They are named the classical formulae in this paper. At present, the researches of the gravity field of the outside earth are paid more attention to in physical geodesy. It is more significant that the gravity field parameters outside the earth are derived using geometrical and/or physical observations on the geoid or the telluroid. Therefore, the formulae derived by this paper are named the general formula. Utilizing operator operations and spherical harmonics, the general inverse Stokes' formula, the general inverse Vening-Meinesz and the general Molodensky's formula are derived in this paper.

1 Operator definition

Based on the definitions of the disturbing potential T, the gravity anomaly Δg, the deflection of the vertical ε as well as the undulation of the geoid N, under the condition of spherical approximation, the relations between the disturbing potential and the gravity anomaly, the deflection of the vertical, the undulation of the geoid can be expressed as

$$N = \frac{T}{\gamma}\Big|_{\rho=R}, \tag{1}$$

$$\Delta g = -\left(\frac{\partial T}{\partial \rho} + \frac{2T}{\rho}\right)\Big|_{\rho=R}, \tag{2}$$

$$\varepsilon = \frac{1}{\rho\gamma}\frac{\partial T}{\partial \psi}\Big|_{\rho=R}, \tag{3}$$

where ρ denotes the geocentric distance, R the radius of the boundary spherical surface, γ the normal gravity, ψ arbitrary direction on the sphere. Here, eq. (1) is the famous Bruns' formula[7]. Eq. (2) is called fundamental differential equation in the physical geodesy[11]. It is an important boundary value condition in studying the gravity field outside the earth. On the basis of definition of the deflection of the vertical[3], eq. (3) denotes the deflection of the vertical in direction ψ. They are all the functional of the disturbing potential. Eq. (1), eq. (2) and eq. (3) are expressed in the form of operator operating:

$$B_1 T = N, \tag{4}$$

$$B_2 T = \Delta g, \tag{5}$$

$$B_3 T = \varepsilon, \tag{6}$$

where B_1, B_2 and B_3 are operator signs.

2 Spherical approximate solution for disturbing potential

When the boundary surface is a sphere, the boundary value problem of the physical geodesy can be solved by using spherical harmonics under the spherical approximate condition. For this fixed boundary value problem, the existence and uniqueness about the solution of disturbing potential in out space of the earth can be found in Stokes theorem[1] and Dirichlet theorem[3].

If the disturbing potential T on the boundary surface is known in eq. (1), the disturbing potential outside the boundary surface can be solved. On the basis of the solution of the first boundary value problem of the potential theory outside the sphere[11], we can obtain

$$T_{(\rho,\varphi,\lambda)} = \frac{\gamma_0}{4\pi}\iint_\sigma K(\rho,\psi)N\big|_\sigma \,\mathrm{d}\sigma, \tag{7}$$

where (ρ,φ,λ) is a point to be computed, $N\big|_\sigma$ the function on the sphere, γ_0 the normal gravity on the sphere, $K(\rho,\psi)$ the kernel function of the solution about Laplace's equation under the boundary value condition defined by eq. (1). The series expansion of $K(\rho,\psi)$ is expressed as

$$K(\rho,\psi) = \sum_{n=0}^{\infty}(2n+1)\frac{R^{n-1}}{\rho^{n+1}}P_n(\cos\psi). \tag{8}$$

When the linear combination of the disturbing potential and its normal direction derivative on the sphere is known in eq. (2), the disturbing potential outside the boundary surface can be also solved. According to the solution of the third boundary value problem for the potential theory outside the sphere[11], we can obtain

$$T_{(\rho,\varphi,\lambda)} = \frac{1}{4\pi}\iint_{\sigma} S(\rho,\psi)\Delta g\big|_{\sigma}\,\mathrm{d}\sigma, \tag{9}$$

where $\Delta g\big|_{\sigma}$ is the function on the sphere, $S(\rho,\psi)$ the kernel function of the solution about Laplace's equation under the boundary value condition defined by eq. (2). The series expansion of $S(\rho,\psi)$ is expressed as

$$S(\rho,\psi) = \sum_{n=2}^{\infty}\frac{2n+1}{n-1}\frac{R^n}{\rho^{n+1}}P_n(\cos\psi). \tag{10}$$

If the direction derivative for the disturbing potential on the sphere is known in eq. (3), the disturbing potential outside the boundary surface can be solved. On the basis of the spherical harmonics expansion theory, we can obtain

$$T_{(\rho,\varphi,\lambda)} = -\frac{1}{4\pi}\iint_{\sigma} M(\rho,\psi)\frac{\partial T}{\partial\psi}\Big|_{\sigma}\,\mathrm{d}\sigma, \tag{11}$$

where $\frac{\partial T}{\partial\psi}\Big|_{\sigma}$ is the function on the sphere, $M(\rho,\psi)$ the kernel function of the solution about Laplace's equation under the boundary value condition defined by eq. (3). The series expansion of $M(\rho,\psi)$ is expressed as

$$M(\rho,\psi) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\frac{R^{n-1}}{\rho^{n+1}}P_{n,1}(\cos\psi). \tag{12}$$

According to the representations of the generating function of Legendre's polynomial[12] and the results given by ref. [13]. The kernel function $K(\rho,\psi)$ can be expressed in an enclosed form as

$$K(\rho,\psi) = \frac{\rho^2 - R^2}{Rl^3}, \tag{13}$$

where

$$l = \left(\rho^2 + R^2 - 2\rho R\cos\psi\right)^{\frac{1}{2}}. \tag{14}$$

It can be seen that $K(\rho,\psi)$ is actually Poisson's kernel function, eq. (7) is called Poisson's integral formula. In the same way, we can get the enclosed expression of kernel function $S(\rho,\psi)$ as

$$S(\rho,\psi) = \frac{2}{l} + \frac{1}{\rho} - 5\frac{R}{\rho^2}\cos\psi - 3\frac{l}{\rho^2} - 3\frac{R}{\rho^2}\cos\psi\ln\frac{l+\rho-R\cos\psi}{2\rho}. \tag{15}$$

Eq. (15) is called general Stokes' function[4], and eq. (9) is called Stokes' integral formula. Similarly, the enclosed expression of the kernel function $M(\rho,\psi)$ is expressed as

$$M(\rho,\psi) = \cot\psi\left(\frac{2}{Rl} + \frac{1}{\rho R}\right) + \frac{1}{R^2\sin\psi}\left(1 + \frac{l}{\rho} - \frac{2\rho}{l}\right) - \cot\psi\left(\frac{1}{\rho R} - \frac{1}{R^2}\right)\ln\frac{l+\rho-R\cos\psi}{2\rho}. \tag{16}$$

The deflection of the vertical on the boundary surface is mapped as the disturbing potential through kernel function $M(\rho,\psi)$ in eq. (11). For this reason, eq. (16) is called the general direction transformation function. Eq. (11) is called the general direction transformation integral equation.

3 General integral formulae

3.1 General inverse Stokes formula

According to the operator operation defined by eq. (5), the disturbing potential is mapped as gravity anomaly through the operator B_2. Let B_2 apply to eq. (7) and define

$$B_2 K(\rho,\psi) = \frac{1}{\rho}Q(\rho,\psi), \tag{17}$$

where

$$Q(\rho,\psi) = -\frac{4\rho^2}{Rl^3} + \frac{3\left(\rho^2 - \rho R\cos\psi\right)\left(\rho^2 - R^2\right)}{Rl^5} + \frac{2R}{l^3}. \tag{18}$$

We get

$$\Delta g_{(\rho,\varphi,\lambda)} = \frac{\gamma_0}{4\pi\rho}\iint_{\sigma} Q(\rho,\psi)N\big|_{\sigma}\,\mathrm{d}\sigma. \tag{19}$$

Eq. (19) is namely the general Stokes' formula. The undulation N of the level plane for gravity potential on the boundary surface is mapped as gravity anomaly Δg

on the identical boundary surface or outside. If the boundary surface is a virtual sphere inside the earth, the disturbing potential outside the earth can be obtained through solving boundary value problem defined by this virtual sphere based on the Stokes theorem[1] and Dirichlet theorem[3]. The contents above is the main part of Bjerhammar's theory researching the shape and gravity field of the earth[14]. The boundary surface with regard to eq. (19) can be not only a virtual sphere of inside the earth, but also the quasi-geoid under spherical approximate condition. If the boundary surface is the geoid on which only the gravity anomaly is computed, eq. (19) degenerates into classical inverse Stokes' formula by letting $\rho = R$ in eq. (19). The kernel function $Q(R,\psi)$ is singular at $\psi = 0$, so we simply transform eq. (19),

$$\Delta g_{(R,\varphi_p,\lambda_p)} = \frac{\gamma_0}{4\pi R}\iint_\sigma Q(R,\psi)N(\varphi_p,\lambda_p)\mathrm{d}\sigma + \frac{\gamma_0}{4\pi R}\iint_\sigma Q(R,\psi)\left[N|_\sigma - N(\varphi_p,\lambda_p)\right]\mathrm{d}\sigma, \tag{20}$$

where (φ_p,λ_p) is a computation point, $N(\varphi_p,\lambda_p)$ the undulation of the geoid at this point. The kernel function $Q(R,\psi)$ in the first integral to the right of eq. (20) is expressed as infinite series expansion:

$$Q(R,\psi) = \frac{1}{R^2}\sum_{n=0}^{\infty}(2n+1)(n-1)P_n(\cos\psi). \tag{21}$$

According to the orthogonal property of the Legendre's function, and taking into account $N(\varphi_p,\lambda_p)$ being constant in integral expression, the first integral expression to the right of eq. (20) can be written as $-\frac{\gamma_0}{R}N(\varphi_p,\lambda_p)$. The kernel function $Q(R,\psi)$ in the second integral to the right of eq. (20) is expressed as enclosed expression:

$$Q(R,\psi) = -\frac{1}{4R^2\sin^3\frac{\psi}{2}}. \tag{22}$$

Combining the above-mentioned results, we obtain

$$\Delta g_{(R,\varphi_p,\lambda_p)} = -\frac{\gamma_0}{R}N(\varphi_p,\lambda_p) - \frac{\gamma_0}{16\pi R^3}\iint_\sigma \frac{1}{\sin^3\frac{\psi}{2}}\left[N|_\sigma - N(\varphi_p,\lambda_p)\right]\mathrm{d}\sigma, \tag{23}$$

which is exactly classical inverse Stokes' formula.

3.2 General inverse Vening-Meinesz formula

Similarly, according to operator operation defined by eq. (5), we apply B_2 to eq. (11) and define

$$B_2M(\rho,\psi) = \frac{1}{\rho}V(\rho,\psi), \tag{24}$$

where

$$V(\rho,\psi) = \frac{2(\rho^2\cos\psi - \rho R)}{l^3}\left(\frac{1}{\rho}\cot\psi - \frac{1}{R\sin\psi}\right) - \frac{1}{R}\left(\frac{1}{l}+\frac{1}{\rho}\right)\cot\psi + \frac{1}{R^2\sin\psi}\left(\frac{3\rho}{l} - \frac{l}{\rho} - 2\right) + \frac{1}{R}\left(\frac{1}{\rho}-\frac{1}{R}\right)\cot\psi\ln\frac{l+\rho-R\cos\psi}{2\rho}, \tag{25}$$

then

$$\Delta g_{(\rho,\varphi,\lambda)} = -\frac{1}{4\pi\rho}\iint_\sigma V(\rho,\psi)\frac{\partial T}{\partial\psi}\Big|_\sigma\mathrm{d}\sigma. \tag{26}$$

Eq. (26) is a general inverse Vening-Meinesz formula. The deflection of the vertical ε on the boundary surface is converted into gravity anomaly Δg on the identcal boundary surface or outside of the boundary surface by using eq. (26). Furthermore, if the boundary surface is the geoid on which only the gravity anomaly is computed, eq. (26) degenerates into classical inverse Vening-Meinesz formula by letting $\rho = R$ in eq. (26). The kernel function $V(R,\psi)$ is expressed as

$$V(R,\psi) = -\frac{1}{R^2}\left(3\csc\psi - \csc\psi\csc\frac{\psi}{2} - \tan\frac{\psi}{2}\right). \tag{27}$$

Substituting eq. (27) into eq. (26), we obtain an expression that is identical with the classical inverse Vening-Meinesz formula. We have a good reason to say that the classical inverse Vening-Meinesz formula is only a special case of the general inverse Vening-Meinesz formula.

3.3 General Molodensky formula

The disturbing potential can be transformed into the undulation of the geoid by operator operation B_1. Therefore, letting B_1 apply to eq. (11), we can obtain

$$N_{(\rho,\varphi,\lambda)}=-\frac{1}{4\pi\gamma}\iint\limits_{\sigma}M(\rho,\psi)\frac{\partial T}{\partial\psi}\Big|_{\sigma}\,\mathrm{d}\sigma, \qquad (28)$$

where $M(\rho,\psi)$ is the general direction transformation function (tentative). We call eq. (28) a general Molodensky formula. Eq. (28) transfers the deflection of the vertical ε on the boundary surface into level plane undulation for the gravity potential on the identical boundary surface or into the geoidal undulation. Eq. (28) degenerates into the classical Molodensky's formula if the boundary surface is the geoid on which only the gravity anomaly is computed. Letting $\rho=R$ in eq. (28), the kernel function (R,ψ) is expressed as

$$M(R,\psi)=\frac{1}{R^2}\cot\frac{\psi}{2}. \qquad (29)$$

Substituting eq. (29) into eq. (28), we obtain a classical Molodensky's formula. It must be pointed out that the kernel function $M(R,\psi)$ given by this paper is identical with classical Molodensky's formula under the condition, $\rho=R$. The kernel function $M(R,\psi)$ is slightly different from $C(\psi)$ given by ref. [10] since the initial value of the series is different.

3.4 The deflection of the vertical outside the level plane

The disturbing potential can be transformed into the deflection of the vertical ε by operator operation B_3. Applying B_3 to eq. (7), we obtain

$$\varepsilon_{(\rho,\varphi,\lambda)}=\frac{\gamma_0}{4\pi\rho\gamma}\iint\limits_{\sigma}B(\rho,\psi)N\big|_{\sigma}\,\mathrm{d}\sigma, \qquad (30)$$

where the kernel function $B(\rho,\psi)$ is defined as

$$B(\rho,\psi)=\rho\gamma\bullet B_3K(\rho,\psi), \qquad (31)$$

where the kernel function $B(\rho,\psi)$ is expressed as

$$B(\rho,\psi)=\frac{3\rho\left(\rho^2-R^2\right)\sin\psi}{l^5}. \qquad (32)$$

If the boundary surface is a virtual sphere inside the earth, the deflection of the vertical outside the earth can be computed at arbitrary points by using eq. (32). If the boundary surface is the geoid and the undulation of the geoid is available, we can compute the deflection of the vertical outside the earth. It is significant for checking the deflection of the vertical on flight path.

4 Analysis and conclusion

On the basis of the functional relationships between the disturbing potential and the undulation of the geoid, the gravity anomaly as well as the deflection of the vertical, the operator operations are defined in this paper. By using the regenerated characteristic of the kernel function and the orthogonal property of the spherical harmonics, the general inverse Stokes' formula, the general inverse Vening-Meinesz formula, the general Molodensky's formula and the expression computing the deflection of the vertical outside the level plane are derived under sphere approximation condition. Through analyzing the general formulae, we can obtain some conclusions:

Because all the general formulae include factor ρ, they can be applied to computing the undulation of the geoid, the gravity anomaly and the deflection of the vertical not only on geoid, like the classical formulas, but also at arbitrary points outside the earth. The general formulae supplement the theory of the shape and gravity field of the earth and provide a new idea in physical geodesy.

The general formulae naturally degenerate into the classical formulae if the boundary surface is the geoid on which only the undulation of the geoid, the gravity anomaly and the deflection of the vertical are computed. It can be said that the classical formulae are only the special case of the general formulae.

The concept of Bjerhammar's virtual sphere is adopted in this paper. The kernel functions are easily expressed by the series of spherical harmonics. The derivation processing is simplified.

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