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A GENERALIZED VARIATIONAL PRINCIPLE FOR THE LIMIT ANALYSIS IN SOLID MECHANICS*

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I. INTRODUCTION

Since the establishment of the well-known theorems of upper and lower bounds, considerable advance has been made in the limit analysis as a branch of applied plasticity in solid mechanics. Now, exact determination of the plastic limit load is feasible with no appreciable difficulty for rigid frame structures consisting of the systems of members subjected mainly to bending action. In the field of two- or three-dimensional structures, especially in plate and shell problems, although numerous results have been found, yet further progress seems to be very difficult in tackling more complicate problems. Progress is restricted by the fact that the limit theorems cannot in these cases give sufficiently approached upper and lower bounds. Moreover, it is especially difficult to apply the lower bound theorems to most of problems to satisfaction.

The present paper suggests a generalized variational principle which is different from the bound theorems in that both the stress state σ_{ij} and the velocity field v_i are introduced and vary independently in the functional of the variational principle. This variational principle is equivalent mathematically to the whole set of equations, which must be satisfied by the limit analysis: equilibrium, mechanism, yield condition, flow law, and appropriate boundary conditions. It is proved that with independently assumed kinematically admissible velocity field and statically admissible stress distribution, the generalized variational principle gives the approximate plastic limit load, lying between the upper and lower bounds obtained from bound theorems. Moreover, numerical examples for circular plate and others[3,4,5] show that the generalized variational principle gives rather stable answers for different combinations of properly assumed stress distribution and velocity field.

It is remarked, furthermore, that the generalized variational principle here proposed can be applied to the limit analysis dealing with the nonhomogeneous as well as the anisotropic perfect plastic material.

The first generalized variational principle in elasticity was established in 1950 by E. Reissner[1]. Later, in 1954, Hu Hai-chang[2] obtained some variational principles in elasticity in the most general form. The generalized variational principle provides effective measures to solve approximately many complex problems with the aid of direct method procedures. But the estimation of the errors in the approximate solution is yet more difficult task as matters stand in other variational methods.

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II. Basic Equations

We shall use the term plastic limit load to mean the load at which plastic deformations of arbitrarily large magnitude would take place under constant load if the body or structure possessed the following properties:

1. The material exhibits rigid perfect plasticity, i. e., no deformation at all can occur until the stress state has reached the yield condition when uncontained plastic flow may start.

2. Changes in geometry of the structure that occur at the limit load are insignificant; hence, the geometrical description of the structure remains unchanged during deformation at the limit load.

In other words, the plastic limit load is defined as the plastic collapse load of a hypothetical structure replacing the actual one and having ideal properties listed above. Let:

$v_i(i = 1, 2, 3)$ be velocity components;

$\varepsilon_{ij}(i, j = 1, 2, 3)$ be strain rate components;

$\sigma_{ij}(i, j = 1, 2, 3)$ be stress components;

$X_i(i = 1, 2, 3)$ be body force components;

$\nu T_i(i = 1, 2, 3)$ be surface loads, T_i being denominated as "basic surface loads" and ν as "multiplier load factor" whose limit value is to be sought in limit analysis;

$f(\sigma_{ij}) - \sigma_T^2 = 0$ be the yielding condition of material, in which σ_T is the yield stress in simple tension and $f(\sigma_{ij})$ is a quadratic form of stress components, and σ_T may be taken as varying from point to point for the nonhomogeneous material;

$x_i(i = 1, 2, 3)$ be coordinates;

$n_i(i = 1, 2, 3)$ be the outward-drawn unit normal vector to a surface element.

Consider a structure that has been loaded to the plastic limit state. In general, a part of the body will be plastic and the remaining part will remain in rigid state under the rigid-plastic assumption of material. Let

V_p denote the plastic region where $f(\sigma_{ij}) - \sigma_T^2 = 0$,

V_r denote the rigid region where $f(\sigma_{ij}) - \sigma_T^2 < 0$,

$V = V_p + V_r$ denote the total volume of structure,

$S_{\sigma,p}, S_{\sigma,r}$ denote the parts of boundary surface of body belonging respectively to the plastic region and to the rigid region where the surface loads T_i are prescribed,

$S_{v,p}, S_{v,r}$ denote the parts of boundary surface of body belonging respectively to the plastic region and to the rigid region where the velocities v_i are prescribed to be zero,

$S_\sigma = S_{\sigma,r} + S_{\sigma,p}$ denote the total part of boundary surface where the surface loads T_i are prescribed,

$S_v = S_{v,r} + S_{v,p}$ denote the total part of boundary surface where the velocities v_i are prescribed to be zero,

$S_{r,p}$ denote the surface of separation between the rigid region and the plastic region.

In the plastic limit state, the stress field σ_{ij} and the velocity field v_i must satisfy the following requirements:

(a) Equilibrium equations:

$$\frac{\partial \sigma_{ij}}{\partial x_i} + X_i = 0; \tag{2.1}$$

(b) Flow law in plastic region V_p:

$$\varepsilon_{ij} - \lambda \frac{\partial f}{\partial \sigma_{ij}} = 0, \tag{2.2}$$

where λ is a positive scalar factor and ε_{ij} are related to v_i by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right); \tag{a}$$

(c) Yield condition in plastic region V_p:

$$f(\sigma_{ij}) = \sigma_T^2, \tag{2.3}$$

whereas in the rigid region V_r

$$f(\sigma_{ij}) \leqslant \sigma_T^2; \tag{2.4}$$

(d) Boundary condition on S_σ:

$$\sigma_{ij} n_j = \nu T_i; \tag{2.5}$$

(e) Boundary condition on S_v:

$$v_i = 0. \tag{2.6}$$

The limit analysis of structures consists of solving exactly or approximately the value of ν in Equations (2.1)—(2.6), which yield a unique solution according to the uniqueness theorem.

III. Theorems

Theorem 1. *The plastic limit state of structure is attained when the multiplier load factor ν takes the stationary value of the following expression for arbitrary variations of σ_{ij} and v_i:*

$$\nu = \text{ext} \frac{\int_V \left(\sigma_{ij}\varepsilon_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f} - X_i v_i\right) dV - \int_{S_v} \sigma_{ij} n_j v_i dS}{\int_{S_\sigma} T_i v_i dS}, \tag{3.1}$$

where ε_{ij} is related to v_i by the expression (a); σ_{ij} and v_i are arbitrary independent functions with the only restrictions that $\varepsilon_{ij} = 0$ in V_r, $\sigma_{ij}\varepsilon_{ij} > 0$ in V_p, $f(\sigma_{ij})$ has its maximum value in V_p, σ_{ij} satisfies the equilibrium conditions in V_r and the boundary conditions on $S_{\sigma,r}$ in the accuracy within a constant multiplier.

Proof. With the arbitrary variations $\delta\sigma_{ij}$, δv_i, and $\delta\varepsilon_{ij}$, the condition for stationary ν: $\delta\nu = 0$ is

$$\int_V \sigma_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta\varepsilon_{ij} dV + \int_V \left(\varepsilon_{ij} - \frac{\sigma_{mk}\varepsilon_{mk}}{\sigma_T^2 + f} \frac{\partial f}{\partial \sigma_{ij}}\right) \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta\sigma_{ij} dV -$$

$$- \int_V X_i \delta v_i dV - \nu \int_{S_\sigma} T_i \delta v_i dS - \int_{S_v} (\sigma_{ij} n_j \delta v_i + v_i n_j \delta\sigma_{ij}) dS = 0. \tag{3.2}$$

Noting that $V = V_r + V_p$, $S_p = S_{\sigma,p} + S_{v,p} + S_{r,p}$, and $\varepsilon_{ij} = v_i = 0$ in V_r, we transform the first integral in the above equation by the Gauss theorem to obtain

$$\int_V \sigma_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta\varepsilon_{ij} dV = \int_{V_p} \sigma_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f} \frac{1}{2} \left(\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i}\right) dV =$$

$$= - \int_{V_p} \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f}\right) \delta v_i dV + \int_{S_{\sigma,p}+S_{v,p}+S_{r,p}} \sigma_{ij} n_j \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta v_i dS. \tag{3.3}$$

Substitution of this expression in (3.2) yields

$$- \int_{V_p} \left[\frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f}\right) + X_i\right] \delta v_i dV + \int_{V_p} \left(\varepsilon_{ij} - \frac{\sigma_{mk}\varepsilon_{mk}}{\sigma_T^2 + f} \frac{\partial f}{\partial \sigma_{ij}}\right) \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta\sigma_{ij} dV -$$

$$- \int_{S_{\sigma,p}} \left[\nu T_i - \sigma_{ij} n_j \frac{2\sigma_T^2}{\sigma_T^2 + f}\right] \delta v_i dS - \int_{S_{v,p}} \sigma_{ij} n_j \left(1 - \frac{2\sigma_T^2}{\sigma_T^2 + f}\right) \delta v_i dS -$$

$$- \int_{S_v} v_i n_j \delta\sigma_{ij} dS - \int_{S_{v,r}} \sigma_{ij} n_j \delta v_i dS - \int_{S_{\sigma,r}} \nu T_i \delta v_i dS - \int_{V_r} X_i \delta v_i dV +$$

$$+ \int_{S_{r,p}} \sigma_{ij} n_j \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta v_i dS = 0. \tag{3.4}$$

Since the variations $\delta\sigma_{ij}$ and δv_i are arbitrary, we must have

$$\frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{2\sigma_T^2}{\sigma_T^2 + f}\right) + X_i = 0 \qquad (\text{in } V_p), \tag{3.5}$$

$$\varepsilon_{ij} - \frac{\sigma_{mk}\varepsilon_{mk}}{\sigma_T^2 + f} \frac{\partial f}{\partial \sigma_{ij}} = 0 \qquad (\text{in } V_p), \tag{3.6}$$

$$\sigma_{ij} n_j \frac{2\sigma_T^2}{\sigma_T^2 + f} = \nu T_i \qquad (\text{on } S_{\sigma,p}), \tag{3.7}$$

$$1 - \frac{2\sigma_T^2}{\sigma_T^2 + f} = 0 \qquad (\text{on } S_{v,p}), \tag{3.8}$$

$$v_i = 0 \qquad (\text{on } S_v), \tag{3.9}$$

$$\int_{S_{v,r}} \sigma_{ij} n_j \delta v_i dS + \int_{S_{\sigma,r}} \nu T_i \delta v_i dS - \int_{S_{r,p}} \sigma_{ij} n_j \frac{2\sigma_T^2}{\sigma_T^2 + f} \delta v_i dS + \int_{V_r} X_i \delta v_i dV = 0. \tag{3.10}$$

Multiplying Eq. (3.6) by σ_{ij}, we obtain

$$\sigma_{mk}\varepsilon_{mk} \left(1 - \frac{1}{\sigma_T^2 + f} \frac{\partial f}{\partial \sigma_{ij}} \sigma_{ij}\right) = 0 \quad (\text{in } V_p).$$

Note that $\frac{\partial f}{\partial \sigma_{ij}} \sigma_{ij} = 2f$. Since $f(\sigma_{ij})$ is a quadratic form of σ_{ij}, we are sure that the above equation gives

$$1 - \frac{2f}{\sigma_T^2 + f} = \frac{\sigma_T^2 - f}{\sigma_T^2 + f} = 0, \quad \text{or} \quad f(\sigma_{ij}) = \sigma_T^2(\text{in } V_p).$$

This equation and Eq. (3.8) fulfil the actual yield condition in plastic region. Since the theorem prescribes that $f(\sigma_{ij})$ has its maximum value in plastic region v_p, it follows that $f(\sigma_{ij}) < \sigma_T^2$ in the rigid region. Thus the yield conditions (2.3) and (2.4) are satisfied completely in the whole structure.

Substituting $f = \sigma_T^2$ in Eqs. (3.5), (3.6), (3.7), and (3.9), we arrive at the same equations as (2.1), (2.2), (2.5), and (2.6) respectively. Equation (3.10) gives the equilibrium of the rigid region as a whole. Furthermore, with the prescriptions made in the theorem on σ_{ij} in V_r and on its boundary, the whole equilibrium requirements of the rigid region are satisfied. It can be concluded, therefore, that the condition for stationary value of ν (Eq. 3.1) formulates the limit analysis problem completely.

The above proof can be extended without difficulty to the case in which there are several rigid regions and plastic regions in the limit state.

An important practical advantage of this variational theorem is that it provides much freedom to select the functions σ_{ij} and v_i in applying the Ritz approximation procedure. But more rationally the functions σ_{ij} and v_i are selected, and a more accurate answer can be expected. Thus, when possible, one is recommanded to assign σ_{ij} to satisfy the equilibrium and v_i to satisfy boundary restraint condition so that the yield condition and the flow law are left to fulfil approximately by the variational procedure. When σ_{ij} and v_i are so selected, we can formulate the following

Theorem 2. *If a stress field σ_{ij}^* is selected to satisfy the equilibrium requirements as in the lower bound theorem, and if a velocity field v_i° is selected to satisfy the mechanism requirement as in the upper bound theorem, the plastic limit load deduced from Theorem 1 will be lying between the lower and upper bounds given by the bound theorems.*

Proof. With σ_{ij}^* and v_i° as defined in Theorem 2, Theorem 1 yields

$$\nu = \frac{\int_V \sigma_{ij}^* \varepsilon_{ij}^\circ \frac{2\sigma_T^2}{\sigma_T^2 + f^*} dV - \int_V X_i v_i^\circ dV}{\int_{S_\sigma} T_i v_i^\circ dS}, \tag{3.11}$$

where $f^* = f(\sigma_{ij}^*)$. Here no variational procedure is necessary because σ_{ij}^* and v_i° are supposed to be given definitely, and the last integral in the numerator of Eq. (3.1) is omitted owing to the fact that v_i° satisfies the boundary restraint condition.

With the same σ_{ij}^* and v_i°, the upper and lower bound theorems give respectively

$$\nu_{\text{upper}} = \frac{\int_V \sigma_{ij}^\circ \varepsilon_{ij}^\circ dV - \int_V X_i v_i^\circ dV}{\int_{S_\sigma} T_i v_i^\circ dV}, \tag{3.12}$$

$$\nu_{\text{lower}} = \frac{\int_V \sigma_{ij}^* \varepsilon_{ij}^\circ dV - \int_V X_i v_i^\circ dV}{\int_{S_\sigma} T_i v_i^\circ dV}, \tag{3.13}$$

where σ_{ij}° in (3.12) is related to ε_{ij}° according to the flow law and the yield condition.

For comparison of these three values of ν, we need only compare the magnitude of the first integral in the numerator of Eqs. (3.11), (3.12), and (3.13).

We observe that the stress field $\left(\sigma_{ij}^{*}\dfrac{\sigma_T}{\sqrt{f^*}}\right)$ does satisfy the yield condition but not the flow law with respect to the stain rates ε_{ij}°. With regard to the well-known property of the yield condition — the convexity of the yield surface—we must have

$$\left(\sigma_{ij}^{*}\frac{\sigma_T}{\sqrt{f^*}}\right)\varepsilon_{ij}^{\circ} \leqslant \sigma_{ij}^{\circ}\varepsilon_{ij}^{\circ}. \tag{3.14}$$

On the other hand, it is evident that

$$\sigma_T(\sigma_T - \sqrt{f^*})^2 = \sigma_T^3 - 2\sigma_T^2\sqrt{f^*} + \sigma_T f^* \geqslant 0;$$

hence,

$$\frac{2\sigma_T^2}{\sigma_T^2 + f^*} \leqslant \frac{\sigma_T}{\sqrt{f^*}}. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\frac{2\sigma_T^2}{\sigma_T^2 + f^*}\sigma_{ij}^{*}\varepsilon_{ij}^{\circ} \leqslant \sigma_{ij}^{\circ}\varepsilon_{ij}^{\circ}. \tag{3.16}$$

With this inequality, Eqs. (3.11) and (3.12) yield

$$\nu \leqslant \nu_{\text{upper}}. \tag{3.17}$$

It should be noted that the equality in (3.17) takes place only in the case where σ_{ij}^{*} satisfies furthermore the yield condition as well as the flow law in relation to ε_{ij}°, i. e., $\nu = \nu_{\text{upper}} = \nu_{\text{exact}}$.

We proceed now to compare (3.11) and (3.13). Obviously we have $f^* = f(\sigma_{ij}^{*}) \leqslant \sigma_T^2$, or $f^* + \sigma_T^2 \leqslant 2\sigma_T^2$. Hence

$$\frac{2\sigma_T^2}{f^* + \sigma_T^2}\sigma_{ij}^{*}\varepsilon_{ij}^{\circ} \geqslant \sigma_{ij}^{*}\varepsilon_{ij}^{\circ}, \tag{3.18}$$

which leads from (3.11) and (3.13) to the inequality:

$$\nu \geqslant \nu_{\text{lower}}. \tag{3.19}$$

By combination of (3.17) and (3.19), it follows that

$$\nu_{\text{lower}} \leqslant \nu \leqslant \nu_{\text{upper}}. \tag{3.20}$$

Thus Theorem 2 is proved.

It should be noted that the symbol "ext" in (3.1) could be reasonably replaced by "$\min\limits_{(v_i)}\ \max\limits_{(\sigma_{ij})}$". This symbol means that ν takes the maximum value for all arbitrary variations of σ_{ij} and then takes the minimum value for all arbitrary variations of v_i. Actually, in virtue of the inequality (3.16), the first procedure of taking ν maximum with respect to σ_{ij} insures that $\sigma_{ij} = \sigma_{ij}^{\circ}$ and that the yield condition and the flow law would be satisfied, while the second procedure of taking ν minimum with respect to v_i is equivalent to the well-known theorem of upper bound. Therefore the stationary property of ν corresponds to the saddle point in the space σ_{ij} and v_i.

Theoretically, if σ_{ij} and v_i are chosen to be complete sets of functions, and if the variational procedure follows strictly to the "$\min\limits_{(v_i)} \max\limits_{(\sigma_{ij})}$" cited above, all of the restrictions prescribed in Theorem 1 with respect to σ_{ij} could be omitted because they are not required in the upper bound theorem.

Theorem 3. *Let $\tilde{\sigma}_{ij}$ be a stress field in equilibrium with the basic surface loads T_i (the body forces X_i assumed to be zero). Putting in Eq.* (3.1) *of Theorem 1:* (a) $\sigma_{ij} = \beta\tilde{\sigma}_{ij}$ *where β is a variational parameter and* (b) $v_i = v_i^\circ$, *a kinematically admissible velocity field, provided $\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ > 0$, we get*

$$\nu = \operatorname{ext} \frac{\displaystyle\int_V \beta\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ \frac{2\sigma_T^2}{\sigma_T^2 + \beta^2\tilde{f}}\, dV}{\displaystyle\int_{S_\sigma} T_i v_i^\circ dS}, \tag{3.21}$$

where $\tilde{f} = f(\tilde{\sigma}_{ij})$. The multiplier load factor ν should be not lower than that obtained from the lower bound theorem with stress field $\tilde{\sigma}_{ij}$ and not higher than that obtained from the upper bound theorem with velocity field v_i°, i.e.,

$$\nu_{\text{lower}} \leqslant \nu \leqslant \nu_{\text{upper}}. \tag{3.22}$$

Proof. The second inequality of (3.22): $\nu \leqslant \nu_{\text{upper}}$ has already been established in Theorem 2. We have only to prove that $\nu \geqslant \nu_{\text{lower}}$.

In the right hand of Eq. (3.21), only the numerator, $F(\beta) = \displaystyle\int_V \beta\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ \frac{2\sigma_T^2}{\sigma_T^2 + \beta^2\tilde{f}}\, dV$, is subjected to the variational procedure with respect to the parameter β. From $\dfrac{\partial F}{\partial \beta} = 0$, we obtain

$$\frac{\partial F}{\partial \beta} = \int_V \tilde{\sigma}_{ij}\varepsilon_{ij}^\circ \left[\frac{2\sigma_T^2}{\sigma_T^2 + \beta^2\tilde{f}} - \frac{4\beta^2\sigma_T^2\tilde{f}}{(\sigma_T^2 + \beta^2\tilde{f})^2}\right] dV = 0. \tag{3.23}$$

Obviously $\lim\limits_{\beta\to 0} \dfrac{\partial F}{\partial \beta} > 0$ and $\lim\limits_{\beta\to\infty} \dfrac{\partial F}{\partial \beta} < 0$, hence the rest roots β of (3.23) exist actually. We search the root β which makes ν maximum. Suppose that the lower bound theorem gives ν_{lower} so that $\sigma_{ij}^* = \nu_{\text{lower}}\tilde{\sigma}_{ij}$ is the corresponding stress field. Then we must have $\nu_{\text{lower}} < \beta$, hence

$$\nu = \frac{\displaystyle\int_V \beta\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ \frac{2\sigma_T^2}{\sigma_T^2 + \beta^2\tilde{f}}\, dV}{\displaystyle\int_{S_\sigma} T_i v_i^\circ dS} \geqslant \frac{\displaystyle\int_V \nu_{\text{lower}}\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ \frac{2\sigma_T^2}{\sigma_T^2 + \nu_{\text{lower}}^2\tilde{f}}\, dV}{\displaystyle\int_{S_\sigma} T_i v_i^\circ dS}. \tag{3.24}$$

On the other hand, we have from (3.18)

$$\nu_{\text{lower}}\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ = \sigma_{ij}^*\varepsilon_{ij}^\circ \leqslant \frac{2\sigma_T^2}{\sigma_T^2 + f^*}\sigma_{ij}^*\varepsilon_{ij}^\circ = \frac{2\sigma_T^2}{\sigma_T^2 + \nu_{\text{lower}}^2\tilde{f}}\nu_{\text{lower}}\tilde{\sigma}_{ij}\varepsilon_{ij}^\circ. \tag{3.25}$$

Then

$$\nu_{\text{lower}} = \frac{\int_V \nu_{\text{lower}}\tilde{\sigma}_{ij}\varepsilon_{ij}^{\circ}dV}{\int_{S_\sigma} T_i v_i^{\circ} dS} \leqslant \frac{\int_V \nu_{\text{lower}}\tilde{\sigma}_{ij}\varepsilon_{ij}^{\circ}\dfrac{2\sigma_T^2}{\sigma_T^2 + \nu_{\text{lower}}^2\tilde{f}}dV}{\int_{S_\sigma} T_i v_i^{\circ} dS}. \tag{3.26}$$

By comparison with (3.24), it follows that

$$\nu_{\text{lower}} \leqslant \nu,$$

hence the proof of Theorem 3 is completed.

IV. Illustrative Solutions

The variational theorems described in the previous sections will now be illustrated by application to a specific problem. The problem selected is a simply supported circular plate subjected to uniformly distributed loading. It is known that with the material obeying the Tresca yield condition, the plastic limit load is

$$p = 6\frac{M_T}{a^2}, \tag{4.1}$$

and that the corresponding carrying capacity under the Mises yield condition is

$$p = 6.5\frac{M_T}{a^2}, \tag{4.2}$$

where a is the radius of plate and $M_T = \frac{\sigma_T h^2}{4}$ is the plastic limit bending moment per unit width of cross section of the plate.

We shall apply the variational equation (3.1) to determine the plastic limit load under the Mises yield condition. In the first, we have to transform (3.1) into an appropriate form for bending plate problems. The transformations are:

$$f \to M_r^2 + M_\theta^2 - M_r M_\theta, \quad \sigma_T^2 \to M_T^2,$$

$$v_i \to w, \quad T_i \to \frac{M_T}{a^2},$$

$$\sigma_{ij}\varepsilon_{ij} \to M_r\kappa_r + M_\theta\kappa_\theta,$$

where $\kappa_r = \frac{\partial^2 w}{\partial r^2}$ and $\kappa_\theta = \frac{1}{r}\frac{\partial w}{\partial r}$. We neglect the dead weight of the plate, i.e., $X_i = 0$. The velocity field w will be chosen to satisfy the boundary restraint. Therefore Eq. (3.1) will take the following form:

$$\nu = \text{ext}\frac{\int_0^{2\pi}\int_0^a (M_r\kappa_r + M_\theta\kappa_\theta)\dfrac{2M_T^2}{M_T^2 + f}rdrd\theta}{\dfrac{M_T}{a^2}\int_0^{2\pi}\int_0^a wrdrd\theta}. \tag{4.3}$$

First, we assume that

$$\left.\begin{aligned} w &= \left(1-\frac{r}{a}\right), \\ M_r &= \beta\left(1-\frac{r^2}{a^2}\right), \quad M_\theta = \beta, \end{aligned}\right\} \tag{4.4}$$

which are the actual distributions of velocity and bending moments in the case where the material obeys the Tresca yield condition. Substituting (4.4) in (4.3), we obtain

$$\nu = \text{ext}\,\frac{\beta a \displaystyle\int_0^a \frac{2M_T^2 dr}{M_T^2+\beta^2\left(1-\dfrac{r^2}{a^2}+\dfrac{r^4}{a^4}\right)}}{M_T \displaystyle\int_0^a \left(1-\frac{r}{a}\right) r dr}. \tag{4.5}$$

The variational equation $\dfrac{\partial \nu}{\partial \beta} = 0$ yields

$$\int_0^a \frac{M_T^2-\beta^2\left(1-\dfrac{r^2}{a^2}+\dfrac{r^4}{a^4}\right)}{\left[M_T^2+\beta^2\left(1-\dfrac{r^2}{a^2}+\dfrac{r^4}{a^4}\right)\right]^2} dr = 0. \tag{4.6}$$

We solve this equation by numerical method to get $\beta = 1.079\ M_T$. Substitution of β in (4.5) gives finally

$$\nu = 6.46, \tag{4.7}$$

which differs only by 0.6% from the existing solution (4.2) for the plate obeying the Mises yield condition.

Table 1

	w	M_r	M_θ	ν			β
				By (3.1)	By Upper Bound Theorem	By Lower Bound Theorem	
1	$\left(1-\frac{r}{a}\right)$	$\beta\left(1-\frac{r^2}{a^2}\right)$	β	6.46	6.91	6.00	$1.079M_T$
2	$\left(1-\frac{r^2}{a^2}\right)$	$\beta\left(1-\frac{r^2}{a^2}\right)$	β	6.59	8.00	6.00	$1.098M_T$
3	$\left(1-\frac{r}{a}\right)$	$\beta\left(1-\frac{r}{a}\right)$	β	6.59	6.91	—	$1.080M_T$
4	$\left(1-\frac{r^2}{a^2}\right)$	$\beta\left(1-\frac{r}{a}\right)$	β	5.92	8.00	—	$1.102M_T$

We proceed to assume different sets of w, M_r, and M_θ, which are more or less irrational than that in (4.4). The results are summarized in Table 1, which contains also the corresponding results given by the upper bound theorem, or by the lower bound theorem.

It is seen that the results given by the generalized variational principle are rather stable for different sets of assumed w, M_r, and M_θ. On the other hand, the results from the bound theorems are more sensible to the assumed functions.

The procedure of calculations in this illustration follows actually that described in Theorem 3, which may be considered as some specialization of Theorem 1. It should be noted that the functions of M_r and M_θ in lines 3 and 4 in Table 1 do not satisfy the equilibrium equation of plate, and hence they are not suitable to application of the lower bound theorem.

There are three theorems proved in this paper. The first one states in a general form the generalized variational principle in limit analysis. A comparison with the bound theorems is given by the second theorem. The third theorem is proposed for practical use. With a statically admissible stress field $\beta\tilde{\sigma}^*_{ij}$ and a kinematically admissible velocity field v_i^0, Theorem 3 supplies what is needed to satisfy the yield condition and the flow law through variational procedure.

Finally, a remark may be made with respect to the yield condition $f - \sigma_T^2 = 0$. If appropriate functions $f(\sigma_{ij})$ or $\sigma_T(x_i)$ are used, the variational theorems here proposed could be applied to the cases of an anisotropic or a nonhomogeneous material.

Remark. In the course of discussion after the first presentation of this paper at the Symposium on Limit Analysis and Theory of Plasticity held at Dairen in August 1963, several discussers point out that the multiplier $\dfrac{2\sigma_T^2}{\sigma_T^2 + f}$ in the variational expression (3.1) may be not unique. It is found later that this multiplier can take the form: $\dfrac{3\sigma_T^2 - f}{2\sigma_T^2}$ with the advantage to facilitate the numerical applications and to keep unaltered all the three theorems established in this paper.

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