A three-dimensional diffusion equation of suspended sediment by waves and currents

DING Pingxing, SHI Fengyan and KONG Yazhen

State Key Laboratory of Estuarine and Coastal Research, East China Normal University, Shanghai 200062, China

Abstract Based on the law of mass conservation, a general three-dimensional diffusion equation of suspended sediment due to waves and currents, adaptable to estuarial and coastal areas, is derived by decomposing the instantaneous velocities and concentrations into three-different-time-scale components respectively. A three-dimensional suspended sediment diffusion equation adaptable to actual calculations is available as the result of the parameterizations of turbulent diffusion term and wave diffusion term. Different from the former diffusion equations, the influence of waves and currents on suspended sediment diffusion can be simultaneously reflected in the newly derived equation.

Keywords: **estuarial and coastal areas, waves, currents, suspended diffusion equation.**

As we all know, waves and currents are the main hydrodynamic factors which lead to sediment transport in estuarial and coastal areas, so quite a lot of scholars at home and abroad have studied sediment movement due to waves and currents. Early research mainly followed the orientation of the study on river sediment transport by replacing the shear stress in river sediment transport formula with the bed shear stress under the influence of waves and currents[1, 2]. Nowadays some scholars use tidal current models in which they use the capacity of sediment due to waves and currents in the suspended diffusion equation to explore sediment transport caused by waves and currents[3—6]. Others introduce radiation stress into tidal currents equation to discuss sediment diffusion due to waves and currents in combination with two-dimensional suspended diffusion equation[7—9]. To some extent, these studies have improved the resolution in calculating sediment transport in estuarial and coastal areas. However, at present, most of these sediment transport mathematical models are two-dimensional, and in addition, in these models, the traditional diffusion equation only due to currents is most commonly used and the influence of waves is considered only in source function. Sometimes the effect of waves is considered by using mixing coefficients in order to express the combined influence of waves and currents on suspended sediment[10—12]. There are few studies on three-dimensional sediment transport due to waves and currents. So in theory as well as in application it is very important to set up a suspended sediment diffusion model that can truly reflect the influence of waves and currents. In this note a three-dimensional suspended sediment diffusion equation is theoretically derived from the law of mass conservation.

1 Derivation of a three-dimensional suspended sediment diffusion equation

Based on the law of mass conservation, continuity equation can be gained easily:

$$\frac{\partial c_s}{\partial t} + \frac{\partial u c_s}{\partial x} + \frac{\partial v c_s}{\partial y} + \frac{\partial (w - \omega_s) c_s}{\partial z} = 0, \tag{1}$$

where c_s is suspended sediment concentration, u, v, w are the components of velocity in x, y, z directions respectively, ω_s is the settling velocity of sediment particle. In order to study suspended sediment transport due to waves and currents, the velocity field and suspended sediment concentration field can be decomposed into

$$\begin{cases} u = \overline{U} + u_w + u', \\ v = \overline{V} + v_w + w', \\ w = w_w + w', \\ c_s = \overline{C} + c_w + c', \end{cases} \tag{2}$$

where $\overline{U}$, $\overline{V}$, u_w, v_w, w_w and u', v', w' denote horizontal large-scale background velocity field (such as tidal currents, river flow), wave particle velocity field and turbulent fluctuation velocity field respectively; $\overline{C}$, c_w, c' denote suspended sediment concentrations corresponding to the background velocity field, wave and turbulence respectively.

Substituting (2) into (1), then implementing time average by wave period that is much longer than turbulent feature time scale, a general form of three-dimensional suspended sediment diffusion equation due to waves and currents can be yielded as follows:

$$\frac{\partial C}{\partial t} + \frac{\partial UC}{\partial x} + \frac{\partial VC}{\partial y} + \frac{\partial (W - \omega_s)}{\partial z} = \frac{\partial \tau_{Rx}}{\partial x} + \frac{\partial \tau_{Ry}}{\partial y} + \frac{\partial \tau_{Rz}}{\partial z} + \frac{\partial \tau_{wx}}{\partial x} + \frac{\partial \tau_{wy}}{\partial y} + \frac{\partial \tau_{wz}}{\partial z}, \tag{3}$$

where

$$U = \overline{U} + \bar{u}_w, \quad V = \overline{V} + \bar{v}_w, \quad W = \bar{w}_w, \quad C = \overline{C} + \bar{c}_w \tag{4}$$

denote velocity field and suspended sediment concentration field under the combined action of waves and currents; τ_{Rx}, τ_{Ry}, τ_{Rz} are suspended sediment diffusion caused by turbulent fluctuation, and their forms are as follows:

$$\tau_{Rx} = -\overline{u'c'}, \quad \tau_{Ry} = -\overline{v'c'}, \quad \tau_{Rz} = -\overline{w'c'}, \tag{5}$$

τ_{wx}, τ_{wy}, τ_{wz} are suspended sediment diffusion induced by waves with the definitions:

$$\begin{cases} \tau_{wx} = -(\overline{u_w c_w} - \bar{u}_w \bar{c}_w), \\ \tau_{wy} = -(\overline{v_w c_w} - \bar{v}_w \bar{c}_w), \\ \tau_{wz} = -(\overline{w_w c_w} - \bar{w}_w \bar{c}_w). \end{cases} \tag{6}$$

If there are no waves in the estuarial and coastal areas, or the influence of waves on suspended sediment diffusion is neglected, that is to say, $u_w = v_w = w_w = c_w = 0$ in eq. (2), eq. (3) can be simplified to a three-dimensional suspended sediment diffusion equation only due to currents:

$$\frac{\partial C}{\partial t} + \frac{\partial UC}{\partial x} + \frac{\partial VC}{\partial y} - \frac{\partial \omega_s C}{\partial z} = \frac{\partial \tau_{Rx}}{\partial x} + \frac{\partial \tau_{Ry}}{\partial y} + \frac{\partial \tau_{Rz}}{\partial z}. \tag{7}$$

Suspended sediment diffusion induced by turbulent fluctuation has less order of magnitude compared with that induced by waves in relatively shallow area, especially in the surf zone. Therefore the turbulent term in eq. (3) can be neglected as the following:

$$\frac{\partial C}{\partial t} + \frac{\partial UC}{\partial x} + \frac{\partial VC}{\partial y} + \frac{\partial (W - \omega_s)}{\partial z} = \frac{\partial \tau_{wx}}{\partial x} + \frac{\partial \tau_{wy}}{\partial y} + \frac{\partial \tau_{wz}}{\partial z}. \tag{8}$$

By integrating eq. (3) from water surface to bed and using the boundary conditions, a two-dimensional diffusion equation of suspended sediment by waves and currents can be yielded:

$$\frac{\partial DC}{\partial t} + \frac{\partial DUC}{\partial x} + \frac{\partial DVC}{\partial y} = -F_s + \frac{\partial T_{cx}}{\partial x} + \frac{\partial T_{cy}}{\partial y} + \frac{\partial T_{wx}}{\partial x} + \frac{\partial T_{wy}}{\partial y}, \tag{9}$$

which has been deduced before[13].

2 Parameterizations of turbulent diffusion term and wave diffusion term

In order to resolve the three-dimensional suspended sediment diffusion equation, sediment diffusion terms induced by turbulence and waves must be determined before the velocity field reflecting the co-action of waves and currents is given. The parameterization of sediment diffusion term due to turbulence can be represented as:

$$\tau_{Rx} = \varepsilon_{cx} \frac{\partial C}{\partial x}, \ \tau_{Ry} = \varepsilon_{cy} \frac{\partial C}{\partial y}, \ \tau_{Rz} = \varepsilon_{cz} \frac{\partial C}{\partial z}, \tag{10}$$

where mixing diffusion coefficients ε_{cx}, ε_{cy}, ε_{cz} are related to friction velocity. By neglecting the influence of wave on suspended sediment and substituting eq. (10) into eq. (7) while omitting "—", a well-known and commonly used three-dimensional suspended sediment diffusion equation due to currents alone is yielded:

$$\frac{\partial C}{\partial t} + \frac{\partial UC}{\partial x} + \frac{\partial VC}{\partial y} - \frac{\partial \omega_s C}{\partial z} = \frac{\partial}{\partial x}\left(\varepsilon_{cx} \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_{cy} \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varepsilon_{cz} \frac{\partial C}{\partial z}\right). \tag{11}$$

Equivalently, the parameterization of suspended sediment diffusion term induced by waves is as follows:

$$\tau_{wx} = \varepsilon_{wx} \frac{\partial C}{\partial x}, \ \tau_{wy} = \varepsilon_{wy} \frac{\partial C}{\partial y}, \ \tau_{wz} = \varepsilon_{wz} \frac{\partial C}{\partial z}, \tag{12}$$

where the wave mixing diffusion coefficients ε_{wx}, ε_{wy}, ε_{wz} should be related with wave feature elements such as wave height, wave period or wave particle velocity. And then eq. (8) can be written as

$$\frac{\partial C}{\partial t} + \frac{\partial UC}{\partial x} + \frac{\partial VC}{\partial y} + \frac{\partial (W - \omega_s) C}{\partial z} = \frac{\partial}{\partial x}\left(\varepsilon_{wx} \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_{wy} \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varepsilon_{wz} \frac{\partial C}{\partial z}\right). \tag{13}$$

In general, both wave and turbulence can influence suspended sediment diffusion, so by substituting eqs. (10) and (12) into (3), a three-dimensional suspended sediment diffusion equation can be yielded:

$$\frac{\partial C}{\partial t} + \frac{\partial UC}{\partial x} + \frac{\partial VC}{\partial y} + \frac{\partial (W - \omega_s) C}{\partial z} = \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varepsilon_z \frac{\partial C}{\partial z}\right), \tag{14}$$

where

$$\varepsilon_x = \varepsilon_{cx} + \varepsilon_{wx}, \ \varepsilon_y = \varepsilon_{cy} + \varepsilon_{wy}, \ \varepsilon_z = \varepsilon_{cz} + \varepsilon_{wz} \tag{15}$$

are mixing diffusion coefficients involving turbulence fluctuation and waves. At present, a diffusion equation similar to eq. (14) is used to describe suspended sediment transport due to waves and currents. In the equation the effects of turbulence and waves are involved in mixing diffusion coefficients.

This note derived a three-dimensional diffusion equation suspended sediment by waves and currents, in which the effects of waves and currents on suspended sediment concentration can be embodied clearly. A three-dimensional coupling model, in correspondence to the three-dimensional diffusion equation which reflects wave and current interaction, will be given in another paper. Now we are studying how to determine the wave mixing diffusive coefficients. It is to be wished that the results presented in the note could be helpful for those who investigate sediment transport in estuarial and coastal areas.

Acknowledgement The authors would like to thank Assoc. Prof. Quan Chen, Institute of Estuarine and Coastal Research for checking the manuscript and giving some good Comments. The work was supported by the National Nature Science Foundation of China (Grant No. 49776279) and the National Excellent Youth Foundation of China (Grant No. 49825161).

References

1 Bijker, K. W., Longshore transport computations, *J. of the Waterways, Harb. and Coastal Eng. Div.*, ASCE, 1971, 97: 678.

2 Swart, D. H., Coastal sediment transport, Computation of Longshore Transport, Delft Hydaulics Lab, Report R968, 1976.

3 Dou Xibing, Luo Zhaosen, A two-dimensional suspended sediment mathematical model due to currents and waves, *Hydraulic and Water Transport Science Research* (in Chinese), 1992, 4: 331.

4 Dou Guoren, Dong Fenwu, A sediment mathematical model due to currents and waves at Huanghua Harbor, *The 7th Coastal Engineer Academy Forum* (in Chinese), Beijing: Ocean Press, 1993, 500—506.

5 Dou Guoren, Dong Foenwu, Do Xiping *et al.*, Sediment mathematical model research in estuarial and coastal areas, *Science in China*, Ser. A (in Chinese), 1995, 25(9): 995.

6 Ding Pingxing, He Songlin, Zhang Guoan *et al.*, Predictable analysis of erode and deposit after coastal engineer in semi-closure harbor II mathematical calculate of erode and deposit, *Acta Oceanologica Sinica* (in Chinese), 1997, 19(1): 64.

7 Cao Zude, Wang Guifen, Numerical simulations of wave lifting sediment and tide transporting sediment, *Acta Oceanologica Sinica (in Chinese), 1993, 15(1): 107.*

8 Cao Zude, Wang Guifen, Three-dimensional numerical simulations of tide, wave and sediment, *The 7th Coastal Engineer Academy Forum* (in Chinese), Beijing: Ocean Press, 1993, 584—590.

9 Cao Zude, Wang Yunhong, *Hydrodynamic and Sediment Numerical Simulation* (in Chinese), Tianjin: Tianjin University Press, 1994.

10 O'Connor, B. A., Three-dimensional sediment-transport model, *Coastal, Estuarial and Habour Engineer's Reference Book*, 1994, 199—214.

11 De Vriend, H. J., Two-dimensional horizontal and weakly three-dimensional models of sediment transport due to waves and currents, *Coastal, Estuarial and Harbour Engineer's Reference Book*, 1994, 215—238.

12 Kotopodi, I., Ribberink, J. S., Quasi-3D modeling of suspended sediment transport by currents and waves, *Coastal Eng.*, 1992, 18: 83.

13 Ding Pinxing, Shi Fenyan, Kong Yazhen, The research of sediment diffusion model due to currents and waves at tidal flat, *The 8th Coastal Engineer Academy Forum* (in Chinese), Beijing, Ocean Press, 1997, 754—761.