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Hörmander oscillatory integral operators: A revisit

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Abstract In this paper, we present new proofs for both the sharp L^p estimate and the decoupling theorem for the Hörmander oscillatory integral operator. The sharp L^p estimate was previously obtained by Stein (1986) and Bourgain and Guth (2011) via the TT^* and multilinear methods, respectively. We provide a unified proof based on the bilinear method for both odd and even dimensions. The strategy is inspired by Barron's work (2022) on the restriction problem. The decoupling theorem for the Hörmander oscillatory integral operator can be obtained by the approach in Beltran et al. (2020), where the key observation can be roughly formulated as follows: in a physical space of sufficiently small scale, the variable setting can be essentially viewed as translation-invariant. In contrast, we reprove the decoupling theorem for the Hörmander oscillatory integral operator through the Pramanik-Seeger approximation approach (Pramanik and Seeger (2007)). Both proofs rely on a scale-dependent induction argument, which can be used to deal with perturbation terms in the phase function.

Keywords Hörmander oscillatory integral operators, bilinear method, decoupling inequality, induction argument, Broad-Narrow analysis

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1 Introduction

Let $n \geqslant 2$, $a \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^{n-1})$ be non-negative and supported in $B_1^n(0) \times B_1^{n-1}(0)$, and

$$\phi\colon B_1^n(0) \times B_1^{n-1}(0) \to \mathbb{R}$$

be a smooth function. For any $\lambda \geqslant 1$, define

$$T^\lambda f(x) := \int_{B_1^{n-1}(0)} \mathrm{e}^{2\pi \mathrm{i} \phi^\lambda(x,\xi)} a^\lambda(x,\xi) f(\xi)\, d\xi, \tag{1.1}$$

where $f\colon B_1^{n-1}(0) \to \mathbb{C}$ and

$$a^\lambda(x,\xi) := a(x/\lambda,\xi), \quad \phi^\lambda(x,\xi) := \lambda\phi(x/\lambda,\xi). \tag{1.2}$$

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We say that the operator T^λ is a Hörmander oscillatory integral operator if ϕ satisfies the following Carleson-Sjölin conditions:

(H1) $\operatorname{rank} \partial^2_{x'\xi}\phi(x,\xi) = n-1$ for all $(x,\xi) \in B_1^n(0) \times B_1^{n-1}(0)$ and $x = (x', x_n)$.

(H2) For each $x_0 \in \operatorname{supp}_x a$, the hypersurface

$$\{\partial_x \phi(x_0,\xi) : \xi \in \operatorname{supp} a(x_0,\cdot)\}$$

has non-vanishing Gaussian curvature.

A typical example for the Hörmander oscillatory integral operators is the following extension operator E defined by

$$Ef(x) := \int_{B_1^{n-1}(0)} \mathrm{e}^{2\pi \mathrm{i}(x'\cdot\xi + x_n\psi(\xi))} f(\xi) d\xi \tag{1.3}$$

with

$$\operatorname{rank}\left(\frac{\partial^2\psi}{\partial\xi_i\partial\xi_j}\right)_{(n-1)\times(n-1)} = n-1, \tag{1.4}$$

and it is straightforward to verify that the phase function $\phi(x,\xi) := x'\cdot\xi + x_n\psi(\xi)$ satisfies the conditions (H1) and (H2). For the Hörmander oscillatory integral operators, we revisit the following two important problems: the sharp L^p estimate and the decoupling inequality.

Sharp L^p estimate. Hörmander [12] conjectured that if ϕ satisfies the conditions (H1) and (H2), then

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(B_1^{n-1}(0))} \tag{1.5}$$

for $p > \frac{2n}{n-1}$, and he proved this conjecture for $n = 2$. For the higher-dimensional cases, Stein [17] proved (1.5) for $p \geqslant \frac{2(n+1)}{n-1}$. Later, Bourgain [3] disproved Hörmander's conjecture and showed that Stein's result is sharp in the odd dimensions. For the even dimensions, up to an endpoint, Bourgain and Guth [5] established the sharp result. In summary, we may state the results as follows.

Theorem 1.1 (See [5,17]). *Let $n \geqslant 3$ and T^λ be a Hörmander oscillatory integral operator as in* (1.1). *For all $\varepsilon > 0$ and $\lambda \geqslant 1$,*

$$\|T^\lambda f\|_{L^p(\mathbb{R}^n)} \lesssim_{\varepsilon,\phi,a} \lambda^\varepsilon \|f\|_{L^p(B_1^{n-1}(0))} \tag{1.6}$$

holds whenever

$$p \geqslant \begin{cases} \dfrac{2(n+1)}{n-1} & \text{for } n \text{ odd}, \\ \dfrac{2(n+2)}{n} & \text{for } n \text{ even}. \end{cases} \tag{1.7}$$

Stein's proof is based on the TT^* method and gives the range $p \geqslant \frac{2(n+1)}{n-1}$ in all the dimensions. However, this result is not sharp in even dimensions. Bourgain and Guth [5] resolved the even-dimensional cases up to the endpoints using the Broad-Narrow approach. Bourgain-Guth's method can also be applied to the odd-dimensional cases (see [11] for details). We give another proof based on the bilinear approach. We take the extension operator as a model case to illustrate how one can derive the linear estimate for the oscillatory integral operator from its bilinear counterpart. To this end, we first recall a sharp bilinear restriction theorem of Lee [15].

Theorem 1.2 (See [15]). *Suppose that $\xi \in B_1^{n-1}(0)$ and the* Hessian matrix *of ϕ is nondegenerate, i.e.,*

$$\det \mathcal{H}\phi(\xi) \neq 0.$$

Additionally, let V_1 and V_2 be two sufficiently small balls contained in $B_1^{n-1}(0)$, and suppose that for all $\xi' \in V_1$, $\xi'' \in V_2$, and $\xi_i \in V_i$, $i = 1, 2$,

$$|\langle (\mathcal{H}\phi)^{-1}(\xi_i)(\nabla\phi(\xi') - \nabla\phi(\xi'')), \nabla\phi(\xi') - \nabla\phi(\xi'')\rangle| \geqslant c > 0. \tag{1.8}$$

Then,

$$\||Ef_1 Ef_2|^{\frac{1}{2}}\|_{L^p(\mathbb{R}^n)} \leqslant R^\varepsilon \|f_1\|_{L^2}^{\frac{1}{2}} \|f_2\|_{L^2}^{\frac{1}{2}} \tag{1.9}$$

for $p \geqslant \frac{2(n+2)}{n}$.

To derive the linear estimate from Theorem 1.2, an important step is to identify the exceptional set, where the condition (1.8) fails. When the Hessian of ϕ has eigenvalues of the same sign, the separation of V_1 and V_2 is sufficient to guarantee the condition (1.8). However, this fact does not hold when the Hessian of ϕ has eigenvalues with different signs. For example, when $n = 2$, if $\phi_{\rm hyp}(\xi) = \xi_1\xi_2$, the exceptional set may be contained in a small neighborhood of coordinates. For the general phase $\phi_{\rm M}$ which can be viewed as a small perturbation of $\phi_{\rm hyp}$, identifying the exceptional set is a bit tricky. There are a number of papers by Buschenhenke et al. [6–9] which are dedicated to the study of the restriction estimate associated with the phase $\phi_{\rm M}$. However, it is still murky to find the exceptional set for the phase $\phi_{\rm M}$ in the higher-dimensional cases. To circumvent this issue, inspired by the work of [10], we consider a class of scale-dependent phase functions. Their exceptional set can be connected with the quadratic cases of which the exceptional set is clear.

Decoupling theorem. Assume that $\{\theta\}$ is a collection of finitely overlapping balls in $\mathbb{R}^{n-1}$ of radius $R^{-1/2}$ which form a cover of $B_1^{n-1}(0)$. Define

$$f_\theta := f\kappa_\theta, \quad \sum_\theta \kappa_\theta = 1, \quad \forall\,\xi \in B_1^{n-1}(0),$$

where $\{\kappa_\theta\}$ is a family of smooth functions which are subjecting to $\{\theta\}$. Correspondingly, we decompose $T^\lambda f$ into

$$T^\lambda f := \sum_\theta T^\lambda f_\theta.$$

We have the following decoupling theorem for the Hörmander oscillatory operator.

Theorem 1.3. *Let T^λ be a Hörmander oscillatory integral operator as in* (1.1). *If $p \geqslant \frac{2(n+1)}{n-1}$, then*

$$\Big\|\sum_\theta T^\lambda f_\theta\Big\|_{L^p(B_R^n(x_0))} \leqslant C_\varepsilon R^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon}\bigg(\sum_\theta \|T^\lambda f_\theta\|^p_{L^p(w_{B_R^n(x_0)})}\bigg)^{\frac{1}{p}} + \mathrm{RapDec}(R)\|f\|_2, \tag{1.10}$$

where $w_{B_R^n(x_0)}$ is a non-negative weight function adapted to the ball $B_R^n(x_0)$ such that

$$w_{B_R^n(x_0)}(x) \lesssim (1 + R^{-1}|x - x_0|)^{-L}$$

for some large constant $L \in \mathbb{N}$.

The decoupling theorem for the extension operator was established by Bourgain and Demeter [4]. When the phase function satisfies the cinematic curvature condition, the associated variable version of the decoupling theorem was established by Beltran et al. [2] (see also [13]). Their method can also be applied to the Hörmander oscillatory integral operator. A key observation in [2] can be roughly formulated as follows: at the small scale of physical space, the variable setting is essentially translation invariant. Hence, the decoupling theorem for the flat version can be brought into play directly at the level of a small scale of physical space.

We present an alternative proof of Theorem 1.3 based on Pramanik-Seeger's approach [16]. To be more precise, we first conduct a localization procedure in frequency space. In this setting, the key is to effectively control the error term so that we can directly use the techniques in the translation-invariant setting.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries which are useful for the proof of the main theorem. In Section 3, we prove the sharp L^p estimate for the Hörmander oscillatory integral operator. In Section 4, we provide the proof of the decoupling theorem for Hörmander oscillatory integral operators.

Notations. For nonnegative quantities X and Y, we write $X \lesssim Y$ to denote the inequality $X \leqslant CY$ for some $C > 0$. If $X \lesssim Y \lesssim X$, we write $X \sim Y$. We write $x \mapsto y$ to mean that we replace x by y. Dependence of implicit constants on the spatial dimensions or integral exponents such as p will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, $X \lesssim_u Y$

indicates $X \leqslant CY$ for some $C = C(u)$. We write $A(R) \leqslant \mathrm{RapDec}(R)B$ to mean that for any power $\beta \in \mathbb{N}$, there is a constant C_β such that

$$|A(R)| \leqslant C_\beta R^{-\beta} B \quad \text{for all } R \geqslant 1.$$

We also often abbreviate $\|f\|_{L^r_x(\mathbb{R}^n)}$ to $\|f\|_{L^r}$. For $1 \leqslant r \leqslant \infty$, we use r' to denote the dual exponent to r such that $\frac{1}{r} + \frac{1}{r'} = 1$. Throughout the paper, χ_E is the characteristic function of the set E. We usually denote by $B^n_r(a)$ a ball in $\mathbb{R}^n$ with center a and radius r. We also denote by B^n_R a ball of radius R and an arbitrary center in $\mathbb{R}^n$. For a function $\varphi \in C^\infty(\mathbb{R}^n)$ and $r > 0$, we define $\varphi_r(x) = r^{-n}\varphi(x/r)$.

We define the Fourier transform on $\mathbb{R}^n$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} f(x)\, dx := \mathcal{F}f(\xi),$$

and the inverse Fourier transform by

$$\check{g}(x) := \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi} g(\xi) d\xi := (\mathcal{F}^{-1}g)(x).$$

2 Basic reductions

Let $m \in \mathbb{N}$ be the number of the positive eigenvalues of the hypersurfaces

$$\{\partial_x \phi(x,\xi) : (x,\xi) \in B^n_1(0) \times B^{n-1}_1(0)\}.$$

Instead of dealing with the phase ϕ directly, we actually reduce it to a special class of functions. Let M be a diagonal matrix with its entries being either -1 or 1 in the diagonal. Analytically, we can express M as follows:

$$M = -I_{n-1-m} \oplus I_m$$

for some $1 \leqslant m \leqslant \lfloor \frac{n-1}{2} \rfloor$[1)].

Definition 2.1. Let $K \geqslant 1$ and $\phi_K : B^n_1(0) \times B^{n-1}_1(0) \mapsto \mathbb{R}$ with

$$\phi_K(x,\xi) = x' \cdot \xi + x_n \langle M\xi, \xi\rangle + \mathrm{E}_K(x,\xi). \tag{2.1}$$

We say that the phase function $\phi_K(x,\xi)$ is *asymptotically flat* if

$$|\partial^\alpha_x \partial^\beta_\xi \mathrm{E}_K(x,\xi)| \leqslant C_{\alpha,\beta} K^{-2}, \quad (\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}, \quad |\alpha| \leqslant N_{\mathrm{ph}}, \quad |\beta| \leqslant N_{\mathrm{ph}}, \tag{2.2}$$

where $N_{\mathrm{ph}} \in \mathbb{N}$ is a large integer and $C_{\alpha,\beta} > 0$ is a constant depending on α and β but not on K.

Remark 2.2. The phase function $\phi_K(x,\xi)$ in (2.1) depends on the scale of the ambient space. We can exploit the properties in (2.1) and (2.2) in the process of induction on scales argument since the balls shrink after the parabolic rescaling transformation.

In the following part, let $R \gg 1$, $K = K_0 R^\delta$ for some constants $K_0 > 0$ and $\delta > 0$ to be chosen later, and define the operator T^λ_K as follows:

$$T^\lambda_K f(x) := \int_{B^{n-1}_1(0)} \mathrm{e}^{\mathrm{i}\phi^\lambda_K(x,\xi)} \mathfrak{a}^\lambda(x,\xi) f(\xi) d\xi, \tag{2.3}$$

where ϕ^λ_K and $\mathfrak{a}^\lambda$ are defined in the same way with (1.2) and $\mathfrak{a}$ is a smooth cut function in $\mathbb{R}^n \times \mathbb{R}^{n-1}$ satisfying: supp $\mathfrak{a}(x,\xi) \subset B^n_1(0) \times B^{n-1}_1(0)$ and

$$|\partial^\alpha_x \partial^\beta_\xi \mathfrak{a}(x,\xi)| \leqslant \bar{C}_{\alpha,\beta}, \quad (\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^{n-1}, \quad |\alpha| \leqslant N_{\mathrm{am}}, \quad |\beta| \leqslant N_{\mathrm{am}} \tag{2.4}$$

for an appropriate large constant $N_{\mathrm{am}} \in \mathbb{N}$.

1) $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Lemma 2.3. *Let T^λ be a Hörmander oscillatory integral operator defined by* (1.1) *and $\delta \ll \varepsilon$. Then there exists a function ϕ_K which is asymptotically flat and an input function $\tilde{f}$ defined by*

$$\tilde{f}(\xi) := K^{-(n-1)} f(\bar{\xi} + K^{-1}\xi) \quad \text{for some } \bar{\xi} \in B_1^{n-1}(0) \tag{2.5}$$

such that

$$\|T^\lambda f\|_{L^p(B_R^n(0))}^p \lesssim_{\phi,\varepsilon} R^\varepsilon \sum_{B_{\tilde{R}}^n \subset \square_R} \|T_{\tilde{K}}^{\tilde{\lambda}} \tilde{f}\|_{L^p(B_{\tilde{R}}^n)}^p, \tag{2.6}$$

where $\tilde{R} := R/K^2$, $\tilde{K} := K_0 \tilde{R}^\delta$, $\square_R$ is a rectangular box of dimensions $R/K \times \cdots \times R/K \times R/K^2$, and $\{B_{\tilde{R}}^n\}$ is a finitely overlapping partition of $\square_R$.

Proof. Without loss of generality, we may assume

$$|\partial_x^\alpha \partial_\xi^\beta \phi(x,\xi)| \leqslant C_{\alpha,\beta} K^{-2}, \quad 2 \leqslant |\alpha| \leqslant N_{\mathrm{ph}}, \quad |\beta| \leqslant N_{\mathrm{ph}}.$$

Otherwise, we may replace $\phi(x,\xi)$ by $\phi(x/A,\xi)$, where A is a sufficiently large constant depending on K. It should be noted that the support of $a(x/A,\xi)$ may be not contained in $B_1^n(0) \times B_1^{n-1}(0)$, but this can be fixed by a partition of unity argument.

Covering $B_1^{n-1}(0)$ by a collection of balls $\{\tau\}$ of radius K^{-1} and define $f_\tau := f\chi_\tau$. By the triangle inequality, we have

$$\|T^\lambda f\|_{L^p(B_R^n(0))} \leqslant \sum_\tau \|T^\lambda f_\tau\|_{L^p(B_R^n(0))}.$$

Thus, there exits a τ_0 such that

$$\sum_\tau \|T^\lambda f_\tau\|_{L^p(B_R^n(0))} \lesssim K^{n-1} \|T^\lambda f_{\tau_0}\|_{L^p(B_R^n(0))}.$$

Without loss of generality, we may assume ξ_{τ_0} is the center of τ_0 and

$$\partial_x^\alpha \phi^\lambda(x,\xi_{\tau_0}) = 0, \quad \partial_\xi^\beta \phi^\lambda(0,\xi) = 0, \quad \alpha \in \mathbb{N}^n, \quad \beta \in \mathbb{N}^{n-1}.$$

Otherwise, we take ϕ^λ to be

$$\phi^\lambda(x,\xi) + \phi^\lambda(0,\xi_{\tau_0}) - \phi^\lambda(0,\xi) - \phi^\lambda(x,\xi_{\tau_0}).$$

By an affine transformation in x, we may also assume the unit normal vector of the hypersurface $\{\partial_x \phi^\lambda(0,\xi) : \xi \in \tau_0\}$ at $\xi = \xi_{\tau_0}$ equals $(0,\ldots,0,1)$ and $\partial_{\xi\xi}\partial_{x_n}\phi^\lambda(0,\xi_{\tau_0}) = M$. Thus, we have

$$\mathrm{rank}\partial_{x'}\partial_\xi \phi^\lambda(x,\xi) = n-1, \quad (x,\xi) \in B_1^n(0) \times B_{K^{-1}}^{n-1}(\xi_{\tau_0}).$$

Then, by the inverse function theorem, there exists a function $\Phi^\lambda(x',x_n)$ such that

$$\partial_\xi \phi^\lambda(\Phi^\lambda(x',x_n), x_n, \xi_{\tau_0}) = x'.$$

By a change of variables in ξ, i.e., $\xi \mapsto \xi + \xi_{\tau_0}$, and Taylor's formula, we have

$$\begin{aligned}\phi^\lambda(x,\xi+\xi_{\tau_0}) = {}& \partial_\xi \phi^\lambda(x,\xi_{\tau_0}) \cdot \xi + \frac{1}{2}\langle \partial_{\xi\xi}^2 \phi^\lambda(x,\xi_{\tau_0})\xi, \xi\rangle \\ & + 3\sum_{|\beta|=3} \frac{\xi^\beta}{\beta!} \int_0^1 (1-t)^2 \partial_\xi^\beta \phi^\lambda(x,\xi_{\tau_0}+t\xi)dt.\end{aligned} \tag{2.7}$$

We make another change of variables in x:

$$x' \mapsto \Phi^\lambda(x',x_n), \quad x_n \mapsto x_n,$$

such that in the new coordinates, the phase becomes

$$\langle x',\xi\rangle+\frac{1}{2}\langle\partial^2_{\xi\xi}\phi^\lambda(\Phi^\lambda(x',x_n),x_n,\xi_{\tau_0})\xi,\xi\rangle$$
$$+3\sum_{|\beta|=3}\frac{\xi^\beta}{\beta!}\int_0^1(1-t)^2\partial_\xi^\beta\phi^\lambda(\Phi^\lambda(x',x_n),x_n,\xi_{\tau_0}+t\xi)dt.$$

Let

$$\mathcal{A}^\lambda_\phi(x,\xi_{\tau_0}):=\lambda\mathcal{A}_\phi(x/\lambda,\xi):=\partial^2_{\xi\xi}\phi^\lambda(\Phi^\lambda(x',x_n),x_n,\xi_{\tau_0}).$$

Then a Taylor expansion in x yields

$$\langle x',\xi\rangle+\frac{1}{2}x_n(\partial_{x_n}\langle\mathcal{A}^\lambda_\phi(x,\xi_{\tau_0})\xi,\xi\rangle)\Big|_{x=0}+\frac{1}{2}x'\cdot(\partial_{x'}\langle\mathcal{A}^\lambda_\phi(x,\xi_{\tau_0})\xi,\xi\rangle)\Big|_{x=0}$$
$$+2\sum_{|\alpha|=2}\frac{x^\alpha}{\alpha!}\int_0^1(1-t)\partial_z^\alpha\langle\mathcal{A}^\lambda_\phi(z,\xi_{\tau_0})\xi,\xi\rangle\Big|_{z=tx}dt$$
$$+3\sum_{|\beta|=3}\frac{\xi^\beta}{\beta!}\int_0^1(1-t)^2\partial_\xi^\beta\phi^\lambda(\Phi^\lambda(x',x_n),\xi_{\tau_0}+t\xi)dt.$$

We make a further diffeomorphic change of variables in $\xi\mapsto\rho(\xi)$ such that in the new coordinates, $\langle x',\xi\rangle+\frac{1}{2}x'\cdot(\partial_{x'}\langle\mathcal{A}^\lambda_\phi(x,\xi_{\tau_0})\xi,\xi\rangle)|_{x=0}$ becomes $\langle x',\xi\rangle$. It is obvious that $\rho(0)=0$, and thus a further Taylor expansion in ξ for $\frac{1}{2}x_n(\partial_{x_n}\langle\mathcal{A}^\lambda_\phi(x,\xi_{\tau_0})\rho(\xi),\rho(\xi)\rangle)|_{x=0}$, up to an affine transformation in ξ, we have

$$\frac{1}{2}x_n(\partial_{x_n}\langle\mathcal{A}^\lambda_\phi(x,\xi_{\tau_0})\rho(\xi),\rho(\xi)\rangle)\Big|_{x=0}=\frac{1}{2}x_n\langle M\xi,\xi\rangle+x_nr(\xi),$$

where $r(\xi)=O(|\xi|^3)$. Define

$$\mathrm{E}(x,\xi):=2\sum_{|\alpha|=2}\frac{x^\alpha}{\alpha!}\int_0^1(1-t)\partial_z^\alpha\langle\mathcal{A}_\phi(z,\xi_{\tau_0})\rho(\xi),\rho(\xi)\rangle\Big|_{z=tx}dt$$
$$+3\sum_{|\beta|=3}\frac{(\rho(\xi))^\beta}{\beta!}\int_0^1(1-t)^2\partial_\xi^\beta\phi(\Phi(x',x_n),\xi_{\tau_0}+t\rho(\xi))dt+x_nr(\xi).$$

Correspondingly, the phase function becomes

$$\langle x',\xi\rangle+\frac{1}{2}x_n\langle M\xi,\xi\rangle+\mathrm{E}^\lambda(x,\xi),$$

where $\mathrm{E}^\lambda(x,\xi):=\lambda\mathrm{E}(x/\lambda,\xi)$. Define $\tilde\lambda:=\lambda/K^2$, $\tilde R:=R/K^2$, and $\tilde K:=K_0(\tilde R)^\delta$. We perform a parabolic rescaling

$$\xi\mapsto K^{-1}\xi,\quad x'\mapsto Kx',\quad x_n\mapsto K^2x_n.$$

The phase function becomes

$$\phi^{\tilde\lambda}_{\tilde K}(x,\xi):=\langle x',\xi\rangle+x_n\langle M\xi,\xi\rangle+\mathrm{E}^{\tilde\lambda}_{\tilde K}(x,\xi),$$

where

$$\mathrm{E}_{\tilde K}(x,\xi):=K^2\mathrm{E}(K^{-1}x',x_n,K^{-1}\xi)$$

and $\mathrm{E}^{\tilde\lambda}_{\tilde K}(x,\xi):=\tilde\lambda\mathrm{E}_{\tilde K}(x/\tilde\lambda,\xi)$. Finally, we have

$$T^{\tilde\lambda}_{\tilde K}\tilde f(x)=\int_{B_1^{n-1}(0)}\mathrm{e}^{2\pi\mathrm{i}\phi^{\tilde\lambda}_{\tilde K}(x,\xi)}\mathfrak{a}^{\tilde\lambda}(x,\xi)\tilde f(\xi)d\xi.\tag{2.8}$$

Note our assumption on $\tilde\phi$, and it is straightforward to verify that $\mathrm{E}_{\tilde K}(x,\xi)$ satisfies the condition

$$|\partial_x^\alpha\partial_\xi^\beta\mathrm{E}_{\tilde K}(x,\xi)|\leqslant C_{\alpha,\beta}\tilde K^{-2},\quad(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^{n-1},\quad|\alpha|\leqslant N_{\mathrm{ph}},\quad|\beta|\leqslant N_{\mathrm{ph}}.\tag{2.9}$$

Thus, $\phi_{\tilde{K}}$ is asymptotically flat. Under the new coordinates, the phase function becomes $\phi^{\tilde{\lambda}}_{\tilde{K}}$. By tracking the change of variables of ξ and x, it is easy to see that the ball B^n_R is transformed into another region which is contained in a box $\square_R$ of dimensions $R/K \times \cdots \times R/K \times R/K^2$, and by choosing K_0 sufficiently large, the conditions (2.2) and (2.4) can be ensured. □

3 Proof of the sharp L^p estimate

Reduction. To prove Theorem 1.1, it suffices to show that for each $1 \leqslant R \leqslant \lambda$,

$$\|T^\lambda f\|_{L^p(B^n_R(0))} \leqslant C_\varepsilon R^\varepsilon \|f\|_{L^p(B^{n-1}_1(0))} \tag{3.1}$$

under the assumption (1.7). The dependence of the implicit constant on n, p, and ϕ is compressed. By Lemma 2.3, it is reduced to showing that for each $1 \leqslant R \leqslant \lambda$,

$$\|T^\lambda_K f\|_{L^p(B^n_R)} \leqslant C_\varepsilon R^\varepsilon \|f\|_{L^p(B^{n-1}_1(0))} \tag{3.2}$$

for all T^λ_K as in (2.3). Indeed, by Lemma 2.3, we have

$$\|T^\lambda f\|_{L^p(B^n_R(0))} \lesssim_{\phi,\varepsilon} R^\varepsilon \sum_{B^n_{\tilde{R}} \subset \square_R} \|T^{\tilde{\lambda}}_{\tilde{K}} \tilde{f}\|^p_{L^p(B^n_{\tilde{R}})},$$

where

$$T^{\tilde{\lambda}}_{\tilde{K}} \tilde{f}(x) = \int_{B^{n-1}_1(0)} \mathrm{e}^{2\pi \mathrm{i} \phi^{\tilde{\lambda}}_{\tilde{K}}(x,\xi)} \mathfrak{a}^{\tilde{\lambda}}(x,\xi) \tilde{f}(\xi) d\xi$$

and $\tilde{f}$ is defined by (2.5). Note that $K = K_0 R^\delta$, and there exists a $\bar{B}^n_{\tilde{R}} \subset \square_R$ such that

$$\sum_{B^n_{\tilde{R}} \subset \square_R} \|T^{\tilde{\lambda}}_{\tilde{K}} \tilde{f}\|^p_{L^p(B^n_{\tilde{R}})} \lesssim K^{n-1} \|T^{\tilde{\lambda}}_{\tilde{K}} \tilde{f}\|^p_{L^p(\bar{B}^n_{\tilde{R}})}.$$

From (3.2), it follows that

$$\|T^{\tilde{\lambda}}_{\tilde{K}} \tilde{f}\|^p_{L^p(\bar{B}^n_{\tilde{R}})} \leqslant C_\varepsilon R^\varepsilon \|\tilde{f}\|_{L^p(B^{n-1}_1(0))}.$$

By choosing $\delta = \varepsilon^2 \ll 1$, we obtain the desired result (3.1).

Let $1 \leqslant R \leqslant \lambda$ and $Q_p(\lambda, R)$ be the optimal constant such that

$$\|T^\lambda_K f\|_{L^p(B^n_R)} \leqslant Q_p(\lambda, R) \|f\|_{L^p(B^n_1(0))} \tag{3.3}$$

holds for all asymptotically flat phase ϕ_K in Definition 2.1 and for all $\mathfrak{a}$ satisfying (2.4), and uniformly for all $f \in L^p(B^{n-1}_1(0))$. Then, (3.2) is reduced to showing

$$Q_p(\lambda, R) \leqslant C_\varepsilon R^\varepsilon. \tag{3.4}$$

We proceed to prove (3.4) via an induction on scale argument. For this purpose, we first set up some basic preparatory tools.

3.1 Parabolic rescaling and flat decoupling

In this subsection, we establish the parabolic rescaling lemma which connects the estimates at different scales and plays a critical role in the induction argument. To that end, we first prove an auxiliary proposition.

Proposition 3.1. *Let $\mathcal{D}$ be a maximal R^{-1}-separated discrete subset of $\Omega \subset B^{n-1}_1(0)$. Then,*

$$\Big\| \sum_{\xi_\theta \in \mathcal{D}} \mathrm{e}^{2\pi \mathrm{i} \phi^\lambda_K(\cdot,\xi_\theta)} F(\xi_\theta) \Big\|_{L^p(B^n_R(0))} \lesssim Q_p(\lambda, R) R^{\frac{n-1}{p'}} \|F\|_{\ell^p(\mathcal{D})} \tag{3.5}$$

for all $F:\mathcal{D}\to\mathbb{C}$*, where*

$$\|F\|_{\ell^p(\mathcal{D})}:=\bigg(\sum_{\xi_\theta\in\mathcal{D}}|F(\xi_\theta)|^p\bigg)^{\frac{1}{p}}$$

for $1\leqslant p<\infty$*.*

Proof. Let η be a bump smooth function on $\mathbb{R}^{n-1}$, which is supported on $B_2^{n-1}(0)$ and equals 1 on $B_1^{n-1}(0)$. For each $\xi_\theta\in\mathcal{D}$, we set $\eta_\theta(\xi):=\eta(10R(\xi-\xi_\theta))$. In exactly the same way as in the proof of [11, Lemma 11.8], we have

$$\bigg|\sum_{\xi_\theta\in\mathcal{D}}\mathrm{e}^{2\pi\mathrm{i}\phi_K^\lambda(\cdot,\xi_\theta)}F(\xi_\theta)\bigg|\lesssim R^{n-1}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)}|T_K^\lambda f_k(x)|,\tag{3.6}$$

where

$$f_k(\xi):=\sum_{\xi_\theta\in\mathcal{D}}F(\xi_\theta)c_{k,\theta}(\xi)\eta_\theta(\xi)$$

with $\|c_{k,\theta}(\xi)\|_\infty\leqslant 1$. By the definition of $Q_p(\lambda,R)$ and (3.6), we get

$$\bigg\|\sum_{\xi_\theta\in\mathcal{D}}\mathrm{e}^{2\pi\mathrm{i}\phi_K^\lambda(\cdot,\xi_\theta)}F(\xi_\theta)\bigg\|_{L^p(B_R^n(0))}\lesssim Q_p(\lambda,R)R^{n-1}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)}\|f_k\|_{L^p(B_1^{n-1}(0))}.$$

The supports of $\{\eta_\theta\}$ are pairwise disjoint, for any $q\geqslant 1$, we have

$$\|f_k\|_{L^q(B_2^{n-1}(0))}\lesssim R^{-\frac{n-1}{q}}\|F\|_{\ell^q(\mathcal{D})}.$$

Thus, we get

$$\begin{aligned}\bigg\|\sum_{\xi_\theta\in\mathcal{D}}\mathrm{e}^{2\pi\mathrm{i}\phi_K^\lambda(\cdot,\xi_\theta)}F(\xi_\theta)\bigg\|_{L^p(B_R^n(0))}&\lesssim Q_p(\lambda,R)R^{n-1}\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-(n+1)}R^{-\frac{n-1}{p}}\|F\|_{\ell^p(\mathcal{D})}\\&\lesssim Q_p(\lambda,R)R^{\frac{n-1}{p'}}\|F\|_{\ell^p(\mathcal{D})}.\end{aligned}$$

This completes the proof. □

Lemma 3.2 (Parabolic rescaling)**.** *Let* $1\leqslant R\leqslant\lambda$*, and* f *be supported in a ball of radius* K^{-1}*, where* $1\leqslant K\leqslant R$*. Then, for all* $p\geqslant 2$ *and* $\delta>0$*, we have*

$$\|T_K^\lambda f\|_{L^p(B_R^n(0))}\lesssim_\delta Q_p\bigg(\frac{\lambda}{K^2},\frac{R}{K^2}\bigg)R^\delta K^{\frac{2n}{p}-(n-1)}\|f\|_{L^p(B_1^{n-1}(0))}.\tag{3.7}$$

Proof. Without loss of generality, we may assume $\operatorname{supp}f\subset B_{K^{-1}}^{n-1}(\bar\xi)$. In the same argument as in Section 2, we obtain

$$\|T_K^\lambda f\|_{L^p(B_R^n(0))}\lesssim_\delta K^{\frac{n+1}{p}}\|\widetilde{T}_{\tilde K}^{\tilde\lambda}\tilde f\|_{L^p(\square_R)},$$

where $\square_R$ and $\tilde f$ are defined in Lemma 2.3 and

$$\tilde\lambda=K^{-2}\lambda,\quad \tilde K=K^{1-2\varepsilon^2}.\tag{3.8}$$

Note that for $q\geqslant 1$,

$$\|\tilde f\|_{L^q(B_1^{n-1}(0))}\leqslant K^{-(n-1)+(n-1)/q}\|f\|_{L^q(B_1^{n-1}(0))},$$

and it suffices to show that

$$\|\widetilde{T}_{\tilde K}^{\tilde\lambda}\tilde f\|_{L^p(\square_R)}\lesssim_\delta Q_p(\tilde\lambda,\tilde R)R^\delta\|\tilde f\|_{L^p(B_1^{n-1}(0))}.$$

To simplify notations, we just need to show

$$\|T_K^\lambda f\|_{L^p(\square(R,R'))}\lesssim_\delta Q_p(\lambda,R)R^\delta\|f\|_{L^p(B_1^{n-1}(0))}\tag{3.9}$$

for all $1 \ll R \leqslant R' \leqslant \lambda$ and $\delta > 0$, where

$$\square(R,R') := \left\{ x = (x', x_n) \in \mathbb{R}^n : \left(\frac{|x'|}{R'}\right)^2 + \left(\frac{|x_n|}{R}\right)^2 \leqslant 1 \right\}.$$

Choosing a collection of essentially disjoint R^{-1}-balls θ which covers $B_1^{n-1}(0)$, we denote the center of θ by ξ_θ and decompose f into $f = \sum_\theta f_\theta$. Set

$$T_{K,\theta}^\lambda f(x) := \mathrm{e}^{-2\pi \mathrm{i} \phi_K^\lambda(x,\xi_\theta)} T_K^\lambda f(x),$$

and we rewrite

$$T_K^\lambda f(x) = \sum_\theta \mathrm{e}^{2\pi \mathrm{i} \phi_K^\lambda(x,\xi_\theta)} T_{K,\theta}^\lambda f_\theta(x).$$

For sufficiently small $\delta > 0$, we may also write

$$T_{K,\theta}^\lambda f_\theta(x) = T_{K,\theta}^\lambda f_\theta * \eta_{R^{1-\delta}}(x) + \mathrm{RapDec}(R)\|f\|_{L^2(B^{n-1})}, \tag{3.10}$$

where η is a Schwartz function on $\mathbb{R}^n$ and has Fourier support on $B_2^n(0)$, and $\hat{\eta} = 1$ on $B_1^n(0)$. Then, $|\eta|$ admits a smooth rapidly decreasing majorant $\zeta : \mathbb{R}^n \to [0, +\infty)$, which satisfies

$$\zeta_{R^{1-\delta}}(x) \lesssim R^\delta \zeta_{R^{1-\delta}}(y) \quad \text{if } |x-y| \lesssim R. \tag{3.11}$$

Cover $\square(R,R')$ by a finitely-overlapping R-balls $\{B_R^n\}$. For any $B_R^n(\bar{x})$ in this cover and for $z \in B_R^n(0)$, we have

$$|T_K^\lambda f(\bar{x}+z)| \lesssim R^\delta \int_{\mathbb{R}^n} \left| \sum_\theta \mathrm{e}^{2\pi \mathrm{i} \phi_K^\lambda(\bar{x}+z,\xi_\theta)} T_{K,\theta}^\lambda f_\theta(y) \right| \zeta_{R^{1-\delta}}(\bar{x}-y)dy.$$

Taking the L^p-norm in z and using Proposition 3.1 for the phase $\phi_K^\lambda(\bar{x}+\cdot,\xi_\theta)$, we have

$$\begin{aligned} \|T_K^\lambda f(\bar{x}+\cdot)\|_{L^p(B_R^n(0))} &\lesssim R^\delta \int_{\mathbb{R}^n} \left\| \sum_\theta \mathrm{e}^{2\pi \mathrm{i} \phi_K^\lambda(\bar{x}+z,\xi_\theta)} T_{K,\theta}^\lambda f_\theta(y) \right\|_{L^p(B_R^n(0))} \zeta_{R^{1-\delta}}(\bar{x}-y)dy \\ &\lesssim Q_p(\lambda,R) R^{\frac{n-1}{p'}} R^\delta \int_{\mathbb{R}^n} \|T_{K,\theta}^\lambda f_\theta(y)\|_{\ell^p(\theta)} \zeta_{R^{1-\delta}}(\bar{x}-y)dy, \end{aligned}$$

where $\|a_\theta\|_{\ell^p(\theta)}$ is denoted by $(\sum_\theta |a_\theta|^p)^{1/p}$.

By the property (3.11), for $z \in B_R^n(0)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \|T_{K,\theta}^\lambda f_\theta(y)\|_{\ell^p(\theta)} \zeta_{R^{1-\delta}}(\bar{x}-y)dy \\ &\quad = \int_{\mathbb{R}^n} \|T_{K,\theta}^\lambda f_\theta(\bar{x}+z-y)\|_{\ell^p(\theta)} \zeta_{R^{1-\delta}}(y-z)dy \\ &\quad \lesssim R^\delta \int_{\mathbb{R}^n} \|T_{K,\theta}^\lambda f_\theta(\bar{x}+z-y)\|_{\ell^p(\theta)} \zeta_{R^{1-\delta}}(y)dy \\ &\quad \lesssim R^{O(\delta)} \left(\int_{\mathbb{R}^n} \|T_{K,\theta}^\lambda f_\theta(\bar{x}+z-y)\|_{\ell^p(\theta)}^p \zeta_{R^{1-\delta}}(y)dy \right)^{1/p}. \end{aligned}$$

Then, we deduce that for all $z \in B_R^n(0)$,

$$\begin{aligned} \|T_K^\lambda f(\bar{x}+\cdot)\|_{L^p(B_R^n(0))} &\lesssim Q_p(\lambda,R) R^{\frac{n-1}{p'}} R^{O(\delta)} \\ &\quad \times \left(\int_{\mathbb{R}^n} \|T_{K,\theta}^\lambda f_\theta(\bar{x}+z-y)\|_{\ell^p(\theta)}^p \zeta_{R^{1-\delta}}(y)dy \right)^{1/p}. \end{aligned}$$

Raising both sides of this estimate to the p-th power, averaging in z, and summing over all the balls B_R^n in the covering, we conclude that $\|T_K^\lambda f\|_{L^p(\square(R,R'))}$ is dominated by

$$Q_p(\lambda,R) R^{\frac{n-1}{p'} - \frac{n}{p}} R^{O(\delta)} \left(\int_{\mathbb{R}^n} \sum_\theta \|T_{K,\theta}^\lambda f_\theta\|_{L^p(\square(R,R')-y)}^p \zeta_{R^{1-\delta}}(y)dy \right)^{1/p}.$$

Using the trivial estimate

$$\|T^{\lambda}_{K,\theta}f_\theta\|_{L^\infty(\square(R,R')-y)} \lesssim \|f_\theta\|_{L^1(B^{n-1}_1(0))} \lesssim R^{-(n-1)}\|f_\theta\|_{L^\infty(B^{n-1}_1(0))} \tag{3.12}$$

and

$$\|T^{\lambda}_{K,\theta}f_\theta\|_{L^2(\square(R,R')-y)} \lesssim R^{1/2}\|f_\theta\|_{L^2(B^{n-1}_1(0))}, \tag{3.13}$$

we have

$$\|T^{\lambda}_{K,\theta}f_\theta\|_{L^p(\square(R,R')-y)} \lesssim R^{-(2n-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}}\|f_\theta\|_{L^p(B^{n-1}_1(0))}.$$

Hence, $\|T^\lambda_K f\|_{L^p(\square(R,R'))}$ is dominated by $Q_p(\lambda,R)R^{O(\delta)}\|f\|_{L^p(B^{n-1}_1(0))}$. □

Lemma 3.3. *Suppose that* $\operatorname{supp} f \subset B^{n-1}_1(0)$. *Then, the Fourier transform of* $T^\lambda_K f$ *is essentially supported on the* K^{-2}*-neighborhood of the surface* $S := \{(\xi,\langle M\xi,\xi\rangle) : \xi \in \operatorname{supp} f\}$ *in the sense that*

$$|\widehat{T^\lambda_K f}(\omega)| \leqslant \mathrm{RapDec}(\lambda)\|f\|_{L^p(B^{n-1}_1(0))} \quad \text{for all} \ \ \omega \notin \mathcal{N}_{CK^{-2}}S. \tag{3.14}$$

Proof. Define

$$G_\lambda(\xi,\omega) := \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}(\phi^\lambda_K(x,\xi)-x\cdot\omega)}\mathfrak{a}^\lambda(x,\xi)dx.$$

Then, we have

$$\widehat{T^\lambda_K f}(\omega) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\mathrm{i}x\cdot\omega}T^\lambda_K f(x)dx = \int_{B^{n-1}_1(0)} f(\xi)G_\lambda(\xi,\omega)d\xi. \tag{3.15}$$

We rewrite as

$$G_\lambda(\xi,\omega) = \lambda^n\int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}\lambda(\phi_K(y,\xi)-y\cdot\omega)}\mathfrak{a}(y,\xi)dy.$$

From integration by parts and the assumption (2.4) of $\mathfrak{a}$, it follows that

$$|G_\lambda(\xi,\omega)| \leqslant \mathrm{RapDec}(\lambda), \tag{3.16}$$

provided that

$$|\omega - \nabla_y\phi_K(y,\xi)| \geqslant CK^{-2}. \tag{3.17}$$

Since ϕ_K is asymptotically flat, (3.17) holds obviously if $\omega \notin \mathcal{N}_{CK^{-2}}S$. Combining (3.15) and (3.16), we have

$$|\widehat{T^\lambda_K f}(\omega)| \leqslant \mathrm{RapDec}(\lambda)\|f\|_{L^p(B^{n-1}_1(0))} \quad \text{for all} \ \ \omega \notin \mathcal{N}_{CK^{-2}}S.$$

This completes the proof. □

To prove (3.4), we also need a flat decoupling estimate for T^λ_K.

Lemma 3.4 (Flat decoupling)**.** *Let* $\{\tau\}$ *be a collection of finitely-overlapping* K^{-1}*-balls covering* $B^{n-1}_1(0)$ *with* $1 \leqslant K \leqslant R$. *Then, we can decompose* f *as*

$$f = \sum_\tau f_\tau.$$

For $2 \leqslant p \leqslant \infty$, *one has*

$$\|T^\lambda_K f\|_{L^p(B_R)} \lesssim (\#\{\tau\})^{\frac{1}{2}-\frac{1}{p}}\bigg(\sum_\tau \|T^\lambda_K f_\tau\|^2_{L^p(\omega_{B_R})}\bigg)^{\frac{1}{2}} + \mathrm{RapDec}(\lambda)\|f\|_{L^2(B^{n-1}_1(0))}. \tag{3.18}$$

Proof. For $p=\infty$, the estimate (3.18) is trivial by Hölder's inequality. By interpolation, we just need to show (3.18) for $p=2$. Using Lemma 3.3 for each f_τ, we get

$$T^\lambda_K f_\tau = \chi_{\mathcal{N}_{CK^{-2}}(S_\tau)}(D)T^\lambda_K f_\tau + \mathrm{RapDec}(\lambda)\|f\|_{L^2(B^{n-1}_1(0))}, \tag{3.19}$$

where $S_\tau := \{(\xi,\langle M\xi,\xi\rangle) : \xi \in \tau\}$. Note that the CK^{-2}-neighborhoods of S_τ are finitely overlapping, and then we complete the proof using Plancherel's theorem. □

3.2 Bilinear restriction estimate

Assume that ϕ satisfies the Carleson-Sjölin conditions. Let U_1 and U_2 be two balls contained in $B_1^{n-1}(0)$ and $\xi_i \in U_i, i = 1, 2$. By the assumption (H1), the map

$$\xi \mapsto \partial_{x'}\phi(x, \cdot)$$

is a diffeomorphism. Define

$$q(x, \xi) := \partial_{x_n}\phi(x, (\partial_{x'}\phi(x, \cdot))^{-1}(\xi)),$$

i.e.,

$$q(x, \partial_{x'}\phi(x, \xi)) = \partial_{x_n}\phi(x, \xi). \tag{3.20}$$

Theorem 3.5 (See [14])**.** *Let* $\phi(x, \xi_i), i = 1, 2$ *satisfy the conditions* (H1) *and* (H2). *Assume that* $(x, \xi_i) \in \text{supp } a_i$. *If* $\partial^2_{\xi\xi} q$ *satisfies*

$$\det \partial^2_{\xi\xi} q(x, \partial_{x'}\phi(x, \xi_i)) \neq 0 \quad \textit{if } \xi_i \in \text{supp } a_i(x, \cdot),$$

and

$$|\langle \partial^2_{x'\xi}\phi(x, \xi)\delta(x, \xi_1, \xi_2), [\partial^2_{x'\xi}\phi(x, \xi_i)]^{-1}[\partial^2_{\xi\xi} q(x, u_i)]^{-1}\delta(x, \xi_1, \xi_2)\rangle| \geqslant c > 0 \tag{3.21}$$

for $i = 1, 2$, *where* $u_i = \partial_{x'}\phi(x, \xi_i)$ *and* $\delta(x, \xi_1, \xi_2) = \partial_\xi q(x, u_1) - \partial_\xi q(x, u_2)$, *then*

$$\||T^\lambda f_1 T^\lambda f_2|^{\frac{1}{2}}\|_{L^p(B_R^n)} \lesssim_{\phi,\varepsilon} R^\varepsilon \prod_{i=1}^{2} \|f_i\|_{L^2}^{\frac{1}{2}} \tag{3.22}$$

for $p \geqslant \frac{2(n+2)}{n}$.

To apply Theorem 3.5 to study the oscillatory operator T_K^λ, we first introduce a notion of the *strongly separated condition*.

Definition 3.6 (Strongly separated condition)**.** Let τ_1 and τ_2 be two balls of dimension K^{-1}. We say that τ_1 and τ_2 satisfy the strongly separated condition if for each $\xi_i \in \tau_i$, the condition

$$|\langle \partial^2_{x'\xi}\phi(x, \xi)\delta(x, \xi_1, \xi_2), [\partial^2_{x'\xi}\phi(x, \xi_i)]^{-1}[\partial^2_{\xi\xi} q(x, u_i)]^{-1}\delta(x, \xi_1, \xi_2)\rangle| \geqslant CK^{-1} \tag{3.23}$$

holds.

The next proposition concerns a geometric lemma associated with the phase ϕ_K^λ.

Proposition 3.7. *Let* $\{\tau\}$ *be a family of finitely-overlapping balls of radius* K^{-1}. *Then, we have the following two dichotomies*:

(i) *There exists an* m*-dimensional affine subspace* V *such that every* τ *is contained in an* $O(K^{-\frac{1}{2n}})$ *neighbourhood of* V.

(ii) *There are two* K^{-1}*-balls* τ *and* τ', *that satisfy the strongly separated condition associated with* ϕ_K.

Barron [1] proved the above proposition for the standard phase $\phi(x, \xi) = x' \cdot \xi + x_n\langle M\xi, \xi\rangle$. Note that

$$\phi_K(x, \xi) = x' \cdot \xi + x_n\langle M\xi, \xi\rangle + \mathrm{E}_K(x, \xi)$$

can be viewed as a small perturbation of the standard case, and the perturbation is sufficiently small compared with K^{-1}, and thus the strongly separated condition under the phase ϕ_K can be essentially identified as the same as the standard phase $x \cdot \xi + \langle M\xi, \xi\rangle$.

3.3 Broad-Narrow analysis.

Let $\delta = \varepsilon^2 \ll 1$, and set

$$K_2 = K_1^{2\delta}, \quad K_1 = K^\alpha, \quad K = K_0 R^\delta, \tag{3.24}$$

where $\alpha = \frac{1}{2n}$. Let $\mathfrak{T}$ be a collection of finitely-overlapping K^{-1}-balls τ covering $\operatorname{supp} f$, and we can fix a collection $\mathscr{Q}$ of finitely-overlapping K^2-cubes that cover $B^n_R(0)$. For each $Q \in \mathscr{Q}$, we define its *significant set*

$$\mathcal{S}_p(Q) := \left\{\tau \in \mathfrak{T} : \|T^\lambda_K f_\tau\|_{L^p(Q)} \geqslant \frac{1}{100\#\mathfrak{T}} \|T^\lambda_K f\|_{L^p(Q)}\right\}.$$

We say that a K^2-cube $Q \in \mathscr{Q}$ is *narrow* and write $Q \in \mathscr{N}$ if and only if there exists an m-dimensional linear subspace $V \subset \mathbb{R}^n$ such that

$$\angle(G^\lambda(x,\tau), V) \leqslant CK_1^{-1} \tag{3.25}$$

for all $\tau \in \mathcal{S}_p(Q)$; here, for given $x \in B^n_R(0)$, $G^\lambda(x,\tau)$ denotes the set of the unit normal vectors of the hypersurface $\{\partial_x \phi^\lambda_K(x,\eta) : \eta \in \tau\}$. If a K^2-cube $Q \in \mathscr{Q}$ is not narrow, then we call it *broad* and write $Q \in \mathscr{B}$. Thus,

$$\|T^\lambda_K f\|^p_{L^p(B^n_R)} \leqslant \sum_{Q\in\mathscr{B}} \|T^\lambda_K f\|^p_{L^p(Q)} + \sum_{Q\in\mathscr{N}} \|T^\lambda_K f\|^p_{L^p(Q)}.$$

We call it the broad case if

$$\|T^\lambda_K f\|^p_{L^p(B_R)} \leqslant 2 \sum_{Q\in\mathscr{B}} \|T^\lambda_K f\|^p_{L^p(Q)},$$

otherwise the narrow case if

$$\|T^\lambda_K f\|^p_{L^p(B_R)} \leqslant 2 \sum_{Q\in\mathscr{N}} \|T^\lambda_K f\|^p_{L^p(Q)}.$$

Now, we are going to prove (3.4). Obviously, (3.4) holds for $1 \leqslant \lambda \leqslant 1{,}000$, so let us suppose that (3.4) holds for $1 \leqslant r \leqslant \lambda' \leqslant \lambda/2$. In the following part, we deal with the broad and narrow cases, respectively. Then, we balance the two cases and close the whole induction for (3.4).

3.4 Narrow estimate

Suppose that $Q \in \mathscr{Q}$ is a narrow cube, and by Proposition 3.7, there exists an m-dimensional affine subspace $V \subset \mathbb{R}^{n-1}$ such that

$$\bigcup_{\tau\in S_p(Q)} \tau \subset N_{CK_1^{-1}} V.$$

We decompose $B^{n-1}_1(0)$ into K_1^{-1}-balls $\{\pi\}$. Let Π_V be a minimal collection of $\{\pi\}$ covering $B^{n-1}_1(0) \cap N_{CK_1^{-1}} V$ and $\mathfrak{I}$ be a collection of finitely-overlapping K_1^{-1}-balls $\{\pi\}$ covering $\operatorname{supp} f$. Note that Π_V contains CK_1^m many balls π. Using Hölder's inequality and Lemma 3.4, we obtain

$$\begin{aligned}
\|T^\lambda_K f\|_{L^p(Q)} &\leqslant CK_1^{m(\frac12-\frac1p)} \left(\sum_{\pi\in\Pi_V} \|T^\lambda_K f_\pi\|^2_{L^p(\omega_Q)}\right)^{\frac12}\\
&\leqslant CK_1^{2m(\frac12-\frac1p)} \left(\sum_{\pi\in\Pi_V} \|T^\lambda_K f_\pi\|^p_{L^p(\omega_Q)}\right)^{\frac1p}\\
&\leqslant CK_1^{2m(\frac12-\frac1p)} \left(\sum_{\pi\in\mathfrak{I}} \|T^\lambda_K f_\pi\|^p_{L^p(\omega_Q)}\right)^{\frac1p}.
\end{aligned}$$

By Lemma 3.2 and our induction assumption, we have

$$\begin{aligned}
\left(\sum_{Q\in\mathscr{N}} \|T^\lambda_K f\|^p_{L^p(Q)}\right)^{\frac1p} &\leqslant CK_1^{2m(\frac12-\frac1p)} \left(\sum_{\pi\in\mathfrak{I}} \|T^\lambda_K f_\pi\|^p_{L^p(\omega_{B_R})}\right)^{\frac1p}\\
&\leqslant \bar{C} C_\varepsilon R^\varepsilon K_1^{-\varepsilon} K_1^{2m(\frac12-\frac1p)-(n-1)+\frac{2n}{p}} \left(\sum_{\pi\in\mathfrak{I}} \|f_\pi\|^p_{L^p(B^{n-1}_1(0))}\right)^{\frac1p}\\
&\leqslant \bar{C} C_\varepsilon R^\varepsilon K_1^{-\varepsilon} K_1^{2m(\frac12-\frac1p)-(n-1)+\frac{2n}{p}} \|f\|_{L^p(B^{n-1}_1(0))},
\end{aligned}$$

where $\bar{C}$ is a large constant.

If $p \geqslant \frac{2(n-m)}{n-m-1}$, we obtain

$$\left(\sum_{Q\in\mathscr{N}} \|T_K^\lambda f\|_{L^p(Q)}^p\right)^{\frac{1}{p}} \leqslant C_\varepsilon R^\varepsilon \|f\|_{L^p(B_1^{n-1}(0))}. \tag{3.26}$$

3.5 Broad estimate

We show the broad estimate using the bilinear arguments.

Proposition 3.8 (Broad estimate)**.** *Let $p \geqslant \frac{2(n+2)}{n}$. We have*

$$\sum_{Q\in\mathscr{B}} \|T_K^\lambda f\|_{L^p(Q)}^p \leqslant CK^{O(1)} \|f\|_{L^2(B_1^{n-1}(0))}^p. \tag{3.27}$$

To prove Proposition 3.8, we naturally need to obtain the bounds of $\|T_K^\lambda f\|_{L^p(Q)}^p$ for each Q first, and then sum them together. For this purpose, we first present two lemmas.

Lemma 3.9. *For any $Q \in \mathscr{B}$, there are two K^{-1}-balls $\tau_1, \tau_2 \in \mathcal{S}_p(Q)$ satisfying the strongly separated condition* (3.23) *such that*

$$\|T_K^\lambda f\|_{L^p(Q)} \leqslant CK^{O(1)} \|T_K^\lambda f_{\tau_1}\|_{L^p(Q)}^{\frac{1}{2}} \|T_K^\lambda f_{\tau_2}\|_{L^p(Q)}^{\frac{1}{2}}. \tag{3.28}$$

Proof. Let $Q \in \mathscr{B}$. Then, $\#\mathcal{S}_p(Q) \geqslant 2$. Suppose that there does not exist two K^{-1}-balls $\tau_1, \tau_2 \in \mathcal{S}_p(Q)$ satisfying the strongly separated condition (3.23). Applying the Proposition 3.7 to $\mathcal{S}_p(Q)$, we get

$$\tau \subset N_{CK_1^{-1}} V \quad \text{for all } \tau \in \mathcal{S}_p(Q)$$

for some m-dimensional affine subspace V. This forces all $G^\lambda(x,\tau)$ to be in the neighborhood $\mathcal{N}_{CK_1^{-1}} W$ of some m-dimensional subspace W. Thus, Q is a narrow cube, which contradicts our assumption. Thus, we can find $\tau_1, \tau_2 \in \mathcal{S}_p(Q)$ satisfying the strongly separated condition (3.23) such that

$$\begin{aligned}\|T_K^\lambda f\|_{L^p(Q)} &\leqslant (100\#\mathfrak{T}) \|T_K^\lambda f_{\tau_1}\|_{L^p(Q)}^{\frac{1}{2}} \|T_K^\lambda f_{\tau_2}\|_{L^p(Q)}^{\frac{1}{2}} \\ &\leqslant CK^{O(1)} \|T_K^\lambda f_{\tau_1}\|_{L^p(Q)}^{\frac{1}{2}} \|T_K^\lambda f_{\tau_2}\|_{L^p(Q)}^{\frac{1}{2}}.\end{aligned} \tag{3.29}$$

This completes the proof. □

Lemma 3.10. *Suppose that $f \in L^2(\mathbb{R}^{n-1})$ with support* $\operatorname{supp} f \subset B_{K^{-1}}^{n-1}(\bar{\xi}) \subset B_1^{n-1}(0)$*. Then, we have*

$$|T_K^\lambda f(x)| = |(\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(\cdot,\bar{\xi})} T_K^\lambda f) * \psi_{K/C}(x)| + \mathrm{RapDec}(\lambda)\|f\|_{L^2(\mathbb{R}^{n-1})}, \tag{3.30}$$

where $\psi_{K/C}(x) = C^n K^{-n} \psi(CK^{-1}x)$ with $\operatorname{supp}\hat{\psi} \subset B_2^n(0)$ and $\hat{\psi} = 1$ on $B_1^n(0)$.

Proof. We observe that

$$\begin{aligned}\mathcal{F}(\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(\cdot,\bar{\xi})} T_K^\lambda f(\cdot))(\omega) &= \int_{\mathbb{R}^n} \int_{B_1^{n-1}(0)} \mathrm{e}^{2\pi\mathrm{i}(\phi_K^\lambda(x,\xi) - \phi_K^\lambda(x,\bar{\xi}) - x\cdot\omega)} \mathfrak{a}^\lambda(x,\xi) f(\xi) d\xi dx \\ &= \int_{B_1^{n-1}(0)} F_\lambda(\xi,\omega) f(\xi) d\xi,\end{aligned}$$

where

$$F_\lambda(\xi,\omega) := \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}(\phi_K^\lambda(x,\xi) - \phi_K^\lambda(x,\bar{\xi}) - x\cdot\omega)} \mathfrak{a}^\lambda(x,\xi) dx.$$

We can rewrite as

$$F_\lambda(\xi,\omega) := \lambda^n \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}\lambda(\phi_K(x,\xi) - \phi_K(x,\bar{\xi}) - x\cdot\omega)} \mathfrak{a}(x,\xi) dx.$$

For $|\omega| \geqslant CK^{-1}$ and $x \in B_1^n(0)$, we have

$$|\nabla_x \phi_K(x,\xi) - \nabla_x \phi_K(x,\bar{\xi}) - \omega| \geqslant K^{-1}.$$

From integration by parts, it follows that

$$|F_\lambda(\xi,\omega)| \leqslant \mathrm{RapDec}(\lambda)(1+|\omega|)^{-(n+1)}. \tag{3.31}$$

Thus, we have

$$\mathcal{F}(\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(\cdot,\bar{\xi})}T_K^\lambda f(\cdot))(\omega) = \widehat{\psi_{K/C}}(\omega)\mathcal{F}(\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(\cdot,\bar{\xi})}T_K^\lambda f(\cdot))(\omega) + U(f,\lambda)(\omega),$$

where

$$|U(f,\lambda)(\omega)| \leqslant \mathrm{RapDec}(\lambda)(1+|\omega|)^{-(n+1)}\|f\|_{L^2(B_1^{n-1}(0))}.$$

Using the Fourier inversion, we obtain

$$\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(x,\bar{\xi})}T_K^\lambda f(x) = \psi_{K/C} * (\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(\cdot,\bar{\xi})}T_K^\lambda f(\cdot))(x) + \mathrm{RapDec}(\lambda)\|f\|_{L^2(B_1^{n-1}(0))}.$$

Then, (3.30) holds obviously. □

Lemma 3.11. *For any two K^{-1}-balls τ_1 and τ_2 satisfying the strongly separated condition* (3.23)*, it holds that*

$$\sum_{Q\in\mathscr{B}} \|T_K^\lambda f_{\tau_1}\|_{L^p(Q)}^{\frac{p}{2}} \|T_K^\lambda f_{\tau_2}\|_{L^p(Q)}^{\frac{p}{2}} \leqslant K^{O(1)}\|f\|_{L^2(B_1^{n-1}(0))}^p + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}. \tag{3.32}$$

Proof. Without loss of generality, we may assume $\|f\|_{L^2(B_1^{n-1}(0))} = 1$. By Lemma 3.10, we have

$$|T_K^\lambda f_\tau(x)| = |(\mathrm{e}^{-2\pi\mathrm{i}\phi_K^\lambda(\cdot,\bar{\xi})}T_K^\lambda f_\tau(\cdot)) * \psi_{K/C}(x)| + \mathrm{RapDec}(\lambda)$$

for each τ. To prove (3.32), we just need to show

$$\begin{aligned}\sum_{Q\in\mathscr{B}} &\|(\mathrm{e}^{-2\pi\mathrm{i}\phi^\lambda(\cdot,\xi_{\tau_1})}T_K^\lambda f_{\tau_1}) * \psi_{K/C}\|_{L^\infty(Q)}^{\frac{p}{2}} \|(\mathrm{e}^{-2\pi\mathrm{i}\phi^\lambda(\cdot,\xi_{\tau_2})}T_K^\lambda f_{\tau_2}) * \psi_{K/C}\|_{L^\infty(Q)}^{\frac{p}{2}} \\ &\leqslant K^{O(1)}\|f\|_{L^2(B_1^{n-1}(0))}^p.\end{aligned} \tag{3.33}$$

Define

$$\zeta_K(x) := \sup_{|y-x|\leqslant K^2} |\psi_{K/C}(x)|.$$

By the locally constant property, one can choose some cube $I_Q \subset Q$ with $|I_Q| \lesssim 1$ such that

$$\begin{aligned}&\|(\mathrm{e}^{-2\pi\mathrm{i}\phi^\lambda(\cdot,\xi_{\tau_1})}T_K^\lambda f_{\tau_1}) * \psi_{K/C}\|_{L^\infty(Q)} \|(\mathrm{e}^{-2\pi\mathrm{i}\phi^\lambda(\cdot,\xi_{\tau_2})}T_K^\lambda f_{\tau_2}) * \psi_{K/C}\|_{L^\infty(Q)} \\ &\quad\leqslant \int_{I_Q}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |T_K^\lambda f_{\tau_1}(x-y)\zeta_K(y)T_K^\lambda f_{\tau_2}(x-z)\zeta_K(z)|dydzdx.\end{aligned}$$

Then, we only need to show

$$\sum_{Q\in\mathscr{B}} \left(\int_{I_Q}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |T_K^\lambda f_{\tau_1}(x-y)\zeta_K(y)T_K^\lambda f_{\tau_2}(x-z)\zeta_K(z)|dydzdx\right)^{\frac{p}{2}} \leqslant K^{O(1)}\|f\|_{L^2(B_1^{n-1}(0))}^p.$$

Using Hölder's inequality, for $p \geqslant \frac{2(n+2)}{n}$, we have

$$\sum_{Q\in\mathscr{B}} \left(\int_{I_Q}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |T_K^\lambda f_{\tau_1}(x-y)\zeta_K(y)T_K^\lambda f_{\tau_2}(x-z)\zeta_K(z)|dydzdx\right)^{\frac{p}{2}}$$

$$
\begin{aligned}
&\leqslant K^{O(1)} \sum_{Q\in\mathscr{B}} \int_{I_Q} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |T_K^\lambda f_{\tau_1}(x-y) T_K^\lambda f_{\tau_2}(x-z)|^{\frac{p}{2}} \zeta_K(y)\zeta_K(z) dydzdx \\
&\leqslant K^{O(1)} \sup_{y,z} \left(\int_{B_R^n(0)} |T_K^\lambda f_{\tau_1}(x-y)|^{\frac{p}{2}} |T_K^\lambda f_{\tau_2}(x-z)|^{\frac{p}{2}} dx \right) \\
&\leqslant K^{O(1)} \|f\|^p_{L^2(B_1^{n-1}(0))},
\end{aligned}
$$

where we have used Theorem 3.5 in the last inequality. Hence, we complete the proof of Lemma 3.11. □

Finally, we use the three lemmas above to give the proof of Proposition 3.8.

Proof of the broad estimate. By Lemmas 3.9–3.11, we have

$$
\begin{aligned}
\sum_{Q\in\mathscr{B}} \|T_K^\lambda f\|^p_{L^p(Q)} &\leqslant CK^{O(1)} \sum_{Q\in\mathscr{B}} \sum_{\substack{\tau_1,\tau_2\in\mathcal{S}_p(Q) \\ \tau_1 \text{ and } \tau_2 \text{ satisfy (3.23)}}} \|T_K^\lambda f_{\tau_1}\|^{\frac{p}{2}}_{L^p(Q)} \|T_K^\lambda f_{\tau_2}\|^{\frac{p}{2}}_{L^p(Q)} \\
&= CK^{O(1)} \sum_{\tau_1 \text{ and } \tau_2 \text{ satisfy (3.23)}} \sum_{Q\in\mathscr{B}:\tau_1,\tau_2\in\mathcal{S}_p(Q)} \|T_K^\lambda f_{\tau_1}\|^{\frac{p}{2}}_{L^p(Q)} \|T_K^\lambda f_{\tau_2}\|^{\frac{p}{2}}_{L^p(Q)} \\
&\leqslant CK^{O(1)} \|f\|^p_{L^2(B_1^{n-1}(0))},
\end{aligned}
$$

where we have used the fact that $\mathfrak{T} \leqslant K^{O(1)}$ in the last inequality. Then, we finish the proof of Proposition 3.8.

For all $m \leqslant \lfloor (n-1)/2 \rfloor$, we prove Theorem 1.1 using the narrow estimate (3.26) and the broad estimate (3.27).

Recall that (3.4) holds for $1 \leqslant \lambda' \leqslant \lambda/2$, and thus we have $Q_p(\tilde{\lambda}, \tilde{R}) \leqslant C_\varepsilon \tilde{R}^\varepsilon$, where $\tilde{\lambda} = K^{-2}\lambda < \lambda/2$ and $\tilde{R} = K^{-2}R \leqslant \tilde{\lambda}$. Thanks to the relation of K, K_1, and R in (3.24), we conclude that

$$
Q_p(\lambda, R) \leqslant C_\varepsilon R^\varepsilon K_1^{-\varepsilon} + CK^{O(1)} \leqslant C_\varepsilon R^\varepsilon
$$

holds for

$$
p \geqslant \max_{0\leqslant m\leqslant \lfloor \frac{n-1}{2} \rfloor} \left\{ \frac{2(n+2)}{n}, \frac{2(n-m)}{n-m-1} \right\}. \tag{3.34}
$$

This inequality is equivalent to

$$
p \geqslant \begin{cases} \dfrac{2(n+1)}{n-1} & \text{for } n \text{ odd}, \\ \dfrac{2(n+2)}{n} & \text{for } n \text{ even}. \end{cases} \tag{3.35}
$$

Then, we finish the proof of Theorem 1.1. □

4 Proof of the decoupling theorem

4.1 Reduction

First, we recall the decoupling theorem of Bourgain and Demeter [4]. Let S be a compact hypersurface with nonvanishing Gaussian curvature, and $\mathcal{N}_\delta(S)$ be the δ-neighborhood of S. Decompose $\mathcal{N}_\delta(S)$ into a collection of finitely-overlapping slabs $\{\triangle\}$ of dimension $\delta^{1/2}$ in the tangent direction and δ in the normal direction. We have the decomposition

$$
f = \sum_{\triangle} f_\triangle,
$$

where $\mathrm{supp}\widehat{f_\triangle} \subset \triangle$. A classical decoupling result associated with this decomposition is as follows.

Theorem 4.1 (See [4]). *Let S be a compact smooth hypersurface in $\mathbb{R}^n$ with nonvanishing Gaussian curvature. If $\mathrm{supp}\hat{f} \subset \mathcal{N}_\delta(S)$, then for $p \geqslant \frac{2(n+1)}{n-1}$ and $\varepsilon > 0$,*

$$
\|f\|_{L^p(\mathbb{R}^n)} \leqslant_\varepsilon \delta^{\frac{n}{p} - \frac{n-1}{2} - \varepsilon} \left(\sum_{\triangle} \|f_\triangle\|^p_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}}. \tag{4.1}
$$

They also have a local version of decoupling

$$\|f\|_{L^p(B_R^n)} \leqslant_\varepsilon \delta^{\frac{n}{p}-\frac{n-1}{2}-\varepsilon}\left(\sum_{\Delta}\|f_\Delta\|_{L^p(\omega_{B_R^n})}^p\right)^{\frac{1}{p}}. \tag{4.2}$$

For $1 \ll R \leqslant \lambda$, let $D_p(\lambda, R)$ be the optimal constant such that

$$\|T_K^\lambda f\|_{L^p(B_R^n)} \leqslant D_p(\lambda, R)\left(\sum_\theta \|T_K^\lambda f_\theta\|_{L^p(w_{B_R^n})}^p\right)^{1/p} + \mathrm{RapDec}(\lambda)\|f\|_{L^p} \tag{4.3}$$

holds for all asymptotically flat phase ϕ_K and for all $\mathfrak{a}$ satisfying (2.4), and uniformly for all $f \in L^p(B_1^{n-1}(0))$. To prove Theorem 1.3, it suffices to show that

$$D_p(\lambda, R) \leqslant C_\varepsilon R^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon}. \tag{4.4}$$

Indeed, by Hölder's inequality, we have

$$\|T^\lambda f\|_{L^p(B_R^n)} \leqslant K^{(n-1)/p'}\left(\sum_\tau \|T^\lambda f_\tau\|_{L^p(B_R^n)}^p\right)^{1/p}. \tag{4.5}$$

For each τ, performing the similar procedure as in the proof of Lemma 2.3, we have

$$\|T^\lambda f_\tau\|_{L^p(B_R^n)}^p \leqslant CK^{O(1)} \sum_{B_{\tilde{R}}^n \subset \square_R} \|T_{\tilde{K}}^{\tilde{\lambda}} \tilde{f}\|_{L^p(B_{\tilde{R}}^n)}^p,$$

where $\tilde{f}(\cdot) = K^{-(n-1)} f(K^{-1}\cdot + \xi_\tau)$, $\tilde{R} = R/K^2, \tilde{K} = K_0 \tilde{R}^{\varepsilon^2}$, $\tilde{\lambda} = \lambda/K^2$, and $\square_R$ is a rectangle of dimensions $R/K \times \cdots \times R/K \times R/K^2$. Then, by (4.4), we have

$$\|T_{\tilde{K}}^{\tilde{\lambda}} \tilde{f}\|_{L^p(B_{\tilde{R}}^n)} \leqslant C_\varepsilon (\tilde{R})^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon}\left(\sum_{\tilde{\theta}} \|T_{\tilde{K}}^{\tilde{\lambda}} \tilde{f}_{\tilde{\theta}}\|_{L^p(w_{B_{\tilde{R}}^n})}^p\right)^{1/p} + \mathrm{RapDec}(\tilde{\lambda})\|f\|_{L^p(B_1^{n-1}(0))}, \tag{4.6}$$

where $\tilde{\theta}$ is a ball of dimension $\tilde{R}^{-1/2}$. By reversing the change of variables, we finally obtain

$$\|T^\lambda f\|_{L^p(B_R^n)} \leqslant C_\varepsilon R^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon}\left(\sum_\theta \|T^\lambda f_\theta\|_{L^p(w_{B_R^n})}^p\right)^{1/p} + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}.$$

4.2 Proof of (4.4)

Let $\tau \subset B_1^{n-1}(0)$ be a ball of radius K^{-1}. For convenience, define

$$\mathbb{H} := \{(\xi, \langle M\xi, \xi\rangle) : \xi \in B_1^n(0)\},$$

and denote by $\mathbb{H}_\tau$ a cap on $\mathbb{H}$, i.e.,

$$\mathbb{H}_\tau := \{(\xi, \langle M\xi, \xi\rangle) : \xi \in \tau\}.$$

If ω does not belong to a CK^{-1}-neighborhood of $\mathbb{H}$, by Lemma 3.3, we have

$$\widehat{T_K^\lambda f}(\omega) = \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}.$$

Therefore,

$$T_K^\lambda f = \chi_{CK^{-1}}(\mathbb{H})(D) T_K^\lambda f + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}. \tag{4.7}$$

Similarly,

$$T_K^\lambda f_\tau = \chi_{CK^{-1}}(\mathbb{H}_\tau)(D) T_K^\lambda f_\tau + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}. \tag{4.8}$$

Applying the local decoupling inequality (4.2) to (4.7) and (4.8), we have

$$\|T_K^\lambda f\|_{L^p(B_R^n)} \leqslant C_{\bar\delta} K^{\frac{n-1}{2}-\frac{n}{p}+\bar\delta}\bigg(\sum_\tau \|T_K^\lambda f_\tau\|^p_{L^p(\omega_{B_R^n})}\bigg)^{1/p} + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))},$$

where $\bar\delta > 0$ is a small constant to be chosen later. By a similar way as in the proof of Lemma 2.3, we have

$$\|T_K^\lambda f_\tau\|^p_{L^p(\omega_{B_R^n})} \leqslant C(K) \sum_{B^n_{\tilde R}\subset \square_R} \|\tilde T_{\tilde K}^{\tilde\lambda} \tilde f\|^p_{L^p(B^n_{\tilde R})} + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^n(0))}, \tag{4.9}$$

where $\tilde f(\xi) = K^{-(n-1)} f(\xi_\tau + K^{-1}\xi)$ and ξ_τ is the center of τ. For each $B^n_{\tilde R}$, by the definition of $D_p(\lambda, R)$, we have

$$\|\tilde T_{\tilde K}^{\tilde\lambda} \tilde f\|_{L^p(B^n_{\tilde R})} \leqslant D_p(\tilde\lambda, \tilde R)\bigg(\sum_{\tilde\theta} \|\tilde T_{\tilde K}^{\tilde\lambda} \tilde f_{\tilde\theta}\|^p_{L^p(w_{B^n_{\tilde R}})}\bigg)^{1/p} + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}, \tag{4.10}$$

where $\{\tilde\theta\}$ is a collection of finitely-overlapping balls of radius $\tilde R^{-1/2}$. By reversing the change of variables, we finally have

$$\|T_K^\lambda f\|_{L^p(B_R^n)} \leqslant C_{\bar\delta} K^{\frac{n-1}{2}-\frac{n}{p}+\bar\delta} D_p(\tilde\lambda, \tilde R)\bigg(\sum_\theta \|T_K^\lambda f_\theta\|^p_{L^p(w_{B_R^n})}\bigg)^{1/p} + \mathrm{RapDec}(\lambda)\|f\|_{L^p(B_1^{n-1}(0))}. \tag{4.11}$$

Recalling the definition of $D_p(\lambda, R)$, we have

$$D_p(\lambda, R) \leqslant C_{\bar\delta} K^{\frac{n-1}{2}-\frac{n}{p}+\bar\delta} D_p(\tilde\lambda, \tilde R). \tag{4.12}$$

The inequality (4.12) yields, by the induction hypothesis, that

$$D_p(\lambda, R) \leqslant C_\varepsilon R^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon} C_{\bar\delta} K^{\bar\delta-2\varepsilon}. \tag{4.13}$$

Choosing $\bar\delta = \varepsilon^2$ and K_0 sufficiently large such that

$$K_0^{\varepsilon^2-2\varepsilon} C_{\bar\delta} \leqslant 1,$$

from (4.13), we can complete the induction procedure, i.e.,

$$D_p(\lambda, R) \leqslant C_\varepsilon R^{\frac{n-1}{2}-\frac{n}{p}+\varepsilon}.$$

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