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Cohomology of a class of Kadison-Singer algebras

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Abstract Let $\mathcal{L}$ be the complete lattice generated by a nest $\mathcal{N}$ on an infinite-dimensional separable Hilbert space $\mathcal{H}$ and a rank one projection P_ξ given by a vector ξ in $\mathcal{H}$. Assume that ξ is a separating vector for $\mathcal{N}''$, the core of the nest algebra $\mathrm{Alg}(\mathcal{N})$. We show that $\mathcal{L}$ is a Kadison-Singer lattice, and hence the corresponding algebra $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer algebra. We also describe the center of $\mathrm{Alg}(\mathcal{L})$ and its commutator modulo itself, and show that every bounded derivation from $\mathrm{Alg}(\mathcal{L})$ into itself is inner, and all n-th bounded cohomology groups $H^n(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ of $\mathrm{Alg}(\mathcal{L})$ with coefficients in $B(\mathcal{H})$ are trivial for all $n \geqslant 1$.

Keywords Kadison-Singer algebra, Kadison-Singer lattice, nest algebra, cohomology group

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1 Introduction

Triangular algebras, introduced by Kadison and Singer [8], and reflexive algebras are two important classes of non-selfadjoint operator algebras. They are closely related to the study of structural properties of bounded linear operators, such as the invariant subspace problem for operators. The intersections of these two are nest algebras [2, 11]. Many people have tried to extend selfadjoint theory, as well as its techniques and invariants, to non-selfadjoint algebras. Recently, combining triangularity, reflexivity and von Neumann algebra property into one consideration, Ge and Yuan introduced a new class of non-selfadjoint algebras which they call Kadison-Singer algebras or KS-algebras for simplicity [3]. These algebras are reflexive, maximal triangular with respect to their "diagonal subalgebras". A more direct connection of Kadison-Singer algebras and von Neumann algebras is through the lattice of invariant projections of a KS-algebra. The lattice is reflexive and "minimally generating" in the sense that it generates the commutant of the diagonal as a von Neumann algebra. Nest algebras are KS-algebras with "abelian cores" and commutative lattices of invariant projections.

In [3], using the tensor product structure of hyperfinite factors, Ge and Yuan constructed the examples of Kadison-Singer algebras with hyperfinite ones as their diagonals. In [4], making use of von Neumann algebra techniques, they proved that the reflexive lattice generated by a double triangular lattice with three nontrivial projections is, in general, isomorphic to the two-dimensional sphere $\mathbb{S}^2$ (plus two distinct points corresponding to 0 and I), and the corresponding reflexive algebra is a Kadison-Singer algebra. Wang and Yuan showed that the reflexive algebra, corresponding to the one point extension of a maximal nest on an infinite-dimensional separable Hilbert space by a rank one projection which is determined by a separating vector for the diagonal of the nest algebras, is a KS-algebra [13].

KS-algebras are generalizations of nest algebras and are a much broader class of (reflexive) non-selfadjoint operator algebras, whose lattices of invariant projections may be noncommutative and non distributive [3,4]. Nest algebras have many nice properties such as the existence of rank one operators, the innerness of bounded derivations and vanishing bounded cohomology groups. It becomes an interesting problem whether all or some of these properties are still true for some or all KS-algebras.

In this paper, using some techniques in [13], we show that the reflexive algebra $\mathrm{Alg}(\mathcal{L})$, with $\mathcal{L}$ generated by a nest on a separable Hilbert space and a rank one projection determined by a separating vector for the core of the nest algebra, is a KS-algebra. Here we do not require that the nest is maximal. Our result generalizes the main result in [13]. We will determine its center, the commutator of this algebra modulo itself. We also show that every bounded derivation from $\mathrm{Alg}(\mathcal{L})$ into itself is inner, and all n-th bounded cohomology groups $H^n(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ of $\mathrm{Alg}(\mathcal{L})$ with coefficients in $B(\mathcal{H})$ are trivial for all $n \geqslant 1$.

Now we recall the definitions of some well-known classes of non-selfadjoint operator algebras. For details on triangular algebras and nest algebras, we refer to [2, 8].

Let $\mathcal{H}$ be a separable Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a set $\mathcal{L}$ of orthogonal projections in $B(\mathcal{H})$, we denote by $\mathrm{Alg}(\mathcal{L})$ the set of all bounded linear operators on $\mathcal{H}$ leaving each element in $\mathcal{L}$ invariant. Then $\mathrm{Alg}(\mathcal{L})$ is a unital weak-operator closed subalgebra of $B(\mathcal{H})$. Similarly, for a subset $\mathcal{S}$ of $B(\mathcal{H})$, we let $\mathcal{L}at(\mathcal{S})$ be the invariant projection lattice of $\mathcal{S}$ consisting of all projections invariant under each operator in $\mathcal{S}$. Then $\mathcal{L}at(\mathcal{S})$ is a strong-operator closed lattice of projections. A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is said to be reflexive if $\mathcal{A} = \mathrm{Alg}(\mathcal{L}at(\mathcal{A}))$. Similarly, a lattice $\mathcal{L}$ of projections in $B(\mathcal{H})$ is called reflexive if $\mathcal{L} = \mathcal{L}at(\mathrm{Alg}(\mathcal{L}))$. A nest $\mathcal{N}$ is a totally ordered family of projections on $\mathcal{H}$ which contains the zero operator 0 and the identity operator I on $\mathcal{H}$ and is closed in strong operator topology. If $\mathcal{N}$ is a nest, then $\mathcal{N}$ is reflexive and $\mathrm{Alg}(\mathcal{N})$ is called a nest algebra. The von Neumann algebra $\mathcal{N}''$ generated by $\mathcal{N}$ is called the *core* of $\mathrm{Alg}(\mathcal{L})$; $\mathrm{Alg}(\mathcal{L}) \cap \mathrm{Alg}(\mathcal{L})^* = \mathcal{N}'$ is called the *diagonal* of $\mathrm{Alg}(\mathcal{L})$. Obviously, $\mathcal{N}'' \subseteq \mathcal{N}'$.

Definition 1.1. *A subalgebra $\mathcal{A}$ of $B(\mathcal{H})$ is called a Kadison-Singer (operator) algebra (or KS-algebra) if $\mathcal{A}$ is reflexive and maximal with respect to the diagonal subalgebra $\mathcal{A} \cap \mathcal{A}^*$ of $\mathcal{A}$, in the sense that if there is another reflexive subalgebra $\mathcal{B}$ of $B(\mathcal{H})$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \cap \mathcal{B}^* = \mathcal{A} \cap \mathcal{A}^*$, then $\mathcal{A} = \mathcal{B}$. When the diagonal of a KS-algebra $\mathcal{A}$ is a factor, we say $\mathcal{A}$ is a Kadison-Singer factor (or KS-factor).*

A lattice $\mathcal{L}$ of projections in $B(\mathcal{H})$ is called a Kadison-Singer lattice (or KS-lattice) if $\mathcal{L}$ is a minimal reflexive lattice that generates the von Neumann algebra $\mathcal{L}''$, or equivalently, $\mathcal{L}$ is reflexive and $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer algebra.

Clearly nest algebras are KS-algebras. When $\mathcal{A}$ is a KS-algebra and $\mathcal{A} \cap \mathcal{A}^*$ is a von Neumann algebra (or factor) of type I, II and III, then $\mathcal{A}$ is called a KS-algebra (or KS-factor) of the same type. In the same way, one can further classify KS-factors into II_1, II_∞, etc., similar to usual factors. Since a nest generates an abelian von Neumann algebra, the nest algebras are "type I" KS-algebras.

Next we recall some notions in Hochschild cohomology theory. For details on cohomology theory of Banach algebras, von Neumann algebras and nest algebras, we refer to [6,9,12]. Let $\mathcal{A}$ be a unital Banach algebra, and $\mathcal{M}$ a unital Banach $\mathcal{A}$-bimodule. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a derivation if $\delta(AB) = A\delta(B) + \delta(A)B$ for all pairs A, B in $\mathcal{A}$; if there exists M in $\mathcal{M}$ such that $\delta(A) = AM - MA$ for each A in $\mathcal{A}$, then δ is called an inner derivation. Every derivation from a nest algebra acting on a separable Hilbert space $\mathcal{H}$ into $B(\mathcal{H})$, or into itself, is inner, and hence, automatically bounded [1].

For $n = 1, 2, 3, \ldots$, we denote by $C^n(\mathcal{A}, \mathcal{M})$ the complex vector space of all bounded n-linear mappings (also called n-cochains) of $\mathcal{A} \times \cdots \times \mathcal{A}$ into $\mathcal{M}$. By convention, we let $C^0(\mathcal{A}, \mathcal{M})$ be $\mathcal{M}$. The coboundary operator $\partial^n : C^n(\mathcal{A}, \mathcal{M}) \to C^{n+1}(\mathcal{A}, \mathcal{M})$ is given by $\partial^0(M)(A) = AM - MA$ for $A \in \mathcal{A}$ and $M \in \mathcal{M}$; for $n \geqslant 1$, $\varphi \in C^n(\mathcal{A}, \mathcal{M})$, $A_1, \ldots, A_{n+1} \in \mathcal{A}$, let

$$\begin{aligned}\partial^n \varphi(A_1, A_2, \ldots, A_{n+1}) = {} & A_1 \varphi(A_2, \ldots, A_{n+1}) \\ & + \sum_{j=1}^{n} (-1)^j \varphi(A_1, \ldots, A_j A_{j+1}, \ldots, A_{n+1}) \\ & + (-1)^{n+1} \varphi(A_1, \ldots, A_n) A_{n+1}.\end{aligned}$$

The kernel of ∂^n in $C^n(\mathcal{A},\mathcal{M})$ is denoted by $Z^n(\mathcal{A},\mathcal{M})$ and called the space of n-cocycles. The image of ∂^{n-1} in $C^{n-1}(\mathcal{A},\mathcal{M})$, denoted by $B^n(\mathcal{A},\mathcal{M})$, is called the space of n-coboundaries. It is standard that $\partial^n\partial^{n-1}=0$, so $B^n(\mathcal{A},\mathcal{M})\subseteq Z^n(\mathcal{A},\mathcal{M})$. (We shall simply denote all these coboundary operators by the same symbol ∂.) The bounded n-th Hochschild cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{M}$ is defined as the quotient vector space:

$$H^n(\mathcal{A},\mathcal{M})=Z^n(\mathcal{A},\mathcal{M})/B^n(\mathcal{A},\mathcal{M}),\qquad n\geqslant 1.$$

By convention, $H^0(\mathcal{A},\mathcal{M})=\{M\in\mathcal{M}|\ AM=MA,\ \text{for each } A\in\mathcal{A}\}$. In [9], Lance proved that all n-th bounded cohomology groups $H^n(\mathrm{Alg}(\mathcal{N}),\mathcal{M})$ of each nest algebra on a separable Hilbert space $\mathcal{H}$ with coefficients in $\mathcal{M}$ are trivial for all $n\geqslant 1$, where $\mathcal{M}$ is an ultraweakly bimodule of $\mathrm{Alg}(\mathcal{N})$ satisfying that $\mathrm{Alg}(\mathcal{N})\subseteq\mathcal{M}\subseteq B(\mathcal{H})$.

2 One point extension of a nest

In the rest, let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $B(\mathcal{H})$ be the space of all the bounded linear operators acting on $\mathcal{H}$. In this paper, we do not distinguish a projection with its range, so write $\gamma\in P$ for an orthogonal projection P to mean that γ belongs to the range of P. Let $P^{\perp}$ denote the orthogonal complement $I-P$ of a projection P. For projections $P<Q$, we write $Q\ominus P$ for the projection $Q-P$.

Let $\mathcal{N}$ be a nontrivial nest of projections on $\mathcal{H}$, and $\mathrm{Alg}(\mathcal{N})$ be the corresponding nest algebra. Since the core $\mathcal{N}''$ of nest algebra $\mathrm{Alg}(\mathcal{N})$ is abelian, it has a separating vector, say ξ, which means the mapping $T\to T\xi$, from $\mathcal{N}''$ into $\mathcal{H}$, is injective [7]. We assume that $\|\xi\|=1$. Let P_ξ be the orthogonal projection from $\mathcal{H}$ onto the one-dimensional subspace of $\mathcal{H}$ generated by ξ. Then for each projection $P\in\mathcal{N}$ with $P\neq 0,I$, we have $\xi\notin P$, $\xi\notin P^{\perp}$ and hence $P\wedge P_\xi=0$. Obviously, $P\vee P_\xi$ is just the orthogonal projection from $\mathcal{H}$ onto the closed subspace $P(\mathcal{H})+\mathbb{C}\xi$. Hence for each pair P,Q in $\mathcal{N}$, $P\wedge(Q\vee P_\xi)=P\wedge Q$. Let $\mathcal{L}$ be the complete lattice of projections generated by $\mathcal{N}$ and P_ξ, which is called an *one point extension* of $\mathcal{N}$ by P_ξ. It is not difficult to show that $\mathcal{L}=\{0,I,P,P_\xi,P\vee P_\xi:\ P\in\mathcal{N},P\neq 0,I\}$.

Similarly, we could consider the dual of the nest $\mathcal{N}$. Let $\widetilde{\mathcal{N}}=\{0,I,P^{\perp}:\ P\in\mathcal{N}\}$. Then $\widetilde{\mathcal{N}}$ is a nest such that $\mathrm{Alg}(\mathcal{N})^*=\mathrm{Alg}(\widetilde{\mathcal{N}})$, so these two nest algebras have the same diagonal and core. Let $\widetilde{\mathcal{L}}$ be the one point extension of $\widetilde{\mathcal{N}}$ by P_ξ, i.e., $\widetilde{\mathcal{L}}=\{0,I,P,P_\xi,P\vee P_\xi:\ P\in\widetilde{\mathcal{N}},P\neq 0,I\}$.

Question. Are $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ KS-lattices?

Remark. Notice that the core $\mathcal{N}''$ is contained in the diagonal $\mathcal{N}'$ of nest algebra $\mathrm{Alg}(\mathcal{N})$, i.e., $\mathcal{N}''\subseteq\mathcal{N}'$. If we assume that ξ is a separating vector for $\mathcal{N}'$, then ξ is a generating vector for $\mathcal{N}''$, which implies that $\mathcal{N}''$ is a maximal abelian self-adjoint subalgebra of $B(\mathcal{H})$, and hence $\mathcal{N}'=\mathcal{N}''$ [7].

Theorem 2.1 [13]. *Suppose that ξ is a separating vector for $\mathcal{N}'$. Then $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are KS-lattices, and hence* $\mathrm{Alg}(\mathcal{L})$ *and* $\mathrm{Alg}(\widetilde{\mathcal{L}})$ *are KS-algebras*; *Moreover, they have the same diagonals equal to $\mathbb{C}I$.*

Inspecting the proof in [13], we can show that $\mathcal{L}$ is a KS-lattice, even though ξ is a separating vector only for $\mathcal{N}''$. Firstly, we recall some notions on complete projection lattices. Let $\mathcal{F}$ be a complete lattice of projections on $\mathcal{H}$. For $P\in\mathcal{F}$, we let

$$P_-^{\mathcal{F}}=\vee\{Q\in\mathcal{L}:\ Q\ngeq P\}\qquad \text{for } P\neq 0,\ 0_-^{\mathcal{F}}=0.$$

If $\mathcal{F}$ is a nest, then $P_-^{\mathcal{F}}=\vee\{Q\in\mathcal{F}:\ Q<P\}$ for $P\neq 0$, and we call $P_-^{\mathcal{F}}$ the *immediate predecessor* of P in $\mathcal{F}$ if there is one; otherwise $P_-^{\mathcal{F}}=P$. If there is no *immediate predecessor* for each nonzero projection in $\mathcal{F}$, then $\mathcal{F}$ is called a continuous nest. Similarly, for a projection P in a nest $\mathcal{F}$, we define

$$P_+^{\mathcal{F}}=\wedge\{Q\in\mathcal{F}:\ P<Q\}\qquad \text{for } P\neq I,\ I_+^{\mathcal{F}}=I,$$

and call $P_+^{\mathcal{F}}$ the *immediate successor* of P in $\mathcal{F}$ if there is one; otherwise $P_+^{\mathcal{F}}=P$.

For nonzero vectors γ and η in $\mathcal{H}$, we denote by $\gamma\otimes\eta$ the rank one operator, defined by $(\gamma\otimes\eta)(z)=\langle z,\eta\rangle\gamma$ for all $z\in\mathcal{H}$. The following Lemma 2.2 comes from [13], and the main ideas in the proof of the following Proposition 2.3 and Theorem 2.4 also come from [13], although we use the langue of nest algebras.

Lemma 2.2 [13]. *Suppose $E \in \mathcal{L}at(\mathrm{Alg}(\mathcal{L}))$, $E \neq 0, I$. For $P \in \mathcal{N}$ with $P \neq 0, I$, if there exists a nonzero vector ζ in $E(\mathcal{H})$ such that $P^\perp\zeta$ and $P^\perp\xi$ are linearly independent, then $P \leqslant E$.*

Proof. Let $\eta = P^\perp\zeta - \frac{\langle\zeta, P^\perp\xi\rangle}{\|P^\perp\xi\|^2}P^\perp\xi$. Then $\eta \neq 0$, $\eta \in (P \vee P_\xi)^\perp$. It is easy to show that for each nonzero vector β in P, the rank one operator $\beta \otimes \eta$ is in $\mathrm{Alg}(\mathcal{L})$ (see also Corollary 3.3). Hence $(\beta \otimes \eta)\zeta \in E$. By calculation, we have

$$(\beta \otimes \eta)\zeta = \left(\|P^\perp\zeta\|^2 - \frac{|\langle P^\perp\zeta,\ P^\perp\xi\rangle|^2}{\|P^\perp\xi\|^2}\right)\beta.$$

Since $P^\perp\zeta$ and $P^\perp\xi$ are linearly independent, we have

$$\|P^\perp\zeta\|^2 - \frac{|\langle P^\perp\zeta,\ P^\perp\xi\rangle|^2}{\|P^\perp\xi\|^2} \neq 0,$$

which implies that $\beta \in E$. Consequently, $P \leqslant E$.

Proposition 2.3. *Suppose that ξ is a separating vector for $\mathcal{N}''$. Then $\mathcal{L}$ is a reflexive lattice.*

Proof. Obviously, $\mathcal{L} \subseteq \mathcal{L}at(\mathrm{Alg}(\mathcal{L}))$. In order to obtain the reflexivity of $\mathcal{L}$, it suffices to show that $\mathcal{L}at(\mathrm{Alg}(\mathcal{L})) \subseteq \mathcal{L}$. Let E be an arbitrary projection in $\mathcal{L}at(\mathrm{Alg}(\mathcal{L})$ with $E \neq 0, I, P_\xi$. Define

$$Q = \vee\{P \in \mathcal{N}:\ P \leqslant E\}.$$

Obviously, $Q \in \mathcal{N}$ and $0 \leqslant Q \leqslant E < I$. If $Q = E$, then $E \in \mathcal{L}$. So we assume that $Q < E$. Now we show that $E = Q \vee P_\xi$. If it is true, then $E \in \mathcal{L}$.

We take two separate cases, depending on whether Q has an immediate successor in $\mathcal{N}$ or not.

Suppose that $Q_+^{\mathcal{N}} = Q$. Then there exists a strictly decreasing sequence $\{Q_n\}$ in $\mathcal{N}$: $I > Q_1 > Q_2 > \cdots > Q_n > \cdots > Q$, such that $\lim_n Q_n = Q$ in the strong operator topology. For each $n \geqslant 1$, it follows that $Q_n \nleqslant E$. Then, by Lemma 2.2, for each nonzero vector ζ in E, we have $Q_n^\perp\zeta$ and $Q_n^\perp\xi$ are linearly dependent, i.e.,

$$Q_n^\perp\zeta = \lambda_n Q_n^\perp\xi \quad \text{for each } n \geqslant 1 \text{ and for some } \lambda_n \in \mathbb{C}.$$

Notice that $Q_n < Q_1$ for each $n > 1$. Hence $Q_1^\perp(Q_n^\perp\zeta) = Q_1^\perp(\lambda_n Q_n^\perp\xi)$, and then $Q_1^\perp\zeta = \lambda_n Q_1^\perp\xi$. Also since $Q_1^\perp\zeta = \lambda_1 Q_1^\perp\xi$, it follows from $Q_1^\perp\xi \neq 0$ that we have $\lambda_n = \lambda_1$ for all $n \geqslant 1$. Consequently, we have

$$Q_n^\perp\zeta = \lambda_1 Q_n^\perp\xi \quad \text{for each } n \geqslant 1 \text{ and } \lambda_1 \in \mathbb{C}.$$

Let $n \to \infty$. We have $\zeta - Q\zeta = \lambda_1(\xi - Q\xi)$, which yields that $\zeta = (Q\zeta - \lambda_1 Q\xi) + \lambda_1\xi \in Q \vee P_\xi$ for each nonzero vector ζ in E. Hence $E \leqslant (Q \vee P_\xi)$. On the other hand, since $Q < E$, we choose a unit vector $\zeta_0 \in E$ such that $\zeta_0 \in Q^\perp$. Since we have proved that $\zeta_0 - Q\zeta_0 = \mu(\xi - Q\xi)$ for some $\mu \in \mathbb{C}$, we have $\zeta_0 = \mu(\xi - Q\xi) \neq 0$, and hence $\xi = \frac{1}{\mu}\zeta_0 + Q\xi \in E$, which implies that $E \geqslant (Q \vee P_\xi)$. Consequently, $E = Q \vee P_\xi$.

Suppose that $Q_+^{\mathcal{N}} > Q$. Now we claim that, for each nonzero vector ζ in E, $Q^\perp\zeta$ and $Q^\perp\xi$ are linearly dependent. For, otherwise, there exists a nonzero vector ζ_0 in E such that $Q^\perp\zeta_0$ and $Q^\perp\xi$ are linearly independent. Then $Q^\perp\zeta_0 \neq 0$. If we let $\beta = Q^\perp\zeta_0 - \frac{\langle\zeta_0, Q^\perp\xi\rangle}{\|Q^\perp\xi\|^2}Q^\perp\xi$, then $\beta \neq 0$ and $\beta \in (Q \vee P_\xi)^\perp$. Hence $y \otimes \beta$ is a rank one operator in $\mathrm{Alg}(\mathcal{L})$ for each nonzero vector y in $Q_+^{\mathcal{N}}$ (see Corollary 3.3(i)); in particular, $(y \otimes \beta)\zeta_0 \in E$, and hence $\langle\zeta_0, \beta\rangle y \in E$, for each nonzero vector y in $Q_+^{\mathcal{N}}$. It follows from the independence of $Q^\perp\zeta_0$ and $Q^\perp\xi$ that $\langle\zeta_0, \beta\rangle = \|Q^\perp\zeta_0\|^2 - \frac{|\langle Q^\perp\zeta_0, Q^\perp\xi\rangle|^2}{\|Q^\perp\xi\|^2} \neq 0$. Hence $y \in E$ for each nonzero vector y in $Q_+^{\mathcal{N}}$, and thus $Q_+^{\mathcal{N}} \subseteq E$, which contradicts the maximality of Q. We have established the claim. In other words, we have shown that, for each nonzero vector ζ in E, there is $\mu \in \mathbb{C}$ such that $Q^\perp\zeta = \mu Q^\perp\xi$, which implies that $\zeta = (Q\zeta - \mu Q\xi) + \mu\xi \in Q \vee P_\xi$. Hence $E \leqslant (Q \vee P_\xi)$. By the same argument as in above paragraph, we can prove that $\xi \in E$, so $E \geqslant (Q \vee P_\xi)$. Consequently, $E = (Q \vee P_\xi)$.

Theorem 2.4. *Suppose that ξ is a separating vector for $\mathcal{N}''$. Then $\mathcal{L}$ is a KS-lattice, and hence $\mathrm{Alg}(\mathcal{L})$ is a KS-algebra.*

Proof. Suppose $\mathcal{N}$ is a trivial nest, i.e., $\mathcal{N} = \{0, I\}$. Then $\mathcal{L}$ is a nest, and hence a KS-lattice. We assume that $\mathcal{N}$ is nontrivial. In this case, $\mathcal{L}$ is noncommutative, since for each nontrivial projection P in $\mathcal{N}$, $PP_\xi \neq P_\xi P$ (otherwise, $P\xi = P_\xi P\xi$, so $P\xi = \lambda\xi$ for some $\lambda \in \mathbb{C}$, which is a contradiction).

Let $\mathcal{L}_0$ be a reflexive sublattice of $\mathcal{L}$ such that $\mathcal{L}_0' = \mathcal{L}'$. Noticing $\mathcal{L}''$ $(= \mathcal{L}_0'')$ is a non-abelian von Neumann algebra, we have $\mathcal{L}_0 \not\subseteq \mathcal{N}$ and $\mathcal{L}_0 \not\subseteq \{P \vee P_\xi : P \in \mathcal{N}\}$. In other words, there exist $P_0, Q_0 \in \mathcal{N}$ with $0 < P_0 < I$, $0 \leqslant Q_0 < I$ such that $Q_0 \vee P_\xi \neq I$, $P_0 \in \mathcal{L}_0$ and $Q_0 \vee P_\xi \in \mathcal{L}_0$.

Claim 1. $P_\xi \in \mathcal{L}_0$.

We let $Q = \wedge\{P \in \mathcal{N} : (P \vee P_\xi) \in \mathcal{L}_0\}$. Then $Q \in \mathcal{N}$, $Q \leqslant Q_0 < I$ and $(Q \vee P_\xi) = \wedge\{P \vee P_\xi \in \mathcal{L}_0 : P \in \mathcal{N}\} \in \mathcal{L}_0$. Suppose $Q > 0$. Then $0 < Q \leqslant Q_0 < I$. For each $M \in \mathcal{L}_0$, if $M \in \mathcal{N}$ then $MQ = QM$; if $M \notin \mathcal{N}$ then $M = P \vee P_\xi$ for some $P \in \mathcal{N}$ with $0 \leqslant P < I$, and hence $Q \leqslant P < M$, so, $MQ = QM$. Hence $Q \in \mathcal{L}_0'$. It follows from $\mathcal{L}_0' = \mathcal{L}'$ that $QP_\xi = P_\xi Q$, which is a contradiction. Consequently, $Q = 0$ and hence $P_\xi \in \mathcal{L}_0$.

Claim 2. For each $P \in \mathcal{N}$ with $0 < P < I$, we have $P \in \mathcal{L}_0$.

Suppose, on the contrary, there exists $P_1 \in \mathcal{N}$ such that $0 < P_1 < I$ and $P_1 \notin \mathcal{L}_0$. We have two cases.

Case 1. Suppose $P \notin \mathcal{L}_0$ for each projection $P \in \mathcal{N}$ with $P_1 \leqslant P < I$.

We let

$$P_2 = \vee\{P \in \mathcal{N} : P \neq I, P \in \mathcal{L}_0\}.$$

Then $P_2 \in \mathcal{L}_0$ and $0 < P_0 \leqslant P_2 < P_1 < I$. Obviously, $P_2 \vee P_\xi \neq I$ (see Lemma 3.2). Now we show that $M_0 := P_2 \vee P_\xi$ is in $\mathcal{L}_0'$, but $M_0 \notin \mathcal{L}'$. If so, this case cannot occur.

In fact, for each $M \in \mathcal{L}_0$ with $0 < M < I$, if M has the form $P \vee P_\xi$ for some $P \in \mathcal{N}$, then M and M_0 are commutative; if $M \in \mathcal{N}$, then $M \leqslant P_2 < M_0$, so $MM_0 = M_0M$. Hence $M_0 = P_2 \vee P_\xi \in \mathcal{L}_0'$. However, $(P_2 \vee P_\xi)P_1 \neq P_1(P_2 \vee P_\xi)$, for otherwise, $(P_2 \vee P_\xi)P_1\xi = P_1(P_2 \vee P_\xi)\xi = P_1\xi$, which yields that $P_1\xi = \zeta + \lambda\xi$ for some $\zeta \in P_2$ and $\lambda \in \mathbb{C}$. Using $P_2^\perp$ acts on both sides of the equation and noticing that $P_2 < P_1$, we get $(P_2^\perp P_1 - \lambda P_2^\perp)\xi = 0$, which is a contradiction, for $(P_2^\perp P_1 - \lambda P_2^\perp)$ is a nontrivial operator in $\mathcal{N}''$. Hence $M_0 \notin \mathcal{L}'$.

Case 2. Suppose that there is a projection P_3 in $\mathcal{N}$ such that $P_1 < P_3 < I$ and $P_3 \in \mathcal{L}_0$.

We let

$$Q_1 = \vee\{M \in \mathcal{N} : M \in \mathcal{L}_0, M < P_1\}, \quad Q_2 = \wedge\{M \in \mathcal{N} : M \in \mathcal{L}_0, M > P_1\}.$$

Then $Q_1, Q_2 \in \mathcal{L}_0 \cap \mathcal{N}$, $0 \leqslant Q_1 < P_1 < Q_2 \leqslant P_3 < I$. By the definitions of Q_1 and Q_2, for each P in $\mathcal{N}$ with $Q_1 < P < Q_2$, we have $P \notin \mathcal{L}_0$, and hence $(P \vee P_\xi) \notin \mathcal{L}_0$ by noticing that $(P \vee P_\xi) \wedge Q_2 = P$ and $Q_2 \in \mathcal{L}_0$. Since $(P_1 - Q_1)\xi \neq 0$ and $(Q_2 - P_1)\xi \neq 0$, we let

$$\eta = (Q_2 - P_1)\xi - \frac{\|(Q_2 - P_1)\xi\|^2}{\|(P_1 - Q_1)\xi\|^2}(P_1 - Q_1)\xi.$$

Then $\eta \neq 0$, $\eta \in Q_2$, $\eta \in Q_1^\perp$, $\eta \in P_\xi^\perp$, $\eta \notin P_1$, $\eta \notin P_1^\perp$. Let $A = \eta \otimes \eta$. Since $P_1\eta$ and η are linearly independent, we have $AP_1 \neq P_1A$, so $A \notin \mathcal{L}'$. Now we show that $A \in \mathcal{L}_0'$.

For each $M \in \mathcal{L}_0$ with $0 < M < I$, if $M \in \mathcal{N}$ then $M \leqslant Q_1$ or $M \geqslant Q_2$, and hence $MA = AM = 0$ or $AM = MA = A$; if M has the form $P \vee P_\xi$ for some $P \in \mathcal{N}$ with $0 < P < I$, then $P \leqslant Q_1$ or $P \geqslant Q_2$, and hence $MA = AM = 0$ or $AM = MA = A$. Hence $A \in \mathcal{L}_0'$. Consequently, we have constructed a rank operator A in $\mathcal{L}_0'$, but not in $\mathcal{L}'$, which is a contradiction. This case also could not occur.

By Cases 1 and 2, we have established Claim 2. By Claims 1 and 2, for each reflexive sublattice of $\mathcal{L}$ such that $\mathcal{L}_0' = \mathcal{L}'$, we have that $\mathcal{L}_0$ and $\mathcal{L}$ are equal. Hence $\mathcal{L}$ is a KS-lattice.

The following are examples of KS-algebras.

Example 2.1 [13]**.** Suppose $\mathcal{H}$ is an infinite dimensional separable Hilbert space with an orthogonal basis $\{e_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let P_n be the orthogonal projection of $\mathcal{H}$ onto the linear subspace of $\mathcal{H}$ generated by $\{e_1, e_2, \ldots, e_n\}$. Then $\mathcal{N} = \{0, I, P_n : n = 1, 2, \ldots\}$ is an $\mathbb{N}$-ordered nest with $I_-^{\mathcal{N}} = I$ and $0_+^{\mathcal{N}} = P_1$. Let $\xi = \sum_{n=1}^{\infty} \frac{1}{n} e_n \in \mathcal{H}$. Then ξ is a separating vector of $\mathcal{N}'$. Hence $\mathcal{L} = \{0, I, P_n, P_\xi, P_n \vee P_\xi : n \in \mathbb{N}\}$ is a KS-lattice, and $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer algebra.

Remark. If let $\widetilde{\mathcal{N}} = \{0, I, P_n^\perp : n = 1, 2, \ldots\}$. Then $\widetilde{\mathcal{N}}$ also is a nest with $0_+^{\widetilde{\mathcal{N}}} = 0$, $I_-^{\widetilde{\mathcal{N}}} = P_1^\perp < I$, and $I_-^{\widetilde{\mathcal{N}}} \vee P_\xi = I$. Hence $\widetilde{\mathcal{L}} = \{0, I, P_n^\perp, P_\xi, P_n^\perp \vee P_\xi : n \in \mathbb{N}\}$ is a KS-lattice.

Suppose that $\mathcal{H}$ has an orthogonal basis $\{e_k : k \in \mathbb{Z}\}$. For each $k \in \mathbb{Z}$, let Q_k be the orthogonal projection of $\mathcal{H}$ onto the closed subspace of $\mathcal{H}$ generated by $\{e_l : l \in \mathbb{Z}, l \leqslant k\}$. Then $\mathcal{N} = \{0, I, Q_k : k \in \mathbb{Z}\}$ is a $\mathbb{Z}$-ordered nest with $I_-^{\mathcal{N}} = I$ and $0_+^{\mathcal{N}} = 0$. Let $\xi = \sum_{-\infty}^{+\infty} \frac{1}{2^{|k|}} e_k \in \mathcal{H}$. Then ξ is a separating vector of $\mathcal{N}'$, and hence $\mathcal{L} = \{0, I, Q_k, P_\xi, Q_k \vee P_\xi : k \in \mathbb{Z}\}$ is a KS-lattice, $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer algebra.

Example 2.2. Let $\mathcal{H}$, P_n, $n = 1, 2, \ldots$, and ξ be as in Example 2.1. For a nontrivial subsequence $\{n_k\}$ of $\mathbb{N}$: $1 \leqslant n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim_k n_k = \infty$, we let $Q_k = P_{n_k}$. Then $\mathcal{N} = \{0, I, Q_k : k = 1, 2, \ldots\}$ is a nest and ξ is a separating vector for $\mathcal{N}''$, but not for $\mathcal{N}'$. Then the complete lattice generated by $\mathcal{N}$ and P_ξ is a KS-lattice.

Example 2.3 [13]**.** Let $\mathcal{H} = L^2[0,1]$ be the Hilbert space consisting of all measurable complex-valued functions f on $[0,1]$ for which $\int_0^1 |f(x)|^2 dx < \infty$ (with respect to the Lebesgue measure on $[0,1]$). For any $t \in (0,1)$, let $P_t = M_{\chi_{[0,t]}}$ be the multiplication operator by the characteristic function $\chi_{[0,t]}$ of $[0,t]$ on $\mathcal{H}$, i.e., $P_t(g) = \chi_{[0,t]} g$ for each g in $\mathcal{H}$. Then $\mathcal{N} = \{0, I, P_t : t \in (0,1)\}$ is a continuous nest. Let ξ be the constant function 1. Then ξ is a separating vector of $\mathcal{N}'$. Hence the complete projection lattice $\mathcal{L}$ generated by the nest $\mathcal{N}$ and the projection P_ξ is a Kadison-Singer lattice, and thus $\mathrm{Alg}(\mathcal{L})$ is a Kadison-Singer algebra.

3 Commutant

Let $\mathcal{L}$ be a one point extension of a nontrivial nest $\mathcal{N}$ on $\mathcal{H}$ by P_ξ, defined as in Section 2, where ξ is a separating vector for $\mathcal{N}''$. In this section, we consider the rank one operators in $\mathrm{Alg}(\mathcal{L})$ and their applications. For nonzero vectors γ and η in $\mathcal{H}$, we denote by $\gamma \otimes \eta$ the rank one operator, defined by $(\gamma \otimes \eta)(z) = \langle z, \eta \rangle \gamma$ for all $z \in \mathcal{H}$. The following lemma is a well-known fact. For completeness, we give a proof.

Lemma 3.1 [10]**.** *For a complete lattice $\mathcal{F}$ of projections on $\mathcal{H}$, a rank one operator $\gamma \otimes \eta \in \mathrm{Alg}(\mathcal{F})$ if and only if there exists $P \in \mathcal{F}$ such that $\gamma \in P$ and $\eta \in (P_-^{\mathcal{F}})^\perp$.*

Proof. Suppose that $0 \neq \gamma \in P$ and $0 \neq \eta \in (P_-^{\mathcal{F}})^\perp$ for a nonzero projection P in $\mathcal{F}$. Let $0 \neq Q \in \mathcal{F}$, and x be an arbitrary vector in Q. If $Q \geqslant P$, then $(\gamma \otimes \eta)(x) = \langle x, \eta \rangle \gamma \in P \subseteq Q$; if $Q \ngeqslant P$, then $Q \leqslant P_-^{\mathcal{F}}$, and thus $\langle x, \eta \rangle = 0$, which implies that $(\gamma \otimes \eta)(x) = 0$. Both cases yield that $(\gamma \otimes \eta)(Q) \subseteq Q$. Hence $\gamma \otimes \eta \in \mathrm{Alg}(\mathcal{F})$.

On the other direction, let $\gamma \otimes \eta$ be a rank one operator in $\mathrm{Alg}(\mathcal{F})$. Let $M_\gamma = \wedge\{P \in \mathcal{F} : \gamma \in P\}$. Then $0 \neq M_\gamma \in \mathcal{F}$ and $\gamma \in M_\gamma$. Now we show that $\eta \in ((M_\gamma)_-^{\mathcal{F}})^\perp$. Let $Q \in \mathcal{F}$ with $Q \ngeqslant M_\gamma$. Then $\gamma \notin Q$. It follows that $\eta \in Q^\perp$, for otherwise, there exists a nonzero vector x in Q such that $\langle x, \eta \rangle \neq 0$. Since $\gamma \otimes \eta \in \mathrm{Alg}(\mathcal{F})$, we have $(\gamma \otimes \eta)(x) = \langle x, \eta \rangle \gamma \in Q$, and thus, $\gamma \in Q$, which is a contradiction. Hence $\eta \in Q^\perp$. Since Q is arbitrary, we have $\eta \in ((M_\gamma)_-^{\mathcal{F}})^\perp$.

In the rest, we replace $M_-^{\mathcal{L}}$ with M_- for a nonzero projection in $\mathcal{L}$; if M is also in $\mathcal{N}$, we still denote by $M_-^{\mathcal{N}}$ predecessor of M in $\mathcal{N}$.

Lemma 3.2 (i) *If $P, Q \in \mathcal{N}$ with $0 < P < Q < I$, then $(P \vee P_\xi) < (Q \vee P_\xi)$ and $(P \vee P_\xi) \ngeqslant Q$. In particular, for each pair $P, Q \in \mathcal{N}$ with $P, Q \neq 0, I$, we have $P \vee P_\xi = Q \vee P_\xi$ if and only if $P = Q$;*

(ii) *$I_- = I_-^{\mathcal{N}} \vee P_\xi$; Moreover, if ξ is a separating vector for $\mathcal{N}'$, then $I_-^{\mathcal{N}} \vee P_\xi = I$, no matter whether I has an immediate predecessor in $\mathcal{N}$ or not;*

(iii) *$P_- = P_-^{\mathcal{N}} \vee P_\xi$ for a projection P in $\mathcal{N}$ with $0 < P < I$;*

(iv) *$(P_\xi)_- = I_-^{\mathcal{N}}$, $(P \vee P_\xi)_- = I_-^{\mathcal{N}} \vee P_\xi$ for each nonzero projection P in $\mathcal{N}$; Moreover, if ξ is a separating vector for $\mathcal{N}'$, then $(P \vee P_\xi)_- = I$, no matter whether I has an immediate predecessor in $\mathcal{N}$ or not.*

Proof. (i) Let $P, Q \in \mathcal{N}$ with $0 < P < Q < I$. Then $(P \vee P_\xi) \leqslant (Q \vee P_\xi)$. If $(P \vee P_\xi) = (Q \vee P_\xi)$ or $(P \vee P_\xi) \geqslant Q$, choose a unit vector y_0 in $Q \ominus P$. Hence $y_0 \in (P \vee P_\xi)$, which implies that there exist a nonzero vector $z_0 \in P$ and a nonzero complex number λ such that $y_0 = z_0 + \lambda\xi$. Consequently, $\xi = \frac{1}{\lambda}(y_0 - z_0) \in Q$, which contradicts with the assumption that ξ is a separating vector. Hence $(P \vee P_\xi) < (Q \vee P_\xi)$ and $(P \vee P_\xi) \ngeqslant Q$.

(ii) By the definition, we have $I_- = \vee\{M \in \mathcal{L} :\ M < I\} = I_-^{\mathcal{N}} \vee P_\xi$. Suppose ξ is a separating vector for $\mathcal{N}'$. If $(I_-^{\mathcal{N}} \vee P_\xi) < I$, we let $M = (I_-^{\mathcal{N}} \vee P_\xi)^\perp$, and then M is a nonzero projection in $\mathcal{N}'$, but $M\xi = 0$, which contradicts with the assumption that ξ is a separating vector for $\mathcal{N}'$. Hence $(I_-^{\mathcal{N}} \vee P_\xi) = I$.

(iii) Let $P \in \mathcal{N}$, $P \neq 0, I$. For each $M \in \mathcal{L}$ with $M \ngeqslant P$, if $M \in \mathcal{N}$ then $M < P$, which implies that $M \leqslant P_-^{\mathcal{N}} \leqslant (P_-^{\mathcal{N}} \vee P_\xi)$; if $M \notin \mathcal{N}$ then there exists $Q \in \mathcal{N}$ such that $Q < P$ and $M = Q \vee P_\xi$, which implies that $M \leqslant (P_-^{\mathcal{N}} \vee P_\xi)$. Hence it follows from the arbitrariness of M that $P_- \leqslant (P_-^{\mathcal{N}} \vee P_\xi)$. On the other hand, since $P_- \geqslant P_-^{\mathcal{N}}$ and $P_- \geqslant P_\xi$, we have $(P_-^{\mathcal{N}} \vee P_\xi) \leqslant P_-$. Hence $P_- = P_-^{\mathcal{N}} \vee P_\xi$.

(iv) By the definition, we have $(P_\xi)_- = \vee\{M \in \mathcal{L} :\ M \ngeqslant P_\xi\} = \vee\{M \in \mathcal{L} :\ \xi \notin M\} = \vee\{M \in \mathcal{N} : \xi \notin M\} = \vee\{M \in \mathcal{N} :\ M < I\} = I_-^{\mathcal{N}}$.

Similarly, for $P \in \mathcal{N}$ with $P \neq 0, I$, we have $(P \vee P_\xi)_- = \vee\{M \in \mathcal{L} :\ M \ngeqslant (P \vee P_\xi)\} = (\vee\{Q \in \mathcal{N} : Q < I\}) \vee (\vee\{Q \vee P_\xi :\ Q \in \mathcal{N}, Q < P\}) = I_-^{\mathcal{N}} \vee P_\xi$.

Using Lemmas 3.1 and 3.2, we have the following corollaries.

Corollary 3.3. *For nonzero vectors x and y in $\mathcal{H}$, the rank one operator $x \otimes y \in \mathrm{Alg}(\mathcal{L})$ if and only if one of the following statements holds*:

(i) *there exists $P \in \mathcal{N}$ with $P \neq 0, I$ such that $x \in P$ and $y \in (P_-^{\mathcal{N}} \vee P_\xi)^\perp$*;

(ii) *$I_-^{\mathcal{N}} < I$, and $x \in \mathbb{C}\xi$, $y \in (I_-^{\mathcal{N}})^\perp$*;

(iii) *$(I_-^{\mathcal{N}} \vee P_\xi) < I$, and $x \in \mathcal{H}$, $y \in (I_-^{\mathcal{N}} \vee P_\xi)^\perp$.*

Corollary 3.4. *Suppose that $I_-^{\mathcal{N}} = I$. Then for each rank one operator $x \otimes y$, $(x \otimes y)(\xi) = 0$. Hence if let $\mathcal{R}_1(\mathrm{Alg}(\mathcal{L}))$ be the linear span generated by all the rank one operators in* $\mathrm{Alg}(\mathcal{L})$*, then $\mathcal{R}_1(\mathrm{Alg}(\mathcal{L}))$ is not dense in* $\mathrm{Alg}(\mathcal{L})$ *under the ultra-weak topology.*

Corollary 3.5. *Suppose that $I_-^{\mathcal{N}} = I$. Then every rank n operator F in* $\mathrm{Alg}(\mathcal{L})$ *can be written a sum of n rank one operators in* $\mathrm{Alg}(\mathcal{L})$.

Proof. For a rank n operator F in $\mathrm{Alg}(\mathcal{L})$, we have $F \in \mathrm{Alg}(\mathcal{N})$. Hence $F = e_1 \otimes f_1 + \cdots + e_n \otimes f_n$, where $e_i \otimes f_i \in \mathrm{Alg}(\mathcal{N})$, $f_i \neq 0$ for each i, and $e_1, \ldots, e_n$ are linearly independent in the range of F. So there exists $P_i \in \mathcal{N}$ such that $e_i \in P_i$ and $f_i \in (P_{i-}^{\mathcal{N}})^\perp$ for each i. Since $I_-^{\mathcal{N}} = I$, we have $P_i \neq I$ for every i. Also since $F(\xi) \in \mathbb{C}\xi$, we get that $\langle \xi, f_1\rangle e_1 + \cdots + \langle \xi, f_n\rangle e_n = \lambda\xi$ for some $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, then $\xi \in P_1 \vee \cdots \vee P_n (\neq I)$, which contradicts with the assumption that ξ is a separating vector. Hence $\langle \xi, f_1\rangle e_1 + \cdots + \langle \xi, f_n\rangle e_n = 0$. Using the linear independence of $e_i's$, we have $\langle \xi, f_i\rangle = 0$, which implies that $e_i \otimes f_i \in \mathrm{Alg}(\mathcal{L})$ for every i. Hence F can be written as a sum of n rank one operators in $\mathrm{Alg}(\mathcal{L})$.

Theorem 3.6. (i) *If $I_-^{\mathcal{N}} = I$ or $(I_-^{\mathcal{N}} \vee P_\xi) < I$, then $(\mathrm{Alg}(\mathcal{L}))' = \mathbb{C}I$. In particular, the center of* $\mathrm{Alg}(\mathcal{L})$ *is trivial.*

(ii) *If $I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}} \vee P_\xi) = I$, then $T \in \mathrm{Alg}(\mathcal{L})'$ if and only if there exist $\lambda, \mu \in \mathbb{C}$ such that $TQ = \lambda Q$ and $T\xi = \mu\xi$, where $Q = I_-^{\mathcal{N}}$. Hence $(\mathrm{Alg}(\mathcal{L}))' \subseteq \mathrm{Alg}(\mathcal{L})$ is a two-dimensional subalgebra. In particular, the center of* $\mathrm{Alg}(\mathcal{L})$ *has dimension two.*

Proof. (i) Let T be a nonzero operator in $B(\mathcal{H})$ such that $TS = ST$ for each S in $\mathrm{Alg}(\mathcal{L})$.

Suppose $I_-^{\mathcal{N}} = I$. Then there exists a strictly increasing sequence $\{Q_n\}$ in $\mathcal{N}$ such that $Q_1 > 0$ and $\lim_{n\to\infty} Q_n = I$ in the strong operator topology. By Lemma 3.2, we have $(Q_n \vee P_\xi) < I$, so that we can choose a unit vector $y_n \in (Q_n \vee P_\xi)^\perp$ for each n. It follows from Corollary 3.3 that $(Q_n x) \otimes y_n \in \mathrm{Alg}(\mathcal{L})$ for each $x \in \mathcal{H}$ and each n. Hence $T((Q_n x) \otimes y_n) = ((Q_n x) \otimes y_n)T$, which implies that

$$TQ_n x = \langle Ty_n, y_n\rangle Q_n x. \tag{1}$$

For a given nonzero vector x in $\mathcal{H}$, since $\lim_n Q_n x = x$, we can assume that $Q_n x \neq 0$ for each n. Since $|\langle Ty_n, y_n\rangle| \leqslant \|T\|$ for each n, there exists a convergence subsequence in $\{\langle Ty_n, y_n\rangle\}$. Without loss of generality, we assume that $\{\langle Ty_n, y_n\rangle\}$ converges to $\lambda \in \mathbb{C}$. Let $n \to \infty$ in both sides of (1). We have $Tx = \lambda x$. Since x is arbitrary, we have $T \in \mathbb{C}I$. Hence $\mathrm{Alg}(\mathcal{L})' = \mathbb{C}I$.

Suppose $(I_-^{\mathcal{N}} \vee P_\xi) < I$. Choose a unit vector $y_0 \in (I_-^{\mathcal{N}} \vee P_\xi)^\perp$. Then for each nonzero vector x in $\mathcal{H}$, it follows from Corollary 3.3 that $x \otimes y_0 \in \mathrm{Alg}(\mathcal{L})$. Hence $T(x \otimes y_0) = (x \otimes y_0)T$, which implies that $Tx = \langle Ty_0, y_0\rangle x$. Consequently, $T \in \mathbb{C}I$.

(ii) Let $Q = I_-^{\mathcal{N}}$. Since $I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}} \vee P_\xi) = I$, we have $\mathcal{H} = Q(\mathcal{H}) + \mathbb{C}\xi$ and $Q(\mathcal{H})$ has codimension 1. For $T \in B(\mathcal{H})$, suppose that there exist $\lambda, \mu \in \mathbb{C}$ such that $TQ = \lambda Q$ and $T\xi = \mu\xi$. For an arbitrary $S \in \mathrm{Alg}(\mathcal{L})$ and a nonzero vector $x \in \mathcal{H}$, we let $S\xi = \alpha\xi$ and $x = x_0 + \beta\xi$ for $x_0 \in Q(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$. Hence

$$TSx = TS(x_0 + \beta\xi) = TSQx_0 + \alpha\beta\mu\xi = TQ(SQx_0) + \alpha\beta\mu\xi = \lambda Sx_0 + \alpha\beta\mu\xi$$

and

$$STx = ST(x_0 + \beta\xi) = STQx_0 + \alpha\beta\mu\xi = \lambda Sx_0 + \alpha\beta\mu\xi.$$

Consequently, $ST = TS$ for each S in $\mathrm{Alg}(\mathcal{L})$, which implies that $T \in \mathrm{Alg}(\mathcal{L})'$. At this time, since $TP = \lambda P$ for each $P \in \mathcal{N}$ with $P < I$ and $T\xi = \mu\xi$, we have $T \in \mathrm{Alg}(\mathcal{L})$.

On the other hand, suppose that $T \in \mathrm{Alg}(\mathcal{L})'$. By Corollary 3.3, we have that $\xi \otimes Q^\perp\xi \in \mathrm{Alg}(\mathcal{L})$. Hence $T(\xi \otimes Q^\perp\xi) = (\xi \otimes Q^\perp\xi)T$, which yields that

$$T\xi = \frac{\langle TQ^\perp\xi, Q^\perp\xi\rangle}{\|Q^\perp\xi\|^2}\xi = \mu\xi,$$

where

$$\mu = \frac{\langle TQ^\perp\xi, Q^\perp\xi\rangle}{\|Q^\perp\xi\|^2}.$$

In order to show that $TQ = \lambda Q$ for some $\lambda \in \mathbb{C}$, we have two cases.

If $Q_-^{\mathcal{N}} < Q$, then using Lemma 3.2, we have $(Q_-^{\mathcal{N}} \vee P_\xi) < I$. Choose a unit vector y_0 in $(Q_-^{\mathcal{N}} \vee P_\xi)^\perp$. Then for each nonzero vector x in Q, Corollary 3.3 yields that $x \otimes y_0 \in \mathrm{Alg}(\mathcal{L})$. Hence $T(x \otimes y_0) = (x \otimes y_0)T$, which implies that $Tx = \langle Ty_0, y_0\rangle x$. Consequently, $TQ = \lambda Q$, where $\lambda = \langle Ty_0, y_0\rangle \in \mathbb{C}$.

If $Q_-^{\mathcal{N}} = Q$, then there exists a sequence $\{Q_n\}$ in $\mathcal{N}$ such that $0 < Q_1 < Q_2 < \cdots < Q_n < \cdots < Q$ and $\lim_n Q_n = Q$ in the strong operator topology. Then by Lemma 3.2, we have $(Q_n \vee P_\xi) < I$, so that we can choose a unit vector $y_n \in (Q_n \vee P_\xi)^\perp$ for each n. Then for each $x \in Q$, we have $Q_n x \otimes y_n \in \mathrm{Alg}(\mathcal{L})$. By a similar way to (i), we can show that there exists a complex number λ such that $TQ = \lambda Q$.

Remark. Suppose that $Q = I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}} \vee P_\xi) = I$. If let

$$\eta = \frac{Q^\perp\xi}{\|Q^\perp\xi\|},$$

and P_η be the orthogonal projection onto $\mathbb{C}\eta$, then $Q^\perp = P_\eta$. Relative to $I = Q + Q^\perp$, $T \in (\mathrm{Alg}(\mathcal{L}))'$ if and only if T has the matrix representation $\left(\begin{smallmatrix}\lambda & T_{12}\\ 0 & \mu\end{smallmatrix}\right)$, where $\lambda, \mu \in \mathbb{C}$ and $T_{12}\eta = \frac{\mu-\lambda}{\|Q^\perp\xi\|}Q\xi$. So

$$\mathrm{Alg}(\mathcal{L})' = \left\{\lambda Q + \mu Q^\perp + \frac{\mu - \lambda}{\|Q^\perp\xi\|^2}Q(\xi \otimes \xi)Q^\perp :\ \lambda, \mu \in \mathbb{C}\right\}.$$

If $\mathcal{A}$ is a subalgebra of $B(\mathcal{H})$ and $\mathcal{M}$ is a bimodule of $\mathcal{A}$ in $B(\mathcal{H})$, we denote by $C(\mathcal{A}, \mathcal{M})$ the commutant of $\mathcal{A}$ module $\mathcal{M}$, i.e., $C(\mathcal{A}, \mathcal{M}) = \{T \in B(\mathcal{H}) :\ TS - ST \in \mathcal{M}\ \text{for each}\ S\ \text{in}\ \mathcal{A}\}$. In [5], Han proved that, for any ultraweakly closed bimodule of a commutative subspace lattice (CSL) algebra $\mathcal{A}$ satisfying $\mathcal{A} \subseteq \mathcal{M} \subseteq B(\mathcal{H})$, the commutant of $\mathcal{A}$ module $\mathcal{M}$ is equal to $\mathcal{M}$, i.e., $C(\mathcal{A}, \mathcal{M}) = \mathcal{M}$.

Proposition 3.7. $C(\mathrm{Alg}(\mathcal{L}), \mathrm{Alg}(\mathcal{L})) = \mathrm{Alg}(\mathcal{L})$.

Proof. Obviously, $C(\mathrm{Alg}(\mathcal{L}), \mathrm{Alg}(\mathcal{L})) \supseteq \mathrm{Alg}(\mathcal{L})$. Let T be an arbitrary nonzero element in $C(\mathrm{Alg}(\mathcal{L}),$ $\mathrm{Alg}(\mathcal{L}))$. Then $TS - ST \in \mathrm{Alg}(\mathcal{L})$ for each S in $\mathrm{Alg}(\mathcal{L})$. We first claim that $T \in \mathrm{Alg}(\mathcal{N})$, i.e., $TP \subseteq P$ for each P in $\mathcal{N}$ satisfying $0 < P < I$.

Suppose that $0 \leqslant P_-^{\mathcal{N}} < P$. Then using Lemma 3.2, we have $(P_-^{\mathcal{N}} \vee P_\xi) < I$ and $P \nleqslant (P_-^{\mathcal{N}} \vee P_\xi)$. Choose $y_0 \in (P_-^{\mathcal{N}} \vee P_\xi)^\perp$ and $z_0 \in P$ such that $\langle z_0, y_0\rangle = 1$. By Corollary 3.3, since $x \otimes y_0$ is in $\mathrm{Alg}(\mathcal{L})$ for each nonzero vector x in P, we obtain that $T(x \otimes y_0) - (x \otimes y_0)T \in \mathrm{Alg}(\mathcal{L})$, which yields that $[T(x \otimes y_0) - (x \otimes y_0)T]z_0 \in P$. Hence $Tx - \langle Tz_0, y_0\rangle x \in P$, and thus, $Tx \in P$ for each x in P.

Suppose that $P_-^{\mathcal{N}} = P$. Then there exists a sequence $\{P_n\}$ in $\mathcal{N}$ such that $0 < P_1 < P_2 < \cdots < P_n < \cdots < P$ and $\lim_n P_n = P$ in the strong operator topology. Then Lemma 3.2 yields that $(P_n \vee P_\xi) < I$ and $P \nleqslant (P_n \vee P_\xi)$ for each n. Choose $y_n \in (P_n \vee P_\xi)^\perp$ and $z_n \in P$ such that $\langle z_n, y_n\rangle = 1$ for each n. Then for each $x \in P$, we have $P_n x \otimes y_n \in \mathrm{Alg}(\mathcal{L})$. Hence $T(P_n x \otimes y_n) - (P_n x \otimes y_n)T \in \mathrm{Alg}(\mathcal{L})$, which implies that $[T(P_n x \otimes y_n) - (P_n x \otimes y_n)T]z_n \in P$, and thus $TP_n x - \langle Tz_n, y_n\rangle P_n x \in P$. So, $TP_n x \in P$ for each n and each $x \in P$. Also since $\lim_n TP_n x = Tx$ for each $x \in P$, we have $Tx \in P$. We have established the claim.

Now we show that $T\xi \in \mathbb{C}\xi$. If so, we can obtain that $T \in \mathrm{Alg}(\mathcal{L})$. We have two cases.

Suppose that $0_+^{\mathcal{N}} = 0$. Then there exists a sequence $\{Q_n\}$ in $\mathcal{N}$ such that $I > Q_0 > Q_1 > \cdots > Q_n > \cdots$ and $\bigwedge_{n\geqslant 1} Q_n = 0$. It follows from Lemma 3.2 that $(Q_n \vee P_\xi) < I$ for each $n \geqslant 1$. Using Corollary 3.3, we have $Q_n\xi \otimes y \in \mathrm{Alg}(\mathcal{L})$ for each n and each $y \in (Q_n \vee P_\xi)^\perp$, which implies that $T(Q_n\xi \otimes y) - (Q_n\xi \otimes y)T \in \mathrm{Alg}(\mathcal{L})$. Hence $[T(Q_n\xi \otimes y) - (Q_n\xi \otimes y)T]\xi \in \mathbb{C}\xi$, and then, $\langle T\xi, y\rangle Q_n\xi \in \mathbb{C}\xi$ for each n and each $y \in (Q_n \vee P_\xi)^\perp$. Since $Q_n\xi \notin \mathbb{C}\xi$ for each $n \geqslant 1$, we have $\langle T\xi, y\rangle = 0$ for each n and each $y \in (Q_n \vee P_\xi)^\perp$, which yields that $T\xi \in (Q_n \vee P_\xi)$ for each $n \geqslant 1$. Let $T\xi = x_1 + \lambda_1\xi$ for some $x_1 \in Q_1$ and $\lambda_1 \in \mathbb{C}$. Since for each $n > 1$, there are $x_n \in Q_n$ and $\lambda_n \in \mathbb{C}$ such that $T\xi = x_n + \lambda_n\xi$, we have $x_1 - x_n = (\lambda_n - \lambda_1)\xi \in Q_1$. It follows that $\lambda_1 = \lambda_n$, and then $x_1 = x_n \in Q_n$. Hence $x_1 \in \bigwedge_{n\geqslant 1} Q_n = 0$. Consequently, $T\xi = \lambda_1\xi$.

Suppose that $0_+^{\mathcal{N}} \neq 0$. Let $P = 0_+^{\mathcal{N}}$. Then $x \otimes y \in \mathrm{Alg}(\mathcal{L})$ for each $x \in P$ and each $y \in P_\xi^\perp$. Fix $x_0 \in P$ with $x_0 \neq 0$. Hence for each nonzero vector $y \in P_\xi^\perp$, $T(x_0 \otimes y) - (x_0 \otimes y)T \in \mathrm{Alg}(\mathcal{L})$, and then $[T(x_0 \otimes y) - (x_0 \otimes y)T]\xi \in \mathbb{C}\xi$. Consequently, $\langle T\xi, y\rangle x_0 \in \mathbb{C}\xi$ for all y in $P_\xi^\perp$, which implies that $\langle T\xi, y\rangle = 0$ for all y in $P_\xi^\perp$, so, $T\xi \in \mathbb{C}\xi$.

4 Cohomology

In this section, we study the innerness of bounded derivations from $\mathrm{Alg}(\mathcal{L})$ into itself and into $B(\mathcal{H})$. Using the techniques in [9], we calculate n-th Hochschild cohomology group $H^n(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ of $\mathrm{Alg}(\mathcal{L})$ with coefficients in $B(\mathcal{H})$ for each $n \geqslant 1$.

Lemma 4.1. *Suppose that $I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}} \vee P_\xi) = I$. Let $T_0 : I_-^{\mathcal{N}}(\mathcal{H}) \to \mathcal{H}$ be a bounded linear operator, and $\eta_0 \in \mathcal{H}$. Then T_0 can be uniquely extended to a bounded linear operator on $\mathcal{H}$, denoted by T, such that $T\xi = \eta_0$ and $\|T\| \leqslant 4\max(\|T_0\|, \|\eta_0\|)/\|(I_-^{\mathcal{N}})^\perp\xi\|$.*

Proof. For convenience, we let $Q = I_-^{\mathcal{N}}$. Then $\mathcal{H} = Q(\mathcal{H}) + \mathbb{C}\xi$, so $Q(\mathcal{H})$ has codimension 1. Since each $x \in \mathcal{H}$ has a unique decomposition $x_0 + \lambda\xi$, where $x_0 \in Q(\mathcal{H})$, $\lambda \in \mathbb{C}$, the mapping T on $\mathcal{H}$, defined by

$$Tx = T_0x_0 + \lambda\eta_0$$

is well-defined. Obviously, T is a linear operator on $\mathcal{H}$, $T|_{Q(\mathcal{H})} = T_0$ and $T\xi = \eta_0$. Next we show that T is bounded and estimate its norm.

Since $Q(\mathcal{H})$ has codimension one, if let $\eta = \frac{Q^\perp\xi}{\|Q^\perp\xi\|}$ and P_η be the orthogonal projection onto $\mathbb{C}\eta$, we have $Q^\perp = P_\eta$. For $x \in \mathcal{H}$, we let $x = x_0 + \lambda\xi = x_1 + \mu\eta$, where $x_0, x_1 \in Q(\mathcal{H})$ and $\lambda, \mu \in \mathbb{C}$. Then $\|x\|^2 = \|x_1\|^2 + |\mu|^2$, $x_0 = x_1 - \frac{\mu}{\|Q^\perp\xi\|}Q\xi$ and $\lambda = \frac{\mu}{\|Q^\perp\xi\|}$. Hence

$$\begin{aligned}\|Tx\| = \|T_0x_0 + \lambda\eta_0\| &\leqslant \max(\|T_0\|, \|\eta_0\|)(\|x_0\| + |\lambda|)\\ &\leqslant \max(\|T_0\|, \|\eta_0\|)(\|x_1\| + 2|\mu|/\|Q^\perp\xi\|)\\ &\leqslant [2\max(\|T_0\|, \|\eta_0\|)/\|Q^\perp\xi\|](\|x_1\| + |\mu|)\\ &\leqslant [2\max(\|T_0\|, \|\eta_0\|)/\|Q^\perp\xi\|]\sqrt{2}\sqrt{\|x_1\|^2 + |\mu|^2}\\ &\leqslant [4\max(\|T_0\|, \|\eta_0\|)/\|Q^\perp\xi\|] \cdot \|x\|.\end{aligned}$$

Consequently, $\|T\| \leqslant 4\max(\|T_0\|, \|\eta_0\|)/\|Q^{\perp}\xi\|$.

Theorem 4.2. *Every bounded derivation δ from $\mathrm{Alg}(\mathcal{L})$ into $B(\mathcal{H})$ is inner, i.e., there exists $T \in B(\mathcal{H})$ such that $\delta(A) = TA - AT$ for each $A \in \mathrm{Alg}(\mathcal{L})$. Hence $H^1(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H})) = \{0\}$.*

Proof. We consider three cases.

(i) Suppose that $(I_-^{\mathcal{N}} \vee P_\xi) < I$. Choose a unit vector $y_0 \in (I_-^{\mathcal{N}} \vee P_\xi)^{\perp}$. Hence it follows from Corollary 3.3 that for each $x \in \mathcal{H}$, we have $x \otimes y_0 \in \mathrm{Alg}(\mathcal{L})$. Define a mapping T on $\mathcal{H}$ by

$$Tx = \delta(x \otimes y_0)y_0, \quad x \in \mathcal{H}.$$

Then $T \in B(\mathcal{H})$ and $TAx = \delta(A)x + ATx$ for each $A \in \mathrm{Alg}(\mathcal{L})$ and $x \in \mathcal{H}$. Hence δ is inner.

(ii) Suppose that $I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}} \vee P_\xi) = I$. If let $Q = I_-^{\mathcal{N}}$, then $\mathcal{H} = Q(\mathcal{H}) + \mathbb{C}\xi$ and $Q(\mathcal{H})$ has codimension 1. Now we claim that there exists a bounded linear operator T_0 from $Q(\mathcal{H})$ into $\mathcal{H}$ such that, for each $A \in \mathrm{Alg}(\mathcal{L})$, $\delta(A)|_{Q(\mathcal{H})} = (T_0A - AT_0)|_{Q(\mathcal{H})}$.

If $Q_-^{\mathcal{N}} < Q$, then using Lemma 3.2, we have $(Q_-^{\mathcal{N}} \vee P_\xi) < I$. Choose a unit vector y_0 in $(Q_-^{\mathcal{N}} \vee P_\xi)^{\perp}$. Then for each nonzero vector x in Q, Corollary 3.3 yields that $x \otimes y_0 \in \mathrm{Alg}(\mathcal{L})$. Let T_0 be a mapping from $Q(\mathcal{H})$ into $\mathcal{H}$, defined by

$$T_0x = \delta(x \otimes y_0)y_0, \quad x \in Q(\mathcal{H}). \tag{2}$$

Then T_0 is a bounded linear operator on $Q(\mathcal{H})$. For each $A \in \mathrm{Alg}(\mathcal{L})$, by calculation, we have $T_0Ax = \delta(Ax \otimes y_0)y_0 = \delta(A)x + AT_0x$ for each $x \in Q(\mathcal{H})$. Hence $\delta(A)|_{Q(\mathcal{H})} = (T_0A - AT_0)|_{Q(\mathcal{H})}$ for each $A \in \mathrm{Alg}(\mathcal{L})$.

If $Q_-^{\mathcal{N}} = Q$, then there exists a sequence $\{Q_n\}$ in $\mathcal{N}$ such that $0 < Q_1 < Q_2 < \cdots < Q_n < \cdots < Q$ and $\lim_n Q_n = Q$ in the strong operator topology. Let $n \geqslant 1$ be fixed. Then Lemma 3.2 yields that $(Q_n \vee P_\xi) < I$, so that we can choose a unit vector $y_n \in (Q_n \vee P_\xi)^{\perp}$. Hence for each $x \in Q$, we have $Q_nx \otimes y_n \in \mathrm{Alg}(\mathcal{L})$. Define a mapping T_n on $Q(\mathcal{H})$ by

$$T_nx = \delta(Q_nx \otimes y_n)y_n, \quad x \in Q(\mathcal{H}).$$

Then for each n, T_n is a bounded linear operator from $Q(\mathcal{H})$ into $\mathcal{H}$ and $\|T_n\| \leqslant \|\delta\|$. Let $k \geqslant n$. Then for each $A \in \mathrm{Alg}(\mathcal{L})$ and each $x \in Q(\mathcal{H})$, we have $T_kAQ_nx = \delta(Q_kAQ_nx \otimes y_k)y_k = \delta(AQ_nx \otimes y_k)y_k = \delta(A)(Q_nx \otimes y_k)y_k + A\delta(Q_kQ_nx \otimes y_k)y_k = \delta(A)Q_nx + AT_kQ_nx = (\delta(A) + AT_k)Q_nx$. Hence

$$T_kAQ_n|_{Q(\mathcal{H})} = (\delta(A) + AT_k)Q_n|_{Q(\mathcal{H})} \quad \text{for each } n \geqslant 1 \text{ and each } k \geqslant n. \tag{3}$$

Since $\{T_k\}$ is a bounded sequence in $B(Q(\mathcal{H}), \mathcal{H})$, it has a convergence subsequence under the weak operator topology. We can assume that $\{T_k\}$ weakly converges to $T_0 \in B(Q(\mathcal{H}), H)$. Let $k \to \infty$, and then, let $n \to \infty$ in (3). We have $\delta(A)|_{Q(\mathcal{H})} = (T_0A - AT_0)|_{Q(\mathcal{H})}$ for each $A \in \mathrm{Alg}(\mathcal{L})$. We have established the claim.

By Corollary 3.3, we remark that $\xi \otimes \eta \in \mathrm{Alg}(\mathcal{L})$, where $\eta = \frac{Q^{\perp}\xi}{\|Q^{\perp}\xi\|}$. Using Lemma 4.1, we can extend the mapping T_0 to be a bounded linear operator T on $\mathcal{H}$ such that

$$T|_{Q(\mathcal{H})} = T_0, \quad T\xi = \delta(\xi \otimes \eta)\eta. \tag{4}$$

For each $A \in \mathrm{Alg}(\mathcal{L})$, if let $A\xi = \lambda\xi$, then

$$TA\xi = \lambda T(\xi) = \lambda\delta(\xi \otimes \eta)\eta = \delta(A\xi \otimes \eta)\eta = \delta(A)\xi + AT\xi.$$

Hence $\delta(A) = TA - AT$, so δ is inner.

(iii) Suppose $I_-^{\mathcal{N}} = I$. Then there is a strictly increasing sequence $\{P_m\}_m$ in $\mathcal{N}$ such that $\lim_{m\to\infty} P_m = I$ in the strong operator topology. For each $m \geqslant 1$, since $(P_m \vee P_\xi) < I$, we could choose a unit vector $y_m \in (P_m \vee P_\xi)^{\perp}$. Then $(P_mx) \otimes y_m \in \mathrm{Alg}(\mathcal{L})$ for every $x \in \mathcal{H}$. Define a mapping T_m on $\mathcal{H}$ by

$$T_mx = \delta(P_mx \otimes y_m)y_m, \quad x \in \mathcal{H}.$$

Then $T_m \in B(\mathcal{H})$, $T_m = T_m P_m$ and $\|T_m\| \leqslant \|\delta\|$. Hence for each $x \in \mathcal{H}$ and each $S \in \mathrm{Alg}(\mathcal{L})$, if $k \geqslant m$, we have $T_k S P_m x = \delta(P_k S P_m x \otimes y_k) y_k = \delta(S P_m x \otimes y_k) y_k = \delta(S)(P_m x \otimes y_k) y_k + S\delta(P_k P_m x \otimes y_k) y_k = \delta(S) P_m x + S T_k P_m x = (\delta(S) + S T_k) P_m x$. That is,

$$T_k S P_m = (\delta(S) + S T_k) P_m \quad \text{for each } m \geqslant 1 \text{ and each } k \geqslant m.$$

Since $\{T_k\}$ is a bounded sequence in $B(\mathcal{H})$, it has a convergence subsequence under the weak operator topology. We can assume that $\{T_k\}$ weakly converges to $T \in B(\mathcal{H})$. Hence $\delta(S) = TS - ST$ for each $S \in \mathrm{Alg}(\mathcal{L})$.

Corollary 4.3. *Every bounded derivation δ from $\mathrm{Alg}(\mathcal{L})$ into itself is inner, i.e.,*

$$H^1(\mathrm{Alg}(\mathcal{L}), \mathrm{Alg}(\mathcal{L})) = \{0\}.$$

Proof. By Theorem 4.2, there exists $T \in B(\mathcal{H})$ such that $\delta(A) = TA - AT$ for each $A \in \mathrm{Alg}(\mathcal{L})$. Note that $T \in C(\mathrm{Alg}(\mathcal{L}), \mathrm{Alg}(\mathcal{L}))$. Proposition 3.7 yields that T is in $\mathrm{Alg}(\mathcal{L})$. Hence δ is inner.

Theorem 4.4. $H^n(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H})) = \{0\}$ *for each* $n \geqslant 2$.

Proof. Let $\sigma \in Z^n(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ be a nonzero bounded n-cocycle. In order to show σ is a coboundary, we consider four separate cases.

Case 1. Suppose $(I_-^{\mathcal{N}} \vee P_\xi) < I$.

Choose a unit vector $y_0 \in (I_-^{\mathcal{N}} \vee P_\xi)^\perp$. It follows from Corollary 3.3 that for each $x \in \mathcal{H}$, we have $x \otimes y_0 \in \mathrm{Alg}(\mathcal{L})$. Define a bounded $(n-1)$-linear mapping $\varphi \in C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ by

$$\varphi(A_1, A_2, \ldots, A_{n-1})x = (-1)^n \sigma(A_1, A_2, \ldots, A_{n-1}, x \otimes y_0) y_0, \quad A_i \in \mathrm{Alg}(\mathcal{L}),\ x \in \mathcal{H}.$$

As in [9], we could show $\sigma = \partial\varphi$. In fact, for each $x \in \mathcal{H}$ and $A_i \in \mathrm{Alg}(\mathcal{L})$, we have

$$\begin{aligned}
&\partial\varphi(A_1, \ldots, A_n)x \\
&\quad= A_1\varphi(A_2, \ldots, A_n)x + \sum_{i=1}^{n-1}(-1)^i\varphi(A_1, \ldots, A_iA_{i+1}, \ldots, A_n)x + (-1)^n\varphi(A_1, \ldots, A_{n-1})A_n x \\
&\quad= (-1)^n\bigg[A_1\sigma(A_2, \ldots, A_n, x \otimes y_0)y_0 + \sum_{i=1}^{n-1}(-1)^i\sigma(A_1, \ldots, A_iA_{i+1}, \ldots, A_n, x \otimes y_0)y_0 \\
&\qquad+ (-1)^n\sigma(A_1, \ldots, A_{n-1}, A_n x \otimes y_0)y_0\bigg] \\
&\quad= (-1)^n[\partial\sigma(A_1, \ldots, A_n, x \otimes y_0)y_0 + (-1)^n\sigma(A_1, \ldots, A_n)x] \\
&\quad= \sigma(A_1, \ldots, A_n)x.
\end{aligned}$$

Hence $\sigma = \partial\varphi$, i.e., σ is an n-coboundary.

Case 2. Suppose $I_-^{\mathcal{N}} = I$.

Then there is a strictly increasing sequence $\{P_m\}_m$ in $\mathcal{N}$ such that $\lim_{m\to\infty} P_m = I$ in the strong operator topology. For each $m = 1, , 2, \ldots$, since $(P_m \vee P_\xi) < I$, we choose a unit vector $y_m \in (P_m \vee P_\xi)^\perp$. Then $(P_m x) \otimes y_m \in \mathrm{Alg}(\mathcal{L})$ for every $x \in \mathcal{H}$. Define an $(n-1)$-linear mapping φ_m on $\mathrm{Alg}(\mathcal{L})$ into $B(\mathcal{H})$ by

$$\varphi_m(A_1, \ldots, A_{n-1})x = (-1)^n\sigma(A_1, \ldots, A_{n-1}, (P_m x) \otimes y_m)y_m, \quad \text{for each } x \in \mathcal{H},$$

for all $A_i's$ in $\mathrm{Alg}(\mathcal{L})$. Then $\varphi_m \in C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ and $\|\varphi_m\| \leqslant \|\sigma\|$. Consequently, $\{\varphi_m : m = 1, 2, \ldots\}$ is a bounded sequence in $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$.

Note that space $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ is isometrically isomorphic to the dual of the (Banach space) projective tensor product

$$\overbrace{\mathrm{Alg}(\mathcal{L})\widehat{\otimes}\mathrm{Alg}(\mathcal{L})\widehat{\otimes}\cdots\widehat{\otimes}\mathrm{Alg}(\mathcal{L})}^{n-1}\widehat{\otimes}B(\mathcal{H})_*,$$

and the weak-* topology on the dual space corresponds to the topology of pointwisely ultraweak convergence on $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$, where $B(\mathcal{H})_*$ is the predual of $B(\mathcal{H})$. It follows from the weak-* compactness of the bounded set in the dual space of a Banach space that there is a subsequence of $\{\varphi_m|\ m = 1, 2, \ldots\}$ which converges in the pointwisely ultraweak topology to an element φ in $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$. Without loss of generality, we assume that $\{\varphi_m|\ m = 1, 2, \ldots\}$ converges in the topology to φ, which implies that $\{\partial\varphi_m\}$ converges in the topology to $\partial\varphi$.

Next we show that $\sigma = \partial\varphi$, which implies that σ is a coboundary. Let $m \geqslant 1$ and $k \geqslant m$ be fixed. For arbitrary $A_1, \ldots, A_n \in \mathrm{Alg}(\mathcal{L})$ and each $x \in P_m(\mathcal{H})$, we have

$$\begin{aligned}
&\partial\varphi_k(A_1, \ldots, A_n)x \\
&\quad = A_1\varphi_k(A_2, \ldots, A_n)x + \sum_{i=1}^{n-1}(-1)^i\varphi_k(A_1, \ldots, A_iA_{i+1}, \ldots, A_n)x + (-1)^n\varphi_k(A_1, \ldots, A_{n-1})A_nx \\
&\quad = (-1)^n\Bigg[A_1\sigma(A_2, \ldots, A_n, P_kx \otimes y_k)y_k + (-1)^n\sigma(A_1, \ldots, A_{n-1}, P_kA_nx \otimes y_k)y_k \\
&\qquad + \sum_{i=1}^{n-1}(-1)^i\sigma(A_1, \ldots, A_iA_{i+1}, \ldots, A_n, P_kx \otimes y_k)y_k\Bigg] \\
&\quad = (-1)^n[\partial\sigma(A_1, \ldots, A_n, P_kx \otimes y_k)y_k - (-1)^{n+1}\sigma(A_1, \ldots, A_n)P_kx] \\
&\quad = \sigma(A_1, \ldots, A_n)x,
\end{aligned}$$

where we use the fact that $P_kA_nx = A_nP_kx$ for each x in P_m. Hence for each $k \geqslant m$, we have $\partial\varphi_k(A_1, \ldots, A_n)P_m = \sigma(A_1, \ldots, A_n)P_m$. Let $k \to \infty$, and then let $m \to \infty$. Then $\partial\varphi = \sigma$.

Case 3. Suppose $(I_-^{\mathcal{N}} \vee P_\xi) = I$, $I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}})_-^{\mathcal{N}} < I_-^{\mathcal{N}}$.

If let $Q = I_-^{\mathcal{N}}$, then $Q_-^{\mathcal{N}} < Q < I$, $\mathcal{H} = Q(\mathcal{H}) + \mathbb{C}\xi$ and $Q(\mathcal{H})$ has codimension 1. Using Lemma 3.2, we have $(Q_-^{\mathcal{N}} \vee P_\xi) < I$. Choose a unit vector y_0 in $(Q_-^{\mathcal{N}} \vee P_\xi)^\perp$. Then for each nonzero vector x in Q, we have $x \otimes y_0 \in \mathrm{Alg}(\mathcal{L})$. Let $\eta = \frac{Q^\perp\xi}{\|Q^\perp\xi\|}$. Then $\xi \otimes \eta \in \mathrm{Alg}(\mathcal{L})$. For arbitrary operators $A_1, \ldots, A_{n-1}$ in $\mathrm{Alg}(\mathcal{L})$, we define a bounded linear operator $\varphi(A_1, A_2, \ldots, A_{n-1}) \in B(\mathcal{H})$ by the following two conditions:

$$\varphi(A_1, A_2, \ldots, A_{n-1})x = (-1)^n\sigma(A_1, A_2, \ldots, A_{n-1}, x \otimes y_0)y_0, \quad x \in Q(\mathcal{H}),$$

and

$$\varphi(A_1, A_2, \ldots, A_{n-1})\xi = (-1)^n\sigma(A_1, A_2, \ldots, A_{n-1}, \xi \otimes \eta)\eta.$$

By Lemma 4.1, we have $\|\varphi(A_1, A_2, \ldots, A_{n-1})\| \leqslant (4/\|Q^\perp\xi\|)\|\sigma\|\|A_1\| \cdots \|A_{n-1}\|$. Hence the mapping $\varphi : (A_1, \ldots, A_{n-1}) \to \varphi(A_1, A_2, \ldots, A_{n-1})$ is a bounded $(n-1)$-linear operator in $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$. Note that $A\xi \in \mathbb{C}\xi$ for each A in $\mathrm{Alg}(\mathcal{L})$. By similar calculations to Case 1, we have

$$\partial\varphi(A_1, A_2, \ldots, A_n)x = \sigma(A_1, A_2, \ldots, A_n)x, \quad \text{for each } x \in Q(\mathcal{H})$$

and

$$\partial\varphi(A_1, A_2, \ldots, A_n)\xi = \sigma(A_1, A_2, \ldots, A_n)\xi,$$

for all A_i's in $\mathrm{Alg}(\mathcal{L})$. Hence $\sigma = \partial\varphi$, i.e., σ is a coboundary.

Case 4. Suppose $(I_-^{\mathcal{N}} \vee P_\xi) = I$, $I_-^{\mathcal{N}} < I$ and $(I_-^{\mathcal{N}})_-^{\mathcal{N}} = I_-^{\mathcal{N}}$.

If we let $Q = I_-^{\mathcal{N}}$, then $Q_-^{\mathcal{N}} = Q < I$, $\mathcal{H} = Q(\mathcal{H}) + \mathbb{C}\xi$ and $Q(\mathcal{H})$ has codimension 1. Since $Q_-^{\mathcal{N}} = Q$, there exists a sequence $\{Q_m\}$ in $\mathcal{N}$ such that $0 < Q_1 < Q_2 < \cdots < Q_m < \cdots < Q$ and $\lim_m Q_m = Q$ in the strong operator topology. Let $m \geqslant 1$ be fixed. Then Lemma 3.2 yields that $(Q_m \vee P_\xi) < I$. Choose a unit vector $y_m \in (Q_m \vee P_\xi)^\perp$. Hence by Corollary 3.3, for each $x \in Q$, we have $Q_mx \otimes y_m \in \mathrm{Alg}(\mathcal{L})$. Notice that $\xi \otimes \eta \in \mathrm{Alg}(\mathcal{L})$, where $\eta = \frac{Q^\perp\xi}{\|Q^\perp\xi\|}$.

For each $m \geqslant 1$, we define φ_m in $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$ by the following two conditions:

$$\varphi_m(A_1, A_2, \ldots, A_{n-1})x = (-1)^n\sigma(A_1, A_2, \ldots, A_{n-1}, Q_mx \otimes y_m)y_m, \quad x \in Q(\mathcal{H}),$$

and

$$\varphi_m(A_1, A_2, \ldots, A_{n-1})\xi = (-1)^n \sigma(A_1, A_2, \ldots, A_{n-1}, \xi \otimes \eta)\eta.$$

By Lemma 4.1, we have $\|\varphi_m(A_1, A_2, \ldots, A_{n-1})\| \leqslant (4/\|Q^{\perp}\xi\|)\|\sigma\|\|A_1\| \cdots \|A_{n-1}\|$ for all A_i's in $\mathrm{Alg}(\mathcal{L})$. Hence $\{\varphi_m : m = 1, 2, \ldots\}$ forms a bounded sequence in $C^{n-1}(\mathrm{Alg}(\mathcal{L}), B(\mathcal{H}))$.

Using a similar argument to Case 2, we can assume that $\{\varphi_m |\ m = 1, 2, \ldots\}$ converges in the pointwisely ultraweak topology to φ, which implies that $\{\partial\varphi_m\}$ converges in the topology to $\partial\varphi$. Using a similar way to the proof in Case 2, we can show that for each $m \geqslant 1$ and all A_i's $\in \mathrm{Alg}(\mathcal{L})$,

$$\partial\varphi_k(A_1, \ldots, A_n)Q_m = \sigma(A_1, \ldots, A_n)Q_m \quad \text{for each } k \geqslant m;$$
$$\partial\varphi_k(A_1, \ldots, A_n)\xi = \sigma(A_1, \ldots, A_n)\xi \quad \text{for each } k \geqslant m.$$

Let $k \to \infty$, and then let $m \to \infty$. Then $\partial\varphi = \sigma$. Hence σ is a coboundary.

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