SCIENTIA SINICA
Vol. XII, No. 5, 1963

GEOPHYSICS

RADAR EQUATION BY TAKING INTO CONSIDERATION THE COHERENT SCATTERING OF RADAR WAVES FROM CLOUD AND RAINDROPS*

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ABSTRACT

Attention has been paid to the fact that grouping association between particles exists in clouds as well as in rain. Coherent scattering of radar waves from cloud and raindrops dependent on each other has been evaluated and a new radar equation derived. The equation involves the old one as its special form in the case of incoherent scattering. It is found that the effect of coherent scattering in some cases is not quite small and should not be neglected.

I. INTRODUCTION

As one of the important tools in modern meteorology, radar has become more and more a part in the equipment for observation of precipitation weather system as well as for the study of cloud physics. At present, radar meteorology has developed into an independent branch of modern meteorology.

Serving as the theoretical basis of radar meteorology, the radar equation deals with the correlations between the physical characteristics of precipitation and the power of received waves, thus making it possible to deduce the physical characteristics of precipitation processes by the aid of radar observations.

The radar equation extensively used today is as follows[1]:

$$\bar{P}_r = \frac{P_t A_p^2 \theta \cdot \Phi \cdot \tau \cdot c}{72\lambda^2 R^2} \sigma \bar{n}, \tag{1}$$

where $\bar{P}_r$ represents the mean power of the received waves, P_t, A_p, λ, θ, ϕ, τ, and c respectively the transmitting power, the antenna area, the transmitted wavelength, the vertical beam width, the horizontal beam width, the pulse duration, and the velocity of light; while σ and $\bar{n}$ are respectively the radar cross-section, and the mean number of drops per unit volume of the scattering particles, and R is the distance of the irradiated volume from the radar.

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* First published in Chinese in *Acta Meteorologica Sinica*, Vol. 32, No. 2, pp. 119—128, 1962.

In the present paper it is intended to discuss the relation between $\bar{P}_r$ and the number of scattering particles. For the sake of convenience (1) is here simplified into

$$\bar{P}_r = C \cdot \sigma \cdot N, \tag{1'}$$

where C is the factor determined by both the radar parameters and the distance R, while N is the total number of scattering particles in the irradiated volume.

Equation (1) is derived on the basis of a very important assumption that the scattering particles are independent of each other, and hence there is no fixed phase relation between the waves scattered by them, and that the scattered waves do not mutually interfere[2]. It thus becomes clear that equation (1) holds only when the conditions of independent or incoherent scattering are satisfied.

Since equation (1) is very important, quite a number of workers have made attempts to verify its accuracy experimentally. One way to achieve this is to make simultaneous accurate measurements of the power of received waves, the radar parameters and the rain droplet-size distribution. But up to date no satisfactory results have been obtained because it is difficult to measure at the same time both the power of received waves and the droplet-size distribution of the rain from which the echo comes. Another way to verify equation (1) is to conduct measurements simultaneously with a number of radars having a series of different wavelengths: from (1′) the theoretical value of $\frac{\bar{P}_r(\lambda_1)}{\bar{P}_r(\lambda_2)}$ is represented by the ratio $\frac{C(\lambda_1)}{C(\lambda_2)} \times \frac{\sigma(\lambda_1)}{\sigma(\lambda_2)}$ which can be exactly determined, and $\frac{\bar{P}_r(\lambda_1)}{\bar{P}_r(\lambda_2)}$ can be measured experimentally; thus by comparing these the accuracy of equation (1) is verified. This method appears to be more practicable.

Led by the latter scheme, J. E. N. Hooper and A. A. Kippax[3] made some observations with radars having three wavelengths and obtained the following results:

	$\lambda_1 = 9.1$ cm $\lambda_2 = 3.2$ cm	$\lambda_1 = 9.1$ cm $\lambda_2 = 1.25$ cm
$\bar{P}_r(\lambda_1)/\bar{P}_r(\lambda_2)$ Theoretical value	−6.4 db	−3.4 db
$\bar{P}_r(\lambda_1)/\bar{P}_r(\lambda_2)$ Observed value	−6.0 — −6.3 db	−2.3 — −2.5 db

The observations showed that the increase of $\bar{P}_r$ with decrease of λ is smaller than that calculated from equation (1). This fact was also pointed out by H. Goldstein[2]. It should be noted that this discrepency can not be explained as due to negligence of the possible Mie scattering effect in the calculation of $\frac{\sigma(\lambda_1)}{\sigma(\lambda_2)}$, on the contrary, the difference would be greater if corrections for the

Mie scattering are introduced[1]. This discrepency can only be explained by the inexactness of equation (1).

On the other hand, A. N. Dingle found that the rain droplet distribution in space is in no way identical with the Poison distribution of independent particles[4]. He discovered that rain droplets have a tendency towards clustering, and the volume of each cluster appears to be smaller than 1 litre (the exact size was not determined on account of the insufficient resolving power of the instruments used); on the average there are several droplets in each cluster and the clusters are separated from one another by rather large intervals. A. N. Dingle suggested that such a structure of rainy cloud may arise from the mechanism of raindrop growth and break-up, atmospheric turbulence and wind shear, electric effects, etc. This discovery, showing that the clustered raindrops can not be entirely independent of each other and that among them there must be some definite connections, is of great significance. This is also true in the case of cloud drops[5]. Observations of the microstructure of clouds indicate that clouds are made up of blob structures of the kind similar to that described by the statistic theory of turbulence, an effect that may be due to the inevitable existence of turbulence in clouds. Cloud drops in the same blob are not mutually independent. Consequently, the assumption, under which equation (1) was derived, that the scattering particles are independent of each other, may not completely correspond to reality, and therefore the equation fails to hold.

In the present paper attempts will be made to consider the relations among the scattering particles, to estimate the effect of coherent scattering, and on this basis to derive a more accurate radar equation.

II. Coherent Scattering of Raindrops

In solving this problem we shall first base ourselves on the observations made by A. N. Dingle on the microstructure of showers, and then make a reasonable generalization of the characteristics of rainy clouds to obtain the following model:

1. The N raindrops in an irradiated volume are divided into m groups $(m \gg 1)$, of which the lth group has a scale a_l and there are n_l raindrops, evidently $\sum_{l=1}^{m} n_l = N$;
2. All the groups are independent of each other and are distributed at random in space;
3. Raindrops in one and the same group possess the same statistical characteristics;
4. All the raindrops are equal in radius (in calculations the mean value is taken);
5. Coordinate variations of the raindrops are subjected to stationary stochastic processes.

The resultant electric field produced by N raindrops at the antenna should be

$$E=\sum_{i=1}^{N}E_i=\frac{p}{\sqrt{4\pi}}\,\frac{E_0(R)}{R}\,e^{i\omega t}\sum_{i=1}^{N}e^{-i\frac{4\pi}{\lambda}R_i},$$

and the mean power of the received waves $\bar{P}_r$ should be

$$\bar{P}_r=\frac{c}{8\pi}\langle EE^*\rangle=C\sigma\left\langle\sum_{i=1}^{N}\sum_{j=1}^{N}e^{-\frac{4\pi}{\lambda}(R_i-R_j)}\right\rangle=$$

$$=C'\Big[N+\Big\langle\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}e^{-ig(R_i-R_j)}\Big\rangle\Big],\qquad(2)$$

where the notations $\langle\ \rangle$ and $*$ respectively represent the mean and conjugate complex, $g=\dfrac{4\pi}{\lambda}$.

As a matter of fact, the first term in equation (2) represents incoherent scattering, and the second, coherent scattering. The latter term should be zero in case of incoherent scattering, since the probabilities for R_i-R_j to take various values are equal.

In the following we shall examine the term that expresses coherent scattering. By separating the contributions due to the "particle pairs" belonging to the same group from those due to the "particle pairs" belonging to different groups, we have

$$\Big\langle\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}e^{-ig(R_i-R_j)}\Big\rangle=\sum_{i=1}^{N}\Big\langle\sum_{\substack{j=1\\ j\neq i}}^{N}e^{-ig(R_i-R_j)}\Big\rangle=$$

$$=\sum_{l=1}^{m}\sum_{p=1}^{n_l}\Big[\Big\langle\sum_{\substack{s=1\\ s\neq p}}^{n_l}e^{-ig(R_{l,p}-R_{l,s})}\Big\rangle+\Big\langle\sum_{\substack{k=1\\ k\neq l}}^{m}\sum_{s=1}^{n_k}e^{ig(R_{l,p}-R_{k,s})}\Big\rangle\Big],\qquad(3)$$

where the first suffix of each of the symbols $R_{l,p}$ and $R_{k,s}$ denotes the ordinal number of the groups, and the second suffix, the ordinal number of raindrops in the groups.

In expression (3) the second term in the square brackets represents the contributions due to the "particles pairs" belonging to different groups ($k\neq l$). This term should be equal to zero, for the groups are independent of each other and are subjected to random distribution in space, and so the probabilities for $(R_{l,p}-R_{k,s})$ to take different values are equal. Thus, the whole problem is reduced to a mere calculation of the contributions due to the "particle pairs" belonging to the same group, represented by the first term of equation (3).

Let $W_l(y)$ represent the probability density function of the random variable $R_{l,p}-R_{l,s}=(\Delta_l R)_{p,s}$. Considering that the statistical characteristics of the particles in the same group are similar to each other, we have

$$\left\langle \sum_{\substack{s=1\\ s\neq p}}^{n_l} e^{-ig(R_{l,p}-R_{l,s})} \right\rangle = \sum_{\substack{s=1\\ s\neq p}}^{n_l} \langle e^{-ig(\Delta_l R)_{p,s}} \rangle =$$
$$= (n_l - 1)\int_{-\infty}^{+\infty} e^{-igy} W_l(y)dy. \tag{4}$$

To calculate (4), it is necessary to know the expression of the probability density function $W_l(y)$. For this we take into account the following two possible cases: (i) the group is a cylinder with radius a, and (ii) the group is a sphere with radius a. No matter which shape is taken, the probability of appearance of raindrops in the group can still be considered everywhere the same. The distribution function $W_l(y)$, which corresponds to the two cases mentioned above, are denoted by $W_l^{(1)}(y)$ and $W_l^{(2)}(y)$.

1. *Cylindrical Groups*

Let $\eta_l^{(1)}(R-\bar{R}_l)$ represent the distribution function of the raindrop with coordinate taking the value R, here the central coordinate of the group is expressed by $\bar{R}_l$ (as shown in Fig. 1), then apparently

$$\begin{cases} \eta_l^{(1)}(R-\bar{R}_l) = \dfrac{1}{\pi a_l^2}\dfrac{ds(R)}{dR} \\ \qquad\qquad\quad = \dfrac{2}{\pi a_l^2}\sqrt{a_l^2-(R-\bar{R}_l)^2} & \text{when} \quad |R-\bar{R}_l| \leqslant a, \\ \eta_l^{(1)}(R-\bar{R}_l) = 0, & \text{when} \quad |R-\bar{R}_l| > a. \end{cases} \tag{5}$$

The theory of probability tells us that between the probability density function of the difference ΔR of random variables and the probability density function $\eta(R-\bar{R}_l)$ of random variable R, there exists a relation as follows[6]:

$$W_l^{(1)}(y) = \int_{-\infty}^{+\infty} \eta_l^{(1)}(z)\eta_l^{(1)}(z+y)dz. \tag{6}$$

Substituting (6) into (5), and when $2a_l \geqslant y \geqslant 0$, we have

Fig. 1

$$W_l^{(1)}(y) = \left(\frac{2}{\pi a_l^2}\right)^2 a_l^3 \int_{-1}^{1-y/a_l} [1-u^2]^{1/2}\left[1-\left(u+\frac{y}{a_l}\right)^2\right]^{1/2} du; \tag{7a}$$

when $-2a_l \leqslant y \leqslant 0$,

$$W_l^{(1)}(y) = \left(\frac{2}{\pi a_l^2}\right)^2 a_l^3 \int_{-1}^{+1-\left(\frac{-y}{a_l}\right)} [1-u^2]^{1/2}\left[1-\left(u+\frac{-y}{a_l}\right)^2\right]^{1/2} du; \tag{7b}$$

when $|y| > 2a_l$

$$W_l^{(1)}(y) = 0. \tag{7c}$$

Combining (7a), (7b), and (7c) we get

$$\begin{cases} W_l^{(1)}(y) = W_l^{(1)}(-y) = \dfrac{4}{\pi^2 a_l}\displaystyle\int_{-1}^{1-\left|\frac{y}{a_l}\right|} [1-u^2]^{1/2} \times \\ \qquad \times \left[1-\left(u+\left|\dfrac{y}{a_l}\right|\right)^2\right]^{1/2} du, \quad y \leqslant 2a, & (8a) \\ W_l^{(1)}(y) = 0, \quad |y| > 2a. & (8b) \end{cases}$$

By expanding the integrand into power series, the integral of (8a) is evaluated as follows:

$$\begin{cases} W_l^{(1)}(y) = \dfrac{4}{\pi^2 a_l} P^{(1)}\left(\left|\dfrac{y}{a}\right|\right), \quad |y| \leqslant 2a_l; & (9a) \\ W_l^{(1)}(y) = 0, \quad |y| > 2a_l; & (9b) \end{cases}$$

where $P^{(1)}\left(\left|\frac{y}{a_l}\right|\right)$ represents the infinite series $P^{(1)}\left(\left|\frac{y}{a_l}\right|\right) = \sum_{n=0}^{\infty} c_n \left|\frac{y}{a_l}\right|^n$,

$$c_0 = \frac{4}{3}, \qquad c_1 = 0, \qquad c_2 = \frac{-23}{15},$$

$$c_3 = \frac{4}{3}, \qquad c_4 = -\frac{5}{6}, \qquad c_5 = \frac{5}{12},$$

$$c_6 = -\frac{1}{8}, \qquad c_7 = \frac{1}{60}, \qquad c_8 = \cdots$$

The characteristics of $W_l^{(1)}(y)$ is shown in Fig. 2. In the following discussion the infinite series will be substituted by the linear functions expressed by the dotted line in the figure, namely,

$$\begin{cases} W_l^{(1)}(y) \doteqdot \dfrac{1}{a_l} \times 0.6 \times \left(1-\dfrac{1}{1.7}\left|\dfrac{y}{a_l}\right|\right), & \text{when } |y| \leqslant 1.7; \quad (9a') \\ W_l^{(1)}(y) = 0, & \text{when } |y| > 1.7. \quad (9b') \end{cases}$$

It can be seen from the figure that the error introduced by such an approximation is not great.

2. *Spherical Groups*

Let $\eta_l^{(2)}(R-\bar{R}_l)$ represent the distribution function of the raindrop with coordinate taking the value R, then evidently

$$\begin{cases} \eta_l^{(2)}(R-\bar{R}_l) = \dfrac{1}{\dfrac{4}{3}\pi a_l^3} \cdot \dfrac{dV(R)}{dR} = \dfrac{3}{4a_l^3}\,[a_l^2 - (R-\bar{R}_l)^2], \\ \qquad\qquad\qquad\qquad\qquad \text{when} \quad |R-\bar{R}_l| \leqslant a; \\ \eta_l^{(2)}(R-\bar{R}_l) = 0, \qquad\qquad \text{when} \quad |R-\bar{R}_l| > a. \end{cases}$$

By the same method as used above, $W_l^{(2)}(y)$ can be obtained as follows:

$$\begin{cases} W_l^{(2)}(y) = \left(\dfrac{3}{4}\right)^2 \dfrac{1}{a}\, P^{(2)}\left(\left|\dfrac{y}{a_l}\right|\right), \quad \text{when} \quad |y| \leqslant 2a; & (10a) \\ W_l^{(2)}(y) = 0, \qquad\qquad \text{when} \quad |y| > 2a; & (10b) \end{cases}$$

where $P^{(2)}\left(\left|\dfrac{y}{a_l}\right|\right) = \sum\limits_{n=0}^{5} c_n' \left|\dfrac{y}{a_l}\right|^n$, $c_0' = \dfrac{16}{15}$, $c_1' = 0$, $c_2' = \dfrac{4}{3}$, $c_3' = \dfrac{2}{3}$, $c_4' = 0$, $c_5' = -\dfrac{1}{30}$. The characteristics of $W_l^{(2)}(y)$ are shown in Fig. 3. In this case $W_l^{(2)}(y)$ can be approximated by a linear function:

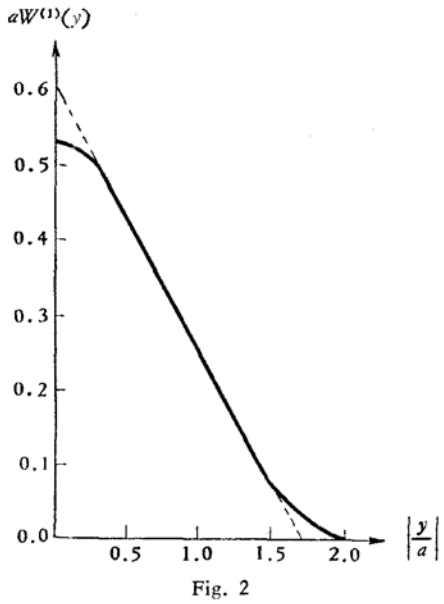

Fig. 2

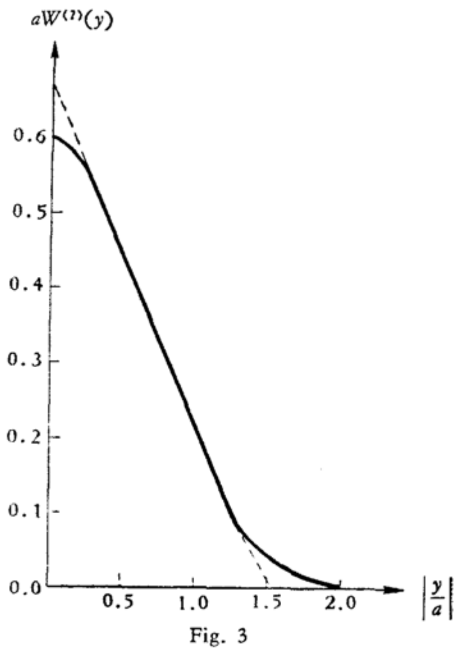

Fig. 3

$$\begin{cases} W_l^{(2)}(y) = \dfrac{1}{a_l} \times 0.675\left(1 - \dfrac{1}{1.5}\left|\dfrac{y}{a_l}\right|\right), \quad \text{when} \quad |y| \leqslant 1.5; & (10a') \\ W_l^{(2)}(y) = 0, \qquad\qquad \text{when} \quad |y| > 1.5. & (10b') \end{cases}$$

Substituting respectively (9′) and (10′) into (4), and then integrating and substituting into (3), we obtain the expression for coherent scattering:

$$\left\langle \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} e^{-ig(R_i - R_j)} \right\rangle = \sum_{l=1}^{m} \sum_{p=1}^{n_l} (n_l - 1) f(h_l) = \sum_{l=1}^{m} n_l (n_l - 1) f(h_l), \quad (11)$$

where

$$h_l = 4\pi \frac{a_l}{\lambda}, \quad f(h) = \begin{cases} f^{(1)}(h) = 0.706 \times (1 - \cos 1.7h) \dfrac{1}{h^2}, \\ \qquad \text{for cylindrical groups,} \\ f^{(2)}(h) = 0.900 \times (1 - \cos 1.5h) \dfrac{1}{h^2}, \\ \qquad \text{for spherical groups.} \end{cases} \quad (12)$$

Equation (11) can be rewritten as $m \cdot \overline{n_l(n_l - 1) f(h_l)}$, where the line means that the mean value is taken over the m groups. Considering $m \gg 1$, this mean value can be replaced by the mathematical expectation of $n(n-1)f(h)$, that is,

$$\overline{n_l(n_l - 1) f(h_l)} = \int_{h_{\min}}^{h_{\max}} \int_{0}^{n_{\max}} n(n-1) f(h) W(h, n) dn\, dh = \overline{n(n-1) f(h)}, \quad (13)$$

where $W(h, n)$ represents the joint probability density function of the number of the particles in the groups n and the value $h = \dfrac{4\pi a}{\lambda}$ of the groups.

Finally, by substituting (11) and (12) into (2) we obtain a new radar equation which takes into account the effect of coherent scattering:

$$\bar{P}_r = c\sigma [N + m \cdot \overline{n(n-1) f(h)}]. \quad (14)$$

In this new equation the first term in the square brackets represents incoherent scattering, while the second term, coherent scattering. It can be seen from (13) that coherent scattering yields additional echoes back to the antenna of the radar, the power of received waves $\bar{P}_r$ being related not only to the total number of scattering particles, but also to how the particles are clustered. A discussion of this new equation is given below:

(1) The term expressing coherent scattering is proportional to $m \cdot \overline{n(n-1) f(h)}$ and the value of this term increases with the number of raindrops n_l in each group. When $n_l = 1$ ($l = 1, \cdots, m = N$), that is, when each group is made up of one particle only, this term disappears, and equation (14) degenerates to (2), indicating that coherent scattering no longer exists. This result is reasonable, because in the above discussion the groups are assumed to be independent of each other, and therefore the fact that $n_l = 1$, ($l = 1, \cdots, m = N$) naturally means the independence of the N particles of each other, which is just the condition utilized in the derivation of the old radar equation. Under this condition no coherent scattering exists and equation (14) naturally takes the form of the old equation (2).

(2) The magnitude and variation of the term of coherent scattering. The magnitude of the effect of coherent scattering depends upon the total number of groups m together with the characteristics of the groups $\left(\text{represented by the joint probability density function } W\left(n, \frac{4\pi a}{\lambda}\right)\right)$. On account of the lack of detailed observation data about $W\left(n, \frac{4\pi a}{\lambda}\right)$ in the evaluation of the effect of coherent scattering, we have, for the time being, to assume that all the groups are of the same radius a and consist of the same number of raindrops n, so $\overline{m \cdot n(n-1)f(h)} = N(n-1)f(h)$. The values of the function $f(h)$ calculated by equation (12) are shown in Fig. 4 and the values of $h = \frac{4\pi a}{\lambda}$ corresponding to wavelengths λ and radius a of the groups are given in the following table.

a \ h \ λ		λ(cm) 10	5	3	1
a (cm)	1	1.3	2.6	4.4	13
	2	2.6	5.2	8.7	26
	3	3.9	7.8	13.0	39
	5	6.5	13.0	21.7	65
	10	13.0	26.0	43.3	130

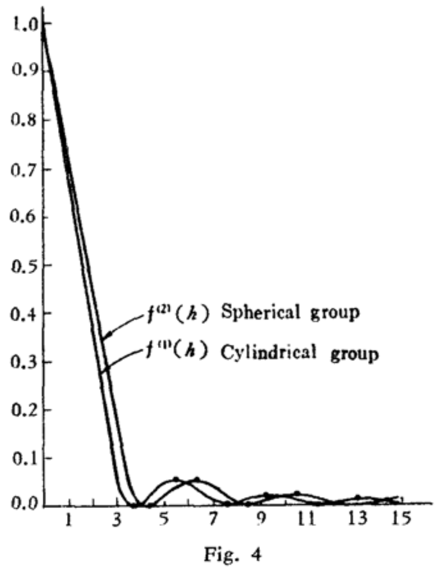

Fig. 4

It is seen from Fig. 4 that $f(h)$ tends to increase when h decreases and that the coherent scattering produced by spherical groups $[f^{(2)}(h)]$ is similar to that produced by cylindrical groups $[f^{(1)}(h)]$. Let α represent the ratio between coherent and incoherent scattering, then $\alpha = (n-1)f(h)$. When $h<7$, $n=6$, α in general exceeds 25%, and when the groups are small, it can even exceed 100% (for instance, when $n=6$, $a=0.75$ cm, $\lambda = 3.2$, or, when $n=6$, $a=2.75$ cm, $\lambda=10$, then $\alpha=1$). According to Dingle's observations, n is less than ten and a is in the order of centimetres, then in case of real rainy clouds the term representing coherent scattering can be comparable with that of incoherent scattering. When the group is large and the wavelength is short, h will be large and then coherent scattering becomes insignificant. Thus it turns out clear that coherent scattering is produced exclusively by groups of the order of centimetres. Moreover, it can be seen from the figure that the value of the term representing coherent scattering goes through a series of maxima and minima and the amplitudes of the maxima and minima decrease as h increases, somewhat like the small-hole diffraction pattern of light. This result is also reasonable, because these two physical processes are essentially of the same nature, i.e., the interference of waves. In real rainy clouds the radius of groups and the number of particles have a distribution $W(n, h)$, therefore the maxima and minima will be averaged out.

3. Since $f\left(\frac{4\pi a}{\lambda}\right)$ is related to wavelength, according to (14) we have

$$\frac{\bar{P}_r(\lambda_1)}{\bar{P}_r(\lambda_2)} = \frac{C(\lambda_1)}{C(\lambda_2)} \times \frac{\sigma(\lambda_1)}{\sigma(\lambda_2)} \times \frac{1 + f\left(\frac{4\pi a}{\lambda_1}\right)(n-1)}{1 + f\left(\frac{4\pi a}{\lambda_2}\right)(n-1)} =$$

$$= \frac{C(\lambda_1)}{C(\lambda_2)} \times \frac{\sigma(\lambda_1)}{\sigma(\lambda_2)} \times \frac{1+\alpha(\lambda_1)}{1+\alpha(\lambda_2)}. \quad (15)$$

Comparing with the results obtained by the old radar equation (1′), equation (15) has in addition a factor $\frac{1+\alpha(\lambda_1)}{1+\alpha(\lambda_2)}$. This factor cannot be neglected when the effect of coherent and incoherent scattering are comparable. If $\lambda_1 > \lambda_2$, then $\frac{\sigma(\lambda_1)}{\sigma(\lambda_2)} < 1$, but $\frac{1+\alpha(\lambda_1)}{1+\alpha(\lambda_2)} > 1$. This just accounts for the above mentioned discrepency existing between the old equation and the observed facts. In accordance with Dingle's observations, taking $n=6$, $a=5$, then when $\lambda_2 = 1.25$ cm and $\lambda_1 = 9.2$ cm, $\frac{1+\alpha(\lambda_1)}{1+\alpha(\lambda_2)} = 1$ db. This shows that the theoretical value calculated by (15) completely agrees with that obtained from observations, and the correctness of the newly derived equation with consideration of coherent scattering is thus indirectly verified.

III. Scattering of Radar Waves from Cloud Drops

The model developed in the previous section for raindrops does not apply to the case of cloud drops, because there are much more cloud droplets per unit volume than raindrops which are divided into discrete groups. The reasonable treatment is to take clouds as continuous media with a "blob" structure (here the term "blob" is used in the sense of blobs in the statistic theory of turbulence). In this way we shall discuss the problem of scattering of radar waves from cloud drops. This treatment can be as well applied to the case when the raindrops are rather dispersed and the group structure is not distinct. Thus the method to be discussed in this section has a comparatively general meaning.

Let the space coordinate be represented by the vector $\mathbf{R}$ and the wave vector of the transmitted waves by $\mathbf{K}$, then the electric field at the receiving antenna due to the waves scattered by all the particles in the irradiated volume V should be

$$E = \frac{1}{\sqrt{4\pi}} \iiint_V \frac{E_0(\mathbf{R})}{R} e^{+i(\omega t - 2\mathbf{K}\cdot\mathbf{R})} \beta(\mathbf{R}) d\mathbf{R}, \tag{16}$$

where β represents the ability of the cloud particles in a unit volume to produce the electric field of scattering waves. If the size-distribution function of the cloud drops is expressed by the function $n(r)$ and the total number of cloud drops in a unit volume by $n = \int_0^\infty n(r)dr$, then

$$\beta = \int_0^\infty p(r)n(r)dr = p(\bar{r})n, \tag{17}$$

where $\bar{r}$ is the mean value of radius in a unit volume.

Because the blob structure of clouds, the number density n, and the mean radius $\bar{r}$ of the particles are subjected to fluctuations, β is a random variable in time and space which fluctuates at random and thus may be expressed as follows:

$$\beta(\mathbf{R}, t) = \bar{\beta} + \Delta\beta(\mathbf{R}, t), \tag{18}$$

where $\Delta\beta(\mathbf{R}, t)$ is also a random variable in time and space. On account of the fact that the time and space involved in radar observation are rather limited, it can be assumed that the field of random variable $\Delta\beta(\mathbf{R}, t)$ is homogeneous and isotropic in space and stationary in time.

Substituting (18) into (16) to calculate the mean strength $\bar{P}_r$ of the echo, we find

$$\bar{P}_r = \frac{C}{8\pi}\langle E\cdot E^*\rangle = C\left\{\left|\bar{\beta}\iiint_V e^{-i\cdot 2\mathbf{K}\cdot\mathbf{R}} d\mathbf{R}\right|^2 + \right.$$

$$+ \left[\left(\bar{\beta}\iiint_V \langle\Delta\beta(\mathbf{R}, t)\rangle e^{-i\cdot 2\mathbf{K}\cdot\mathbf{R}} d\mathbf{R}\right)\iiint_V e^{+i\cdot 2\mathbf{K}\cdot\mathbf{R}'} d\mathbf{R}'\right] +$$

$$+\left[\left(\bar{\beta}\iiint_V \langle\Delta\beta(\mathbf{R}',t)\rangle e^{+i\cdot 2\mathbf{K}'\cdot\mathbf{R}'}d\mathbf{R}'\right)\iiint_V e^{-i\cdot 2\mathbf{K}\cdot\mathbf{R}}d\mathbf{R}\right]+$$

$$+\left[\iiint_V d\mathbf{R}\iiint_V \langle\Delta\beta(\mathbf{R},t)\Delta\beta(\mathbf{R}',t)\rangle e^{i2(\mathbf{K}'\cdot\mathbf{R}'-\mathbf{K}\cdot\mathbf{R})}d\mathbf{R}'\right]\Big\}. \quad (19)$$

The first term of this equation has been calculated by A. J. F. Siegert and H. Goldstein and demonstrated to be negligible[2]. Since $\langle\Delta\beta(\mathbf{R},t)\rangle=0$, the second and third terms are equal to 0. Thus only the last term remains. In this term the covariance of $\Delta\beta(\mathbf{R},t)$ contained in the integration sign can be expressed by the correlation function $C(\mathbf{R}-\mathbf{R}')$:

$$\langle\Delta\beta(\mathbf{R},t)\Delta\beta(\mathbf{R}',t)\rangle=\langle(\Delta\beta)^2\rangle C(\mathbf{R}-\mathbf{R}'). \quad (20)$$

Substituting (20) into (19), introducing the new variable $\boldsymbol{\rho}=\mathbf{R}'-\mathbf{R}$ and considering that the correlation function becomes appreciably different from zero only when $\boldsymbol{\rho}$ is very small, we can take the approximation $\mathbf{K}'=\mathbf{K}$, $\iiint_V[\cdots]d\boldsymbol{\rho}\doteqdot\iiint_\infty[\cdots]\,dp$, and equation (19) is transformed into

$$\bar{P}_r=C\langle(\Delta\beta)^2\rangle\iiint_V d\mathbf{R}\iiint_\infty e^{i\cdot 2\mathbf{k}\cdot\boldsymbol{\rho}}C(\boldsymbol{\rho})d\boldsymbol{\rho}. \quad (21)$$

In accordance with the statistical theory of turbulance, the inverse Fourier transform of the correlation function is the spectrum function $\phi(\mathbf{k})$[7],

$$\Phi(\mathbf{k})=\iiint_\infty e^{i\cdot\mathbf{k}\cdot\boldsymbol{\rho}}C(\boldsymbol{\rho})d\boldsymbol{\rho}, \quad (22)$$

and therefore equation (21) changes into

$$\bar{P}_rC=\langle(\Delta\beta)^2\rangle_{\cdot}\iiint_V d\mathbf{R}\cdot\Phi(2\mathbf{k})=C'V\cdot\langle(\Delta\beta)^2\rangle\,\Phi(k)\Big|_{k=\frac{4\pi}{\lambda}}. \quad (23)$$

This is the radar equation for the clouds (or rain) with blob structure. In (23) it can be seen directly that

$$\bar{P}_r\propto\langle(\Delta\beta)^2\rangle=\langle(np'(\bar{r})\Delta\bar{r}+p(\bar{r})\Delta n)^2\rangle,$$

that is, the greater the values and fluctuations of $\bar{r}$ and n, the stronger the received waves. Besides,

$$\bar{P}_r\propto\Phi(k)\Big|_{k=\frac{4\pi}{\lambda}}.$$

This means that the mean power of the received waves is proportional to the spectrum function of the blob having a wave number $4\pi/\lambda$ and size $\lambda/2$. Such results are in like manner analogous to the small-hole diffraction in optics. Aside from the blobs whose size approximates to half a wavelength, all the other blobs do not contribute to $\bar{P}_r$.

In the following let us have a more thorough discussion of equation (23).

(1) Assuming, as in the derivation of the old radar equation (1), that the particles are independent of each other and have the same radius, we find that the correlation function is just the δ function, $C(\boldsymbol{\rho}) = \delta(\boldsymbol{\rho})$, and, from (22) $\phi(\mathbf{k}) = 1$; on the other hand[8], $\langle(\Delta\beta)^2\rangle = \langle(\Delta n)^2\rangle \cdot |p|^2 = n|p|^2$, equation (23) reduces to (2), and it becomes clear that (23) is more general than (2).

(2) In deriving (23) no specifications were made in relation to the form of the correlation function $C(\boldsymbol{\rho})$ as well as of the spectrum function $\phi(\mathbf{k})$, so the equation is valid for random turbulence fields. In this paper it is shown that $\bar{P}_r$ is proportional to $\phi(4\pi/\lambda)$, but up to date, it seems that the best form with which to express the turbulence function $\phi(k)$ of clouds has not yet been investigated. For free atmosphere the following four forms have been used[9].

Title	Bessel	Exponential	Gaussian	Couchy
$C(\rho)$	$\frac{\rho}{\rho_0}K_1\left(\frac{\rho}{\rho_0}\right)$	$e^{-\rho/\rho_0}$	$e^{-(\rho/\rho_0)^2}$	$\frac{1}{[1+(\rho/\rho_0)^2]^2}$
$\phi(k)$	$\frac{6\pi^2\rho_0^3}{[1+k^2\rho_0^2]^{p/2}}$	$\frac{8\pi^2\rho_0^3}{[1+k^2\rho_0^2]^2}$	$\frac{\pi^{3/2}\rho_0^3}{e^{\frac{k^2\rho_0^2}{4}}}$	$\frac{\pi^2\rho_0^3}{e^{k\rho_0}}$

The ρ_0 in each term corresponds to the average size of the blob (or the scale length). Whether any of the above mentioned forms can be applied to the turbulence of clouds is still open to question.

3. Equation (23) shows that the wavelength factor is contained not only in C, $p_\lambda(\bar{r})$, but also in $\phi(4\pi/\lambda)$. Owing to this result, it might be possible to conduct simultaneous measurements with radars of different wavelengths in the investigation of the state of turbulence in clouds.

VI. Conclusions

1. When cloud or raindrops are clustered into rather small groups or blobs, the effect of coherent scattering is not negligible. In this case the radar equation should take the following form:

for rain
$$\bar{P}_r = \frac{P_t A_p^2 \theta \cdot \Phi \cdot \tau \cdot c}{72\lambda^2 R^2}\sigma \cdot \bar{n}\left[1 + \frac{m}{N}\overline{n(n-1)f\left(\frac{4\pi a}{\lambda}\right)}\right], \tag{24}$$

where $\bar{n}$ is the mean number of raindrops, the other symbols are as above; for cloud

$$\bar{P}_r = \frac{P_t A_p^2 \theta \cdot \Phi \cdot \tau \cdot c}{72\lambda^2 R^2} \cdot \langle (\Delta\beta)^2 \rangle \Phi \left(\frac{4\pi}{\lambda}\right). \tag{25}$$

The equation in the form of (25) is in principle also applicable for the case of rain. In comparison with (24), equation (25) might be more accurate but less convenient.

If the attenuation experienced by radar waves during their passage through the atmosphere, can not be neglected, (24) and (25) should be multiplied by a factor $F(\lambda, R) < 1$.

2. To raise the accuracy of radar measurement, it is necessary to have a further understanding of the microstructures of clouds and rain, such as the characteristics of groups and blobs, etc.

The author wishes to take this opportunity to express his deep gratitude to Professors Jaw Jeou-jan, Hsieh Yi-pin, and Koo Chen-chao for their kind encouragement and guidance in preparing this paper. He is also indebted to Mr. Chao Ber-lin for his valuable suggestions and help.

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