On adjacent-vertex-distinguishing total coloring of graphs

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Abstract In this paper, we present a new concept of the adjacent-vertex-distinguishing total coloring of graphs (briefly, AVDTC of graphs) and, meanwhile, have obtained the adjacent-vertex-distinguishing total chromatic number of some graphs such as cycle, complete graph, complete bipartite graph, fan, wheel and tree.

Keywords: graph, proper total coloring, adjacent-vertex-distinguishing total coloring, adjacent-vertex-distinguishing total chromatic number.

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1 Introduction

Because of its theoretical and practical significance, the coloring problem of graph is one of the primary fields studied by many scholars all over the world. The fundamental coloring problem of graph is to determine the number of various kinds of colorings.

All of the graphs considered in this paper are simple, finite and undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and edges of graph G, respectively.

After Burris and Schelp[1], Bazgan[2] and Balister et al.[3] discussed vertex-distinguishing proper edge-coloring, Zhang et al.[4] presented the concept of adjacent strong edge coloring of graphs, i.e. adjacent-vertex-distinguishing proper edge-coloring of graphs, and obtained some results, especially presented a meaningful conjecture. In this paper, a new concept of adjacent-vertex-distinguishing total coloring of graphs is proposed.

Definition 1.1. Let $G(V, E)$ be a connect graph with order at least 2, k is a positive integer and f is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$. For all $u \in V(G)$, the set $\{f(u)\} \cup \{f(uv)|uv \in E(G)\}$ is denoted by $C(u)$. If

1) for any $uv, vw \in E(G), u \neq w$, we have $f(uv) \neq f(vw)$;

2) for any $uv \in E(G), u \neq v$, we have $f(u) \neq f(v), f(u) \neq f(uv), f(v) \neq f(uv)$, then f is called a k-proper-total-coloring. If f is a k-proper-total-coloring, and

3) for any edge $uv \in E(G)$, we have $C(u) \neq C(v)$, then f is called a k-adjacent-vertex-distinguishing total coloring of graph G(k-AVDTC of G in brief) and the number

$$\chi_{at}(G) = \min\{k \mid G \text{ has a } k\text{-AVDTC}\}$$

is called the adjacent-vertex-distinguishing total chromatic number of G.

$C(u)$ in Definition 1.1 is called the color set of vertex u and $\{1, 2, \cdots, k\} \setminus C(u)$ is denoted by $\overline{C}(u)$.

Lemma 1.1. If graph G has two vertices of maximum degree which are adjacent, then

$$\chi_{at}(G) \geqslant \Delta(G) + 2.$$

Proof. Suppose u, v are the two adjacent-vertices with maximum degree. Then for any k-AVDTC of G, both $C(u)$ and $C(v)$ must have $\Delta(G) + 1$ elements, but $C(u) \neq C(v)$. So it must be true that $k \geqslant \Delta(G) + 2$ for any k-AVDTC f of graph G. So the conclusion is followed.

The following lemma is obvious.

Lemma 1.2. If graph G has k components $G_1, G_2, \cdots, G_k$, and $|V(G_i)| \geqslant 2, i = 1, 2, \cdots, k$, then

$$\chi_{at}(G) = \max\{\chi_{at}(G_1), \chi_{at}(G_2), \cdots, \chi_{at}(G_k)\}.$$

Based on Lemma 1.2, we only discuss the connected graph with order at least 2.

In this paper, we will obtain the adjacent-vertex-distinguishing total chromatic number of cycle, complete graph, complete bipartite graph, fan, wheel and tree and, according to these results, give a conjecture. The other terminologies we refer to refs. [5—8].

2 Main results

Theorem 2.1. Let C_n be a cycle with order $n, n \geqslant 4$, then $\chi_{at}(C_n) = 4$.

Proof. Suppose $C_n = v_1 v_2 \cdots v_n$. From Lemma 1.1, we know that $\chi_{at}(C_n) \geqslant 4$. We now prove $\chi_{at}(C_n) \leqslant 4$, we need only prove that C_n has a 4-AVDTC. There are four cases to be considered.

Case 1. $n \equiv 0(mod 4)$.

A mapping f from $V(C_n) \cup E(C_n)$ to $\{1, 2, 3, 0\}$ is defined as follows:

$$f(v_i v_{i+1}) \equiv i(mod 4), f(v_i) \equiv i + 1(mod 4), i = 1, 2, \cdots, n.$$

Obviously f is a 4-proper-total-coloring of C_n. And for $1 \leqslant j \leqslant n$, we have

$$C(v_j) = \{3, 0, 1\} \text{ when } j \equiv 0(mod 4),$$

$$C(v_j) = \{0, 1, 2\} \text{ when } j \equiv 1(mod 4),$$

$$C(v_j) = \{1, 2, 3\} \text{ when } j \equiv 2(mod 4),$$
$$C(v_j) = \{2, 3, 0\} \text{ when } j \equiv 3(mod 4).$$

So f is a 4-AVDTC of C_n, and $\chi_{at}(C_n) = 4$.

Case 2. $n \equiv 1(mod 4)$.

We define a mapping f from $V(C_n) \cup E(C_n)$ to $\{1, 2, 3, 0\}$ as follows:

$$f(v_i v_{i+1}) \equiv i(mod 4), f(v_i) \equiv i + 1(mod 4), i = 1, 2, \cdots, n - 5;$$
$$f(v_{n-4}v_{n-3}) = 1, f(v_{n-3}v_{n-2}) = 2, f(v_{n-2}v_{n-1}) = 0, f(v_{n-1}v_n) = 3, f(v_n v_1) = 0,$$
$$f(v_{n-4}) = 2, f(v_{n-3}) = 3, f(v_{n-2}) = 1, f(v_{n-1}) = 2, f(v_n) = 1.$$

Obviously f is a 4–proper-total-coloring of C_n. And for $1 \leqslant j \leqslant n - 5$, we have

$$C(v_j) = \{3, 0, 1\} \text{ when } j \equiv 0(mod 4),$$
$$C(v_j) = \{0, 1, 2\} \text{ when } j \equiv 1(mod 4),$$
$$C(v_j) = \{1, 2, 3\} \text{ when } j \equiv 2(mod 4),$$
$$C(v_j) = \{2, 3, 0\} \text{ when } j \equiv 3(mod 4),$$

whereas

$$C(v_{n-4}) = \{0, 1, 2\}, C(v_{n-3}) = \{1, 2, 3\}, C(v_{n-2}) = \{2, 0, 1\},$$
$$C(v_{n-1}) = \{0, 3, 2\}, C(v_n) = \{3, 0, 1\}.$$

So f is a 4-AVDTC of C_n, and $\chi_{at}(C_n) = 4$.

Case 3. $n \equiv 2(mod 4)$.

We define a mapping f from $V(C_n) \cup E(C_n)$ to $\{1, 2, 3, 0\}$ as follows:

$$f(v_i v_{i+1}) \equiv i(mod 4), f(v_i) \equiv i + 1(mod 4), i = 1, 2, \cdots, n - 6;$$
$$f(v_{n-5}v_{n-4}) = 1, f(v_{n-4}v_{n-3}) = 2, f(v_{n-3}v_{n-2}) = 3,$$
$$f(v_{n-2}v_{n-1}) = 0, f(v_{n-1}v_n) = 3, f(v_n v_1) = 0,$$
$$f(v_{n-5}) = 2, f(v_{n-4}) = 3, f(v_{n-3}) = 0,$$
$$f(v_{n-2}) = 1, f(v_{n-1}) = 2, f(v_n) = 1.$$

Obviously f is a 4-proper-total-coloring of C_n. And for $1 \leqslant j \leqslant n - 6$, we also have

$$C(v_j) = \{3, 0, 1\} \text{ when } j \equiv 0(mod\, 4),$$
$$C(v_j) = \{0, 1, 2\} \text{ when } j \equiv 1(mod\, 4),$$
$$C(v_j) = \{1, 2, 3\} \text{ when } j \equiv 2(mod\, 4),$$
$$C(v_j) = \{2, 3, 0\} \text{ when } j \equiv 3(mod\, 4),$$

whereas

$$C(v_{n-5}) = \{0, 1, 2\}, C(v_{n-4}) = \{1, 2, 3\}, C(v_{n-3}) = \{2, 3, 0\},$$
$$C(v_{n-2}) = \{3, 0, 1\}, C(v_{n-1}) = \{0, 3, 2\}, C(v_n) = \{3, 0, 1\}.$$

So f is a 4-AVDTC of C_n, and $\chi_{at}(C_n) = 4$.

Case 4. $n \equiv 3(mod\, 4)$.

A mapping f from $V(C_n) \cup E(C_n)$ to $\{1, 2, 3, 0\}$ is defined as follows:

$$f(v_iv_{i+1}) \equiv i(mod4), f(v_i) \equiv i+1(mod4), i=1,2,\cdots,n-7;$$
$$f(v_{n-6}v_{n-5})=1, f(v_{n-5}v_{n-4})=2, f(v_{n-4}v_{n-3})=3, f(v_{n-3}v_{n-2})=1,$$
$$f(v_{n-2}v_{n-1})=0, f(v_{n-1}v_n)=3, f(v_nv_1)=0;$$
$$f(v_{n-6})=2, f(v_{n-5})=3, f(v_{n-4})=0, f(v_{n-3})=2,$$
$$f(v_{n-2})=3, f(v_{n-1})=2, f(v_n)=1.$$

Obviously f is a 4-proper-total-coloring of C_n. Similar to Case 2 and Case 3, we can verify that f is a 4-AVDTC of C_n. So $\chi_{at}(C_n)=4$.

From all above, theorem 2.1 is true.

Theorem 2.2. Let K_n be a complete graph with order n, $n \geqslant 3$, then

$$\chi(K_n) = \begin{cases} n+1, & n \equiv 0(mod\ 2), \\ n+2, & n \equiv 1(mod\ 2). \end{cases}$$

Proof. Suppose $V(K_n)=\{v_1, v_2, \cdots, v_n\}$. In order to describe conveniently, we identify v_l with v_r when $l \equiv r(mod\ n)$. From Lemma 1.1, we know that $\chi_{at}(K_n) \geqslant n+1$. There are two cases to be considered.

Case 1. $n \equiv 0(mod\ 2)$.

Let $n=2t$. We need only to prove that K_{2t} has a $(2t+1)$-AVDTC. When $t=1$, K_2 has a 3-AVDTC obviously. When $t \geqslant 2$, construct a mapping f from $V(K_{2t}) \bigcup E(K_{2t})$ to $\{1, 2, \cdots, 2t+1\}$ as follows:

$$f(v_i)=i, i=1,2,\cdots,2t; f(v_jv_{2t+1-j})=2t+1, j=1,2,\cdots,t;$$
$$f(v_jv_{2t+3-j})=1, j=3,4,\cdots,t+1;$$
$$f(v_1v_3)=f(v_jv_{2t+5-j})=2, j=5,6,\cdots,t+2;$$
$$f(v_2v_4)=f(v_1v_5)=f(v_jv_{2t+7-j})=3, j=7,8,\cdots,t+3;$$
$$f(v_3v_5)=f(v_2v_6)=f(v_1v_7)=f(v_jv_{2t+9-j})=4, j=9,10,\cdots,t+4;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f(v_{t-2}v_t)=f(v_{t-3}v_{t+1})=f(v_{t-4}v_{t+2})=\cdots$$
$$=f(v_1v_{2t-3})=f(v_{2t-1}v_{2t})=t-1;$$
$$f(v_{t-1}v_{t+1})=f(v_{t-2}v_{t+2})=f(v_{t-3}v_{t+3})=\cdots=f(v_2v_{2t-2})=f(v_1v_{2t-1})=t;$$
$$f(v_tv_{t+2})=f(v_{t-1}v_{t+3})=f(v_{t-2}v_{t+4})=\cdots=f(v_3v_{2t-1})=f(v_2v_{2t})=t+1;$$
$$f(v_{t+1}v_{t+3})=f(v_tv_{t+4})=f(v_{t-1}v_{t+5})=\cdots=f(v_4v_{2t})=f(v_1v_2)=t+2;$$
$$f(v_{t+2}v_{t+4})=f(v_{t+1}v_{t+5})=f(v_tv_{t+6})=\cdots=f(v_6v_{2t})$$
$$=f(v_{5-i}v_i)=t+3, i=1,2;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f(v_{2t-3}v_{2t-1}) = f(v_{2t-4}v_{2t}) = f(v_{2t-5-i}v_i) = 2t-2, i = 1, 2, \cdots, t-3;$$
$$f(v_{2t-2}v_{2t}) = f(v_{2t-3-i}v_i) = 2t-1, i = 1, 2, \cdots, t-2;$$
$$f(v_{2t-1-i}v_i) = 2t, i = 1, 2, \cdots, t-1.$$

It is clear that $\overline{C}(v_{2i}) = \{i\}$, $i = 1, 2, \cdots, t$; $\overline{C}(v_{2i-1}) = \{t+i\}$, $i = 1, 2, \cdots, t$. Thus f is a $(n+1)$-AVDTC of K_n. So $\chi_{at}(K_n) = n+1$.

Case 2. $n \equiv 1 (mod\, 2)$.

Firstly we are going to prove that K_n does not have $(n+1)$-AVDTC, and then to prove that K_n has a $(n+2)$-AVDTC.

Suppose that K_n has a $(n+1)$-AVDTC f. Then $f(v_1), f(v_2), \cdots, f(v_n)$ are distinct. Without loss of generality, let $f(v_i) = i, i = 1, 2, \cdots, n$. We define a $(n+1) \times n$ matrix C_0 as follows:

$$C_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & n+1 \\ 3 & 4 & 5 & \cdots & n+1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n & n+1 & 1 & \cdots & n-3 & n-2 \\ n+1 & 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}.$$

For every $v_i \in V(K_n)$, $C(v_i)$ is a set composed of all elements in some row of C_0. As $C(v_1), C(v_2), \cdots, C(v_n)$ are distinct, therefore different vertices of K_n correspond to different rows of C_0. Thus there is exact one row which does not correspond to any vertices of K_n. Now consider the problem for deleting this particular row.

If delete the first row, then a matrix with order n is obtained. For this matrix, the color $n+1$ appears n (n is an odd number) times. It is impossible, because the color $n+1$ is only used to color edges, but not to vertices.

If delete the ith row ($i = 2, 3, \cdots, n$), then a matrix with order n again is obtained. For this matrix, the color i appears $n-1$ ($n-1$ is an even number) times. It is impossible, because the color i is only used once to color vertex, and the color i which colors vertex appears only once in this matrix, but the color i which colors edges appears even times in this matrix.

If delete the last row, then the color 1 appears $n-1$ times ($n-1$ is an even number), it is impossible, too.

Now we need only to prove that K_n has a $(n+2)$-AVDTC. Let $n = 2t+1$. If $t = 1, 2$, then we can get a $(n+2)$-AVDTC of K_n easily. Assume that $t \geqslant 3$. Construct a mapping f from $V(K_n) \bigcup E(K_n)$ to $\{1, 2, \cdots, n+2\}$ as follows:

$$f(v_j) = f(v_{j+2}v_{j+2t-1}) = f(v_{j+3}v_{j+2t-2}) = \cdots$$
$$= f(v_{j+t-1}v_{j+t+2}) = f(v_{j+t}v_{j+t+1}) = j, \quad j = 1, 2, \cdots, 2t+1.$$

For $j = 1, 2, \cdots, 2t-1$, if $j \equiv 1$ or $2(mod\ 4)$, then let $f(v_j v_{j+2}) = 2t+2$; if $j \equiv 3$ or $0(mod\ 4)$, then let $f(v_j v_{j+2}) = 2t+3$. Let $f(v_{2t}v_1) = 2k+1$, $f(v_{2t+1}v_2) = 1$. Obviously f is a $(n+2)$-proper-total-coloring of K_n. Whereas

$$\overline{C}(v_j) = \{j-1, j+1\}, 3 \leqslant j \leqslant 2t-1; \overline{C}(v_1) = \{2, 2t+3\}, \overline{C}(v_2) = \{3, 2t+3\}.$$

If $t \equiv 1(\text{mod } 2)$, then $\overline{C}(v_{2t}) = \{2t-1, 2t+2\}$, $\overline{C}(v_{2t+1}) = \{2t, 2t+3\}$; if $t \equiv 0(\text{mod } 2)$, then $\overline{C}(v_{2t}) = \{2t-1, 2t+3\}$, $\overline{C}(v_{2t+1}) = \{2t, 2t+2\}$. Thus f is a $(n+2)$-AVDTC of K_n.

The proof of Theorem 2.2 is completed.

It is interesting that $\chi_{at}(K_3 - e) = 3$ for every $e \in E(K_3)$ and $\chi_{at}(K_3) = 5$. For $\chi_{at}(K_{2n+1} - e)$, where $e \in E(K_{2n+1}), n \geqslant 2$, we have

Theorem 2.3. For any $e \in E(K_{2n+1})(n \geqslant 1)$, we have

$$\chi_{at}(K_{2n+1} - e) = \begin{cases} 3, & n = 1; \\ 2n+2, & n = 2, 3, 4; \\ 2n+3, & n \geqslant 5. \end{cases}$$

Proof. Let $V(K_{2n+1} - e) = \{w_1, w_2, \cdots, w_{2n-1}, u, v\}$ and $W = \{w_1, w_2, \cdots, w_{2n-1}\}, e = uv$. If $n = 1, 2$, then the conclusion is correct obviously. If $n = 3$, then $\chi_{at}(K_7 - e) \geqslant 8$ by Lemma 1.1. In order to prove $\chi_{at}(K_7 - e) = 8$, we need only to prove that $K_7 - e$ has an 8-AVDTC. Construct a mapping f from $V(K_7 - e) \bigcup E(K_7 - e)$ to $\{1, 2, \cdots, 8\}$ as follows:

$$f(w_j) = j, j = 1, 2, 3, 4, 5; f(u) = f(v) = 8,$$
$$f(w_2w_5) = f(w_3w_4) = 1, f(w_1u) = f(w_2w_3) = 4,$$
$$f(w_1w_2) = f(w_3v) = 5, f(w_2w_4) = f(w_3w_5) = 8,$$
$$f(w_1w_3) = f(uw_5) = f(vw_4) = 2,$$
$$f(w_1w_4) = f(w_2u) = f(vw_5) = 3, f(w_1v) = f(w_3u) = f(w_4w_5) = 6,$$
$$f(w_1w_5) = f(w_2v) = f(uw_4) = 7.$$

Obviously, f is an 8-AVDTC of $K_7 - e$. So $\chi_{at}(K_7 - e) = 8$.

If $n = 4$, then $\chi_{at}(K_9 - e) \geqslant 10$ from Lemma 1.1. In order to prove $\chi_{at}(K_9 - e) = 10$, we need only to give a 10-AVDTC of $K_9 - e$. Construct a mapping f from $V(K_9 - e) \cup E(K_9 - e)$ to $\{1, 2, \ldots, 10\}$ as follows:

$$f(w_j) = j, j = 1, 2, 3, 4, 5, 6, 7; f(u) = f(v) = 10,$$
$$f(w_2w_5) = f(w_3v) = f(w_4w_6) = f(w_7u) = 1,$$
$$f(w_1u) = f(w_3w_7) = f(w_4v) = f(w_5w_6) = 2,$$
$$f(w_1w_4) = f(w_2v) = f(w_5w_7) = f(w_6u) = 3,$$
$$f(w_1w_7) = f(w_3u) = f(w_4w_5) = f(w_6v) = 8,$$
$$f(w_1w_2) = f(w_4u) = f(w_5v) = f(w_6w_7) = 9,$$

$$f(w_1v) = f(w_2w_7) = f(w_3w_6) = 4, f(w_1w_6) = f(w_2w_3) = f(w_7v) = 5,$$
$$f(w_1w_3) = f(w_2w_4) = f(w_5w_8) = 6, f(w_1w_5) = f(w_2w_8) = f(w_3w_4) = 7,$$
$$f(w_2w_6) = f(w_3w_5) = f(w_4w_7) = 10.$$

Obviously, f is a 10-AVDTC. So $\chi_{at}(K_9 - e) = 10$.

If $n \geqslant 5$, then $\chi_{at}(K_{2n+1} - e) \geqslant 2n+2$ from Lemma 1.1. By Theorem 2.2, we know that $\chi_{at}(K_{2n+1} - e) \leqslant 2n+3$. So we need only to prove that $\chi_{at}(K_{2n+1} - e) \neq 2n+2$. Use reduction to absurdity. Assume that $K_{2n+1} - e$ has a $(2n+2)$-AVDTC f. Let $C = \{1, 2, \cdots, 2n+2\}$ be the set of all $(2n+2)$ colors . Then we have

(i) $n - 1 \leqslant |E_i| \leqslant n$, where $E_i = \{z \in E(K_{2n+1} - e) | f(z) = i\}$, $i = 1, 2, \ldots, 2n+2$;

(ii) there are exact $n+1$ colors s.t. each such color just color n edges, and there are exact $n+1$ colors s.t. each such color just color $n-1$ edges.

Obviously, $|E_i| \leqslant n, i = 1, 2, \cdots, 2n+2$. Suppose $n - 1 \leqslant |E_i|$ does not hold for some $i \in \{1, 2, \cdots, 2n+2\}$. Then there are at most $2(n-2)$ vertices which are in W and are incident to some edge colored with i. So there are at least 3 vertices which are in W and are not incident to any edges colored with i. Thus there are at least two vertices which have the same color set. A contradiction. So (i) holds.

For (ii), suppose there are exact x colors s.t. each such color just color n edges, and there are exact y colors s.t. each such color just color $n-1$ edges. Then

$$\begin{cases} nx + (n-1)y = 2n^2 + n - 1, \\ x + y = 2n + 2. \end{cases}$$

Thus $x = y = n+1$, i.e. (ii) holds.

As $|\overline{C}(u)| = |\overline{C}(v)| = 2$, therefore there are at least $n-3$ colors in B, which are both in the color set of u's and v's, where

$$B = \{c \in C || E_c| = n - 1\}.$$

Suppose $c, c' \in B$, s.t. $c, c' \in C(u) \cap C(v)$, $c \neq c'$.

1) $f(u) \neq c, f(v) \neq c$.

In this time, there are 3 vertices in W which are not incident to any edge with color c. This implies that there are two vertices in W which have the same color set. A contradiction.

2) $f(u) = c$ or $f(v) = c$, $f(u) \neq f(v)$.

In this time, there exist $w_i, w_j \in W, i \neq j$, s.t. $c \notin C(w_i) \cup C(w_j)$. So $C(w_i) = C(w_j)$. A contradiction.

3) $f(u) = f(v) = c$.

In this time, we consider the color c'. There exist $w_i, w_j, w_k \in W$, s.t. c' does not color any edge which is incident to w_i, w_j or w_k. So the two of $C(w_i), C(w_j), C(w_k)$ are the same. A contradiction.

The proof of this theorem is completed.

Theorem 2.4. Let F_n be a fan with order $n+1, n \geqslant 3$, then

$$\chi_{at}(F_3) = 5, \chi_{at}(F_n) = n+1 (n \geqslant 4).$$

Proof. When $n = 3$, we have $\chi_{at}(F_3) \geqslant 5$ by Lemma 1.1. It is easy to give a 5-AVDTC of F_3. So $\chi_{at}(F_3) = 5$.

When $n \geqslant 4$, let $V(F_n) = \{v_0, v_1, v_2, \cdots, v_n\}$, $E(F_n) = \{v_0 v_i | i = 1, 2, \cdots, n\} \bigcup \{v_i v_{i+1} | i = 1, 2, \cdots, n-1\}$. Obviously $\chi_{at}(F_n) \geqslant n+1$ by Lemma 1.1. Construct a mapping f from $V(F_n) \bigcup E(F_n)$ to $\{1, 2, \cdots, n+1\}$ as follows:

$$f(v_0) = n+1, f(v_n) = 1, f(v_i) = i+1 (i = 1, 2, \cdots, n-1),$$
$$f(v_0 v_i) = i (i = 1, 2, \cdots, n), f(v_1 v_2) = n, f(v_i v_{i+1}) = i-1 (i = 2, 3, \cdots, n-1).$$

f is a $(n+1)$-AVDTC of F_n. Thus $\chi_{at}(F_n) = n+1$.

Suppose W_n is the wheel with $n+1$ vertices. When $n = 3$, from Theorem 2.2 we know that $\chi_{at}(W_n) = \chi_{at}(K_4) = 5$.

Theorem 2.5. When $n \geqslant 4, \chi_{at}(W_n) = n+1$.

Proof. Because the center vertex of the wheel W_n is the only vertex with maximum degree, $\chi_{at}(W_n) \geqslant n+1$. Based on the proof of Theorem 2.4, let $f(v_n v_1) = n-1$. Then f is a $(n+1)$-AVDTC of W_n. So $\chi_{at}(W_n) = n+1$.

Theorem 2.6. For complete bipartite graph $K_{m,n}$, if $m \geqslant n \geqslant 1$, then

$$\chi_{at}(K_{m,n}) = \begin{cases} m+1, & m \geqslant n+1, \\ 3, & m = n = 1, \\ n+2, & m = n \geqslant 2. \end{cases}$$

Proof. When $m = n = 1$, we have $\chi_{at}(K_{m,n}) = 3$. This is obvious.

When $m \geqslant n+1$, a proper total coloring of $K_{m,n}$ is also the AVDTC of $K_{m,n}$. Because the total chromatic number of $K_{m,n}$ is $m+1$, $\chi_{at}(K_{m,n}) = m+1$.

When $m = n = 2$, as $K_{m,n}$ is C_4, from Theorem 2.1, $\chi_{at}(K_{2,2}) = 4$.

When $m = n \geqslant 3$, from Lemma 1.1, we know that $\chi_{at}(K_{m,n}) \geqslant n+2$. Now we prove $\chi_{at}(K_{m,n}) \leqslant n+2$. We need only to prove that $K_{m,n}$ has a $(n+2)$-AVDTC. Suppose that $V(K_{m,n}) = V_1 \cup V_2$, where $V_1 \cap V_2 = \phi$, V_1 and V_2 are independent sets of $K_{m,n}$, and both have n vertices in $K_{m,n}$, and each vertex in V_1 is adjacent to each vertex in V_2. Let $V_1 = \{u_1, u_2, \cdots, u_n\}, V_2 = \{v_1, v_2, \cdots, v_n\}$. Construct a mapping f from $V(K_{m,n}) \cup E(K_{m,n})$ to $\{1, 2, \cdots, n+2\}$ as follows:

$$f(u_i) = n+2, i = 1, 2, \cdots, n; f(v_j) = j, j = 1, 2, \cdots, n;$$
$$f(u_i v_j) \in \{1, 2, \cdots, n+1\} \text{ and } f(u_i v_j) \equiv i + j (mod\, n+1).$$

Obviously f is a $(n+2)$-proper-total-coloring of $K_{n,n}$, and $\overline{C}(u_i) = \{i\}, \overline{C}(v_i) = \{n+2\}, i \in \{1, 2, \cdots, n\}$. So f is a $(n+2)$-AVDTC of $K_{n,n}$, and $\chi_{at}(K_{m,n}) = n+2$.

In order to discuss the adjacent-vertex-distinguishing total coloring of tree, we give the following two lemmas firstly.

Lemma 2.1. Let P_n be a path with order $n, n \geqslant 2$. Then

$$\chi(P_n) = \begin{cases} 3, & n = 2, 3; \\ 4, & n \geqslant 4. \end{cases}$$

Proof. Suppose that $P_n = v_1 v_2 \cdots v_n$. It is clear that $\chi_{at}(P_n) = 3$ when $n = 2, 3$.

When $n \geqslant 4$, from Lemma 1.1, we have $\chi_{at}(P_n) \geqslant 4$. We need only to prove $\chi_{at}(P_n) \leqslant 4$. Construct a mapping f from $V(P_n) \bigcup E(P_n)$ to $\{1, 2, 3, 4\}$ as follows:

$$f(v_i v_{i+1}) \in \{1, 2, 3, 4\} \text{ and } f(v_i v_{i+1}) \equiv i(mod 4), i = 1, 2, \cdots, n-1;$$
$$f(v_j) \in \{1, 2, 3, 4\} \text{ and } f(v_j) \equiv j+1(mod 4), j = 1, 2, \cdots, n.$$

Obviously f is a 4-proper-total-coloring. And we also have

$$C(v_j) = \{3, 4, 1\} \text{ when } 2 \leqslant j \leqslant n-1 \text{ and } j \equiv 0(mod 4),$$
$$C(v_j) = \{4, 1, 2\} \text{ when } 2 \leqslant j \leqslant n-1 \text{ and } j \equiv 1(mod 4),$$
$$C(v_j) = \{1, 2, 3\} \text{ when } 2 \leqslant j \leqslant n-1 \text{ and } j \equiv 2(mod 4),$$
$$C(v_j) = \{2, 3, 4\} \text{ when } 2 \leqslant j \leqslant n-1 \text{ and } j \equiv 3(mod 4).$$

Both $C(v_1)$ and $C(v_n)$ are two-elements sets. Then $C(v_i) \neq C(v_{i+1})$ for all $i \in \{1, 2, \cdots, n-1\}$. So f is a 4-AVDTC of P_n, therefore $\chi_{at}(P_n) = 4$.

The following lemma is obvious.

Lemma 2.2. Let S_n be a star with order $n+1, n \geqslant 3$. Then

$$\chi_{at}(S_n) = n + 1.$$

Theorem 2.7. Let T_n be a tree with order $n (n \geqslant 2)$. If there are no two adjacent vertices of maximum degree, then $\chi_{at}(T_n) = \Delta(T_n)+1$; if there are two adjacent vertices of maximum degree, then $\chi_{at}(T_n) = \Delta(T_n) + 2$.

Proof. Let $S = \{x \in V(T) | d(x) \geqslant 2$, there are at least $d(x) - 1$ vertices of degree 1 which adjacent to $x\}$. For $n \geqslant 3$, obviously $|S| \geqslant 1$. When $|S| = 1$, T_n is a star. By Lemma 2.2, conclusion of Theorem 2.7 is valid. In the following we assume that $|S| \geqslant 2$.

We prove the result of Theorem 2.7 by induction on the order n of tree T_n.

If $n = 4$, then $T_4 = P_4$, the conclusion of Theorem 2.7 is valid from Lemma 2.1. Assume that for a tree of order n, the conclusion of Theorem 2.7 is valid. Now we prove that for a tree T_{n+1} of order $n + 1$, the conclusion of Theorem 2.7 is also valid.

If T_{n+1} is a path, then the conclusion of Theorem 2.7 holds by Lemma 2.1. Suppose $\Delta(T_n) \geqslant 3$. Let $u \in S$, and $d(u) =$min$\{d(x) | x \in S\}$. Let $wu \in E(T_{n+1})$, $d(w) \geqslant 2$.

Let $uv \in E(T_{n+1})$, and $d(v) = 1$. Let $T' = T_{n+1} - v$. Then $V(T') = n$. By induction hypothesis, for T', the conclusion of Theorem 2.7 is valid.

Case 1. There are no two adjacent vertices of maximum degree in T_{n+1}.

Case 1.1. $d_{T'}(w) \leqslant d_{T'}(u)$.

In this moment, $\Delta(T') = \Delta(T_{n+1})$(Because if T_{n+1} has only one vertex of maximum degree, then from the selection of u, u is not the vertex of maximum degree). Obviously, we can obtain a $(\Delta(T_{n+1})+1)$-AVDTC of T_{n+1} from the $(\Delta(T_{n+1})+1)$-AVDTC of T'.

Case 1.2. In $d_{T'}(w) > d_{T'}(u)$.

Case 1.2.1. $d_{T'}(w) = d_{T'}(u) + 1$.

Since there are no two adjacent vertices of maximum degree, in T_{n+1}, $d_{T_{n+1}}(u) < \Delta(T_{n+1})$ (Otherwise, in T_{n+1}, w and u are two vertices of maximum degree which are adjacent. This is a contradiction). Let g be a $(\Delta(T_{n+1}) + 1)$-AVDTC of T'. We have $\overline{C}(w) \neq \phi$ (in the meaning of g). If $C(u) \subseteq C(w)$, let $g(v) = g(uw)$, $g(uv) \in \{1, 2, \cdots, \Delta(T_{n+1}) + 1\} - C(w)$; if $C(u) \nsubseteq C(w)$, then let $g(v), g(uv) \in \{1, 2, \cdots, \Delta(T_{n+1}) + 1\} - C(u)$, s.t. $g(v) \neq g(uv)$, where $C(u) = \{g(u)\} \cup \{g(ux) | x \in V(T'), ux \in E(T')\}$, similar for $C(w)$. So g is a $(\Delta(T_{n+1}) + 1)$-AVDTC of T_{n+1}.

Case 1.2.2. $d_{T'}(w) \geqslant d_{T'}(u) + 2$.

In this subcase, we can easily obtain a $(\Delta(T_{n+1})+1)$-AVDTC of T_{n+1} from $(\Delta(T_{n+1}) +1)$-AVDTC of T'.

Case 2. There are two adjacent vertices of maximum degree in T_{n+1}.

Case 2.1. $d_{T'}(u) + 1 = d_{T'}(w)$.

In this moment, $d_{T'}(w) \leqslant \Delta(T') = \Delta(T_{n+1})$. By induction hypothesis, there exists a $(\Delta(T_{n+1}) + 2)$-AVDTC f of T' (Note that if there are no two adjacent vertices of maximum degree, then T' has $(\Delta(T_{n+1}) + 1)$-AVDTC. And the $(\Delta(T_{n+1}) + 1)$-AVDTC of T' is also a $(\Delta(T_{n+1}) + 2)$-AVDTC of T'). Let $C(u) = \{f(u)\} \cup \{f(ux) | x \in V(T'), ux \in E(T')\}$, $C(w) = \{f(w)\} \cup \{f(wy) | y \in V(T'), wy \in E(T')\}$. If $C(u) \subseteq C(w)$, then let $f(uv) \in \{1, 2, \cdots, \Delta(T_{n+1}) + 2\} - C(w)$, $f(v) = f(uw)$. If $C(u) \nsubseteq C(w)$, then let $f(v), f(uv) \in \{1, 2, \cdots, \Delta(T_{n+1}) + 2\} - C(u)$ s.t. $f(v) \neq f(uv)$. In this way, we obtain a $(\Delta(T_{n+1}) + 2)$-AVDTC of T_{n+1}.

Case 2.2. $d_{T'}(u) + 1 \neq d_{T'}(w)$.

In this moment, by induction hypothesis, there is a $(\Delta(T_{n+1}) + 2)$-AVDTC f of T'. For this f, $|C(u)| \leqslant \Delta(T_{n+1})$. Let $f(v), f(uv) \in \{1, 2, \cdots, \Delta(T_{n+1}) + 2\} - C(u)$, s.t. $f(v) \neq f(uv)$. Then f becomes a $(\Delta(T_{n+1}) + 2)$-AVDTC of T_{n+1}.

From the discussion above, the proof of Theorem 2.7 is completed.

3 Conjecture and unsolved problem

From Theorem 2.1 to Theorem 2.7, we give the following conjecture.

Conjecture 3.1. For connected simple graph G with order at least 2, we have

$$\chi_{at}(G) \leqslant \Delta(G) + 3.$$

Let G be an order 4 graph obtained by joining K_3 and K_2 at one vertex. Then $4 = \chi_{at}(G) < \chi_{at}(K_3) = 5$. So we propose the following unsolved problem.

Open Problem 3.1. If H is a subgraph of G, when do we have $\chi_{at}(H) \leqslant \chi_{at}(G)$?

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References

1. Burris, A. C., Schelp, R. H., Vertex-distinguishing proper edge-colorings, J. of Graph Theory, 1997, 26(2): 73—82.
2. Bazgan, C., Harkat-Benhamdine, A., Li, H. et al., On the vertex-distinguishing proper edge-coloring of graphs, J. Combin. Theory, Ser. B, 1999, 75: 288—301.
3. Balister, P. N., Bollobs, B., Schelp, R. H., Vertex distinguishing colorings of graphs with $\Delta(G) = 2$, Discrete Mathmatics, 2002, 252(2): 17—29.
4. Zhang Zhongfu, Liu Linzhong, Wang Jianfang, Adjacent strong edge coloring of graphs, Applied Mathematics Letters, 2002, 15(5): 623—626.
5. Reinhard, D., Graph Theory, New York: Springer-Verlag, Inc., 1997.
6. Chartrand, G., Lesniak-Foster, L., Graph and Digraphs, 2nd ed., Monterey, CA: Wadsworth Brooks/Cole, 1986.
7. Hansen, P., Marcotte, O., Graph coloring and application, Rhode Island USA: AMS Providence, 1999.
8. Bondy, J. A., Murty, U. S. A., Graph Theory with Applications, London: Macmillan; New York: Elsevier, 1976.