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Alternating difference block methods and their difference graphs*

ZHANG Baolin (张宝琳)

(Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, China)

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Abstract The general concept of a class of alternating difference block methods and their difference graphs are introduced. The methods are unconditionally stable, and according to the difference graphs the design of parallel algorithms and programs of the methods are flexible and easy. The traditional alternating explicit-implicit method, the ADI method, the AGE method, the Block ADI method and the ABE-I method are all the special cases of this class of methods. Their difference graphs are presented.

Keywords: two-dimensional diffusion equation, the finite difference method, parallel algorithms, difference graphs.

The appearance and the increasing availability of high performance computers with parallel processing have pushed the development of parallel numerical method forward. For the parallel solution of the parabolic differential equation in two space variables, much work was done in recent years, for example the AGE method[1] and the ABE-I mehtod[2]. Among the traditional difference methods for the two-dimensional diffusion equation the alternating direction implicit (ADI) method[3,4] is the most notable one which reduces the multidimensional problem to a sequence of one-dimensional problems with tridiagonal systems easily solvable by computers. We know that the ADI method is not only suitable for the sequential computations, but also suitable for the parallel computations on the shared memory computers. As for the distributed memory multiprocessors the implementation of ADI method requires the expensive communications. To design the difference methods which may easily be implemented on the distributed memory computers with a minimal cost of communications, we have developed the Block ADI method[5] which transforms the global computations and communications of ADI method into the local ones, and thus the localization strategy of parallel computing[6] becomes a reality there.

This paper presents the general concept to design the alternating difference block methods such as AGE, ABE-I, and Block ADI and others, and describes the difference graphs of these methods. According to the difference graphs, the unconditional stability of methods can be proved in a very simple way, and the design of parallel algorithms and programs will be flexible and easy.

1 Basic difference schemes

The problem is: find the solution $u(x, y, t)$ in the domain $D: \{0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1, t>0\}$ of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{1}$$

with the boundary conditions

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$$u(0,y,t)=f_1(y,t),\quad u(1,y,t)=f_2(y,t),$$
$$u(x,0,t)=f_3(x,t),\quad u(x,1,t)=f_4(x,t), \tag{2}$$

and the initial condition

$$u(x,y,0)=f(x,y). \tag{3}$$

Let Δx, Δy and Δt be the grid spacings in the x, y and t directions, where $\Delta x = 1/(m+1)$, $\Delta y = 1/(n+1)$, m and n are positive integers. The approximate values $u_{i,j,k}$ of the solution $u(x, y, t)$ for (1)—(3) are to be computed at grid points (x_i, y_j, t_k), where $x_i = i\Delta x$, $i=0,1,\cdots,m+1$; $y_j = j\Delta y$, $j=0,1,\cdots,n+1$; $t_k = k\Delta t$, $k=1,2,\cdots$. For simplicity, we take $\Delta x = \Delta y = \Delta s$. At first let us review the following eight basic schemes for (1) (see refs. [3, 7]):

Explicit scheme

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i-1,j,k}-2u_{i,j,k}+u_{i+1,j,k}}{\Delta x^2}+\frac{u_{i,j-1,k}-2u_{i,j,k}+u_{i,j+1,k}}{\Delta y^2}; \tag{4}$$

implicit scheme

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i-1,j,k+1}-2u_{i,j,k+1}+u_{i+1,j,k+1}}{\Delta x^2}+\frac{u_{i,j-1,k+1}-2u_{i,j,k+1}+u_{i,j+1,k+1}}{\Delta y^2}; \tag{5}$$

implicit scheme in x-direction

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i-1,j,k+1}-2u_{i,j,k+1}+u_{i+1,j,k+1}}{\Delta x^2}+\frac{u_{i,j-1,k}-2u_{i,j,k}+u_{i,j+1,k}}{\Delta y^2}; \tag{6}$$

implicit scheme in y-direction

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i-1,j,k}-2u_{i,j,k}+u_{i+1,j,k}}{\Delta x^2}+\frac{u_{i,j-1,k+1}-2u_{i,j,k+1}+u_{i,j+1,k+1}}{\Delta y^2}; \tag{7}$$

asymmetric system (a)

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i+1,j,k}-u_{i,j,k}-u_{i,j,k+1}+u_{i-1,j,k+1}}{\Delta x^2}+\frac{u_{i,j+1,k}-u_{i,j,k}-u_{i,j,k+1}+u_{i,j-1,k+1}}{\Delta y^2}; \tag{8}$$

asymmetric system (b)

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i+1,j,k+1}-u_{i,j,k+1}-u_{i,j,k}+u_{i-1,j,k}}{\Delta x^2}+\frac{u_{i,j+1,k+1}-u_{i,j,k+1}-u_{i,j,k}+u_{i,j-1,k}}{\Delta y^2}; \tag{9}$$

asymmetric system (c)

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i+1,j,k+1}-u_{i,j,k+1}-u_{i,j,k}+u_{i-1,j,k}}{\Delta x^2}+\frac{u_{i,j+1,k}-u_{i,j,k}-u_{i,j,k+1}+u_{i,j-1,k+1}}{\Delta y^2}; \tag{10}$$

asymmetric system (d)

$$\frac{u_{i,j,k+1}-u_{i,j,k}}{\Delta t}=\frac{u_{i+1,j,k}-u_{i,j,k}-u_{i,j,k+1}+u_{i-1,j,k+1}}{\Delta x^2}+\frac{u_{i,j+1,k+1}-u_{i,j,k+1}-u_{i,j,k}+u_{i,j-1,k}}{\Delta y^2} \tag{11}$$

(see figure 1).

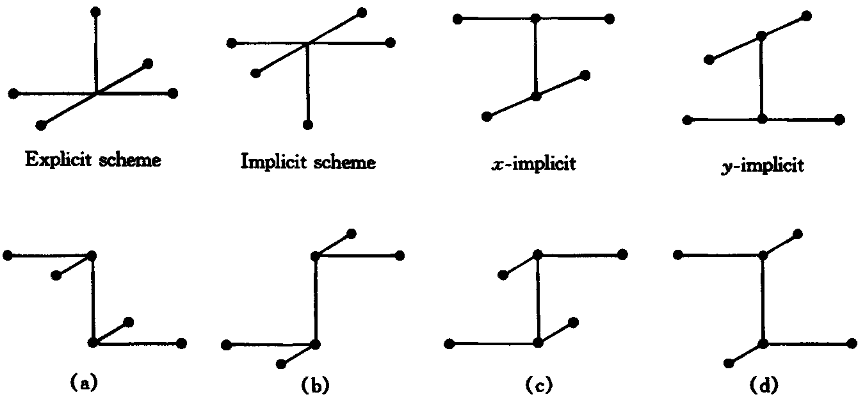

Fig. 1

If for each grid point (x_i, y_j, t_{k+1}) $(1 \leqslant i, j \leqslant n)$ we use precisely one difference equation $l_{i,j}$ from the basic schemes (4)—(11) in a certain way, then for all the grid points the difference equations $l_{i,j}(i, j = 1, 2, \cdots, n)$ form an $n^2 \times n^2$ linear system which gives a difference method for (1)—(3).

2 Difference block method and its graphs

We need some definitions in order to describe the construction and property of our method in geometry, concerning the grid points (x_i, y_j, t_{k+1}) on the $(k+1)$th time level and the grid points (x_i, y_j, t_k) on the kth time level. Let the grid domain $D_d^{(k+1)} = \{(x_i, y_j, t_{k+1}), i, j = 0, 1, \cdots, n+1\}$, $D_d^{(k)} = \{(x_i, y_j, t_k), i, j = 0, 1, \cdots, n+1\}$. When no confusion might occur, simply use the grid points (x_i, y_j) to denote the points in $D_d^{(k+1)}$ or in $D_d^{(k)}$. In addition, the grid points (x_i, y_0), (x_i, y_{n+1}), (x_0, y_j), (x_{n+1}, y_j) are called the boundary points of $D_d^{(k+1)}$ or $D_d^{(k)}$, where $i, j = 0, 1, \cdots, n+1$; the grid points (x_i, y_j) $(i, j = 1, 2, \cdots, n)$ are called the interior points of $D_d^{(k+1)}$ or $D_d^{(k)}$. If (x_i, y_j) is an interior point in $D_d^{(k+1)}$ or in $D_d^{(k)}$, the points (x_{i-1}, y_j), (x_{i+1}, y_j), (x_i, y_{j-1}), (x_i, y_{j+1}) are called the geometric neighbors of (x_i, y_j). Suppose a difference method is well defined by a difference equation system $\{l_{i,j}\}$ $(i, j = 1, 2, \cdots, n)$, then we have the following definition.

Definition 1. The points (x_i, y_j) is called the difference center of the difference equation $l_{i,j}$ from the basic schemes (4)—(11) in $D_d^{(k+1)}$ and in $D_d^{(k)}$ $(i, j = 1, 2, \cdots, n)$; a geometric neighbor (x_{i_1}, y_{j_1}) is called the difference neighbor of the difference center (x_i, y_j) of $l_{i,j}$ in $D_d^{(k+1)}$, if $u_{i_1, j_1, k+1}$ appears on the right hand side of the difference equation $l_{i,j}$. Similarly, a geometric neighbor (x_{i_2}, y_{j_2}) is called the difference neighbor of the difference center (x_i, y_j) of $l_{i,j}$ in $D_d^{(k)}$, if $u_{i_2, j_2, k}$ appears on the right hand side of the difference equation $l_{i,j}$.

Observing the subscripts of the approximate solution u in the difference equations (4)—(11), and by Definition 1 we find in general a difference center has at most four difference neighbors, and we have the following proposition.

Proposition 1. *A grid point (x_{i^*}, y_{j^*}) is a difference neighbor of the grid point (x_i, y_j) in $D_d^{(k+1)}$ if and only if (x_{i^*}, y_{j^*}) is not a difference neighbor of the grid point (x_i, y_j) in $D_d^{(k)}$.*

Definition 2. The grid points (x_i, y_j) and (x_{i^*}, y_{j^*}) are said to have a symmetric difference relation if and only if one of the following conditions is satisfied:

(i) (x_i, y_j) and (x_{i^*}, y_{j^*}) are difference neighbors with each other;

(ii) either of (x_i, y_j) and (x_{i^*}, y_{j^*}) is a boundary point and is also a difference neighbor of the other point.

Difference block method. If for each grid point (x_i, y_j, t_{k+1}) $(i, j = 1, 2, \cdots, n)$ we use precisely one difference equation $l_{i,j}$ from the basic schemes (4)—(11) in such a way that in the grid domain $D_d^{(k+1)}$ if a point (x_{i^*}, y_{j^*}) is a difference neighbor of (x_i, y_j), then there is a symmetric difference relation between (x_{i^*}, y_{j^*}) and (x_i, y_j); thus for all the grid points the difference equations $l_{i,j}(i, j = 1, 2, \cdots, n)$ form a $n^2 \times n^2$ linear system which gives a "difference block method" for (1)—(3).

Lemma 1. *For a given difference block method for* (1)—(3), *in the grid domain* $D_d^{(k)}$ *if a point* (x_{i^*}, y_{j^*}) *is a difference neighbor of* (x_i, y_j), *then there is a symmetric difference relation between* (x_{i^*}, y_{j^*}) *and* (x_i, y_j).

Proof. The conclusion of the lemma holds by Definition 2 (ii) if (x_{i^*}, y_{j^*}) is a boundary point.

Suppose that (x_{i^*}, y_{j^*}) is an interior point and (x_i, y_j) is not a difference neighbor of (x_{i^*}, y_{j^*}) in $D_d^{(k)}$. By Proposition 1 (x_i, y_j) is a difference neighbor of (x_{i^*}, y_{j^*}) in $D_d^{(k+1)}$. Noticing the condition in Lemma 1 and by Proposition 1 again, (x_{i^*}, y_{j^*}) is not a difference neighbor of (x_i, y_j) in $D_d^{(k+1)}$. By Definition 2, there is not a symmetric difference relation between (x_{i^*}, y_{j^*}) and (x_i, y_j) in $D_d^{(k+1)}$, and this contradicts the definition of difference block method. The lemma is proved.

If there is a symmetric difference relation between grid points (x_i, y_j) and (x_{i^*}, y_{j^*}) in $D_d^{(k+1)}$ or in $D_d^{(k)}$, we join the two points by a straight line and say that (x_i, y_j) and (x_{i^*}, y_{j^*}) are connected, where the line segment between (x_i, y_j) and (x_{i^*}, y_{j^*}) is called an edge[9].

Definition 3. The difference graph $G_1(G_2)$ of a difference block method is defined by a finite set which consists of (x_i, y_j) $(i, j = 1, 2, \cdots, n)$ and their difference neighbors among the boundary points, and all the edges in $D_d^{(k+1)}(D_d^{(k)})$. G_1 is called the up-graph and G_2 is called the down-graph of the method.

The connection relation of two points has the symmetric property, i.e. that (x_i, y_j) is connected with (x_{i^*}, y_{j^*}) implies (x_{i^*}, y_{j^*}) is connected with (x_i, y_j). We define that a difference center is connected with itself (the reflexive property) and define that if (x_{i_1}, y_{j_1}) is connected with (x_{i_2}, y_{j_2}) and (x_{i_2}, y_{j_2}) is connected with (x_{i_3}, y_{j_3}), then it is said that (x_{i_1}, y_{j_1}) is connected with (x_{i_3}, y_{j_3}) (the transitive property). So the connection relation is an equivalence relation on the set of the grid points in G_1 and G_2. By a well known theorem on the equivalence classes[10], we have the following result:

Theorem 1. *Suppose* G_1 *and* G_2 *are the difference graphs of a block difference method. Then*

$$G_1 = \bigcup_{\mu=1}^{p_1} G_1^{\mu}, \tag{12}$$

$$G_2 = \bigcup_{\upsilon=1}^{p_2} G_2^{\upsilon}, \tag{13}$$

where G_1^{μ} *and* G_2^{ν} *are respectively the connected components of* G_1 *and* G_2, p_1, $p_2 \geqslant 1$ *are positive integers.*

3 Alternating difference block method

When a block difference method at $(k+1)$th time level for (1)—(3) is given, the "co-difference block method" for (1)—(3) at $(k+2)$th time level is defined by such a way that the linear system is formed by the difference equations $\bar{l}_{i,j}$ generated at each grid point (x_i, y_j, t_{k+2}) by using a basic difference scheme changed from that at the point (x_i, y_j, t_{k+1}) in the given difference block method according to the following rules, where $i, j = 1, 2, \cdots, n$:

(i) The explicit scheme (4) changes into the implicit scheme (5), and the implicit scheme (5) changes into explicit scheme (4).

(ii) The x-implicit scheme changes into the y-implicit scheme. The y-implicit scheme changes into the x-implicit scheme.

(iii) The scheme (a) changes into scheme (b), and the scheme (b) changes into scheme (a).

(iv) The scheme (c) changes into scheme (d), and the scheme (d) changes into scheme (c).

We can easily introduce the difference graphs of the co-difference block method, one of which is G_2 in $D_d^{(k+2)}$ and the other is G_1 in $D_d^{(k+1)}$, where G_1 and G_2 are just the same as that defined for the given difference block method. In other words, G_1 and G_2 are respectively the up-graph and the down-graph of the difference block method, but for its co-difference block method G_2 is the up-graph and G_1 is the down-graph. Alternatively using the difference block method at the $(k+1)$th time level and the co-difference block method at the $(k+2)$th time level while $k = 0, 2, \cdots$ gives the "alternating difference block (ADB) method" for solving (1)—(3), in which we are interested.

Based on the expressions (12), (13) of G_1, G_2 and the difference equations (4)—(11), the ADB method can be described by the system in matrix form:

$$(I + rH_1)u^{k+1} = (I - rH_2)u^k + b_1, \quad k = 0,2,4,\cdots, \tag{14}$$

$$(I + rH_2)u^{k+2} = (I - rH_1)u^{k+1} + b_2, \tag{15}$$

where $r = \Delta t/\Delta s^2$, H_1 and H_2 are matrices of order $n^2 \times n^2$, u^{k+1} is vector of the elements $u_{i,j,k+1}$ ordered by a given manner, b_1 and b_2 are column matrices related to the boundary conditions.

Here we do not give the expressions of H_1 and H_2 in detail, but we can verify that H_1 and H_2 are non-negative definite matrices from the symmetric difference relation in the method. By using the well-known Kellogg's lemma[8], we obtain the following theorem.

Theorem 2. *The alternating difference block method for solving* (1)—(3) *denoted by* (14) *and* (15) *is unconditionally stable.*

Proof. Make an ordering on the grid points (x_i, y_j) $(1 \leqslant i, j \leqslant n)$. Then get u^{k+1} and u^k as follows:

$$u^{k+1} = (u_1^{k+1}, u_2^{k+1}, \cdots, u_{n^2}^{k+1})^{\mathrm{T}},$$

$$u^k = (u_1^k, u_2^k, \cdots, u_{n^2}^k)^{\mathrm{T}}.$$

Suppose (x_{i_1}, y_{j_1}) and (x_{i_2}, y_{j_2}) are difference neighbors with each other in the graph G_1. Their numbers in the above ordering are respectively α_1 and α_2, so as non-diagonal elements of the matrix $H_1 = (h_{ij})_{n^2 \times n^2}$,

$$h_{\alpha_1\alpha_2} = h_{\alpha_2\alpha_1} = -1.$$

The diagonal elements $h_{\alpha_1\alpha_1}$ (or $h_{\alpha_2\alpha_2}$) $=2$ or 4. It can be seen that H_2 is non-negative definite. Similarly, H_2 is also non-negative definite. Hence, the result can be established in the same way as that in ref. [2]. This completes the proof.

Figure 2 (a)—(c) are respectively the difference graphs G_1 and G_2 of AGE method[1], Block ADI method[5] and ABE-I method[2]. When the number of the components of G_1 or G_2, i.e. p_1 or $p_2>1$, the method has the property of parallelism at the time level $(k+1)$ or at the time level $(k+2)$. The components can be designed in various structures which keep the symmetric difference relations among the grid points. We have found in fig. 2 that the component of a difference graph can be an isolated point, a simple path, a cycle[9] or the one having more complicated structures.

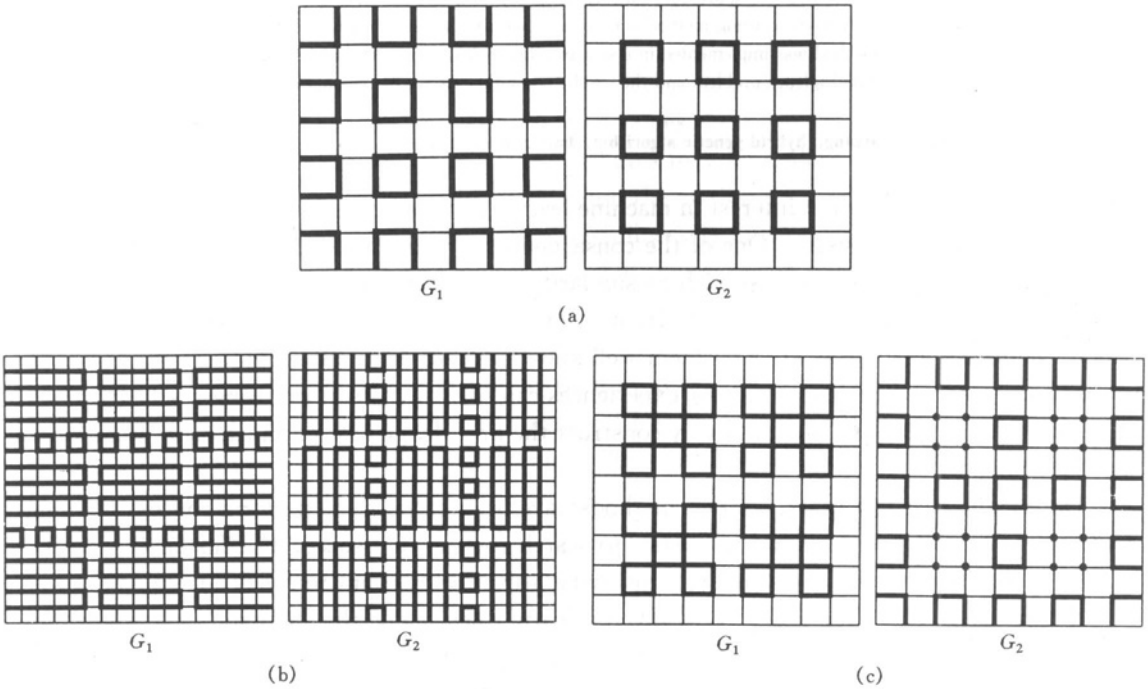

Fig. 2

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