弹性振动的闭环系统

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摘　　要

细长体的飞行器在飞行中考虑了既有刚性运动又有弹性振动的运动. 由于刚性运动对弹性振动的影响，通过安装在飞行器上面的仪表所测得的角速度作为反馈信号输入到控制器，由控制器输出端输出信号到执行机构来实现反馈控制. 把刚性运动飞行器、弹性振动飞行器同时考虑作受控对象. 这里我们研究了由刚性飞行器、弹性飞行器和控制器三者形成的闭环系统的弹性振动问题，得到了求闭环系统的频率和振型的公式. 设计控制器使得闭环系统渐近稳定的条件和能控性、能观测性的条件.

用两端自由梁来描述细长体的飞行器的弹性振动时，只考虑有惯性力和弹性力的自由弹性振动情况下的零阶振型代表飞行器的刚性运动. 即是本征值问题

$$\frac{d^2}{dx^2}\left(E(x)J(x)\frac{d^2\varphi}{dx^2}\right)+\lambda^2 m(x)\varphi(x)=0,\quad 0<x<l,$$

$$\varphi''(x)\Big|_{\substack{x=0\\x=l}}=0,\quad (E(x)J(x)\varphi''(x))'\Big|_{\substack{x=0\\x=l}}=0$$

（其中 $m(x)\geqslant m_0>0$, $E(x)J(x)\geqslant\alpha_0>0$），有本征值 $\lambda=0$ 和属于该本征值 λ 的本征函数是 $\varphi(x)\equiv 1$ 和 $\varphi(x)\equiv x$. 因此属于本征值 $\lambda=0$ 的本征子空间 $\mathfrak{H}_0$ 是 2 维的.

细长体的飞行器在推力、重力、控制力和气动力作用下飞行时的弹性振动方程，在本征子空间 $\mathfrak{H}_0$ 上投影时，就得到飞行器的质心运动方程和绕质心的转动方程的刚性运动方程. 因此，在下面的数学模型中，飞行器的刚性运动方程用常微分方程组来描述. 用两端自由弹性梁来描述飞行器的弹性振动. 用常微分方程组来描述控制器. 只考虑到飞行器刚性运动对弹性振动影响，弹性振动的角速度作反馈信号输入到控制器，由控制器输出信号到执行机构以实现反馈控制飞行器的刚性运动和弹性振动. 经过物理上的简化和数学上的处理，得到了刚性运动飞行器、弹性运动飞行器和控制器三者形成的闭环系统的数学模型. 即是分布参数系统同集中参数系统的耦合模型. 基于这种模型，得到了求闭环系统频率和振型的公式，镇定的条件和能控性、能观测性的条件.

一、数学模型

设 Φ 是 $r\times r$ 定常矩阵，H 是 $r\times n$ 定常矩阵. 飞行器的刚性运动的状态用 r 个参

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数来表示 $\boldsymbol{\phi}=(\phi_1,\phi_2,\cdots,\phi_r)$，飞行器的刚性运动用下面的 r 阶常微分方程来描述，

$$\frac{d\boldsymbol{\phi}}{dt}=\Phi\boldsymbol{\phi}+H\boldsymbol{z},\tag{1}$$

其中 $\boldsymbol{z}=(z_1,z_2,\cdots,z_n)$ 是 n 维列向量，它的分量表示控制器的输出量.

设 $b_i(x),g_j(x)\in L^2(0,l)\ (i=1,\cdots,r;\ j=1,\cdots,n)$. 考虑到飞行器刚性运动对弹性振动的影响和控制器输出量到执行机构对弹性振动的影响，细长体的飞行器的弹性振动方程为：

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(E(x)J(x)\frac{\partial^2 y}{\partial x^2}\right)=\sum_{i=1}^{r}b_i(x)\phi_i(t)+\sum_{i=1}^{n}g_i(x)z_i(t),$$

$$0<x<l,\quad t>0,$$

$$E(x)J(x)\frac{\partial^2 y}{\partial x^2}\bigg|_{x=0,l}=0,\quad \frac{\partial}{\partial x}\left(E(x)J(x)\frac{\partial^2 y}{\partial x^2}\right)\bigg|_{x=0,l}=0,$$

$$y(x,t)|_{t=0}=y_0(x),\qquad \frac{\partial}{\partial t}y(x,t)\bigg|_{t=0}=y_1(x).$$

现在考虑较上述方程更为一般的情况. 设 $\mathfrak{H}$ 是一 Hilbert 空间，A 是 $\mathfrak{H}$ 中自伴正定算子，$b_i,g_j\in\mathfrak{H}(i=1,\cdots,r;\ j=1,\cdots,n)$，$\boldsymbol{\phi}$ 是 r 维欧氏空间 E^r 中一向量，$\boldsymbol{z}$ 是 n 维欧氏空间 E^n 中一向量，用记号

$$\boldsymbol{b}\otimes\boldsymbol{\phi}=\sum_{i=1}^{r}b_i\phi_i\quad(\boldsymbol{b}=(b_1,b_2,\cdots,b_r)),$$

$$\boldsymbol{g}\otimes\boldsymbol{z}=\sum_{j=1}^{n}g_jz_j\quad(\boldsymbol{g}=(g_1,g_2,\cdots,g_n)).$$

考虑 $\mathfrak{H}$ 中的二阶演化方程

$$\frac{d^2y}{dt^2}+Ay=\boldsymbol{b}\otimes\boldsymbol{\phi}+\boldsymbol{g}\otimes\boldsymbol{z}.\tag{2}$$

令

$$\boldsymbol{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}A^{\frac12}y\\\dot{y}\end{pmatrix},\quad \mathscr{A}_0=\begin{pmatrix}0 & A^{\frac12}\\-A^{\frac12} & 0\end{pmatrix}$$

$$\mathscr{B}=\begin{pmatrix}0,&0,&\cdots,&0\\b_1,&b_2,&\cdots,&b_r\end{pmatrix},\quad \mathscr{G}=\begin{pmatrix}0,&0,&\cdots,&0\\g_1,&g_2,&\cdots,&g_n\end{pmatrix}.$$

于是上述二阶演化方程可写成 $\mathfrak{H}\times\mathfrak{H}$ 中的一阶演化方程

$$\frac{d\boldsymbol{v}}{dt}=\mathscr{A}_0\boldsymbol{v}+\mathscr{B}\boldsymbol{\phi}+\mathscr{G}\boldsymbol{z}.\tag{2'}$$

设 F 是 $n\times n$ 定常矩阵，K 是 $n\times r$ 定常矩阵，$\boldsymbol{k}$ 是 E^n 中一非零向量，定义算子 M：$M\boldsymbol{v}=\langle Pv_2,g_0\rangle\boldsymbol{k}$，这里 $0\neq g_0\in\mathfrak{H}$. 如果是前面的自由梁的弹性振动方程，算子

$$P=\frac{d}{dx},\quad M\boldsymbol{v}=\boldsymbol{k}\int_0^l\frac{\partial^2 y(x,t)}{\partial t\,\partial x}g_0(x)dx$$

表示飞行器的角速度的平均值. 考虑到飞行器弹性振动角速度、刚性运动角速度输入到控制器后，控制器的运动方程为：

$$\frac{d\boldsymbol{z}}{dt}=F\boldsymbol{z}+K\boldsymbol{\phi}+M\boldsymbol{v},\tag{3}$$

于是弹性振动闭环系统的方程为:

$$\left.\begin{aligned}\frac{d\boldsymbol{v}}{dt}&=\mathscr{A}_0\boldsymbol{v}+\mathscr{B}\boldsymbol{\phi}+\mathscr{G}\boldsymbol{z},\\ \frac{d\boldsymbol{\phi}}{dt}&=\Phi\boldsymbol{\phi}+H\boldsymbol{z},\\ \frac{d\boldsymbol{z}}{dt}&=M\boldsymbol{v}+K\boldsymbol{\phi}+F\boldsymbol{z}.\end{aligned}\right\}\tag{4}$$

这一闭环系统是分布参数同集中参数的耦合系统.

二、弹性振动闭环系统的频率和振型

令

$$\mathscr{A}_1=\begin{pmatrix}\mathscr{A}_0 & \mathscr{B} & \mathscr{G}\\ 0 & \Phi & H\\ M & K & F\end{pmatrix},\quad \boldsymbol{w}=\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{\phi}\\ \boldsymbol{z}\end{pmatrix}.$$

相应于闭环系统 (4) 的本征方程为:

$$\lambda\boldsymbol{w}=\mathscr{A}_1\boldsymbol{w}.\tag{5}$$

相应于方程 (1)—(3) 的本征方程为:

$$\left.\begin{aligned}&\lambda^2y+Ay=\boldsymbol{b}\otimes\boldsymbol{\phi}+\boldsymbol{g}\otimes\boldsymbol{z},\\ &\lambda\boldsymbol{\phi}=\Phi\boldsymbol{\phi}+H\boldsymbol{z},\\ &\lambda\boldsymbol{z}=\lambda My+K\boldsymbol{\phi}+F\boldsymbol{z}.\end{aligned}\right\}\tag{6}$$

于是可以证明

引理. 假设 A 是 $\mathfrak{H}$ 中自伴正定算子,如果 $(\lambda, y, \boldsymbol{\phi}, \boldsymbol{z})$ 是方程 (6) 的解,令

$$\boldsymbol{v}=\begin{pmatrix}A^{\frac{1}{2}}y\\ \lambda y\end{pmatrix},\quad \boldsymbol{w}=\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{\phi}\\ \boldsymbol{z}\end{pmatrix},$$

则 $(\lambda, \boldsymbol{w})$ 是方程 (5) 的解. 反之,如果 $(\lambda_0, \boldsymbol{w}_0)$ 是方程 (5) 的解,

$$\boldsymbol{w}_0=\begin{pmatrix}\boldsymbol{v}_0\\ \boldsymbol{\phi}_0\\ \boldsymbol{z}_0\end{pmatrix},\quad \boldsymbol{v}_0=\begin{pmatrix}v_0^1\\ v_0^2\end{pmatrix},$$

则 $(\lambda_0, A^{-\frac{1}{2}}v_0^1, \boldsymbol{\phi}_0, \boldsymbol{z}_0)$ 是方程 (6) 的解.

令

$$\boldsymbol{\chi}=\begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{z}\end{pmatrix},\quad \mathscr{F}=\begin{pmatrix}\Phi & H\\ K & F\end{pmatrix},\quad \mathscr{A}=\begin{pmatrix}\mathscr{A}_0 & 0\\ 0 & \mathscr{F}\end{pmatrix},$$

$$\mathscr{G}_1\boldsymbol{v}=\begin{pmatrix}0\\ M\boldsymbol{v}\end{pmatrix},\quad \mathscr{G}_2\boldsymbol{\chi}=\begin{pmatrix}0\\ \boldsymbol{b}\otimes\boldsymbol{\phi}+\boldsymbol{g}\otimes\boldsymbol{z}\end{pmatrix}=\mathscr{B}\boldsymbol{\phi}+\mathscr{G}\boldsymbol{z}.$$

对任一 $\lambda\in\rho(\mathscr{F})\cap\rho(F)\cap\rho(\Phi)$,

$$(\lambda I-\mathscr{F})^{-1}=\begin{pmatrix}D_1(\lambda) & D_2(\lambda)\\ D_3(\lambda) & D_4(\lambda)\end{pmatrix},$$

这里

$$D_1(\lambda)=[(\lambda I-\Phi)-H(\lambda I-F)^{-1}K]^{-1},$$
$$D_3(\lambda)=(\lambda I-F)^{-1}KD_1(\lambda),$$
$$D_4(\lambda)=[(\lambda I-F)-K(\lambda I-\Phi)^{-1}H]^{-1},$$
$$D_2(\lambda)=(\lambda I-\Phi)^{-1}HD_4(\lambda).$$

简记 $D_2(\lambda)=D_2$, $D_4(\lambda)=D_4$, 于是当 $\lambda\in\rho(\mathscr{A}_0)$ 时,

$$(\lambda I-\mathscr{A})^{-1}\begin{pmatrix}0\\ D_2\boldsymbol{k}\otimes\boldsymbol{b}+D_4\boldsymbol{k}\otimes\boldsymbol{g}\end{pmatrix}=\begin{pmatrix}A^{\frac12}R\boldsymbol{b}\otimes D_2\boldsymbol{k}+A^{\frac12}R\boldsymbol{g}\otimes D_4\boldsymbol{k}\\ \lambda R\boldsymbol{b}\otimes D_2\boldsymbol{k}+\lambda R\boldsymbol{g}\otimes D_4\boldsymbol{k}\end{pmatrix},$$

这里 $R=R(\lambda^2;-A)=(\lambda^2I+A)^{-1}$.

定理 1. 假设 A 是 Hilbert 空间 $\mathfrak{H}$ 中自伴正定算子, P 是 $\mathfrak{H}$ 中稠定线性算子,

$$\mathscr{D}(A^{\frac12})\subset\mathscr{D}(P),$$

如果 $\lambda\in\rho(\mathscr{A})\cap\rho(F)\cap\rho(\Phi)$, 并且

$$\lambda\langle PR\boldsymbol{b}\otimes D_2\boldsymbol{k},g_0\rangle+\lambda\langle PR\boldsymbol{g}\otimes D_4\boldsymbol{k},g_0\rangle\neq 1,$$

则 $\lambda\in\rho(\mathscr{A}_1)$. 如果 $\lambda\in\rho(\mathscr{A})\cap\rho(F)\cap\rho(\Phi)$, 并且

$$\lambda\langle PR\boldsymbol{b}\otimes D_2\boldsymbol{k},g_0\rangle+\lambda\langle PR\boldsymbol{g}\otimes D_4\boldsymbol{k},g_0\rangle=1,\tag{7}$$

则 $\lambda\in\sigma_p(\mathscr{A}_1)$. 并且相应于 $\mathscr{A}_1$ 的本征值 λ 的本征矢量是 $\boldsymbol{w}=\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{\chi}\end{pmatrix}$, 本征子空间是一维的. 这里

$$\boldsymbol{v}=\begin{pmatrix}A^{\frac12}R(\lambda^2;-A)\boldsymbol{b}\otimes D_2\boldsymbol{k}+A^{\frac12}R(\lambda^2;-A)\boldsymbol{g}\otimes D_4\boldsymbol{k}\\ \lambda R(\lambda^2;-A)\boldsymbol{b}\otimes D_2\boldsymbol{k}+\lambda R(\lambda^2;-A)\boldsymbol{g}\otimes D_4\boldsymbol{k}\end{pmatrix},\tag{8}$$

$$\boldsymbol{\phi}=D_2(\lambda)\boldsymbol{k},\tag{9}$$

$$\boldsymbol{z}=D_4(\lambda)\boldsymbol{k}.\tag{10}$$

从定理 1 知道，为了求弹性振动闭环系统的最小频率和相应的振型，只要从方程 (7) 中求解满足条件 $\lambda\in\rho(\mathscr{A})\cap\rho(F)\cap\rho(\Phi)$ 的虚部绝对值最小复根 λ, 与这一 λ 相应的振型依公式 (8) 为:

$$\varphi_\lambda=R(\lambda^2;-A)\boldsymbol{b}\otimes D_2(\lambda)\boldsymbol{k}+R(\lambda^2;-A)\boldsymbol{g}\otimes D_4(\lambda)\boldsymbol{k}.$$

证. 设 λ 不是方程 (7) 的根，我们来证明 $\lambda\in\rho(\mathscr{A}_1)$.

考虑算子

$$(\lambda I-\mathscr{A}_1)^*=(\bar\lambda I-\mathscr{A}_1^*)=\begin{pmatrix}\bar\lambda I-\mathscr{A}_0^* & -\mathscr{G}_1^*\\ -\mathscr{G}_2^* & \bar\lambda I-\mathscr{F}^*\end{pmatrix},$$

对任一 $\boldsymbol{f}_1\in\mathfrak{H}\times\mathfrak{H}$, $\boldsymbol{f}_2\in E^r\times E^n$, $\boldsymbol{f}_2=\begin{pmatrix}\boldsymbol{\phi}_2\\ \boldsymbol{z}_2\end{pmatrix}$, 解方程

$$(\lambda I-\mathscr{A}_1^*)\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{\chi}\end{pmatrix}=\begin{pmatrix}\boldsymbol{f}_1\\ \boldsymbol{f}_2\end{pmatrix},$$

即是

$$(\bar\lambda I-\mathscr{A}_0^*)\boldsymbol{v}-\mathscr{G}_1^*\boldsymbol{\chi}=\boldsymbol{f}_1,\tag{11}$$

$$-\mathscr{G}_2^*\boldsymbol{v}+(\bar\lambda I-\mathscr{F}^*)\boldsymbol{\chi}=\boldsymbol{f}_2.\tag{12}$$

因为 $\lambda\in\rho(\mathscr{A})\Longleftrightarrow\lambda\in\rho(\mathscr{A}_0)\cap\rho(\mathscr{F})$, 从 (11) 式解出 $\boldsymbol{v}$ 和从 (12) 式解出 $\boldsymbol{\chi}$, 得

$$\boldsymbol{v} = R^*(\lambda; \mathscr{A}_0)\boldsymbol{f}_1 + R^*(\lambda; \mathscr{A}_0)\mathscr{G}_1^*\boldsymbol{\chi}, \tag{13}$$

$$\boldsymbol{x} = \boldsymbol{f} + R^*(\lambda; \mathscr{F})\mathscr{G}_2^*R^*(\lambda; \mathscr{A}_0)\mathscr{G}_1^*\boldsymbol{\chi}, \tag{14}$$

这里

$$\boldsymbol{f} = R^*(\lambda; \mathscr{F})\boldsymbol{f}_2 + R^*(\lambda; \mathscr{F})\mathscr{G}_2^*R^*(\lambda; \mathscr{A}_0)\boldsymbol{f}_1.$$

但是对任一 $\begin{pmatrix}\boldsymbol{\chi}_1\\ \boldsymbol{\chi}_2\end{pmatrix} = \boldsymbol{\chi} \in E^r \times E^n$,

$$\mathscr{G}_1R(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda; \mathscr{F})\boldsymbol{\chi} = \begin{pmatrix} 0 \\ < \tilde{P}R(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda; \mathscr{F})\boldsymbol{\chi}, \boldsymbol{g}_0 > \boldsymbol{k}\end{pmatrix},$$

这里

$$\tilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & P\end{pmatrix}, \quad \boldsymbol{g}_0 = \begin{pmatrix} 0 \\ g_0\end{pmatrix}.$$

对任一

$$\begin{aligned}\boldsymbol{\phi} = \begin{pmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{pmatrix} \in E^r \times E^n \ &\langle \mathscr{G}_1R(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda;\mathscr{F})\boldsymbol{\chi}, \boldsymbol{\phi}\rangle \\ &= \langle PR(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda; \mathscr{F})\boldsymbol{\chi}, \boldsymbol{g}_0\rangle\langle \boldsymbol{k}, \boldsymbol{\phi}^2\rangle \\ &= \langle \boldsymbol{\chi}, R^*(\lambda; \mathscr{F})\mathscr{G}_2^*R^*(\lambda; \mathscr{A}_0)\mathscr{G}_1^*\boldsymbol{\phi}\rangle.\end{aligned}$$

因此

$$R^*(\lambda; \mathscr{F})\mathscr{G}_2^*R^*(\lambda; \mathscr{A}_0)\mathscr{G}_1^*\boldsymbol{\phi} = \langle \tilde{P}R(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda; \mathscr{F})\cdot, \boldsymbol{g}_0\rangle \times \langle \boldsymbol{k}, \boldsymbol{\phi}^2\rangle,$$

于是 (14) 式可写为:

$$\boldsymbol{\chi} = \boldsymbol{f} + \langle \tilde{P}R(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda; \mathscr{F})\cdot, \boldsymbol{g}_0\rangle\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle. \tag{15}$$

上式两端用 $\boldsymbol{\phi}_0 = \begin{pmatrix} 0\\ \boldsymbol{k}\end{pmatrix}$ 取内积,得

$$\begin{aligned}\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle &= \langle \boldsymbol{\phi}_0, \boldsymbol{f}\rangle + \langle \tilde{P}R(\lambda; \mathscr{A}_0)\mathscr{G}_2R(\lambda; \mathscr{F})\boldsymbol{\phi}_0, \boldsymbol{g}_0\rangle\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle \\ &= \langle \boldsymbol{\phi}_0, \boldsymbol{f}\rangle + [\lambda\langle PR\boldsymbol{b}\otimes D_2\boldsymbol{k}, g_0\rangle + \lambda\langle PR\boldsymbol{g}\otimes D_4\boldsymbol{k}, g_0\rangle]\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle \\ &= \langle \boldsymbol{\phi}_0, \boldsymbol{f}\rangle + W(\lambda)\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle.\end{aligned}$$

这里 $W(\lambda)$ 表示方程 (7) 的左端.

于是

$$[1 - W(\lambda)]\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle = \langle \boldsymbol{\phi}_0, \boldsymbol{f}\rangle,$$

因此

$$\langle \boldsymbol{k}, \boldsymbol{\chi}^2\rangle = \frac{1}{1 - W(\lambda)}\langle \boldsymbol{\phi}_0, \boldsymbol{f}\rangle,$$

代入 (15) 式得唯一解 $\boldsymbol{\chi}$, 再代入 (13) 式得唯一解 $\boldsymbol{v}$.

综上所述,如果 $\lambda \in \rho(F) \cap \rho(\Phi)$, $\lambda \in \rho(\mathscr{A})$, λ 不是方程 (7) 的根,则对任一

$$(\boldsymbol{f}_1, \boldsymbol{f}_2) \in \mathfrak{H} \times \mathfrak{H} \times E^r \times E^n$$

方程 (11), (12) 存在唯一解 $(\boldsymbol{\chi}, \boldsymbol{v})$, 即 $(\lambda I - \mathscr{A}_1)^*$ 的值域是全空间 $\mathfrak{H} \times \mathfrak{H} \times E^r \times E^n$. 于是 $(\lambda I - \mathscr{A}_1)$ 有逆,并且由 (13), (15) 式知 $(\boldsymbol{v}, \boldsymbol{\chi})$ 都连续依赖于 $\boldsymbol{f}$, 从而 $(\lambda I - \mathscr{A}_1)^{-1}$ 有界,故 $\lambda \in \rho(\mathscr{A}_1)$.

如果 λ 是方程 $W(\lambda)=1$ 的根．因为 $\lambda\in\rho(\mathscr{A})$,

$$(\lambda I-\mathscr{A}_1)\boldsymbol{w}=(\lambda I-\mathscr{A})[I-R(\lambda;\mathscr{A})T]\boldsymbol{w}.$$

如果 $\lambda\in\sigma_p(\mathscr{A}_1)$，且 $\boldsymbol{w}$ 是相应本征矢量 $\Longleftrightarrow$

$$\boldsymbol{w}=R(\lambda;\mathscr{A})T\boldsymbol{w},$$

这里

$$T=\begin{pmatrix}0 & \mathscr{G}_2\\ \mathscr{G}_1 & 0\end{pmatrix},$$

于是

$$\boldsymbol{v}=R(\lambda;\mathscr{A}_0)\begin{pmatrix}0\\ \boldsymbol{b}\otimes\boldsymbol{\phi}+\boldsymbol{g}\otimes\boldsymbol{z}\end{pmatrix},$$

$$\boldsymbol{\phi}=\langle P\boldsymbol{v}_2,\ g_0\rangle D_2(\lambda)\boldsymbol{k},$$

$$\boldsymbol{z}=\langle P\boldsymbol{v}_2,\ g_0\rangle D_4(\lambda)\boldsymbol{k}.$$

由此易知，如果 λ 是方程 $W(\lambda)=1$ 的根，则 $\lambda\in\sigma_p(\mathscr{A}_1)$，并且相应本征矢量是(8)—(10)式．而且相应于本征值 λ 的本征矢量是(8)—(10)式乘以同一数，故本征子空间是一维的．定理1证毕．

三、弹性振动的镇定

假设 A 是自伴正定离散算子，A 的每一本征值是单重的，以 $\{\varphi_n\}$ 表示 A 在 $\mathfrak{H}$ 中本征矢量全体．并且在 $\mathfrak{H}$ 中形成规格化直交基，以 $\{\mu_n\}$ 表算子 A 的本征值全体．

$$\mathscr{D}(A^{\frac{1}{2}})\subset\mathscr{D}(P),$$

令

$$w_m=\begin{cases}+\sqrt{\mu_m} & (m=1,2,\cdots)\\ -\sqrt{\mu_{-m}} & (m=-1,-2,\cdots),\end{cases}$$

$$\boldsymbol{\varphi}_m=\frac{1}{\sqrt{2}}\begin{pmatrix}i\varepsilon_m\varphi_m\\ \varphi_m\end{pmatrix}\quad(m=\pm1,\pm2,\cdots),$$

其中

$$\varphi_m=\varphi_{-m},\quad \mu_m=\mu_{-m}\quad(m=-1,-2,\cdots),$$

$$\varepsilon_m=\begin{cases}+1 & (m=-1,-2,\cdots),\\ -1 & (m=1,2,\cdots),\end{cases}$$

则 $\nu_m=iw_m$ 和 $\boldsymbol{\varphi}_m(m=\pm1,\pm2,\cdots)$ 是 $\mathscr{A}_0$ 的本征值和相应本征矢量全体．

设 $(r+n)\times(r+n)$ 矩阵 $\mathscr{F}$ 具有 $r+n$ 个非零本征值，

$$(r_i,\boldsymbol{\chi}_i),\quad \boldsymbol{\chi}_i=\begin{pmatrix}\boldsymbol{\phi}_i\\ \boldsymbol{z}_i\end{pmatrix}\quad(i=1,2,\cdots,r+n)$$

是 $\mathscr{F}$ 的本征值和本征矢量的全体．$\mathscr{F}^*$ 表示矩阵 $\mathscr{F}$ 的转置矩阵．

$$(r_i,\boldsymbol{\phi}_i),\quad \boldsymbol{\phi}_i=\begin{pmatrix}\boldsymbol{\phi}_i^1\\ \boldsymbol{\phi}_i^2\end{pmatrix}$$

表示矩阵 $\mathscr{F}^*$ 的本征值和相应本征矢量全体．令

$$\langle\boldsymbol{\phi}_i,\boldsymbol{\chi}_i\rangle=1\quad(i=1,2,\cdots,r+n).$$

令

$$\lambda_k = \begin{cases} r_k & (k = 1, 2, \cdots, r+n), \\ \nu_k & (k = \pm 1, \pm 2, \cdots), \end{cases}$$

$$\boldsymbol{v}_k = \begin{cases} 0 & (k = 1, 2, \cdots, r+n), \\ \boldsymbol{\varphi}_k & (k = \pm 1, \pm 2, \cdots), \end{cases}$$

$$\boldsymbol{y}_k = \begin{cases} \boldsymbol{\chi}_k & (k = 1, 2, \cdots, r+n), \\ 0 & (k = \pm 1, \pm 2, \cdots), \end{cases}$$

$$\boldsymbol{w}_k = \begin{pmatrix} \boldsymbol{v}_k \\ \boldsymbol{y}_k \end{pmatrix},$$

则

$$(\lambda_k, \boldsymbol{w}_k) \quad (k = 1, 2, \cdots, r+n; \pm 1, \pm 2, \cdots)$$

是算子

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_0 & 0 \\ 0 & \mathscr{F} \end{pmatrix}$$

的本征值和本征矢量的全体.

定理 2. 假定 $\sigma_p(\mathscr{A}_0) \cap \sigma(F) = \varnothing$, $\sigma_p(\mathscr{A}_0) \cap \sigma(\Phi) = \varnothing$, $\sigma_p(\mathscr{A}_0) \cap \sigma(\mathscr{F}) = \varnothing$, 这里 $\varnothing$ 表示空集.

(1) $\left\langle \tilde{P}(r_k I - \mathscr{A}_0)^{-1} \begin{pmatrix} 0 \\ \boldsymbol{b} \otimes \boldsymbol{\phi}_k + \boldsymbol{g} \otimes \boldsymbol{z}_k \end{pmatrix}, \boldsymbol{g}_0 \right\rangle \neq 0 \ (k = 1, 2, \cdots, r+n)$,

(2) $\langle P\varphi_k, g_0 \rangle \neq 0 \ (k = 1, 2, \cdots)$,

(3) $\langle \boldsymbol{k}, \boldsymbol{\phi}_k^2 \rangle \neq 0 \ (k = 1, 2, \cdots, r+n)$,

(4) $\langle \boldsymbol{b} \otimes D_2(\nu_k)\boldsymbol{k} + \boldsymbol{g} \otimes D_4(\nu_k)\boldsymbol{k}, \varphi_k \rangle \neq 0 \ (k = \pm 1, \pm 2, \cdots)$, 则 $\sigma_p(\mathscr{A}) \cap \sigma_p(\mathscr{A}_1) = \varnothing$.

证. 若是 $\sigma_p(\mathscr{A}) \cap \sigma_p(\mathscr{A}_1) \neq \varnothing$, 则有 $\lambda_k \in \sigma_p(\mathscr{A}) \cap \sigma_p(\mathscr{A}_1)$, 于是仅有下面两种可能情况:

1) $(\lambda_k, \boldsymbol{w}_k)$ 是 $\mathscr{A}_1$ 的本征值和相应的本征矢量.

2) 存在 $\boldsymbol{u}_k \in \mathscr{D}(\mathscr{A})$, $\boldsymbol{u}_k \neq 0$, 使得 $(\lambda_k, \boldsymbol{u}_k)$ 是 $\mathscr{A}_1$ 的本征值和相应的本征矢量.

第 1) 种情况是

$$\mathscr{A}_1 \boldsymbol{w}_k = \lambda_k \boldsymbol{w}_k \text{ 和 } \mathscr{A} \boldsymbol{w}_k = \lambda_k \boldsymbol{w}_k,$$

因此

$$0 = \begin{pmatrix} 0 & \mathscr{G}_2 \\ \mathscr{G}_1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_k \\ \boldsymbol{y}_k \end{pmatrix} = \begin{pmatrix} \mathscr{G}_2 \boldsymbol{y}_k \\ \mathscr{G}_1 \boldsymbol{v}_k \end{pmatrix}.$$

因此, $\mathscr{G}_2 \boldsymbol{y}_k = 0$, $\mathscr{G}_1 \boldsymbol{v}_k = 0$. 如果 $k = 1, 2, \cdots, r+n$, 则 $\boldsymbol{v}_k = 0$, $\boldsymbol{y}_k = \boldsymbol{\chi}_k$, $0 = \mathscr{G}_2 \boldsymbol{y}_k = \begin{pmatrix} 0 \\ \boldsymbol{b} \otimes \boldsymbol{\phi}_k + \boldsymbol{g} \otimes \boldsymbol{z}_k \end{pmatrix}$, 故 $\boldsymbol{b} \otimes \boldsymbol{\phi}_k + \boldsymbol{g} \otimes \boldsymbol{z}_k = 0$ 这同定理 2 的假设 (1)矛盾.

如果 $k = \pm 1, \pm 2, \cdots$, 则

$$\boldsymbol{y}_k = 0, \quad \boldsymbol{v}_k = \boldsymbol{\varphi}_k, \quad 0 = \mathscr{G}_1 \boldsymbol{v}_k = \mathscr{G}_1 \boldsymbol{\varphi}_k = \frac{1}{\sqrt{2}} \langle P\varphi_k, g_0 \rangle \boldsymbol{k},$$

因为 $\boldsymbol{k} \neq 0$, 故 $\langle P\varphi_k, g_0\rangle = 0$ 这同条件 (2) 矛盾. 因此第 1) 种情况不能出现.

第 2) 种情况是: 令 $\boldsymbol{u}_k = \begin{pmatrix} \boldsymbol{u}_k^1 \\ \boldsymbol{u}_k^2 \end{pmatrix}$,

$$\mathscr{A}_0\boldsymbol{u}_k^1 + \mathscr{G}_2\boldsymbol{u}_k^2 = \lambda_k\boldsymbol{u}_k^1, \tag{16}$$

$$\mathscr{G}_1\boldsymbol{u}_k^1 + \mathscr{F}\boldsymbol{u}_k^2 = \lambda_k\boldsymbol{u}_k^2. \tag{17}$$

如果 $k = 1, 2, \cdots, r+n$, 则 $\lambda_k = r_k$, 用 $\boldsymbol{\phi}_k$ 于 (17) 式两端取内积得:

$$\begin{aligned} r_k\langle \boldsymbol{u}_k^2, \boldsymbol{\phi}_k\rangle &= \langle \mathscr{G}_1\boldsymbol{u}_k^1, \boldsymbol{\phi}_k\rangle + \langle \mathscr{F}\boldsymbol{u}_k^2, \boldsymbol{\phi}_k\rangle \\ &= \langle \mathscr{G}_1\boldsymbol{u}_k^1, \boldsymbol{\phi}_k\rangle + r_k\langle \boldsymbol{u}_k^2, \boldsymbol{\phi}_k\rangle. \end{aligned}$$

因此 $\langle \mathscr{G}_1\boldsymbol{u}_k^1, \boldsymbol{\phi}_k\rangle = 0$, 令 $\boldsymbol{u}_k^1 = \begin{pmatrix} \boldsymbol{u}_k^{1,1} \\ \boldsymbol{u}_k^{1,2} \end{pmatrix}$. 但是

$$\langle \mathscr{G}_1\boldsymbol{u}_k^1, \boldsymbol{\phi}_k\rangle = \langle P\boldsymbol{u}_k^{1,2}, g_0\rangle\langle \boldsymbol{k}, \boldsymbol{\phi}_k^2\rangle,$$

由条件 (3) 式得 $\langle P\boldsymbol{u}_k^{1,2}, g_0\rangle = 0$, 因此 $\mathscr{G}_1\boldsymbol{u}_k^1 = 0$. 由方程 (17) 得 $\mathscr{F}\boldsymbol{u}_k^2 = r_k\boldsymbol{u}_k^2$, 于是 $\boldsymbol{u}_k^2 = \boldsymbol{\chi}_k$. 由方程 (16) 解出 $\boldsymbol{u}_k^1$, 得

$$\boldsymbol{u}_k^1 = R(\lambda_k; \mathscr{A}_0)\mathscr{G}_2\boldsymbol{u}_k^2 = R(r_k; \mathscr{A}_0)\begin{pmatrix} 0 \\ \boldsymbol{b}\otimes\boldsymbol{\phi}_k + \boldsymbol{g}\otimes\boldsymbol{z}_k \end{pmatrix},$$

因此,

$$0 = \langle P\boldsymbol{u}_k^{1,2}, g_0\rangle = \langle \tilde{P}R(r_k; \mathscr{A}_0)\begin{pmatrix} 0 \\ \boldsymbol{b}\otimes\boldsymbol{\phi}_k + \boldsymbol{g}\otimes\boldsymbol{z} \end{pmatrix}, \boldsymbol{g}_0\rangle$$

这同条件 (1) 矛盾.

如果 $k = \pm 1, \pm 2, \cdots$, $\lambda_k = \nu_k$, 用 $\boldsymbol{\varphi}_k$ 于 (16) 式两端取内积, 得

$$\langle \mathscr{A}_0\boldsymbol{u}_k^1, \boldsymbol{\varphi}_k\rangle + \langle \mathscr{G}_2\boldsymbol{u}_k^2, \boldsymbol{\varphi}_k\rangle = \nu_k\langle \boldsymbol{u}_k^1, \boldsymbol{\varphi}_k\rangle,$$

故 $\langle \mathscr{G}_2\boldsymbol{u}_k^2, \boldsymbol{\varphi}_k\rangle = 0$, 因此

$$0 = \langle \mathscr{G}_2\boldsymbol{u}_k^2, \boldsymbol{\varphi}_k\rangle = \langle \boldsymbol{b}\otimes\boldsymbol{u}_k^{2,1} + \boldsymbol{g}\otimes\boldsymbol{u}_k^{2,2}, \varphi_k\rangle. \tag{18}$$

从方程 (17) 解出 $\boldsymbol{u}_k^2$ 得

$$\boldsymbol{u}_k^2 = R(\nu_k; \mathscr{F})\mathscr{G}_1\boldsymbol{u}_k^1 = \langle P\boldsymbol{u}_k^{1,2}, g_0\rangle\begin{pmatrix} D_2(\nu_k)\boldsymbol{k} \\ D_4(\nu_k)\boldsymbol{k} \end{pmatrix},$$

代入 (18) 式, 得

$$\langle P\boldsymbol{u}_k^{1,2}, g_0\rangle\langle \boldsymbol{b}\otimes D_2(\nu_k)\boldsymbol{k} + \boldsymbol{g}\otimes D_4(\nu_k)\boldsymbol{k}, \varphi_k\rangle = 0,$$

从条件 (4) 式得

$$\langle P\boldsymbol{u}_k^{1,2}, g_0\rangle = 0. \tag{19}$$

因此, $\boldsymbol{u}_k^2 = 0$, 由方程 (16) 式得

$$\mathscr{A}_0\boldsymbol{u}_k^1 = \boldsymbol{\chi}_k\boldsymbol{u}_k^1 = \nu_k\boldsymbol{u}_k^1,$$

故

$$\boldsymbol{u}_k^1 = \begin{pmatrix} \boldsymbol{u}_k^{1,1} \\ \boldsymbol{u}_k^{1,2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i\varepsilon_k\varphi_k \\ \varphi_k \end{pmatrix},$$

把

$$\boldsymbol{u}_k^{1,2} = \frac{1}{\sqrt{2}}\varphi_k,$$

代入 (19) 式得 $\langle P_{\varphi_k}, g_0\rangle = 0$，这同定理 2 的条件 (2) 矛盾．定理 2 证毕．

依定理 1 和定理 2 容易证明

定理 3．假设在满足定理 2 的条件下，$\lambda \notin \sigma(F)$，$\lambda \notin \sigma(\Phi)$，则 $\lambda \in \sigma_p(\mathscr{A}_1)$ 必须且只须 λ 是下面方程的根，

$$\lambda\langle PR\boldsymbol{b}, g_0\rangle \otimes D_2\boldsymbol{k} + \lambda\langle PR\boldsymbol{g}, g_0\rangle \otimes D_4\boldsymbol{k} = 1.$$

在方程 (2) 的右端，在工程上，不仅 $\boldsymbol{b} \otimes \boldsymbol{\phi}$ 这项反映飞行器的刚性运动对弹性振动的影响，而且 $\boldsymbol{g} \otimes \boldsymbol{z}$ 这项也反映一部分刚性运动对弹性振动的影响．考虑一种特殊的情况，即令 $b_1 = b_2 = \cdots = b_r = 0$，$g_2 = g_3 = \cdots = g_n = 0$，$g_1 \neq 0$．对于这种情况有下面的镇定方面定理．

定理 4．在定理 3 的条件下，再设 $\mathscr{F}$ 的每一个本征值 $r_i(i = 1, 2, \cdots, r+n)$ 是负实数．$\langle g_1, \varphi_l\rangle\langle P\varphi_l, g_0\rangle > 0(l = 1, 2, \cdots)$，$z_j^1\langle \boldsymbol{k}, \boldsymbol{\phi}_j^2\rangle < 0(j = 1, 2, \cdots, r + n)$，这里 z_j^1 表示 $\boldsymbol{z}_j$ 的第一个分量，则每一 $\lambda \in \sigma_p(\mathscr{A}_1)$ 具有负实部．并且如果 $\lambda \in \sigma_p(\mathscr{A}_1)$，则 $\bar{\lambda} \in \sigma_p(\mathscr{A}_1)$．

证．依定理 3，$\lambda \in \sigma_p(\mathscr{A}_1)$ 必须且只须 λ 是下面方程的根，

$$1 = \sum_{j=1}^{r+n} \frac{\langle \boldsymbol{k}, \boldsymbol{\phi}_j^2\rangle}{\lambda - r_j} z_j^1 \sum_{l \in J_2} \frac{\langle g_1, \varphi_l\rangle\langle P\varphi_l, g_0\rangle}{\lambda - \nu_l}. \tag{20}$$

令

$$J_1 = \{1, 2, \cdots, r + n\}, \quad J_2 = \{\pm 1, \pm 2, \cdots\},$$

$$\alpha_j = \frac{\langle \boldsymbol{k}, \boldsymbol{\phi}_j^2\rangle}{|\lambda - r_j|^2} z_j^1 < 0 \quad (j \in J_1),$$

$$\beta_l = \frac{\langle g_1, \varphi_l\rangle\langle P\varphi_l, g_0\rangle}{|\lambda - \nu_l|^2} > 0 \quad (l \in J_2).$$

于是方程 (20) 可写成：

$$\sum_{j \in J_1} \alpha_j(\bar{\lambda} - r_j) \sum_{l \in J_2} \beta_l(\bar{\lambda} - \bar{\nu}_l) = 1. \tag{21}$$

令

$$\lambda = \lambda_1 + i\lambda_2,$$

$$a_1 = \Big(\sum_{j \in J_1} \alpha_j\Big)\Big(\sum_{l \in J_2} \beta_l\Big), \quad a_2 = \Big(\sum_{j \in J_1} \alpha_j r_j\Big)\Big(\sum_{l \in J_2} \beta_l\Big),$$

$$b_1 = \Big(\sum_{j \in J_1} \alpha_j\Big), \qquad b_2 = \Big(\sum_{j \in J_1} \alpha_j r_j\Big).$$

于是方程 (21) 可写成两个实方程，

$$a_1\lambda_1^2 - a_2\lambda_1 - a_1\lambda_2^2 + \Big(\sum_{l \in J} \beta_l\omega_l\Big) b_1\lambda_2 - 1 = 0, \tag{22}$$

$$2a_1\lambda_1\lambda_2 - a_2\lambda_2 + [b_2 - b_1\lambda_1]\Big(\sum_{l \in J_l} \beta\omega_{ll}\Big) = 0. \tag{23}$$

如果

$$b_2 - b_1\lambda_1 = 0 \Rightarrow \lambda_1 = \frac{b_2}{b_1} < 0.$$

如果 $b_2 - b_1\lambda_1 \neq 0$，从方程(23)解出 $\left(\sum_{l\in J_2}\beta_l\omega_l\right)$ 代入方程(22)，经整理后，得

$$-a_1b_1\lambda_1^3 + (a_1b_2 + a_2b_1)\lambda_1^2 + (b_1 - a_1b_1\lambda_2^2 - a_2b_2)\lambda_1 - b_2 = 0,$$

由此方程知，$\mathrm{Re}\,\lambda = \lambda_1 < 0$.

如果 $\lambda \in \sigma_p(\mathscr{A}_1)$，$\lambda$ 是方程(20)的根，于是

$$\sum_{j=1}^{r+n}\frac{\langle \boldsymbol{k}, \boldsymbol{\phi}_j^2\rangle}{\bar\lambda - r_j} z_j^1 \sum_{l\in J_2}\frac{\langle g_1, \varphi_l\rangle\langle P\varphi_l, g_0\rangle}{\bar\lambda - \nu_l}$$

$$= \overline{\sum_{j=1}^{r+n}\frac{\langle \boldsymbol{k}, \boldsymbol{\phi}_j^2\rangle}{\lambda - r_j} z_j^1 \sum_{l\in J_2}\frac{\langle g_1, \varphi_l\rangle\langle P\varphi_l, g_0\rangle}{\lambda - \nu_l}} = 1.$$

依定理3，$\bar\lambda \in \sigma_p(\mathscr{A}_1)$. 定理2证毕.

四、能控性、能观测性

在这里讨论细长体的飞行器，既有刚性运动又有弹性振动，把飞行器的刚性体和弹性体都作为受控对象，考虑到刚性运动对弹性振动的影响，飞行器的角速度作为反馈信号输入到控制器这样一个系统的能控性和能观测性问题. 这一系统描述为：

$$\left.\begin{aligned}
&\frac{d^2y}{dt} + Ay = \boldsymbol{b}\otimes\boldsymbol{\phi} + g_1u(t),\\
&\frac{d\boldsymbol{\phi}}{dt} = \Phi\boldsymbol{\phi} + Hu(t),\\
&\frac{d\boldsymbol{z}}{dt} = F\boldsymbol{z} + K\boldsymbol{\phi} + \langle P\dot y, g_0\rangle\boldsymbol{k},
\end{aligned}\right\}\tag{24}$$

这里 H 是 r 维列向量. $u(t)$ 是数值函数是控制量.

量测方程是

$$m(t) = \langle P\dot y, g_0\rangle + \langle\boldsymbol{\phi}(t), \boldsymbol{h}\rangle.$$

令

$$\boldsymbol{v} = \begin{pmatrix} A^{\frac12}y \\ \dot y\end{pmatrix},\quad M\boldsymbol{v} = \langle Pv_2, g_0\rangle\boldsymbol{k},\quad \mathscr{B}_0\boldsymbol{\phi} = \begin{pmatrix} 0 \\ \boldsymbol{b}\otimes\boldsymbol{\phi}\end{pmatrix},\quad \boldsymbol{g}_1 = \begin{pmatrix} 0 \\ g_1\end{pmatrix},$$

$$\mathscr{A}_2 = \begin{pmatrix} \mathscr{A}_0 & \mathscr{B}_0 & 0 \\ 0 & \Phi & 0 \\ M & K & F\end{pmatrix},$$

$$\mathscr{B}u(t) = \begin{pmatrix} \boldsymbol{g}_1 \\ H \\ 0\end{pmatrix}u(t),\quad \boldsymbol{w} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\phi} \\ \boldsymbol{z}\end{pmatrix}.$$

P 是由 $\mathfrak{H}\times\mathfrak{H}\times E^r\times E^n$ 到 $\mathfrak{H}\times E^r\times E^1$ 的投影算子，即 $(v_1, v_2, \boldsymbol{\phi}, \boldsymbol{z})\in\mathfrak{H}\times\mathfrak{H}\times E^r\times E^n$，$P(v_1, v_2, \boldsymbol{\phi}, \boldsymbol{z}) = (v_1, \boldsymbol{\phi}, z_1)$. 这里 z_1 是 $\boldsymbol{z}$ 的第一个分量.

$$C^1 \triangleq \{u(t)\,|\,\exists t_1 > 0,\ u(t)\in C^1[0, t_1]\},$$

依线性算子半群理论，对任一 $\boldsymbol{w}_0\in\mathscr{D}(\mathscr{A}_2)$，方程(24)的解可写为：

$$\boldsymbol{w}(t) = e^{t\mathscr{A}_2}\boldsymbol{w}_0 + \int_0^t e^{(t-\tau)\mathscr{A}_2}\mathscr{B}u(\tau)d\tau.$$

令

$$L_t u = \int_0^t e^{(t-\tau)\mathscr{A}_2} \mathscr{B} u(\tau) d\tau,$$

$$z(t) = \{PL_t u \mid u \in C^1\}.$$

定义 1. 如果 $\overline{\left(\bigcup_{t\geq 0} z(t)\right)} = \mathfrak{H} \times E^r \times E^1$ 称系统 (24) 完全能控.

以 $\tau_i(i = 1, 2, \cdots, r)$ 表示 $r \times r$ 矩阵 Φ 的 r 个不同的本征值，φ_i 表示相应于 τ_i 的本征矢量. $(\bar{\tau}_i, \psi_i)$ 表示矩阵 Φ 的转置矩阵 Φ^* 的本征值和相应本征矢量. $s_i(i = 1, 2, \cdots, n)$ 表示 $n \times n$ 矩阵 F 的 n 个不同本征值. f_i 是相应于本征值 s_i 的本征矢量，$(\bar{s}_i, F_i)$ 表示矩阵 F 的转置矩阵 F^* 的本征值和相应本征矢量.

定理 5. 假设 $\sigma_p(\mathscr{A}_0) \cap \sigma(F) = \emptyset$, $\sigma_p(\mathscr{A}_0) \cap \sigma(\Phi) = \emptyset$, $\sigma(F) \cap \sigma(\Phi) = \emptyset$,

(1) $\langle \psi_i, H\rangle \neq 0 \ (i = 1, 2, \cdots, r)$,

(2) $\langle g_1 + \boldsymbol{b} \otimes R(\nu_l; \Phi)H, \varphi_l\rangle \neq 0 \ (l = \pm 1, \pm 2, \cdots)$,

(3) $\langle M R(s_i; \mathscr{A}_0)\boldsymbol{g}_1 + MR(s_i; \mathscr{A}_0)\mathscr{B}_0 R(s_i; \Phi)H + KR(s_i; \Phi)H, F_i\rangle \neq 0 \ (i = 1, 2, \cdots, n)$,

则系统 (24) 完全能控.

证. 任一 $\boldsymbol{u}^* \in \mathfrak{H} \times E^r \times E^1$,

$$\langle PL_t u, \boldsymbol{u}^*\rangle = \int_0^t \langle Pe^{(t-\tau)\mathscr{A}_2} \mathscr{B} u(\tau), \boldsymbol{u}^*\rangle d\tau = \int_0^t u(\tau) \mathscr{B}^* e^{(t-\tau)\mathscr{A}_2^*} P^* \boldsymbol{u}^* d\tau$$
$$= 0, \quad \forall u \in C^1, \quad t > 0.$$

系统 (24) 完全能控 $\Longleftrightarrow \boldsymbol{u}^* = 0$.

由

$$\mathscr{B}^* e^{t\mathscr{A}_2^*} P^* \boldsymbol{u}^* = 0, \quad \forall t > 0 \tag{25}$$

要证 $\boldsymbol{u}^* = 0$.

$$\lambda \in \rho(\mathscr{A}_0^*) \cap \rho(\Phi^*) \cap \rho(F^*) \Rightarrow \lambda \in \rho(\mathscr{A}_2^*),$$

并且

$$R(\lambda; \mathscr{A}_2^*) = \begin{pmatrix} R(\lambda; \mathscr{A}_0^*) & 0 & R(\lambda; \mathscr{A}_0^*)M^*R(\lambda; F^*) \\ R(\lambda; \Phi^*)\mathscr{B}_0^* R(\lambda; \mathscr{A}_0^*) & R(\lambda; \Phi^*) & R(\lambda; \Phi^*)\mathscr{B}_0^* R(\lambda; \mathscr{A}_0^*)M^*R(\lambda; F^*) + R(\lambda; \Phi^*)\bar{K}^* R(\lambda; F^*) \\ 0 & 0 & R(\lambda; F^*) \end{pmatrix}.$$

由 (25) 式及半群公式得

$$\mathscr{B}^* R(\lambda; \mathscr{A}_2^*) P^* \boldsymbol{u}^* = 0, \quad \forall \lambda \in \rho(\mathscr{A}_2^*).$$

令

$$\boldsymbol{u}^* = \begin{pmatrix} v_1^* \\ \boldsymbol{\phi}^* \\ z_1^* \end{pmatrix}, \quad P^* \boldsymbol{u}^* = \begin{pmatrix} \boldsymbol{v}^* \\ \boldsymbol{\phi}^* \\ \boldsymbol{z}^* \end{pmatrix},$$

$$\begin{aligned} 0 = \mathscr{B}^* R(\lambda; \mathscr{A}_2^*) P^* \boldsymbol{u}^* &= \langle \boldsymbol{g}_1, R(\lambda; \mathscr{A}_0^*)\boldsymbol{v}^* + R(\lambda; \mathscr{A}_0^*)M^*R(\lambda; F^*)\boldsymbol{z}^*\rangle \\ &+ \langle H, R(\lambda; \Phi^*)\mathscr{B}_0^* R(\lambda; \mathscr{A}_0^*)\boldsymbol{v}^*\rangle + \langle H, R(\lambda; \Phi^*)\boldsymbol{\phi}^*\rangle \\ &+ \langle H, R(\lambda; \Phi^*)\mathscr{B}_0^* R(\lambda; \mathscr{A}_0^*)M^*R(\lambda; F^*)\boldsymbol{z}^*\rangle \\ &+ \langle H, R(\lambda; \Phi^*)K^*R(\lambda; F^*)\boldsymbol{z}^*\rangle. \end{aligned} \tag{26}$$

以 C_i 表示以 s_i 为心的足够小的圆，使得在 C_i 内不包含 F^* 的其余本征值和 $\mathscr{A}_0^*$，Φ^* 的任何本征值，从 (26) 式得

$$0=\int_{C_i}\mathscr{B}^*R(\lambda;\mathscr{A}_2^*)P^*\boldsymbol{u}^*d\lambda=2\pi i\langle \boldsymbol{z}^*,f_i\rangle\langle MR(s_i;\mathscr{A}_0)\boldsymbol{g}_1+MR(s_i;\mathscr{A}_0)\mathscr{B}_0R(s_i;\Phi)H+KR(s_i;\Phi)H,F_i\rangle\quad(i=1,2,\cdots,n).$$

由条件 (3) 式得 $\langle \boldsymbol{z}^*,f_i\rangle=0(i=1,\cdots n)$，故 $z_1^*=0$.

以 C_l 表示以 $\bar{\nu}_l$ 为心的足够小的圆，使得在 C_l 内不包含 $\mathscr{A}_0^*$ 的其余本征值和 Φ^*，F^* 的任何本征值，由上面已证 $\boldsymbol{z}^*=0$，从 (26) 式得

$$0=\int_{C_l}\langle \boldsymbol{g}_1,R(\lambda;\mathscr{A}_0^*)\boldsymbol{v}^*\rangle d\lambda+\int_{C_l}\langle H,R(\lambda;\Phi^*)\mathscr{B}_0^*R(\lambda;\mathscr{A}_0^*)\boldsymbol{v}^*\rangle d\lambda$$
$$=2\pi i\langle \boldsymbol{\varphi}_l,\boldsymbol{v}^*\rangle\langle g_1+\boldsymbol{b}\otimes R(\nu_l;\Phi)H,\varphi_l\rangle\quad(l=\pm1,\pm2,\cdots).$$

由条件 (2) 得 $\langle \boldsymbol{\varphi}_l,\boldsymbol{v}^*\rangle=0(l=\pm1,\pm2,\cdots)$，故 $v_1^*=0$.

类似可证 $\boldsymbol{\phi}^*=0$，因此 $\boldsymbol{u}^*=0$. 因此系统 (24) 式完全能控.

定义 2. 系统 (24) 称为完全能观测是指对每一 $\boldsymbol{w}_0\in\mathscr{D}(\mathscr{A}_2)$. 从

$$M_0\boldsymbol{w}(t)=M_0e^{t\mathscr{A}_2}\boldsymbol{w}_0=0,\quad \forall t\geqslant0\Rightarrow \boldsymbol{w}_0=0.$$

这里 M_0 是泛函:

$$M_0=(\langle P\cdot,\boldsymbol{g}_0\rangle,\boldsymbol{h},0)M_0\boldsymbol{w}=\langle Pv_2,g_0\rangle+\langle \boldsymbol{h},\boldsymbol{\phi}\rangle.$$

定理 6. 假设 $\sigma_p(\mathscr{A}_0)\cap\sigma(\Phi)=\emptyset$，如果

(1) $\langle P\varphi_l,g_0\rangle\neq0\ (l=1,2,\cdots)$,

(2) $\langle PR(\tau_j,\mathscr{A}_0)\mathscr{B}_0\varphi_j,\boldsymbol{g}_0\rangle+\langle\varphi_j,\boldsymbol{h}\rangle\neq0\ (j=1,\cdots,r)$，则系统 (24) 完全能观测.

证. 依线性算子半群公式

$$R(\lambda;\mathscr{A}_2)\boldsymbol{w}_0=\int_0^\infty e^{-\lambda t}e^{t\mathscr{A}_2}\boldsymbol{w}_0d\lambda.\qquad \forall\,\mathrm{Re}\,\lambda>w_0.$$

因此

$$M_0R(\lambda;\mathscr{A}_2)\boldsymbol{w}_0=\int_0^\infty e^{-\lambda t}M_0e^{t\mathscr{A}_2}\boldsymbol{w}_0d\lambda.$$

于是系统 (24) 式完全能观测必须且只须

$$M_0R(\lambda;\mathscr{A}_2)\boldsymbol{w}_0=0,\quad \forall\lambda\in\rho(\mathscr{A}_2)\Rightarrow\boldsymbol{w}_0=0.$$

对 $\lambda\in\rho(\mathscr{A}_0)\cap\rho(\Phi)\cap\rho(F)\Rightarrow\lambda\in\rho(\mathscr{A}_2)$，并且

$$R(\lambda;\mathscr{A}_2)=\begin{pmatrix}R(\lambda;\mathscr{A}_0) & R(\lambda;\mathscr{A}_0)\mathscr{B}_0R(\lambda;\Phi) & 0\\ 0 & R(\lambda;\Phi) & 0\\ R(\lambda;F)MR(\lambda;\mathscr{A}_0) & R(\lambda;F)MR(\lambda;\mathscr{A}_0)\mathscr{B}_0R(\lambda;\Phi)+R(\lambda;F)KR(\lambda;\Phi) & R(\lambda;F)\end{pmatrix}.$$

令

$$\boldsymbol{w}_0=\begin{pmatrix}\boldsymbol{v}_0\\ \boldsymbol{\phi}_0\\ \boldsymbol{z}_0\end{pmatrix},$$

$$0=M_0R(\lambda;\mathscr{A}_2)\boldsymbol{w}_0=\langle PR(\lambda;\mathscr{A}_0)\boldsymbol{v}_0+PR(\lambda;\mathscr{A}_0)\mathscr{B}_0R(\lambda;\Phi)\boldsymbol{\phi}_0,\boldsymbol{g}_0\rangle+\langle R(\lambda;\Phi)\boldsymbol{\phi}_0,\boldsymbol{h}\rangle.$$

余下证明类似于能控性证明，从略．定理 6 的条件 (1) 是弹性振动系统能观测性的条件．条件 (2) 是考虑到刚性运动对弹性振动的影响所产生的条件．

ON THE CLOSED-LOOP OF THE ELASTIC VIBRATION OF SPACE VEHICLES WITH SLENDER BODY

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ABSTRACT

In this paper, the motion of a space vehicle with slender body is taken into account of both its rigid body motion and its elastic vibration. Considering the influence of the rigid body motion on the elastic vibration, the angular velocity measured by the instruments mounted on the space vehicle is taken as a feedback signal into the controller, the output of which is transmitted as a signal into the actuator in order to realize the feedback control. The plant in consideration includes therefore both the vehicle in rigid motion and its elastic vibration. In this paper we deal with the problem of elastic vibration of the closed-loop system formed by the vehicle in rigid motion, its elastic vibration and the controller. Expressions for the frequencies and for the modes of vibration with regard to the closed-loop system are obtained. Conditions for the asympotic stability of the closed-loop system and for the controllability and observability are given.