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On the polynomiality and asymptotics of moments of sizes for random $(n, dn \pm 1)$-core partitions with distinct parts

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Abstract Amdeberhan's conjectures on the enumeration, the average size, and the largest size of $(n, n+1)$-core partitions with distinct parts have motivated much research on this topic. Recently, Straub (2016) and Nath and Sellers (2017) obtained formulas for the numbers of $(n, dn-1)$- and $(n, dn+1)$-core partitions with distinct parts, respectively. Let $X_{s,t}$ be the size of a uniform random (s, t)-core partition with distinct parts when s and t are coprime to each other. Some explicit formulas for the k-th moments $\mathrm{E}[X^k_{n,n+1}]$ and $\mathrm{E}[X^k_{2n+1,2n+3}]$ were given by Zaleski and Zeilberger (2017) when k is small. Zaleski (2017) also studied the expectation and higher moments of $X_{n,dn-1}$ and conjectured some polynomiality properties concerning them in arXiv:1702.05634.

Motivated by the above works, we derive several polynomiality results and asymptotic formulas for the k-th moments of $X_{n,dn+1}$ and $X_{n,dn-1}$ in this paper, by studying the β-sets of core partitions. In particular, we show that these k-th moments are asymptotically some polynomials of n with degrees at most $2k$, when d is given and n tends to infinity. Moreover, when $d = 1$, we derive that the k-th moment $\mathrm{E}[X^k_{n,n+1}]$ of $X_{n,n+1}$ is asymptotically equal to $(n^2/10)^k$ when n tends to infinity. The explicit formulas for the expectations $\mathrm{E}[X_{n,dn+1}]$ and $\mathrm{E}[X_{n,dn-1}]$ are also given. The $(n, dn-1)$-core case in our results proves several conjectures of Zaleski (2017) on the polynomiality of the expectation and higher moments of $X_{n,dn-1}$.

Keywords partition, hook length, core partition, average size, distinct part

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1 Introduction

A partition λ is called a *t-core partition* if none of its hook lengths is divisible by t. Core partitions arise naturally in the study of modular representations of finite groups. For example, they label the blocks of irreducible characters of symmetric groups [16]. Furthermore, λ is called a $(t_1, t_2, \ldots, t_m)$-core partition if it is simultaneously a t_1-core, a t_2-core, ..., a t_m-core partition [1, 11]. It is well known that, the number of $(t_1, t_2, \ldots, t_m)$-core partitions is finite if and only if the greatest common divisor $\gcd(t_1, t_2, \ldots, t_m) = 1$ (see, for example, [11, Theorem 1] or [22, Theorem 1.1]).

In 2002, Anderson [3] proved the following result on the number of (t_1, t_2)-core partitions, by studying their connections with certain lattice paths.

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Theorem 1.1 (See [3]). *Let t_1 and t_2 be two coprime positive integers. Then the number of (t_1, t_2)-core partitions equals*

$$\frac{(t_1 + t_2 - 1)!}{t_1!\, t_2!}.$$

Anderson's work has motivated many researches on the enumeration, largest sizes and average sizes of simultaneous core partitions (see [2, 7–9, 15, 19, 23, 27]). For example, when t_1 and t_2 are coprime to each other, it was proved by Olsson and Stanton [16] that the largest size of (t_1, t_2)-core partitions equals $(t_1^2 - 1)(t_2^2 - 1)/24$, in their study of block inclusions of symmetric groups. Armstrong et al. [4] gave the following conjecture on the average size of such partitions, which was first proved by Johnson [10] and later by Wang [21].

Theorem 1.2 (Armstrong's conjecture). *Let t_1 and t_2 be two coprime positive integers. Then the average size of (t_1, t_2)-core partitions equals*

$$\frac{(t_1 - 1)(t_2 - 1)(t_1 + t_2 + 1)}{24}.$$

Recently, the problem on the enumeration of simultaneous core partitions with distinct parts was raised by Amdeberhan [1]. He conjectured explicit formulas for the number, the largest size and the average size of $(n, n+1)$-core partitions with distinct parts, which were first proved by Xiong [25], and later proved independently (and extended) by Straub [20], Nath and Sellers [13], Zaleski [28] and Paramonov [17]. Let $X_{s,t}$ be the size of a uniform random (s, t)-core partition with distinct parts when s and t are coprime to each other. Zaleski [28] derived several explicit formulas for the k-th moment $\mathrm{E}[X^k_{n,n+1}]$ of $X_{n,n+1}$ when $k \leqslant 16$. The number, the largest size and the average size of $(2n+1, 2n+3)$-core partitions with distinct parts were also well studied (see [5, 17, 20, 26, 30]). Several explicit formulas for the k-th (when $k \leqslant 7$) moment $\mathrm{E}[X^k_{2n+1,2n+3}]$ of $X_{2n+1,2n+3}$ were obtained by Zaleski and Zeilberger [30].

In 2016, Straub [20] derived the following generalized Fibonacci recurrence for the number $N_d(n)$ of $(n, dn-1)$-core partitions with distinct parts.

Theorem 1.3 (See [20]). *Let $N_d(1) = 1$, and $N_d(n)$ be the number of $(n, dn-1)$-core partitions with distinct parts for two positive integers $d \geqslant 1$ and $n \geqslant 2$. Then*

$$\begin{aligned} &N_d(1) = 1, \quad N_d(2) = d, \\ &N_d(n) = N_d(n-1) + dN_d(n-2), \quad \textit{if} \quad n \geqslant 3. \end{aligned} \tag{1.1}$$

The $(n, dn+1)$-core analog was obtained later by Nath and Sellers [14].

Theorem 1.4 (See [14]). *Let $M_d(-1) = 0$, $M_d(0) = 1$, and $M_d(n)$ be the number of $(n, dn+1)$-core partitions with distinct parts for two positive integers d and n. Then*

$$\begin{aligned} &M_d(-1) = 0, \quad M_d(0) = 1, \\ &M_d(n) = M_d(n-1) + dM_d(n-2), \quad \textit{if} \quad n \geqslant 1. \end{aligned} \tag{1.2}$$

Table 1 gives the first few values for $N_d(n)$ and $M_d(n)$.

It is easy to derive that, when $d \neq 2$,

$$M_d(n) = \frac{d(d-1)N_d(n) - N_d(n+1)}{d(d-2)} \tag{1.3}$$

Table 1 The number of $(n, dn \pm 1)$-core partitions with distinct parts for $1 \leqslant n \leqslant 6$

n	1	2	3	4	5	6
$N_d(n)$	1	d	$2d$	$d^2 + 2d$	$3d^2 + 2d$	$d^3 + 5d^2 + 2d$
$M_d(n)$	1	$d + 1$	$2d + 1$	$d^2 + 3d + 1$	$3d^2 + 4d + 1$	$d^3 + 6d^2 + 5d + 1$

and

$$M_d(n-1) = \frac{(d-1)N_d(n+1) - dN_d(n)}{d(d-2)}. \tag{1.4}$$

Recently, the largest sizes of the above two kinds of partitions were given by Xiong [24]. Zaleski [29] conjectured an explicit formula for the average size of $(n, dn-1)$-core partitions with distinct parts. Furthermore, Zaleski conjectured some polynomiality properties for higher moments of their sizes.

In this paper, we derive results on moments of sizes for random $(n, dn \pm 1)$-core partitions with distinct parts. The $(n, dn-1)$-core case proves several conjectures of Zaleski [29]. Let $\mathcal{C}_{n,dn+1}$ and $\mathcal{C}_{n,dn-1}$ be the sets of $(n, dn+1)$-core and $(n, dn-1)$-core partitions with distinct parts respectively. Our main results are stated next. The $(n, dn-1)$-core case in Theorems 1.5 and 1.7 are equivalent to Zaleski's Conjectures 3.5 and 3.1 in [29], respectively.

Theorem 1.5 (See [29, Conjecture 3.5]). *Let k be a positive integer. The k-th power sums*

$$\sum_{\lambda \in \mathcal{C}_{n,dn+1}} |\lambda|^k \quad \textit{and} \quad \sum_{\lambda \in \mathcal{C}_{n,dn-1}} |\lambda|^k$$

for sizes of partitions in $\mathcal{C}_{n,dn+1}$ and in $\mathcal{C}_{n,dn-1}$ are of the form

$$A(n,d)M_d(n) + B(n,d)M_d(n+1), \tag{1.5}$$

where $A(n,d)$ and $B(n,d)$ are some polynomials of n with degrees at most $2k$, whose coefficients are rational functions in d.

Remark 1.6. In the above theorem, we use $M_d(n)$ and $M_d(n+1)$ as a basis, while $N_d(n)$ and $N_d(n+1)$ are used in the original statement of Zaleski's conjectures in [29]. As mentioned by Zaleski [29], some of his conjectures are anomalous for the case $d = 2$. The use of the basis $M_d(n)$ and $M_d(n+1)$ avoids this problem, i.e., the form (1.5) always holds for any $d \geqslant 1$. Also, by (1.3) and (1.4) we know, when $d \neq 2$, $M_d(n)$ and $M_d(n+1)$ in (1.5) can be replaced by $N_d(n)$ and $N_d(n+1)$.

Theorem 1.7 (See [29, Conjecture 3.1]). *Let n and k be two given positive integers. Then the k-th power sums*

$$\sum_{\lambda \in \mathcal{C}_{n,dn+1}} |\lambda|^k \quad \textit{and} \quad \sum_{\lambda \in \mathcal{C}_{n,dn-1}} |\lambda|^k$$

are polynomials of d with degrees at most $2k + \lfloor n/2 \rfloor$.

Recall that $X_{n,dn-1}$ and $X_{n,dn+1}$ are sizes of uniform random $(n, dn-1)$-core and $(n, dn+1)$-core partitions with distinct parts, respectively. By Theorems 1.5 and 1.7 we derive the following asymptotic formulas when d is fixed or n is fixed, respectively.

Theorem 1.8. *Let d and k be two given positive integers. Then the k-th moments of $X_{n,dn+1}$ and $X_{n,dn-1}$ are asymptotically some polynomials of n with degrees at most $2k$, when n tends to infinity, i.e., there exist some constants $A_{d,k}$ and $B_{d,k}$ such that*

$$\mathrm{E}[X^k_{n,dn+1}] = A_{d,k}\, n^{2k} + O(n^{2k-1}) \tag{1.6}$$

and

$$\mathrm{E}[X^k_{n,dn-1}] = B_{d,k}\, n^{2k} + O(n^{2k-1}). \tag{1.7}$$

Theorem 1.9. *Let $n \geqslant 2$ and $k \geqslant 1$ be two given integers. Then the k-th moments of $X_{n,dn+1}$ and $X_{n,dn-1}$ are asymptotically some polynomials of d with degrees at most $2k$, when d tends to infinity, i.e., there exist some constants $C_{n,k}$ and $D_{n,k}$ such that*

$$\mathrm{E}[X^k_{n,dn+1}] = C_{n,k}\, d^{2k} + O(d^{2k-1}) \tag{1.8}$$

and

$$\mathrm{E}[X^k_{n,dn-1}] = D_{n,k}\, d^{2k} + O(d^{2k-1}). \tag{1.9}$$

Moreover, when $d=1$, we derive the leading term in the asymptotic formula of $\mathrm{E}[X_{n,n+1}^k]$.

Theorem 1.10. *Let k be a given positive integer. Then the k-th moment of $X_{n,n+1}$ satisfies the following asymptotic formula*:

$$\mathrm{E}[X_{n,n+1}^k] = \left(\frac{1}{10}\right)^k n^{2k} + O(n^{2k-1}).$$

We also derive explicit formulas for the expectations of $X_{n,dn+1}$ and $X_{n,dn-1}$.

Theorem 1.11. *Let d and n be two given positive integers. The expectation of $X_{n,dn+1}$ equals*

$$\begin{aligned}\mathrm{E}[X_{n,dn+1}] &= \frac{d(d+1)(5d+1)(n-1)^2}{24(4d+1)} + \frac{d(d+1)(32d^2+63d+7)(n-1)}{24(4d+1)^2}\\ &\quad + \frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2} - \frac{M_d(n-1)}{M_d(n)}\\ &\quad \times \left(\frac{d(d+1)(d-1)(n-1)^2}{24(4d+1)} + \frac{d(d+1)(14d^2+21d+1)(n-1)}{24(4d+1)^2}\right.\\ &\quad \left. + \frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2}\right).\end{aligned}$$

Example 1.12. Let $d=2$ and $n=4$. Then $M_d(n-1)=M_2(3)=5$ and $M_d(n)=M_2(4)=11$. By the above theorem the expectation of $X_{n,dn+1}$ should be $54/11$. We can check that this is true since the number of $(4,9)$-core partitions with distinct parts equals 11, and the sum of their sizes equals 54:

$$\mathcal{C}_{4,9} = \{\emptyset, (1), (2), (3), (2,1), (4,1), (5,2), (6,3), (3,2,1), (5,2,1), (4,3,2,1)\}.$$

Example 1.13. Let $d=3$ and $n=3$. Then $M_d(n-1)=M_3(2)=4$ and $M_d(n)=M_3(3)=7$. By the above theorem the expectation of $X_{n,dn+1}$ should be $34/7$. We can check that this is true since the number of $(3,10)$-core partitions with distinct parts equals 7, and the sum of their sizes equals 34:

$$\mathcal{C}_{3,10} = \{\emptyset, (1), (2), (3,1), (4,2), (5,3,1), (6,4,2)\}.$$

Theorem 1.14. *Let $d \geqslant 1$ and $n \geqslant 2$ be two given positive integers. The total sum of sizes of partitions in $\mathcal{C}_{n,dn-1}$ is*

$$\begin{aligned}\sum_{\lambda\in\mathcal{C}_{n,dn-1}} |\lambda| &= M_d(n)\cdot\left(\frac{(d^2-1)(5d^2+d-1)n^2}{24d(4\,d+1)} - \frac{(d+1)(8d^4+27d^3+2d^2-1)n}{24d(4\,d+1)^2} + \frac{d^2-1}{12d}\right)\\ &\quad + M_d(n-1)\cdot\left(\frac{(d+1)(-d^3+7d^2+d-1)n^2}{24d(4\,d+1)} - \frac{(d+1)(6d^4-19d^3-7d^2+d+1)n}{24d(4\,d+1)^2}\right.\\ &\quad \left. - \frac{(d+1)(d^4+20d^3-6d^2-8d-1)}{12d(4\,d+1)^2}\right).\end{aligned}$$

Example 1.15. Let $d=1$ and $n=4$. Then $M_d(n-1)=M_1(3)=3$ and $M_d(n)=M_1(4)=5$. By the above theorem the total sum of sizes of $(4,3)$-core partitions with distinct parts should be 3. We can check that this is true since the number of such partitions equals $N_d(n)=N_1(4)=3$, and the sum of their sizes equals 3:

$$\mathcal{C}_{4,3} = \{\emptyset, (1), (2)\}.$$

Example 1.16. Let $d=2$ and $n=5$. Then $M_d(n-1)=M_2(4)=11$ and $M_d(n)=M_2(5)=21$. By the above theorem the total sum of sizes of $(5,9)$-core partitions with distinct parts should be 92. We can check that this is true since the number of such partitions equals $N_d(n)=N_2(5)=16$, and the sum of their sizes equals 92:

$$\begin{aligned}\mathcal{C}_{5,9} = \{&\emptyset, (1), (2), (3), (4), (2,1), (3,1), (3,2), (5,1), (6,2), (7,3),\\ &(4,2,1), (6,2,1), (4,3,1), (5,3,2), (5,4,2,1)\}.\end{aligned}$$

By (1.3) and (1.4) we obtain, Theorem 1.14 implies the following conjecture of Zaleski [29] directly.

Corollary 1.17 (See [29, Conjecture 3.8]). *Let $d \geqslant 1$ and $n \geqslant 2$ be two given positive integers. When $d \neq 2$, the expectation of $X_{n,dn-1}$ equals*

$$\begin{aligned}
\mathrm{E}[X_{n,dn-1}] = &\frac{(5\,d^3+7\,d^2+d-1)n^2}{24(4\,d+1)} - \frac{(8\,d^5+21\,d^4+7\,d^3-d^2+3\,d-2)n}{24(16\,d^3-24\,d^2-15\,d-2)} \\
&+ \frac{17\,d^4+13\,d^3-9\,d^2-7\,d-2}{12(16\,d^3-24\,d^2-15\,d-2)} + \frac{N_d(n+1)}{N_d(n)} \\
&\times \left(-\frac{(d^2-1)n^2}{24(4\,d+1)} - \frac{(2\,d^4-9\,d^3-16\,d^2-3\,d+2)n}{8(16\,d^3-24\,d^2-15\,d-2)} - \frac{d^4+20\,d^3+9\,d^2-20\,d-10}{12(d-2)(4\,d+1)^2} \right).
\end{aligned}$$

The rest of the paper is arranged as follows. In Section 2 we review some basic results on core partitions. The characterizations for the β-sets of $(n, dn-1)$- and $(n, dn+1)$-core partitions with distinct parts are given in Section 3. Then in Section 4 we use these characterizations to translate the problems to study two families of functions $G^{+}_{d,m,a,b}(n)$ and $G^{-}_{d,m,a,b}(n)$, therefore prove the main results. The explicit formulas for expectations of $X_{n,dn+1}$ and $X_{n,dn-1}$ are derived in Section 5. The asymptotic formulas for moments of $X_{n,n+1}$ are given in Section 6. In Section 7, we give further directions.

2 Simultaneous core partitions and their β-sets

A *partition* is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. The numbers λ_i $(1 \leqslant i \leqslant \ell)$ are called the *parts* and $\sum_{1\leqslant i\leqslant \ell} \lambda_i$ the *size* of the partition λ (see [12,18]). Each partition λ is identified with its *Young diagram*, which is an array of boxes arranged in left-justified rows with λ_i boxes in the i-th row. For the (i,j)-box in the i-th row and j-th column in the Young diagram, its *hook length* $h(i,j)$ is defined to be the number of boxes exactly to the right, and exactly below, and the box itself. Recall that a partition λ is called a $(t_1, t_2, \ldots, t_m)$-*core partition* if none of its hook lengths is divisible by $t_1, t_2, \ldots, t_{m-1}$, or t_m (see [1,11]). For example, Figure 1 gives the Young diagram and hook lengths of the partition $(6,3,3,2)$. Therefore, it is a $(7,10)$-core partition since none of its hook lengths is divisible by 7 or 10.

The β-*set* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is denoted by $\beta(\lambda) = \{\lambda_i + \ell - i : 1 \leqslant i \leqslant \ell\}$. In fact, $\beta(\lambda)$ is equal to the set of hook lengths of boxes in the first column of the corresponding Young diagram of λ (see [16,23]). For example, from Figure 1 we know that $\beta((6,3,3,2)) = \{9,5,4,2\}$. It is easy to see that a partition λ is uniquely determined by its β-set $\beta(\lambda)$. The following results on β-sets are well-known.

Lemma 2.1 (See [16,23–25]). *The size of a partition λ is determined by its β-set as the following*:

$$|\lambda| = \sum_{x\in\beta(\lambda)} x - \binom{|\beta(\lambda)|}{2}. \tag{2.1}$$

Lemma 2.2 (See [24,25]). *The partition λ is a partition with distinct parts if and only if there does not exist $x, y \in \beta(\lambda)$ with $x - y = 1$.*

Lemma 2.3 (See [3,6,16,23,25]). *A partition λ is a t-core partition if and only if for any $x \in \beta(\lambda)$ with $x \geqslant t$, we always have $x - t \in \beta(\lambda)$.*

9	8	6	3	2	1
5	4	2			
4	3	1			
2	1				

Figure 1 The Young diagram and hook lengths of the partition $(6,3,3,2)$

3 The β-sets of $(n, dn \pm 1)$-core partitions with distinct parts

In this section we focus on $(n, dn-1)$- and $(n, dn+1)$-core partitions with distinct parts. The following characterizations for β-sets are well-known. We give a short proof here for completeness.

Theorem 3.1 (See [14,20,24,29])**.** *Let n and d be two positive integers. Then a finite subset S of $\mathbb{N}$ is a β-set of some $(n, dn+1)$-core partition with distinct parts if and only if the following conditions hold*:

(i) $S \subseteq \{(i-1)n + j : 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant n-1\}$;

(ii) *if $in + j \in S$ with $1 \leqslant i \leqslant d-1,\ 1 \leqslant j \leqslant n-1$, then $(i-1)n + j \in S$*;

(iii) *if $j \in S$ with $1 \leqslant j \leqslant n-2$, then $j+1 \notin S$.*

Proof. (1) Suppose that λ is an $(n, dn+1)$-core partition with distinct parts and $S = \beta(\lambda)$. By Lemma 2.3 we have $dn+1 \notin S$ and $nx \notin S$ for any $1 \leqslant x \leqslant d$ since $0 \notin S$. For $x \geqslant dn+2$, if $x \in S$, by Lemma 2.3 we know $x - dn$, $x - (dn+1) \in S$. But by Lemma 2.2 this is impossible since λ is a partition with distinct parts. Then the condition (i) holds. Also, (ii) and (iii) hold by Lemmas 2.2 and 2.3.

(2) On the other hand, suppose that the set S satisfies the conditions (i)–(iii). Let λ be the partition with $\beta(\lambda) = S$. Since $\beta(\lambda)$ does not have elements larger than $dn-1$, λ must be a $(dn+1)$-core partition. Also, by (ii) λ must be an n-core partition. Finally by (i)–(iii) and Lemma 2.2 we know λ is a partition with distinct parts. □

Let

$$\mathcal{A}_{d,n} := \{(i,j) : 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant n\}.$$

We say that a subset $I \subseteq \mathcal{A}_{d,n}$ is *nice* if it satisfies the following two conditions:

(1) $(i+1, j) \in I$ and $i \geqslant 1$ imply $(i,j) \in I$;

(2) $(1,j) \in I$ and $1 \leqslant j \leqslant n-1$ imply $(1, j+1) \notin I$.

Let $\mathcal{B}^{+}_{d,n}$ be the set of nice subsets of $\mathcal{A}_{d,n}$. For each n-core partition λ, define

$$\psi_n(\lambda) := \{(i,j) : 1 \leqslant j \leqslant n-1,\ (i-1)n + j \in \beta(\lambda)\}. \tag{3.1}$$

Then by Theorem 3.1 the map ψ_n gives a bijection between the sets $\mathcal{C}_{n,dn+1}$ and $\mathcal{B}^{+}_{d,n-1}$. Furthermore, by Lemma 2.1 we have the following lemma.

Lemma 3.2. *It holds that*

$$\sum_{\lambda \in \mathcal{C}_{n,dn+1}} |\lambda|^k = \sum_{I \in \mathcal{B}^{+}_{d,n-1}} \left(\sum_{(i,j) \in I} ((i-1)n + j) - \frac{|I|^2}{2} + \frac{|I|}{2} \right)^k. \tag{3.2}$$

Example 3.3. Let $d = 3$ and $n = 3$. By Example 1.13 we know there are 7 of $(3,10)$-core partitions with distinct parts $\emptyset, (1), (2), (3,1), (4,2), (5,3,1), (6,4,2)$. The corresponding nice subsets of $\mathcal{A}_{3,2}$ are:

$$\mathcal{B}^{+}_{3,2} = \{\emptyset,\ \{(1,1)\},\ \{(1,2)\},\ \{(1,1),(2,1)\},\ \{(1,2),(2,2)\},$$
$$\{(1,1),(2,1),(3,1)\},\ \{(1,2),(2,2),(3,2)\}\}.$$

Let $k = 2$. It is easy to check that both sides of (3.2) equal 282.

Similarly the following are characterizations for β-sets of $(n, dn-1)$-core partitions with distinct parts. Notice that $dn - 1 \notin S$ in the following condition (iv).

Theorem 3.4 (See [14,20,24,29])**.** *Let $n \geqslant 2$ and $d \geqslant 1$ be two positive integers. Then a finite subset S of $\mathbb{N}$ is a β-set of some $(n, dn-1)$-core partition with distinct parts if and only if the following conditions hold*:

(iv) $S \subseteq \{(i-1)n + j : 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant n-2\} \cup \{in - 1 : 1 \leqslant i \leqslant d-1\}$;

(v) *if $in + j \in S$ with $i \geqslant 1$ and $1 \leqslant j \leqslant n-1$, then $(i-1)n + j \in S$*;

(vi) *if $j \in S$ with $1 \leqslant j \leqslant n-2$, then $j+1 \notin S$.*

Let $\mathcal{B}^{-}_{d,n}$ be the set of nice subsets I of $\mathcal{A}_{d,n}$ with $(d,n) \notin I$. Then by Theorem 3.4 the map ψ_n defined in (3.1) gives a bijection between the sets $\mathcal{C}_{n,dn-1}$ and $\mathcal{B}^{-}_{d,n-1}$. Furthermore, by Lemma 2.1 we obtain the following lemma.

Lemma 3.5. *It holds that*

$$\sum_{\lambda\in\mathcal{C}_{n,dn-1}}|\lambda|^k=\sum_{I\in\mathcal{B}^-_{d,n-1}}\left(\sum_{(i,j)\in I}((i-1)n+j)-\frac{|I|^2}{2}+\frac{|I|}{2}\right)^k. \tag{3.3}$$

Example 3.6. Let $d=3$ and $n=3$. Then there are 6 of $(3,8)$-core partitions with distinct parts: $\emptyset,(1),(2),(3,1),(4,2),(5,3,1)$. The corresponding nice subsets of $\mathcal{A}_{3,2}$ are

$$\mathcal{B}^-_{3,2}=\{\emptyset,\ \{(1,1)\},\ \{(1,2)\},\ \{(1,1),(2,1)\},\ \{(1,2),(2,2)\},\ \{(1,1),(2,1),(3,1)\}\}.$$

Let $k=2$. Then both sides of (3.3) equal 138.

4 Polynomiality of moments of sizes for core partitions

In this section, we will prove the main results. For each nice subset I of $\mathcal{A}_{d,n}$, let $|I|$ be the cardinality of I. Define

$$\sigma_m(I):=\sum_{(i,j)\in I}((i-1)m+j)$$

and

$$G^+_{d,m,a,b}(n):=\sum_{I\in\mathcal{B}^+_{d,n}}\sigma_m(I)^a\,|I|^b$$

for $d,m,a,b,n\geqslant 0$.

To compute the k-th power sum of sizes of partitions in $\mathcal{C}_{n,dn+1}$, by Lemma 3.2 we just need to compute the functions $G^+_{d,n,a,b}(n-1)$ with four variables d,n,a and b. The basic idea is induction on n. To do this, we need one more parameter m here, i.e., we study a more general family of functions $G^+_{d,m,a,b}(n)$ with five variables d,m,a,b and n. First, we derive formulas for generating functions of $G^+_{d,m,a,b}(n)$.

Theorem 4.1. *Assume that a and b are two non-negative integers. For each $1\leqslant i\leqslant 2a+b+1$, there exists some polynomial $P_{a,b,i}(d,m,q)$ of d, m and q with*

$$\deg_m(P_{a,b,i})\leqslant 2a+b+1-i,$$

such that the generating function of $G^+_{d,m,a,b}(n)$ equals:

$$\Psi_{d,m,a,b}:=\sum_{n\geqslant 0}G^+_{d,m,a,b}(n)\,q^n=\sum_{i=1}^{2a+b+1}\frac{P_{a,b,i}(d,m,q)}{(1-q-dq^2)^i}. \tag{4.1}$$

Proof. We will prove this result by induction on $a+b$. When $a+b=0$, we have $a=b=0$. For $n\geqslant 2$,

$$\begin{aligned}G^+_{d,m,0,0}(n)&=\sum_{I\in\mathcal{B}^+_{d,n}}1=|\mathcal{B}^+_{d,n}|\\&=\sum_{I\in\mathcal{B}^+_{d,n-1}}1+\sum_{I\in\mathcal{B}^+_{d,n}\setminus\mathcal{B}^+_{d,n-1}}1\\&=\sum_{I\in\mathcal{B}^+_{d,n-1}}1+\sum_{(1,n)\in I\in\mathcal{B}^+_{d,n}}1.\end{aligned}$$

When $(1,n)\in I\in\mathcal{B}^+_{d,n}$, we know $(1,n-1)\notin I$ and therefore

$$I\cap\mathcal{A}_{d,n-1}\in\mathcal{B}^+_{d,n-2}.$$

Thus for each $1\leqslant i\leqslant d$,

$$|\{I\in\mathcal{B}^+_{d,n}:(i,n)\in I,\ (i+1,n)\notin I\}|=|\mathcal{B}^+_{d,n-2}|.$$

Therefore

$$G^{+}_{d,m,0,0}(n) = |\mathcal{B}^{+}_{d,n-1}| + d\,|\mathcal{B}^{+}_{d,n-2}| = G^{+}_{d,m,0,0}(n-1) + d\,G^{+}_{d,m,0,0}(n-2) \tag{4.2}$$

for $n \geqslant 2$. By definition it is easy to derive

$$G^{+}_{d,m,0,0}(0) = 1, \quad G^{+}_{d,m,0,0}(1) = d+1. \tag{4.3}$$

Therefore

$$\Psi_{d,m,0,0} - (d+1)q - 1 = q(\Psi_{d,m,0,0} - 1) + dq^2\Psi_{d,m,0,0}$$

and thus

$$\Psi_{d,m,0,0} = \frac{dq+1}{1-q-dq^2}. \tag{4.4}$$

Then the theorem is true for $a+b=0$. Next, assume that $a+b>0$ and (4.1) holds for all pairs (a',b') with $a'+b'<a+b$. For $n \geqslant 2$, considering the largest integer i such that $(i,n) \in I$ (or $(1,n) \notin I$), we obtain

$$\begin{aligned} G^{+}_{d,m,a,b}(n) &= \sum_{I\in\mathcal{B}^{+}_{d,n}} \sigma_m(I)^a|I|^b = \sum_{(1,n)\notin I\in\mathcal{B}^{+}_{d,n}} \sigma_m(I)^a|I|^b + \sum_{(1,n)\in I\in\mathcal{B}^{+}_{d,n}} \sigma_m(I)^a|I|^b \\ &= \sum_{I\in\mathcal{B}^{+}_{d,n-1}} \sigma_m(I)^a|I|^b + \sum_{i=1}^{d} \sum_{\substack{(i,n)\in I\in\mathcal{B}^{+}_{d,n}\\ (i+1,n)\notin I}} \sigma_m(I)^a|I|^b \\ &= \sum_{I\in\mathcal{B}^{+}_{d,n-1}} \sigma_m(I)^a|I|^b + \sum_{I\in\mathcal{B}^{+}_{d,n-2}} \sum_{i=1}^{d} \left(\sigma_m(I) + \binom{i}{2}m + in\right)^a (|I|+i)^b \\ &= G^{+}_{d,m,a,b}(n-1) + d\,G^{+}_{d,m,a,b}(n-2) + \sum_{\substack{a'+b'<a+b\\ 0\leqslant a'\leqslant a\\ 0\leqslant b'\leqslant b}} A^{a,b}_{a',b'}(d,m,n)\,G^{+}_{d,m,a',b'}(n-2), \end{aligned} \tag{4.5}$$

where

$$A^{a,b}_{a',b'}(d,m,n) = \binom{a}{a'}\binom{b}{b'} \sum_{i=1}^{d} \left(\binom{i}{2}m + in\right)^{a-a'} i^{b-b'}$$

are polynomials of d, m and n such that

$$\deg_m A^{a,b}_{a',b'} + \deg_n A^{a,b}_{a',b'} \leqslant a - a'.$$

It is obvious that, when $a+b>0$,

$$G^{+}_{d,m,a,b}(0) = 0, \quad G^{+}_{d,m,a,b}(1) = \sum_{i=1}^{d} \left(\binom{i}{2}m + i\right)^a i^b = \sum_{k=0}^{a} B^{a,b}_k(d)\; m^k, \tag{4.6}$$

where

$$B^{a,b}_k(d) = \binom{a}{k} \sum_{i=1}^{d} \binom{i}{2}^k i^{a-k+b}.$$

Considering the generating function, by (4.5) we have

$$\begin{aligned} &\Psi_{d,m,a,b} - q\,G^{+}_{d,m,a,b}(1) \\ &\quad = q\Psi_{d,m,a,b} + dq^2\Psi_{d,m,a,b} + \sum_{\substack{a'+b'<a+b\\ 0\leqslant a'\leqslant a\\ 0\leqslant b'\leqslant b}} \sum_{n\geqslant 2} A^{a,b}_{a',b'}(d,m,n)G^{+}_{d,m,a',b'}(n-2)\; q^n \end{aligned}$$

$$= q\Psi_{d,m,a,b} + dq^2\Psi_{d,m,a,b} + q^2 \sum_{\substack{a'+b'<a+b\\ 0\leqslant a'\leqslant a\\ 0\leqslant b'\leqslant b}} \sum_{n\geqslant 0} A^{a,b}_{a',b'}(d,m,n+2)G^{+}_{d,m,a',b'}(n)\ q^n. \tag{4.7}$$

When $a'+b'<a+b$, by induction hypothesis and

$$\sum_{n\geqslant 0} na_n\, q^n = q\left(\sum_{n\geqslant 0} a_n q^n\right)'$$

we obtain

$$\sum_{n\geqslant 0} n^j G^{+}_{d,m,a',b'}(n)\ q^n = \sum_{i=1}^{2a'+b'+1+j} \frac{C_{j,a',b',i}(d,m,q)}{(1-q-dq^2)^i}$$

for each $j\geqslant 0$, where $C_{j,a',b',i}(d,m,q)$ are some polynomials of d, m and q with

$$\deg_m(C_{j,a',b',i}(d,m,q)) \leqslant 2a'+b'+1+j-i.$$

Therefore for $a'+b'<a+b$, we have

$$\sum_{n\geqslant 0} A^{a,b}_{a',b'}(d,m,n+2)\, G^{+}_{d,m,a',b'}(n)\ q^n = \sum_{i=1}^{2a'+b'+1+a-a'} \frac{D_{a',b',i}(d,m,q)}{(1-q-dq^2)^i}, \tag{4.8}$$

where $D_{a',b',i}(d,m,q)$ are some polynomials of d, m and q with

$$\deg_m(D_{a',b',i}(d,m,q)) \leqslant 2a'+b'+1+a-a'-i = a+a'+b'+1-i \leqslant 2a+b-i.$$

Then by (4.6)–(4.8) we obtain

$$\Psi_{d,m,a,b} = \sum_{n\geqslant 0} G^{+}_{d,m,a,b}(n)q^n = \sum_{i=1}^{2a+b+1} \frac{P_{a,b,i}(d,m,q)}{(1-q-dq^2)^i},$$

where $P_{a,b,i}(d,m,q)$ are some polynomials of d, m and q with

$$\deg_m(P_{a,b,i}(d,m,q)) \leqslant 2a+b+1-i. \qquad \square$$

By the above theorem, to derive the explicit expression for $G^{+}_{d,m,a,b}(n)$, we need to study the expansion of $1/(1-q-dq^2)^k$. Let $x_d=(1+\sqrt{1+4d})/2$ and $y_d=(1-\sqrt{1+4d})/2$ be two roots of x^2-x-d. By the partial fraction decomposition, we obtain the following results.

Lemma 4.2. *Let d and k be given positive integers. Then*

$$\frac{1}{(1-q-dq^2)^k} = \sum_{i=1}^{k} \frac{\binom{2k-1-i}{k-1} d^{k-i}}{(1+4d)^{\frac{2k-i}{2}}} \sum_{n\geqslant 0} \binom{n+i-1}{i-1} (x_d^{n+i} + (-1)^i y_d^{n+i})\, q^n. \tag{4.9}$$

Proof. For $a,b\geqslant 0$, let

$$F_{a,b} = \frac{1}{(1-x_d\,q)^a(1-y_d\,q)^b}.$$

It is easy to see that

$$F_{a+1,b+1} = \frac{x_d}{x_d-y_d}F_{a+1,b} + \frac{y_d}{y_d-x_d}F_{a,b+1}$$

for all $a,b\geqslant 0$. Therefore by induction we derive

$$F_{a,b} = \sum_{i=1}^{a} \frac{(-1)^{a-i}\binom{a+b-1-i}{b-1}\, x_d^b\, y_d^{a-i}}{(x_d-y_d)^{a+b-i}(1-x_d\,q)^i} + \sum_{j=1}^{b} \frac{(-1)^a\binom{a+b-1-j}{a-1}\, x_d^{b-j}\, y_d^a}{(x_d-y_d)^{a+b-j}(1-y_d\,q)^i}$$

for all $a, b \geqslant 1$. Let $a = b = k$. Then by $x_d\, y_d = -d$, $x_d - y_d = \sqrt{1+4d}$, and

$$\frac{1}{(1-zq)^k} = \sum_{n\geqslant 0} \binom{n+k-1}{k-1} z^n q^n,$$

we derive (4.9). □

Lemma 4.3. *Let k be a positive integer. Then*

$$\frac{1}{(1-q-dq^2)^k} = \sum_{n\geqslant 0} c_n\, q^n,$$

where c_n is of the form $A(n,d)M_d(n) + B(n,d)M_d(n+1)$, such that $A(n,d)$ and $B(n,d)$ are polynomials of n with degrees at most $k-1$, whose coefficients are rational functions in d. In particular, we have (notice that $M_d(n+2) = M_d(n+1) + dM_d(n)$)

$$\frac{1}{1-q-dq^2} = \sum_{n\geqslant 0} M_d(n)\, q^n, \tag{4.10}$$

$$\frac{1}{(1-q-dq^2)^2} = \sum_{n\geqslant 0} \frac{1}{4d+1}((n+1)\, M_d(n+2) + (n+3)\, d\, M_d(n))\, q^n \tag{4.11}$$

and

$$\frac{1}{(1-q-dq^2)^3} = \sum_{n\geqslant 0} \left(\left(\frac{3d(n+1)}{(4d+1)^2} + \frac{1}{4d+1}\binom{n+2}{2} \right) \cdot M_d(n+2) + \frac{3d^2(n+3)}{(4d+1)^2}\, M_d(n) \right) q^n. \tag{4.12}$$

Proof. By the recurrence relation (1.2), it is easy to see that

$$M_d(n) = \frac{1}{\sqrt{1+4d}}\, (x_d^{n+1} - y_d^{n+1}) \tag{4.13}$$

and

$$2M_d(n+1) - M_d(n) = x_d^{n+1} + y_d^{n+1}. \tag{4.14}$$

Therefore Lemma 4.2 implies

$$\frac{1}{(1-q-dq^2)^k} = \sum_{n\geqslant 0} c_n q^n,$$

where c_n is of the form

$$\sum_{i=0}^{k} A_i(n,d) M_d(n+i),$$

such that each $A_i(n,d)$ is a polynomial of n with degree at most $k-1$, whose coefficients are rational functions in d. But by (1.2), each $M_d(n+i)$ can be written as some linear combination of $M_d(n)$ and $M_d(n+1)$, whose coefficients are rational functions in d. Therefore we prove the main result of the lemma. In particular, letting $k = 1, 2, 3$ in Lemma 4.2, we derive (4.10)–(4.12). □

Notice that for each $i \in \mathbb{Z}$, $M_d(n+i)$ can be written as some linear combination of $M_d(n)$ and $M_d(n+1)$, whose coefficients are rational functions in d. The next result follows from Theorem 4.1 and Lemma 4.3 directly.

Theorem 4.4. *Let $a, b \geqslant 0$ be some given integers. Then $G^{+}_{d,m,a,b}(n)$ is of the form*

$$A(m,n,d)\, M_d(n) + B(m,n,d)\, M_d(n+1),$$

where $A(m,n,d)$ and $B(m,n,d)$ are polynomials of m and n with degrees at most $2a+b$ (i.e., $\deg_m + \deg_n \leqslant 2a+b$), whose coefficients are rational functions in d.

Next, we give some examples of explicit expressions for $G^+_{d,m,a,b}(n)$ when a and b are small.

Example 4.5. Let $a=1,\ b=0$. We have

$$\begin{aligned}G^+_{d,m,1,0}(n)&=\sum_{I\in\mathcal{B}^+_{d,n}}\sigma_m(I)=\sum_{I\in\mathcal{B}^+_{d,n-1}}\sigma_m(I)+\sum_{I\in\mathcal{B}^+_{d,n-2}}\sum_{i=1}^{d}\left(\sigma_m(I)+\binom{i}{2}m+in\right)\\&=G^+_{d,m,1,0}(n-1)+d\,G^+_{d,m,1,0}(n-2)+\left(\binom{d+1}{3}m+\binom{d+1}{2}n\right)G^+_{d,m,0,0}(n-2).\end{aligned}$$

Also

$$\begin{aligned}&G^+_{d,m,1,0}(0)=0,\\&G^+_{d,m,1,0}(1)=\sum_{j=1}^{d}\sum_{i=1}^{j}(1+(i-1)m)=\binom{d+1}{3}m+\binom{d+1}{2}.\end{aligned}$$

Then by (4.7) we have

$$\begin{aligned}\Psi_{d,m,1,0}&=\frac{\left(\binom{d+1}{3}m+\binom{d+1}{2}\right)q}{1-q-dq^2}+\frac{\left(\binom{d+1}{3}m+2\binom{d+1}{2}\right)q^2(dq+1)}{(1-q-dq^2)^2}+\frac{\binom{d+1}{2}q^3(d^2q^2+2dq+d+1)}{(1-q-dq^2)^3}\\&=\frac{\binom{d+1}{3}mq-\binom{d+1}{2}q}{(1-q-dq^2)^2}+\frac{\binom{d+1}{2}(2q-q^2)}{(1-q-dq^2)^3}.\end{aligned}$$

Therefore by (1.2) and Lemma 4.3,

$$\begin{aligned}G^+_{d,m,1,0}(n)=&\frac{1}{4d+1}\left(\binom{d+1}{3}m+\binom{d+1}{2}\cdot\frac{n+1}{2}\right)n\,M_d(n)\\&+\frac{d}{4d+1}\binom{d+1}{2}\left(\frac{2(d-1)m}{3}+n+1\right)(n+1)\,M_d(n-1).\end{aligned}\tag{4.15}$$

Example 4.6. Let $a=0$ and $b=1$. We have

$$\begin{aligned}G^+_{d,m,0,1}(n)&:=\sum_{I\in\mathcal{B}^+_{d,n}}|I|=\sum_{I\in\mathcal{B}^+_{d,n-1}}|I|+\sum_{I\in\mathcal{B}^+_{d,n-2}}\sum_{i=1}^{d}(|I|+i)\\&=G^+_{d,m,0,1}(n-1)+d\,G^+_{d,m,0,1}(n-2)+\binom{d+1}{2}G^+_{d,m,0,0}(n-2).\end{aligned}$$

Also

$$G^+_{d,m,0,1}(0)=0,\quad G^+_{d,m,0,1}(1)=\sum_{i=1}^{d}i=\binom{d+1}{2}.$$

Then the generating function satisfies

$$\Psi_{d,m,0,1}-qG^+_{d,m,0,1}(1)=q\Psi_{d,m,0,1}+dq^2\Psi_{d,m,0,1}+\binom{d+1}{2}q^2\Psi_{d,m,0,0}.$$

Therefore,

$$\Psi_{d,m,0,1}=\frac{\binom{d+1}{2}q}{1-q-dq^2}+\frac{\binom{d+1}{2}q^2(dq+1)}{(1-q-dq^2)^2}.$$

Finally,

$$G^+_{d,m,0,1}(n)=\frac{1}{4d+1}\binom{d+1}{2}(n\,M_d(n)+d(2n+2)\,M_d(n-1)).\tag{4.16}$$

Example 4.7. Let $a=0$ and $b=2$. We have

$$\begin{aligned}G^+_{d,m,0,2}(n) &= \sum_{I\in\mathcal{B}^+_{d,n}}|I|^2 = \sum_{I\in\mathcal{B}^+_{d,n-1}}|I|^2 + \sum_{I\in\mathcal{B}^+_{d,n-2}}\sum_{i=1}^{d}(|I|+i)^2\\ &= G^+_{d,m,0,2}(n-1) + d\,G^+_{d,m,0,2}(n-2) + 2\binom{d+1}{2}G^+_{d,m,0,1}(n-2)\\ &\quad + \frac{1}{4}\binom{2d+2}{3}G^+_{d,m,0,0}(n-2).\end{aligned}$$

Also

$$\begin{aligned}G^+_{d,m,0,2}(0) &= 0,\\ G^+_{d,m,0,2}(1) &= \sum_{i=1}^{d} i^2 = \frac{1}{4}\binom{2d+2}{3}.\end{aligned}$$

Then the generating function satisfies

$$\Psi_{d,m,0,2} - qG^+_{d,m,0,2}(1) = q\Psi_{d,m,0,2} + dq^2\Psi_{d,m,0,2} + 2\binom{d+1}{2}q^2\Psi_{d,m,0,1} + \frac{1}{4}\binom{2d+2}{3}q^2\Psi_{d,m,0,0}.$$

Therefore,

$$\Psi_{d,m,0,2} = \frac{\frac{1}{4}\binom{2d+2}{3}q}{1-q-dq^2} + \frac{2\binom{d+1}{2}^2q^3 + \frac{1}{4}\binom{2d+2}{3}q^2(dq+1)}{(1-q-dq^2)^2} + \frac{2\binom{d+1}{2}^2q^4(dq+1)}{(1-q-dq^2)^3}.$$

Finally,

$$\begin{aligned}G^+_{d,m,0,2}(n) &= \left(\frac{1}{4}\binom{2d+2}{3}\frac{1}{4d+1} - \binom{d+1}{2}^2\frac{6}{(4d+1)^2}\right)\cdot n\,M_d(n)\\ &\quad + \frac{1}{4}\binom{2d+2}{3}\frac{2d}{4d+1}\cdot(n+1)\,M_d(n-1)\\ &\quad + \binom{d+1}{2}^2\frac{1}{(4d+1)^2}(n^2(4d+1)+3n-4d+2)\cdot M_d(n-1).\end{aligned}\tag{4.17}$$

Next, we show that $G^+_{d,m,a,b}(n)$ is a polynomial of d when other variables are fixed.

Theorem 4.8. *Let $m,n\geqslant 1$ and $a,b\geqslant 0$ be some given integers. Then $G^+_{d,m,a,b}(n)$ is a polynomial of d with degree*

$$2a+b+\left\lfloor\frac{n+1}{2}\right\rfloor.$$

Proof. The $a=b=0$ case is guaranteed by (4.2) and (4.3). Therefore we can assume that $a+b\geqslant 1$. We will prove this result by induction on n. It is easy to see that

$$\begin{aligned}G^+_{d,m,a,b}(1) &= \sum_{i=1}^{d}\left(\binom{i}{2}m+i\right)^a i^b,\\ G^+_{d,m,a,b}(2) &= \sum_{i=1}^{d}\left(\binom{i}{2}m+i\right)^a i^b + \sum_{i=1}^{d}\left(\binom{i}{2}m+2i\right)^a i^b\end{aligned}$$

are polynomials of d with degrees $2a+b+1$, which shows that the theorem is true for $n=1$ and $n=2$.

When $n\geqslant 3$, we assume that this result is true for $n-1$ and $n-2$. Therefore $G^+_{d,m,a,b}(n-1)$ and $d\,G^+_{d,m,a,b}(n-2)$ are polynomials of d with degrees

$$2a+b+\left\lfloor\frac{n}{2}\right\rfloor$$

and

$$2a+b+\left\lfloor\frac{n-1}{2}\right\rfloor+1=2a+b+\left\lfloor\frac{n+1}{2}\right\rfloor,$$

respectively. Also, for $a'\leqslant a$, $b'\leqslant b$ with $a'+b'<a+b$, we have

$$\binom{a}{a'}\binom{b}{b'}\sum_{i=1}^{d}\left(\binom{i}{2}m+in\right)^{a-a'}i^{b-b'}G^{+}_{d,m,a',b'}(n-2)$$

is a polynomial of d with degree

$$2(a-a')+(b-b')+1+2a'+b'+\left\lfloor\frac{n-2+1}{2}\right\rfloor=2a+b+\left\lfloor\frac{n+1}{2}\right\rfloor.$$

Therefore by (4.5) we prove the theorem. □

For $a,b,m,n\geqslant 0$ and $d\geqslant 1$, let

$$G^{-}_{d,m,a,b}(n):=\sum_{I\in\mathcal{B}^{-}_{d,n}}\sigma_m(I)^a|I|^b.$$

Then

$$G^{-}_{d,m,0,0}(n)=N_d(n+1)=M_d(n)+(d-1)M_d(n-1).$$

When $a+b>0$, it is obvious that

$$G^{-}_{d,m,a,b}(0)=0,\quad G^{-}_{d,m,a,b}(1)=G^{+}_{d-1,m,a,b}(1).\tag{4.18}$$

For $n\geqslant 2$, we have

$$\begin{aligned}G^{-}_{d,m,a,b}(n)&=\sum_{I\in\mathcal{B}^{+}_{d,n-1}}\sigma_m(I)^a|I|^b+\sum_{I\in\mathcal{B}^{+}_{d,n-2}}\sum_{i=1}^{d-1}\left(\sigma_m(I)+\binom{i}{2}m+in\right)^a(|I|+i)^b\\&=G^{+}_{d,m,a,b}(n-1)+(d-1)G^{+}_{d,m,a,b}(n-2)\\&\quad+\sum_{a'+b'<a+b}B^{a,b}_{a',b'}(d,m,n)G^{+}_{d,m,a',b'}(n-2),\end{aligned}\tag{4.19}$$

where

$$B^{a,b}_{a',b'}(d,m,n)=\binom{a}{a'}\binom{b}{b'}\sum_{i=1}^{d-1}\left(\binom{i}{2}m+in\right)^{a-a'}i^{b-b'}.$$

Similarly to the $G^{+}_{d,m,a,b}(n)$ case, we obtain the following results for $G^{-}_{d,m,a,b}(n)$.

Theorem 4.9. *Let $a,b\geqslant 0$ be some given integers. Then $G^{-}_{d,m,a,b}(n)$ is of the form*

$$A(m,n,d)M_d(n)+B(m,n,d)M_d(n+1),$$

where $A(m,n,d)$ and $B(m,n,d)$ are polynomials of m and n with degrees at most $2a+b$, whose coefficients are rational functions in d.

Theorem 4.10. *Let $m,n\geqslant 1$ and $a,b\geqslant 0$ be some given integers. Then $G^{-}_{d,m,a,b}(n)$ is a polynomial of d with degree $2a+b+\lfloor\frac{n+1}{2}\rfloor$.*

Now we are ready to prove the main theorems.

Proofs of Theorems 1.5 *and* 1.7. By Lemmas 3.2 and 3.5 we know

$$\sum_{\lambda\in\mathcal{C}_{n,dn+1}}|\lambda|^k\quad\text{and}\quad\sum_{\lambda\in\mathcal{C}_{n,dn-1}}|\lambda|^k$$

can be written as some linear combinations of $G^+_{d,n,a',b'}(n-1)$ and $G^-_{d,n,a',b'}(n-1)$, respectively, where $2a'+b'\leqslant 2k$. Notice that for each $i\in\mathbb{Z}$, $M_d(n+i)$ can be written as some linear combination of $M_d(n)$ and $M_d(n+1)$, whose coefficients are rational functions in d. Replace n by $n-1$, and m by n in Theorems 4.4 and 4.9, we obtain that $G^+_{d,n,a',b'}(n-1)$ and $G^-_{d,n,a',b'}(n-1)$ are of the form

$$A(n,d)M_d(n)+B(n,d)M_d(n+1),$$

where $A(n,d)$ and $B(n,d)$ are polynomials of n with degrees $2a'+b'\leqslant 2k$, whose coefficients are rational functions in d. Therefore Theorem 1.5 is true. Also, Theorem 1.7 follows from Theorems 4.8 and 4.10. □

5 Explicit formulas for expectations of $X_{n,dn+1}$ and $X_{n,dn-1}$

In this section we give proofs of Theorems 1.11 and 1.14.

Proof of Theorem 1.11. Let $k=1$ in Lemma 3.2. We have

$$\begin{aligned}\sum_{\lambda\in\mathcal{C}_{n,dn+1}}|\lambda| &= \sum_{I\in\mathcal{B}^+_{d,n-1}}\left(\sum_{(i,j)\in I}((i-1)n+j)-\frac{|I|^2}{2}+\frac{|I|}{2}\right)\\ &= G^+_{d,n,1,0}(n-1)-\frac{1}{2}G^+_{d,n,0,2}(n-1)+\frac{1}{2}G^+_{d,n,0,1}(n-1).\end{aligned}$$

Then by (4.15)–(4.17) we derive

$$\begin{aligned}\sum_{\lambda\in\mathcal{C}_{n,dn+1}}|\lambda| = M_d(n-1)\cdot&\left(\frac{-d(d+1)(d-1)(n-1)^2}{24(4d+1)}-\frac{d(d+1)(14d^2+21d+1)(n-1)}{24(4d+1)^2}\right.\\ &\left.-\frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2}\right)+M_d(n)\cdot\left(\frac{d(d+1)(5d+1)(n-1)^2}{24(4d+1)}\right.\\ &\left.+\frac{d(d+1)(32d^2+63d+7)(n-1)}{24(4d+1)^2}+\frac{d(d+1)(6d^2+27d+3)}{12(4d+1)^2}\right),\end{aligned}$$

which implies Theorem 1.11. □

Proof of Theorem 1.14. Let $k=1$ in Lemma 3.5. We have

$$\begin{aligned}\sum_{\lambda\in\mathcal{C}_{n,dn-1}}|\lambda| &= \sum_{I\in\mathcal{B}^-_{d,n-1}}\left(\sum_{(i,j)\in I}((i-1)n+j)-\frac{|I|^2}{2}+\frac{|I|}{2}\right)\\ &= G^-_{d,n,1,0}(n-1)-\frac{1}{2}G^-_{d,n,0,2}(n-1)+\frac{1}{2}G^-_{d,n,0,1}(n-1).\end{aligned}$$

But by the definitions of $G^+_{d,m,a,b}$ and $G^-_{d,m,a,b}$ we obtain

$$\begin{aligned}G^-_{d,n,1,0}(n-1) &= G^+_{d,n,1,0}(n-1)-G^+_{d,n,1,0}(n-3)-M_d(n-2)\left(\binom{d}{2}n+d(n-1)\right),\\ G^-_{d,n,0,2}(n-1) &= G^+_{d,n,0,2}(n-1)-\sum_{I\in\mathcal{B}^+_{d,n-3}}(|I|+d)^2\\ &= G^+_{d,n,0,2}(n-1)-G^+_{d,n,0,2}(n-3)-2d\,G^+_{d,n,0,1}(n-3)-d^2M_d(n-2)\end{aligned}$$

and

$$\begin{aligned}G^-_{d,n,0,1}(n-1) &= G^+_{d,n,0,1}(n-1)-\sum_{I\in\mathcal{B}^+_{d,n-3}}(|I|+d)\\ &= G^+_{d,n,0,1}(n-1)-G^+_{d,n,0,1}(n-3)-dM_d(n-2).\end{aligned}$$

Then by (4.15)–(4.17) we derive Theorem 1.14. □

6 Asymptotic formulas for moments of $X_{n,dn+1}$ and $X_{n,dn-1}$

In this section we study the asymptotic behavior for moments of $X_{n,dn+1}$ and $X_{n,dn-1}$. First, we give the proofs of Theorems 1.8 and 1.9.

Proof of Theorem 1.8. By the recurrence relations (1.1) and (1.2) it is easy to derive

$$M_d(n) = \frac{1}{\sqrt{1+4d}}\left(\left(\frac{1+\sqrt{1+4d}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4d}}{2}\right)^{n+1}\right) \tag{6.1}$$

and

$$N_d(n) = M_d(n) - M_d(n-2). \tag{6.2}$$

Then by Theorem 1.5 we have

$$\mathrm{E}[X_{n,dn+1}^k] = A(n,d) + B(n,d)\cdot\frac{(\frac{1+\sqrt{1+4d}}{2})^{n+2} - (\frac{1-\sqrt{1+4d}}{2})^{n+2}}{(\frac{1+\sqrt{1+4d}}{2})^{n+1} - (\frac{1-\sqrt{1+4d}}{2})^{n+1}},$$

where $A(n,d)$ and $B(n,d)$ are some polynomials of n with degrees at most $2k$. Therefore (1.6) holds. Similarly (1.7) follows from Theorem 1.5, (6.1) and (6.2). □

Proof of Theorem 1.9. By the recurrence relations (1.1) and (1.2) it is easy to see that $M_d(n)$ and $N_d(n)$ are polynomials of d with degrees $\lfloor n/2\rfloor$ when n is given. Then Theorem 1.9 follows from Theorem 1.7. □

Next, we consider the asymptotic formula for $G^+_{1,0,a,b}(n)$.

Theorem 6.1. *Suppose that a and b are two given non-negative integers. Let $\alpha := (1+\sqrt{5})/2$. Then*

$$G^+_{1,0,a,b}(n) = 2^{-a}5^{-(a+b+1)/2}n^{2a+b}\alpha^{n+2-a-b} + O(n^{2a+b-1}\alpha^n). \tag{6.3}$$

Proof. We will prove (6.3) by induction on $a+b$. When $a+b=0$, we have $a=b=0$. Let $d=1$ and $m=0$ in (4.4) we derive

$$G^+_{1,0,0,0}(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} = 5^{-1/2}\alpha^{n+2} + O(n^{-1}\alpha^n). \tag{6.4}$$

Next, assume that $a+b>0$, and (6.3) holds for all pairs (a',b') with $a'+b'<a+b$.

By (4.13) and Theorem 4.4, for any $a',b'\geqslant 0$, there exist some constants $C_{a',b'}$ and $D_{a',b'}$ such that

$$G^+_{1,0,a',b'}(n) = C_{a',b'}\; n^{2a'+b'}\;\alpha^n + D_{a',b'}n^{2a'+b'-1}\;\alpha^n + O(n^{2a'+b'-2}\alpha^n). \tag{6.5}$$

Letting $d=1$ and $m=0$ in (4.5) we derive

$$\begin{aligned} G^+_{1,0,a,b}(n) &= G^+_{1,0,a,b}(n-1) + G^+_{1,0,a,b}(n-2) + a\,n\,G^+_{1,0,a-1,b}(n-2) + b\,G^+_{1,0,a,b-1}(n-2) \\ &\quad + \sum_{a'+b'\leqslant a+b-2}\binom{a}{a'}\binom{b}{b'}n^{a-a'}G^+_{1,0,a',b'}(n-2), \end{aligned} \tag{6.6}$$

where $G^+_{d,m,a',b'}(n) := 0$ if $a'<0$ or $b'<0$. But by (6.5), when $a'+b'\leqslant a+b-2$,

$$n^{a-a'}G^+_{1,0,a',b'}(n-2) = O(n^{2a+b-2}\alpha^n).$$

Notice that $\alpha^n = \alpha^{n-1}+\alpha^{n-2}$. Also by (6.5), we have

$$G^+_{1,0,a,b}(n) - G^+_{1,0,a,b}(n-1) - G^+_{1,0,a,b}(n-2) = (\alpha+2)(2a+b)\,C_{a,b}\,n^{2a+b-1}\alpha^{n-2} + O(n^{2a+b-2}\alpha^n)$$

and

$$a\,n\,G^+_{1,0,a-1,b}(n-2) + b\,G^+_{1,0,a,b-1}(n-2) = (a\,C_{a-1,b} + b\,C_{a,b-1})n^{2a+b-1}\alpha^{n-2} + O(n^{2a+b-2}\alpha^n),$$

where $C_{a',b'} := 0$ if $a' < 0$ or $b' < 0$.

Therefore by (6.6), we have

$$((\alpha+2)(2a+b)C_{a,b} - (aC_{a-1,b} + bC_{a,b-1}))\, n^{2a+b-1}\, \alpha^{n-2} = O(n^{2a+b-2}\, \alpha^n),$$

which means that

$$(\alpha+2)(2a+b)C_{a,b} - (aC_{a-1,b} + bC_{a,b-1}) = 0. \tag{6.7}$$

By induction hypothesis we have

$$C_{a-1,\,b} = 2^{-a+1}\ 5^{-(a+b)/2}\ \alpha^{3-a-b} \quad \text{if} \ \ a \geqslant 1;$$

and

$$C_{a,\,b-1} = 2^{-a}\ 5^{-(a+b)/2}\ \alpha^{3-a-b} \quad \text{if} \ \ b \geqslant 1.$$

Notice that $\sqrt{5}\alpha = \alpha + 2$. Then by (6.7) we obtain

$$C_{a,b} = 2^{-a}\ 5^{-(a+b+1)/2}\ \alpha^{2-a-b}.$$

Therefore (6.3) holds. □

Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. By Lemma 3.2 and Theorem 6.1 we have

$$\begin{aligned}\sum_{\lambda\in\mathcal{C}_{n,n+1}} |\lambda|^k &= \sum_{a=0}^{k} \binom{k}{a} \left(-\frac{1}{2}\right)^{k-a} G^{+}_{1,0,a,2(k-a)}(n-1) + O(n^{2k-1}\alpha^n)\\ &= \left(\sum_{a=0}^{k} \binom{k}{a} (-1)^{k-a}\, 2^{-k}\ 5^{-(2k-a+1)/2}\ \alpha^{2-2k+a}\right)(n-1)^{2k}\, \alpha^{n-1} + O(n^{2k-1}\alpha^n)\\ &= 2^{-k}\ 5^{-(2k+1)/2}\, \alpha^{2-2k} \cdot (\sqrt{5}\alpha - 1)^k \cdot n^{2k}\, \alpha^{n-1} + O(n^{2k-1}\alpha^n).\end{aligned}$$

Notice that $\sqrt{5}\alpha - 1 = \alpha^2$. Then the above formula becomes

$$\sum_{\lambda\in\mathcal{C}_{n,n+1}} |\lambda|^k = 2^{-k}\ 5^{-(2k+1)/2}\, \alpha \cdot n^{2k}\, \alpha^n + O(n^{2k-1}\alpha^n).$$

Therefore

$$\begin{aligned}\mathrm{E}(X_{n,n+1}^k) &= \frac{1}{M_1(n)} \sum_{\lambda\in\mathcal{C}_{n,n+1}} |\lambda|^k\\ &= \frac{\sqrt{5}}{\alpha^{n+1} - (1-\alpha)^{n+1}} \cdot (2^{-k}\ 5^{-(2k+1)/2}\, \alpha \cdot n^{2k}\, \alpha^n + O(n^{2k-1}\alpha^n))\\ &= \left(\frac{1}{10}\right)^k n^{2k} + O(n^{2k-1}).\end{aligned}$$

The proof is completed. □

7 Further directions

In this paper, we derive several polynomiality results and asymptotic formulas for moments of sizes of random $(n, dn \pm 1)$-core partitions with distinct parts, which prove several conjectures of Zaleski [29]. In the recent years, the numbers, the largest sizes and the average sizes of $(n, n+1)$- and $(2n+1, 2n+3)$-core partitions with distinct parts were also well studied by many mathematicians (see [5, 13, 17, 20, 25, 26, 28, 30]). But for general (s, t)-core partitions with distinct parts, even for the $(n, n+3)$-core case, we know very little. We hope that the methods used and results obtained in this paper provide some clues for studying the general (s,t)-core case.

Also, Zaleski [29, Conjecture 3.4] conjectured that the distribution of $(n, dn-1)$-core partitions with distinct parts is asymptotically normal as n tends to infinity when d is given. At this moment, we are unable to prove this asymptotic distribution conjecture. By the idea from Zeilberger [31], to try to prove this conjecture, we need to have a better understanding of the leading terms in the asymptotic formulas of $\mathrm{E}[X^k_{n,dn+1}]$ and $\mathrm{E}[X^k_{n,dn-1}]$, which means that we should study the coefficients of the generating functions in (4.1). It would be interesting to find a proof of this distribution conjecture and furthermore study the distribution of general (s,t)-core partitions with distinct parts.

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