MECHANICAL DEDUCTION OF FORMULAS OF DIFFERENTIAL EQUATIONS (III)

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I. PROBLEM AND RESULTS

In our previous paper[1] general description of DEPS is given, and as an example concrete results of the system of the Bautin type are given in [2]. However, the number of parameters of the Bautin type is reduced by one using a special rotation, although it is inconvenient for general purposes.

In this article we retain all six parameters and deduce the V_j's for the following system of equations:

$$\begin{cases} \dfrac{dx}{dt} = y + (L_2 - L_4)x^2 + 2(L_7 - L_5 - L_3)xy + L_4y^2 \equiv X, \\ \dfrac{dy}{dt} = -x - L_5x^2 + 2(L_6 + L_4 - L_2)xy + (L_5 - L_7)y^2 \equiv Y. \end{cases} \tag{1.1}$$

II. SELECTION OF PARAMATER COMBINATIONS

We select System (1.1) of the parameter combinations for the following four reasons.

1. *Independence of the Parameters*

Comparing (1.1) with the following general type of equations

$$\begin{cases} \dfrac{dx}{dt} = y + \lambda_2x^2 + 2\lambda_3xy + \lambda_4y^2, \\ \dfrac{dy}{dt} = -x + \lambda_5x^2 + 2\lambda_6xy + \lambda_7y^2, \end{cases} \tag{2.1}$$

one gets the following relations

$$\lambda_2 = L_2 - L_4, \quad \lambda_3 = L_7 - L_5 - L_3, \quad \lambda_4 = L_4,$$
$$\lambda_5 = -L_5, \quad \lambda_6 = -L_2 + L_4 + L_6, \quad \lambda_7 = L_5 - L_7,$$

and the inverse relations

$$L_2 = \lambda_2 + \lambda_4, \quad L_3 = -\lambda_3 - \lambda_7, \quad L_4 = \lambda_4,$$
$$L_5 = -\lambda_5, \quad L_6 = \lambda_2 + \lambda_6, \quad L_7 = -\lambda_5 - \lambda_7,$$

so that the parameters L_j's ($j = 2, 3, \cdots 7$) are independent.

2. *Asymmetric Property of the Parameters*

Rotating the axes 90° such that (x, y) are replaced by $(y, -x)$, System (1.1) is transformed into

$$\begin{cases} dx/dt = y + (L_7 - L_5)x^2 + 2(-L_2 + L_4 + L_6)xy + L_3y^2, \\ dy/dt = -x + L_4x^2 + 2(L_3 + L_5 - L_7)xy + (-L_4 + L_2)y^2. \end{cases} \tag{2.2}$$

Comparing (1.1) with (2.2), one can see that the parameters

$$L_2,\ L_3,\ L_4,\ L_5,\ L_6 \text{ and } L_7$$

are replaced by

$$L_7,\ -L_6,\ L_5,\ -L_4,\ L_3 \text{ and } -L_2 \text{ respectively,}$$

or in short, L_j is replaced by $(-1)^j L_{9-j}$ $(j = 2, 3, \cdots 7)$. Since the rotation of axes does not affect the stability property of the singularity, so one obtains the relation.

$$V_j(L_2, L_3, L_4, L_5, L_6, L_7) = V(L_7, -L_6, L_5, -L_4, L_3, -L_2).$$

From the relation

$$\frac{d}{dt}\sum_{j=2}^{8} F_j(x, y; L_l) = \sum_{k=2} V_{2k-1}(L_l)y^{2k}$$

one gets

$$\frac{d}{dt}\sum_{j=2}^{8} F_j(y, -x; (-1)^l L_{9-l}) = \sum_{k=2} V_{2k-1}((-1)^l L_{9-l})x^{2k},$$

so that

$$\frac{d}{dt}\sum_{j=2}^{8}\left(\frac{F_j(x, y; L_l) + F_j(y, -x; (-1)^l L_{9-l})}{2}\right)$$

$$= \sum_{k=2}^{4} V_{2k-1}(L_l)(x^{2k} + y^{2k}),$$

where $l = 2, \cdots, 7.$

3. *The Divergence Relation*

When the divergence is identically zero, the singularity of the system is a center. Since

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 2L_6x - 2L_3y \equiv 0,$$

so one comes to that $L_3 = L_6 = 0$ leads to $V_j = 0$ for all j.

4. *The Harmonic Relations*

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} = 0, \text{ i. e. } \quad L_2 = 0,$$

$$\frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} = 0, \text{ i. e. } \quad L_7 = 0.$$

Then $L_2 = L_7 = 0$ leads to $V_j = 0$ for all j.

III. Concrete Results

$V_3 = (2 \text{ terms}) = (2/3)L_2L_3 - (2/3)L_6L_7;$

$V_5 = (21 \text{ terms});$

$\widetilde{V}_5 = V_5|_{V_3=0} = (14 \text{ terms}) = ;$

No. of Terms	Coefficient	Power					
		L_2	L_3	L_4	L_5	L_6	L_7
1	2/3	2			1	1	
2	− 2/1	2				1	1
3	2/1	1		1		1	1
4	− 4/15	1			1	2	
5	32/15	1				2	1
6	4/15		2	1			1
7	4/5		1		1	1	1
8	− 32/15		1			1	2
9	− 2/3		1	1			2
10	8/15		2			1	1
11	− 4/5			1		2	1
12	− 2/1				1	1	2
13	− 8/15					3	1
14	2/1					1	3

$V_7 =$ (129 terms);

$V_7|_{V_3=0} =$ (72 terms);

$V_7|_{V_3=V_5=0} = \widetilde{V}_7 =$ (43 terms) = .

No. of Terms	Coefficient	Power					
		L_2	L_3	L_4	L_5	L_6	L_7
1	8/35	1		2	1	2	
2	−24/35	1		2		2	1
3	16/175	1		1	1	3	
4	−48/175	1		1		3	1
5	152/105	1			2	2	1
6	8/35	1			3	2	
7	−116/21	1			1	2	2
8	−92/35	1				2	3
9	− 8/35		2	3			1
10	−24/35		1	2	1	1	1
11	24/35		1	2		1	2
12	−64/25		1	1	1	2	1
13	−96/175		2	2		1	1
14	64/175		1	1		2	2
15	−92/105		2	1			3
16	−8/35		2	1	2		1
17	16/21		3	1	1		1
18	−16/525		3	1			2
19	−16/25		2	1		2	1
20	16/105		4	1			1
21	−24/35		1		3	1	1
22	−40/7		1		2	1	2
23	64/35		2		2	1	1
24	132/35		1		1	1	3
25	−3488/525		2		1	1	2
26	208/105		3		1	1	1
27	−176/105		1		1	3	1
28	272/525		1			3	2
29	92/35		1			1	4

(to be continued)

(continued)

No. of Terms	Coefficient	Power					
		L_2	L_3	L_4	L_5	L_6	L_7
30	−872/525		2			1	3
31	−272/525		3			1	2
32	−32/105		2			3	1
33	−32/15		2	1	1		2
34	32/105		4			1	1
35	24/35			3		2	1
36	128/175			2		3	1
37	24/35			1	2	2	1
38	32/175			1		4	1
39	92/35			1		2	3
40	32/5			1	1	2	2
41	−32/105				2	3	1
42	1152/175				1	3	2
43	872/525					3	3

The numbers of terms of $F_j (j = 2, 3, \cdots, 7)$ are as follows:

j	2	3	4	5	6	7	8
No. of Terms of F_j	2	8	16	78	213	614	1337

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References

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[3] 刘尊全,秦朝斌,微分方程公式的机器推导 (II), 科学通报, **26** (1981), 257—258.