SCIENCE CHINA
Mathematics

• ARTICLES •

October 2023 Vol. 66 No. 10: 2325–2336
https://doi.org/10.1007/s11425-022-2080-x

Minimal Euler characteristics of 4-manifolds with 3-manifold groups

Hongbin Sun[1,*] & Zhongzi Wang[2,†]

[1]*Department of Mathematics, Rutgers University - New Brunswick, Piscataway, NJ* 08854*, USA;*
[2]*Department of Mathematical Sciences, Tsinghua University, Beijing* 100084*, China*

Email: hongbin.sun@rutgers.edu, wangzz22@stu.pku.edu.cn

Received September 23, 2022; accepted December 19, 2022; published online June 9, 2023

Abstract Let $\pi = \pi_1(M)$ for a compact 3-manifold M, and χ_4, p and q^* be the invariants of Hausmann and Weinberger (1985), Kotschick (1994) and Hillman (2002), respectively. For a certain class of compact 3-manifolds M, including all those not containing two-sided RP^2's, we determine $\chi_4(\pi)$. We address when $p(\pi)$ equals $\chi_4(\pi)$ and when $q^*(\pi)$ equals $\chi_4(\pi)$, and answer a question raised by Hillman (2002).

Keywords 3-manifold, 4-manifold, fundamental group, Euler characteristic

MSC(2020) 57N65, 57M05, 57K40

Citation: Sun H B, Wang Z Z. Minimal Euler characteristics of 4-manifolds with 3-manifold groups. Sci China Math, 2023, 66: 2325–2336, https://doi.org/10.1007/s11425-022-2080-x

1 Introduction

For a cell complex X, we use $H_i(X)$, $H^i(X)$ and $\beta_i(X)$ (resp. $H_i(X;\mathbb{Z}_2)$, $H^i(X;\mathbb{Z}_2)$ and $\beta_i(X;\mathbb{Z}_2)$) to denote its i-th homology group, i-th cohomology group and i-th Betti number with real coefficients (resp. $\mathbb{Z}_2$-coefficients). For a finitely presented group G, denote by $\beta_i(G)$ ($\beta_i(G;\mathbb{Z}_2)$) the i-th Betti number of $K(G,1)$, the classifying space of G. Denote by $\chi(X)$ the Euler characteristic of a finite CW complex X, and by $\sigma(X)$ the signature of a closed oriented 4-manifold X. In this paper, by 4-manifolds, we mean topological 4-manifolds, although all the 4-manifolds we will construct are smooth manifolds.

In 1985, Hausmann and Weinberger [5] introduced the 4-manifold Euler characteristic for a finitely presented group G, defined by

$$\chi_4(G) = \inf\{\chi(X) \mid X \text{ is a closed orientable 4-manifold and } \pi_1(X) \cong G\}.$$

There are also some variations of $\chi_4(G)$. In 1994, Kotschick [12] introduced

$$p(G) = \inf\{\chi(X) - |\sigma(X)| \mid X \text{ is a closed orientable 4-manifold and } \pi_1(X) \cong G\}.$$

In 2002, Hillman [7] introduced

$$q^*(G) = \inf\{\chi(X) \mid X \text{ is a closed 4-manifold and } \pi_1(X) \cong G\}.$$

† Current address: School of Mathematical Sciences, Peking University, Beijing 100871, China
* Corresponding author

It is known that

$$\chi_4(G) \geqslant p(G), \quad \chi_4(G) \geqslant q^*(G). \tag{1.1}$$

Note both $\chi_4(G)$ and $q^*(G)$ were denoted by $q(G)$ in their original definitions (see [5, 7]). Moreover, $q^*(G)$ was originally defined for PD_4-complexes in [7].

Kotschick [12, Theorems 2.8 and 4.2] made a useful observation to estimate the lower bounds of $\chi_4(G)$ and $p(G)$ when $\beta_4(G) = 0$. This observation is crucial for our work in this article (see the approach in Section 3).

More variations and generalizations of $\chi_4(G)$ can be found in [1, 2, 7, 11, 13].

An important family of finitely presented groups which can be classified are groups of compact 3-manifolds, which include all the cyclic groups, free groups, surface groups and knot groups. Many studies have been made for 4-manifolds with 3-manifold groups (see [7–9, 11, 12] and the references therein).

We survey known results on the above invariants of 3-manifold groups in the following theorem.

Theorem 1.1. *Suppose that M is a compact 3-manifold and $\pi_1(M) = \pi$.*

(1) $\chi_4(\pi) = p(\pi) = 2$ *if M is closed, orientable and aspherical* (*see* [12, *Proposition* 5.6]).

(2) $\chi_4(\pi) = 2 - 2q$ *if M is closed and orientable, where q is the maximal rank of the free groups in the free product decomposition of π* (*see* [7, *pp.* 61–62] *and* [11, *Theorem* 3.3]).

(3) $\chi_4(\pi) = 0$ *if π is the group of a knot complement in the 3-sphere S^3* (*see* [7, *Corollary* 3.12.3] *and* [11, *Theorem* 3.4]).

(4) *Suppose that M is a closed aspherical 3-manifold and $\pi_1(X) = \pi$ for a closed 4-manifold X. If X and M have the same orientability, then $\chi(X) > 0$ and $q^*(\pi) \in \{1, 2\}$* (*see* [7, *Theorem* 3.13 *and Corollary* 3.13.1]).

In [7, p. 63], Hillman asked whether the results in Theorem 1.1(4) can be extended to all the closed 3-manifold groups without torsions and without free $\mathbb{Z}$-factors.

In this paper, we try to determine $\chi_4(\pi)$ for fundamental groups of compact 3-manifolds. We also get some results related to $p(\pi)$ and $q^*(\pi)$. It turns out that for non-orientable 3-manifolds, the problem becomes more difficult, and some new approaches are needed.

For undefined terminologies, see [4, 6] about 3-manifolds, see [10] about 4-manifolds, and see [3] about algebraic topology.

The Kneser-Milnor theorem claims that each compact 3-manifold has a prime decomposition, whose prime factors are unique up to homeomorphism (and up to possibly replacing $S^2 \times S^1$ by $S^2 \tilde{\times} S^1$) and permutation (see [6]). So M has a prime decomposition

$$M = (\#_{i=1}^m M_i) \# (\#_{j=1}^n N_j) \# (\#_{l=1}^p Q_l) \# (\#_{e=1}^q S_e). \tag{1.2}$$

Here, each prime factor may or may not be orientable and belongs to one of the following categories:

(i) each M_i is a closed prime 3-manifold with $|\pi_1(M_i)| = \infty$ and is not an S^2- or RP^2-bundle over S^1;

(ii) each N_j is a closed prime 3-manifold with $|\pi_1(N_j)| < \infty$;

(iii) each Q_l is a prime 3-manifold and ∂Q_l is non-empty;

(iv) each S_e is an S^2- or RP^2-bundle over S^1.

Note that in (ii), each N_j is orientable, and in (iv), each RP^2-bundle over S^1 is homeomorphic to $RP^2 \times S^1$. Also each orientable 3-manifold contains no embedded 2-sided RP^2, and each 3-manifold containing embedded RP^2's has 2-torsions in its fundamental group.

We make the following conjecture about $\chi_4(\pi)$ and $p(\pi)$.

Conjecture 1.2. Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). Let $\pi = \pi_1(M)$. Then

$$\chi_4(\pi) = p(\pi) = 2 - 2(p + q) + \chi(\partial M).$$

If Conjecture 1.2 is true, it implies that any closed orientable 4-manifold X realizing $\chi_4(\pi)$ has signature zero.

We state our work on $\chi_4(\pi)$ and related invariants in the following three theorems. Theorem 1.3 is the main result of this paper and is more difficult to prove. Theorems 1.3 and 1.4 and Remark 1.6(1) support Conjecture 1.2.

Theorem 1.3. *Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). Let $\pi = \pi_1(M)$. If each M_i in (i) and Q_l in (iii) contains no two-sided RP^2, then*

$$\chi_4(\pi) = 2 - 2(p+q) + \chi(\partial M). \tag{1.3}$$

In particular, (1.3) holds when M contains no two-sided RP^2.

Theorem 1.4. *Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). Let $\pi = \pi_1(M)$. If each closed 3-manifold M_i in (i) is orientable, then*

$$p(\pi) = \chi_4(\pi) = 2 - 2(p+q) + \chi(\partial M). \tag{1.4}$$

In particular, (1.4) holds when M is orientable.

Theorem 1.4 confirms the question asked by Hillman [7] above, and in fact we prove a stronger result.

Theorem 1.5. *Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). Let $\pi = \pi_1(M)$. If π contains no 2-torsion, then*

$$q^*(\pi) = \chi_4(\pi) = 2 - 2(p+q) + \chi(\partial M). \tag{1.5}$$

In particular, if $p = q = 0$, then $q^*(\pi) = 2$.

Remark 1.6. (1) Let M be a closed, irreducible and non-orientable 3-manifold and $M \neq RP^2 \times S^1$. Let r be the number of disjoint non-parallel 2-sided RP^2's in M. Then $\chi_4(\pi_1(M)) = 2$ if $r = 0$ by Theorem 1.3, and

$$\chi_4(\pi_1(M)) \in \{0, 1, 2\} \quad \text{if } r > 0 \tag{1.6}$$

by the proof of Theorem 1.4 (see Proposition 4.2) since $p = q = \chi(\partial M) = 0$ and the number of non-orientable summands k is 1 in this case. (1.6) suggests that $\chi_4(\pi_1(M))$ should be independent of the number of 2-sided RP^2's in M, and to determine whether $\chi_4(\pi_1(M)) = 2$ holds is the first step to verify the conjecture.

(2) By Theorem 1.5, if $\pi_1(M)$ contains no 2-torsion, then $q^*(\pi)$ can be realized by closed orientable 4-manifolds. Note that $q^*(\mathbb{Z}_2) = 1$ is realized by the RP^4. So the equality $q^*(\pi) = \chi_4(\pi)$ in Theorem 1.5 is not true for all 3-manifold groups (since $\chi_4(\mathbb{Z}_2) = 2$ by Theorem 1.3).

The rest of this paper is organized as follows. In Section 2, we first construct 4-manifolds that provide an upper bound of $\chi_4(\pi)$ and compute Betti numbers of 3-manifold groups. In Section 3, we provide lower bounds of $\chi_4(\pi)$, $p(\pi)$ and $q^*(\pi)$. In Section 4, we prove Theorems 1.3–1.5.

2 An upper bound of $\chi_4(\pi)$ and computations of $\beta_i(\pi)$

We first list several standard facts which will be repeatedly used in our proofs.

Lemma 2.1. *Suppose that N_1 and N_2 are compact n-manifolds.*

(i) $\pi_1(N_1 \# N_2) = \pi_1(N_1) * \pi_1(N_2)$ *for $n \geqslant 3$.*

(ii) $\chi(N_1 \# N_2) = \chi(N_1) + \chi(N_2) - 2$ *for $n = 4$.*

(iii) *If $p : N_1 \to N_2$ is a covering map of degree p, then $\chi(N_1) = p\chi(N_2)$.*

An upper bound of $\chi_4(\pi)$ in Theorems 1.3–1.5 is given in the following proposition.

Proposition 2.2. *Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). Let $\pi = \pi_1(M)$. Then*

$$\chi_4(\pi) \leqslant 2 - 2(p+q) + \chi(\partial M).$$

For a compact 3-manifold M associated with the decomposition (1.2) and (i)–(iv), to prove Proposition 2.2, we only need to construct a closed orientable 4-manifold M^* with $\pi_1(M^*) = \pi_1(M)$ and $\chi(M^*) = 2 - 2(p+q) + \chi(\partial M)$.

Construction of M^*. Suppose that M is a compact 3-manifold and ∂M has k components $\{S_1, \ldots, S_k\}$.

Case 1. M is orientable and $\partial M \neq \emptyset$. Let $M^* = (M \times S^1) \cup (\bigcup_i S_i \times D^2)$, where each component $S_i \times S^1$ of $\partial(M \times S^1)$ is identified with $\partial(S_i \times D^2) = S_i \times S^1$ canonically. By the van Kampen theorem, we can verify that $\pi_1(M^*) = \pi_1(M)$.

Case 2. M is non-orientable and $\partial M \neq \emptyset$. Let $p : \tilde{M} \to M$ be the orientable double cover of M with a fixed-point free orientation reversing involution $\tau : \tilde{M} \to \tilde{M}$ such that $\tilde{M}/\tau = M$. Let $r : S^1 \to S^1$ be an orientation reversing involution on S^1. Now we have an orientation-preserving fixed-point free involution $\tau \times r : \tilde{M} \times S^1 \to \tilde{M} \times S^1$. Then $\tilde{M} \times S^1/\tau \times r$ is an orientable 4-manifold, which indeed is a twisted product $M \tilde{\times} S^1$. Each boundary component of $M \tilde{\times} S^1$ is either $S_j \times S^1$ if S_j is orientable, or $S_j \tilde{\times} S^1$ otherwise. Then we close these components canonically by $S_j \times D^2$ or $S_j \tilde{\times} D^2$, depending on whether S_j is orientable or not. Again we get a closed orientable 4-manifold M^* with $\pi_1(M^*) = \pi_1(M)$.

Case 3. M is closed. Let B^3 be a 3-ball in M. We define $\breve{M} = M \setminus \mathrm{int} B^3$. Then $\partial \breve{M} = S^2$ and $\breve{M}^*$ is defined.

Final construction. For a compact 3-manifold M with a prime decomposition given in (1.2), denote the connected sum of closed 3-manifold pieces in (i) and (ii) by $P = (\#_{i=1}^m M_i) \# (\#_{j=1}^n N_j)$. We suppose that there are q_1 prime factors that are S^2-bundles and q_2 prime factors that are RP^2-bundles with $q_1 + q_2 = q$. Then we define

$$M^* = \breve{P}^* \# (\#_{l=1}^p Q_l^*) \# (\#_{e=1}^{q_1} (S^3 \times S^1)) \# (\#_{f=1}^{q_2} (RP^3 \times S^1)), \tag{2.1}$$

which is a closed orientable 4-manifold.

By Lemma 2.1, it is easily shown that

$$\pi_1(M^*) \cong \pi_1(M) \quad \text{and} \quad \chi(M^*) = 2 - 2(p+q) + \chi(\partial M).$$

Then Proposition 2.2 follows.

The proofs of Theorems 1.3–1.5 use the following proposition.

Proposition 2.3. *Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). Let $\pi = \pi_1(M)$. Then we have*

(1) $\beta_4(\pi) = 0$;

(2) $\beta_2(\pi) = \sum_{i=1}^m \beta_2(M_i) + \sum_{l=1}^p \beta_2(Q_l)$.

Moreover, if π contains no 2-torsion, then

(3) $\beta_4(\pi; \mathbb{Z}_2) = 0$;

(4) $\beta_2(\pi; \mathbb{Z}_2) = \sum_{i=1}^m \beta_2(M_i; \mathbb{Z}_2) + \sum_{l=1}^p \beta_2(Q_l; \mathbb{Z}_2)$.

The following result is used to prove Propositions 2.3 and 3.1.

Proposition 2.4 (See [3, Proposition 3G.1]). *Let $p : \tilde{X} \to X$ be a finite regular cover with a deck transformation group Γ. Then*

(1) *$p^* : H^k(X) \to H^k(\tilde{X})$ is injective for each integer k;*

(2) *the image of p^* is the subspace $H^k(\tilde{X})^\Gamma$ consisting of elements fixed by all $\gamma \in \Gamma$.*

To prove Proposition 2.3, we first work on prime 3-manifolds.

Proposition 2.5. *Let M be a prime 3-manifold that is not an S^2- or RP^2-bundle over S^1, and $\pi = \pi_1(M)$. Then we have $\beta_2(\pi) = \beta_2(M)$.*

Proof. **Case I.** M is an orientable 3-manifold.

If M has a finite fundamental group, since no boundary component of M is S^2, M must be closed. Then the result holds since $\beta_2(M) = 0$ (by the Poincaré duality) and every finite group has zero Betti numbers in all positive dimensions (see, for example, [14, Theorem 6.5.8]).

If M has an infinite fundamental group, since M is not an S^2-bundle, it is an irreducible 3-manifold. By the sphere theorem and the Hurewicz theorem, M is aspherical and a model of $K(\pi,1)$. So $\beta_2(\pi)=\beta_2(M)$ holds.

Case II. M is a non-orientable 3-manifold.

Let $i: M\to X=K(\pi,1)$ be the inclusion that induces an isomorphism on π_1. Let $p_M:\tilde{M}\to M$ be the orientable double cover with $\pi_1(\tilde{M})=\tilde{\pi}<\pi$, and $p_X:\tilde{X}=K(\tilde{\pi},1)\to X=K(\pi,1)$ be the double cover corresponding to $\tilde{\pi}<\pi$. Let $\tau_M:\tilde{M}\to\tilde{M}$ and $\tau_X:\tilde{X}\to\tilde{X}$ be nontrivial deck transformations, and $\tilde{i}:\tilde{M}\to\tilde{X}$ be an inclusion. Then we have the following commutative diagrams:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{i}} & \tilde{X} \\ \downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle p_X} \\ M & \xrightarrow{i} & X, \end{array} \qquad \begin{array}{ccc} \tilde{M} & \xrightarrow{\tau_M} & \tilde{M} \\ \downarrow{\scriptstyle \tilde{i}} & & \downarrow{\scriptstyle \tilde{i}} \\ \tilde{X} & \xrightarrow{\tau_X} & \tilde{X}. \end{array}$$

Let $\mathbb{P}$ be a maximal collection of disjoint non-parallel two-sided RP^2's in M (which exists by an exercise in [4, p. 12]). Let $\mathbb{S}$ be the preimage of $\mathbb{P}$ in $\tilde{M}$. Then each component of $\mathbb{S}$ is a 2-sphere. Moreover, for each component K of $\tilde{M}\setminus\mathbb{S}$, if we cap off each S^2 boundary component of K by a 3-ball to obtain $\hat{K}$, a classical argument in 3-manifold topology implies that $\hat{K}$ is irreducible. So if we pinch each component of $\mathbb{S}$ to a point, we get a space $\tilde{X}^{(3)}$ homotopy-equivalent to a one-point union of orientable irreducible 3-manifolds and S^1. Then we can add cells of dimension at least 4 to $\tilde{X}^{(3)}$ to obtain $\tilde{X}$, which is a model of $K(\tilde{\pi},1)$. Since adding cells of dimension at least 4 does not affect H^2, we have an exact sequence

$$0=H^1(\mathbb{S})\to H^2(\tilde{X})=H^2(\tilde{X}^{(3)})\to H^2(\tilde{M})\to H^2(\mathbb{S}),$$

which is $\mathbb{Z}_2$-equivariant (induced by the action τ_X). Since $\mathbb{Z}_2$ acts on $H^2(\mathbb{S})$ as multiplied by -1, by applying Proposition 2.4 twice, we have

$$\beta_2(\pi)=\beta_2(X)=\dim H^2(\tilde{X})^{\tau_X^*}=\dim H^2(\tilde{M})^{\tau_M^*}=\beta_2(M).$$

This completes the proof. □

Proof of Proposition 2.3. Suppose that the 3-manifold M has a prime decomposition

$$M=(\#_{i=1}^m M_i)\,\#\,(\#_{j=1}^n N_j)\,\#\,(\#_{l=1}^p Q_l)\,\#\,(\#_{e=1}^q S_e)$$

as in (1.2). Here, each M_i is a closed prime 3-manifold with $|\pi_1(M_i)|=\infty$ and is not an S^2- or RP^2-bundle, each N_j is closed and has a finite fundamental group, each Q_l is prime and has non-empty ∂Q_l, and each S_e is an S^2- or RP^2-bundle over S^1.

Among the q prime factors of M that are S^2- or RP^2-bundles, we suppose that q_1 of them are S^2-bundles and q_2 of them are RP^2-bundles. Then a $K(\pi_1(M),1)$ space can be taken to be

$$(\vee_{i=1}^m K(\pi_1(M_i),1))\vee(\vee_{j=1}^n K(\pi_1(N_j),1))\vee(\vee_{l=1}^p K(\pi_1(Q_l),1))\vee(\vee^{q_1}S^1)\vee(\vee^{q_2}RP^\infty\times S^1).$$

We have

$$\begin{aligned}\beta_2(\pi_1(M))&=\sum_{i=1}^m\beta_2(\pi_1(M_i))+\sum_{j=1}^n\beta_2(\pi_1(N_j))+\sum_{l=1}^p\beta_2(\pi_1(Q_l))\\&=\sum_{i=1}^m\beta_2(M_i)+\sum_{j=1}^n\beta_2(N_j)+\sum_{l=1}^p\beta_2(Q_l)=\sum_{i=1}^m\beta_2(M_i)+\sum_{l=1}^p\beta_2(Q_l).\end{aligned}$$

Here, the second equality follows from Proposition 2.5, and the third equality follows from the fact that $\pi_1(N_j)$ is finite.

We can always find an orientable finite cover $\tilde{M}$ of M such that each prime factor $\tilde{M}$ is either aspherical or $S^2\times S^1$ or has finite π_1. Then by the argument we used above, it is easy to see that $\beta_4(\pi_1(\tilde{M}))=0$. Then by Proposition 2.4(1), $\beta_4(\pi_1(M))=0$. We have proved (1) and (2) of Proposition 2.3.

Suppose that π contains no 2-torsion. We conclude that each M_i and Q_l contains no 2-sided projective plane, each S_e is an S^2-bundle over S^1, and the fundamental group of each N_j has an odd order. As we discussed in the proof of Proposition 2.5, each M_i and Q_l is aspherical and

$$K(\pi_1(M),1) = (\vee_{i=1}^m M_i) \vee (\vee_{j=1}^n K(\pi_1(N_j),1)) \vee (\vee_{l=1}^p Q_l) \vee (\vee^q S^1).$$

Since $\pi_1(N_j)$ is odd, $\beta_k(\pi_1(N_j);\mathbb{Z}_2) = 0$ for all $k \geqslant 1$ (see, for example [14, Theorem 6.5.8]). Then (3) and (4) of Proposition 2.3 follow. □

3 Obstructions for lower bounds of $\chi_4(\pi)$, $p(\pi)$ and $q^*(\pi)$

Propositions 3.1, 3.4 and 3.5, the three obstruction results in this section, are for lower bounds of $\chi_4(\pi)$, $p(\pi)$ and $q^*(\pi)$, respectively. Both Proposition 3.1 and its proof are new, Proposition 3.4 is known, and both Proposition 3.5 and its proof are $\mathbb{Z}_2$-versions of known results.

Proposition 3.1. *Let G be a finitely presented group, and $\tilde{G} < G$ be an index-2 subgroup such that $\beta_4(\tilde{G}) = 0$. Then we have*

$$\chi_4(G) \geqslant 1 - \beta_1(\tilde{G}) + \beta_2(\tilde{G}) + |2(\beta_1(G) - \beta_2(G)) - (\beta_1(\tilde{G}) - \beta_2(\tilde{G})) - 1|.$$

Proof. Let X be a closed orientable 4-manifold with $\pi_1(X) \cong G$. Then we have a map $i : X \to K(G,1)$ that induces an isomorphism on π_1 (where we use the fact that any compact manifold is homotopy-equivalent to a CW-complex [3, Corollary A.12]).

Let $p : \tilde{X} \to X$ and $q : K(\tilde{G},1) \to K(G,1)$ be the double covers of X and $K(G,1)$ corresponding to $\tilde{G} < G$, respectively. We get the following commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & K(\tilde{G},1) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{i} & K(G,1). \end{array}$$

Since $\pi_1(\tilde{X}) \cong \tilde{G}$, we have $\beta_1(\tilde{X}) = \beta_1(\tilde{G})$. By the Poincaré duality, we have $\beta_3(\tilde{X}) = \beta_1(\tilde{X}) = \beta_1(\tilde{G})$. So we get

$$\chi(\tilde{X}) = 2 - 2\beta_1(\tilde{G}) + \beta_2(\tilde{X}), \quad \chi(X) = 1 - \beta_1(\tilde{G}) + \frac{1}{2}\beta_2(\tilde{X}). \tag{3.1}$$

To prove this proposition, we only need to bound $\beta_2(\tilde{X})$ from below.

Since $j : \tilde{X} \to K(\tilde{G},1)$ induces an isomorphism on π_1, $j^* : H^1(K(\tilde{G},1)) \to H^1(\tilde{X})$ is an isomorphism. Since $K(\tilde{G},1)$ can be obtained (up to homotopy equivalence) by attaching cells to $\tilde{X}$ of dimension at least 3, $j^* : H^2(K(\tilde{G},1)) \to H^2(\tilde{X})$ is injective.

Let $\tau_X : \tilde{X} \to \tilde{X}$ and $\tau_K : K(\tilde{G},1) \to K(\tilde{G},1)$ be the nontrivial deck transformations of $p : \tilde{X} \to X$ and $q : K(\tilde{G},1) \to K(G,1)$, respectively. By Proposition 2.4, for each n, $q^* : H^n(K(G,1)) \to H^n(K(\tilde{G},1))$ is injective, and we have

$$q^*(H^n(K(G,1))) = (H^n(K(\tilde{G},1)))^{\tau_K^*}.$$

Since $\tau_K^* : H^n(K(\tilde{G},1)) \to H^n(K(\tilde{G},1))$ gives a $\mathbb{Z}_2$-action, we have

$$H^n(K(\tilde{G},1)) = H^n(K(\tilde{G},1))_+ \oplus H^n(K(\tilde{G},1))_-,$$

where $H^n(K(\tilde{G},1))_+$ and $H^n(K(\tilde{G},1))_-$ denote the eigenspaces of τ_K^* corresponding to eigenvalues 1 and -1, respectively. Similarly, by considering the $\mathbb{Z}_2$-action on $H^n(\tilde{X})$ given by τ_X^* and the eigenspaces corresponding to 1 and -1, we have

$$H^n(\tilde{X}) = H^n(\tilde{X})_+ \oplus H^n(\tilde{X})_-.$$

Then we have

$$H^n(K(\tilde{G},1))_+ = H^n(K(\tilde{G},1))^{\tau_K^*} = q^*(H^n(K(G,1))).$$

Since $q^* : H^n(K(G,1)) \to H^n(K(\tilde{G},1))$ is injective, we have

$$\dim H^n(K(\tilde{G},1))_+ = \dim q^*(H^n(K(G,1))) = \dim H^n(K(G,1)) = \beta_n(G)$$

and

$$\dim H^n(K(\tilde{G},1))_- = \dim H^n(K(\tilde{G},1)) - \dim H^n(K(\tilde{G},1))_+ = \beta_n(\tilde{G}) - \beta_n(G).$$

Since $j^* : H^1(K(\tilde{G},1)) \to H^1(\tilde{X})$ is an isomorphism and commutes with the action of the deck transformation, we have

$$\dim H^1(\tilde{X})_+ = \dim j^*(H^1(K(\tilde{G},1))_+) = \dim H^1(K(\tilde{G},1))_+ = \beta_1(G)$$

and

$$\dim H^1(\tilde{X})_- = \dim j^*(H^1(K(\tilde{G},1))_-) = \dim H^1(K(\tilde{G},1))_- = \beta_1(\tilde{G}) - \beta_1(G).$$

So we get

$$\mathrm{tr}(\tau_X^* : H^1(\tilde{X}) \to H^1(\tilde{X})) = 2\beta_1(G) - \beta_1(\tilde{G}). \tag{3.2}$$

Since $\tau_X : \tilde{X} \to \tilde{X}$ is an orientation-preserving homeomorphism, for any $\alpha \in H^1(\tilde{X})$ and $\beta \in H^3(\tilde{X})$, we have $\alpha \cup \beta = \tau_X^*(\alpha) \cup \tau_X^*(\beta) \in H^4(\tilde{X}) \cong \mathbb{R}$. By the Poincaré duality, we have $\dim H^3(\tilde{X})_+ = \dim H^1(\tilde{X})_+ = \beta_1(G)$ and $\dim H^3(\tilde{X})_- = \dim H^1(\tilde{X})_- = \beta_1(\tilde{G}) - \beta_1(G)$, so we get

$$\mathrm{tr}(\tau_X^* : H^3(\tilde{X}) \to H^3(\tilde{X})) = 2\beta_1(G) - \beta_1(\tilde{G}). \tag{3.3}$$

By the Poincaré duality, the cup product $H^2(X) \times H^2(X) \to H^4(X) \cong \mathbb{R}$ is a non-singular bilinear form.

Now we need two lemmas, and the first one is well known.

Lemma 3.2. *Let V be a vector space of dimension n over a field F and q be a non-degenerate symmetric bilinear form on V. If q vanishes on a subspace W of dimension m, then $n \geqslant 2m$.*

Lemma 3.3. (1) *The restrictions of the cup product of $H^*(\tilde{X})$ on both $H^2(\tilde{X})_+ \times H^2(\tilde{X})_+$ and $H^2(\tilde{X})_- \times H^2(\tilde{X})_-$ are non-degenerate.*

(2) *The restrictions of the cup product of $H^*(\tilde{X})$ on both $j^*(H^2(K(\tilde{G},1))_+) \times j^*(H^2(K(\tilde{G},1))_+)$ and $j^*(H^2(K(\tilde{G},1))_-) \times j^*(H^2(K(\tilde{G},1))_-)$ are trivial.*

Proof. (1) For any non-zero elements $\alpha \in H^2(\tilde{X})_+$ and $\beta \in H^2(\tilde{X})_-$, then $\tau_X^*(\alpha) = \alpha$ and $\tau_X^*(\beta) = -\beta$, and hence

$$\tau_X^*(\alpha \cup \beta) = -\alpha \cup \beta.$$

However, since τ_X is orientation-preserving, we have

$$\tau_X^*(\alpha \cup \beta) = \alpha \cup \beta,$$

which implies that $\alpha \cup \beta = 0$.

Since the cup product $H^2(\tilde{X}) \times H^2(\tilde{X}) \to H^4(\tilde{X}) \cong \mathbb{R}$ is a non-singular bilinear form, for any non-zero $\alpha \in H^2(\tilde{X})_+$, there must be a non-zero element $\gamma \in H^2(\tilde{X})_+$ such that $\gamma \cup \alpha \neq 0$. So the restriction of the cup product of $H^*(\tilde{X})$ on $H^2(\tilde{X})_+ \times H^2(\tilde{X})_+$ is non-degenerate. The same argument works for $H^2(\tilde{X})_-$.

(2) Since $\beta_4(\tilde{G}) = 0$, the restrictions of the cup product of $H^*(K(\tilde{G},1))$ on $H^2(K(\tilde{G},1))_+ \times H^2(K(\tilde{G},1))_+$ and $H^2(K(\tilde{G},1))_- \times H^2(K(\tilde{G},1))_-$ are trivial. Since $j^*(H^2(K(\tilde{G},1))_+) \subset H^2(\tilde{X})_+$ and $j^*(H^2(K(\tilde{G},1))_-) \subset H^2(\tilde{X})_-$, the conclusion follows. □

By Lemmas 3.2 and 3.3, we have

$$\dim H^2(\tilde{X})_+ \geqslant 2\dim j^*(H^2(K(\tilde{G},1))_+) = 2\dim H^2(K(\tilde{G},1))_+ = 2\beta_2(G) \tag{3.4}$$

and

$$\dim H^2(\tilde{X})_- \geqslant 2\dim H^2(K(\tilde{G},1))_- = 2\beta_2(\tilde{G}) - 2\beta_2(G). \tag{3.5}$$

The above inequalities imply that

$$\dim H^2(\tilde{X})_+ = 2\beta_2(G) + \Delta_+, \quad \dim H^2(\tilde{X})_- = 2\beta_2(\tilde{G}) - 2\beta_2(G) + \Delta_- \tag{3.6}$$

for some non-negative integers Δ_+ and Δ_-. So we have

$$\mathrm{tr}(\tau_X^* : H^2(\tilde{X}) \to H^2(\tilde{X})) = 4\beta_2(G) - 2\beta_2(\tilde{G}) + (\Delta_+ - \Delta_-). \tag{3.7}$$

Since $\tau_X : \tilde{X} \to \tilde{X}$ has no fixed point, by the Lefschetz fixed-point theorem, (3.2), (3.3) and (3.7), we have

$$\begin{aligned} 0 &= \sum_{i=0}^{4} (-1)^i \mathrm{tr}(\tau_X^* : H^i(\tilde{X}) \to H^i(\tilde{X})) \\ &= 1 - (2\beta_1(G) - \beta_1(\tilde{G})) + (4\beta_2(G) - 2\beta_2(\tilde{G}) + (\Delta_+ - \Delta_-)) - (2\beta_1(G) - \beta_1(\tilde{G})) + 1 \\ &= 2 - 4(\beta_1(G) - \beta_2(G)) + 2(\beta_1(\tilde{G}) - \beta_2(\tilde{G})) + (\Delta_+ - \Delta_-). \end{aligned}$$

Thus

$$\Delta_+ - \Delta_- = 4(\beta_1(G) - \beta_2(G)) - 2(\beta_1(\tilde{G}) - \beta_2(\tilde{G})) - 2. \tag{3.8}$$

By (3.1), we have

$$\begin{aligned} \chi(X) &= 1 - \beta_1(\tilde{G}) + \frac{1}{2}(\dim H^2(\tilde{X})_+ + \dim H^2(\tilde{X})_-) \\ &= 1 - \beta_1(\tilde{G}) + \beta_2(\tilde{G}) + \frac{1}{2}(\Delta_+ + \Delta_-) \ (\text{by (3.6)}) \\ &\geqslant 1 - \beta_1(\tilde{G}) + \beta_2(\tilde{G}) + \frac{1}{2}|\Delta_+ - \Delta_-| \\ &= 1 - \beta_1(\tilde{G}) + \beta_2(\tilde{G}) + |2(\beta_1(G) - \beta_2(G)) - (\beta_1(\tilde{G}) - \beta_2(\tilde{G})) - 1| \ (\text{by (3.8)}). \end{aligned}$$

This completes the proof. □

Proposition 3.4 (See [12, Theorem 4.2])**.** *If* $\beta_4(\pi) = 0$*, then*

$$p(\pi) \geqslant 2 - 2\beta_1(\pi) + 2\beta_2(\pi).$$

Proposition 3.5. *Let* M *be a compact* 3*-manifold whose fundamental group* π *contains no* 2*-torsion, and* X *be a closed* 4*-manifold with* $\pi_1(X) = \pi$*. Then we have*

$$\beta_2(X; \mathbb{Z}_2) \geqslant 2\beta_2(\pi; \mathbb{Z}_2).$$

Since the group π of M contains no 2-torsion, $H^4(\pi; \mathbb{Z}_2)$ is trivial by Proposition 2.3(3). Then the proof of Proposition 3.5 is analogous to that of Proposition 3.4 (or directly see the proof of the inequality (3.4) above) by Lemma 3.2 for $F = \mathbb{Z}_2$.

4 Proofs of theorems

We introduce some notions and conventions to simplify computations in our proofs. For each finitely presented group G, define

$$b(G) = \beta_2(G) - \beta_1(G), \quad b(G;\mathbb{Z}_2) = \beta_2(G;\mathbb{Z}_2) - \beta_1(G;\mathbb{Z}_2).$$

The verification of the following fact is routine.

Lemma 4.1. *For finitely presented groups G_1 and G_2,*

$$b(G_1 * G_2) = b(G_1) + b(G_2), \quad b(G_1 * G_2;\mathbb{Z}_2) = b(G_1;\mathbb{Z}_2) + b(G_2;\mathbb{Z}_2).$$

Convention $(*)$. Suppose that M is a compact 3-manifold with $\pi = \pi_1(M)$ and with a prime decomposition described as in (1.2) and (i)–(iv).

Note that capping off each S^2 boundary component of ∂M by a 3-ball and swapping each S^2-bundle factor by $D^2 \times S^1$ do not affect either the group or the right-hand sides of the formulas (1.3)–(1.5) in Theorems 1.3–1.5. So below we assume that M has neither S^2 boundary component nor S^2-bundle factor.

4.1 $p(\pi) = \chi_4(\pi)$ when closed prime factors of M are orientable

Proof of Theorem 1.4. Suppose that M is a compact 3-manifold with a prime decomposition described as in (1.2) and (i)–(iv). We have

$$2 - 2(p+q) + \chi(\partial M) \geqslant \chi_4(\pi) \geqslant p(\pi) \geqslant 2 - 2\beta_1(\pi) + 2\beta_2(\pi) = 2 + 2b(\pi). \tag{4.1}$$

Here, the first inequality follows from Proposition 2.2, the second inequality follows from (1.1), and the third inequality follows from Propositions 3.4 and 2.3(1).

Theorem 1.4 follows from (4.1) and Proposition 4.2 which gives the explicit value of $2b(\pi)$ in terms of the presentation (1.2) with $k = 0$. □

Proposition 4.2. *Suppose that M is a compact* 3*-manifold with a prime decomposition described as in* (1.2) *and* (i)–(iv) *and let $\pi = \pi_1(M)$. Suppose that the number of closed non-orientable* 3*-manifolds M_i's in* (i) *is k. Then*

$$2 - 2\beta_1(\pi) + 2\beta_2(\pi) = 2 - 2(k+p+q) + \chi(\partial M).$$

Proof. By Convention $(*)$, we can rewrite M as $M = P\#N\#Q\#S$ satisfying the following:

(i) $P = \#_{i=1}^{m+n-k} M_i$ and each M_i is closed orientable and irreducible with either a finite or an infinite fundamental group;

(ii) $W = \#_{j=1}^{k} W_j$, and each W_j is a closed non-orientable 3-manifold and $W_j \neq RP^2 \times S^1$;

(iii) $Q = \#_{l=1}^{p} Q_l$ and each Q_l is a prime 3-manifold with non-empty ∂Q_l;

(iv) $S = \#_{e=1}^{q} S_e$ and each S_e is homeomorphic to $RP^2 \times S^1$.

For each M_i, we have

$$b(\pi_1(M_i)) = \beta_2(\pi_1(M_i)) - \beta_1(\pi_1(M_i)) = \beta_2(M_i) - \beta_1(M_i) = 0. \tag{4.2}$$

The second equality follows from Proposition 2.5, and the last equality follows from the Poincaré duality for closed orientable 3-manifolds.

For each W_j, we have

$$b(\pi_1(W_j)) = \beta_2(\pi_1(W_j)) - \beta_1(\pi_1(W_j)) = \beta_2(W_j) - \beta_1(W_j) = -1. \tag{4.3}$$

The second equality follows from Proposition 2.5, and the last equality follows from the Euler-Poincaré formula for closed non-orientable 3-manifolds.

For each Q_l, we have

$$b(\pi_1(Q_l)) = \beta_2(\pi_1(Q_l)) - \beta_1(\pi_1(Q_l)) = \beta_2(Q_l) - \beta_1(Q_l) = \chi(Q_l) - 1 = \frac{\chi(\partial Q_l)}{2} - 1. \tag{4.4}$$

The second equality follows from Proposition 2.5, and the third equality follows from the Euler-Poincaré formula for compact 3-manifolds with boundaries.

Since $H^*(RP^\infty \times S^1) = H^*(S^1)$, we have

$$b(\pi_1(RP^2 \times S^1)) = -1. \tag{4.5}$$

Since $b(*)$ is additive, by (4.2)–(4.5), we have

$$\begin{aligned} b(\pi) &= \sum_{i=1}^{m} b(\pi_1(M_i)) + \sum_{j=1}^{k} b(\pi_1(W_j)) + \sum_{l=1}^{p} b(\pi_1(Q_l)) + \sum_{e=1}^{q} b(\pi_1(S_e)) \\ &= \sum_{i=1}^{m} 0 + \sum_{j=1}^{k} (-1) + \sum_{l=1}^{p} \left(\frac{\chi(\partial Q_l)}{2} - 1 \right) + \sum_{e=1}^{q} (-1) = \frac{\chi(\partial M)}{2} - k - p - q. \end{aligned}$$

So $2b(\pi) = -2(k+p+q) + \chi(\partial M)$. □

4.2 $q^*(\pi) = \chi_4(\pi)$ for π containing no 2-torsion

Proof of Theorem 1.5. By Proposition 2.2 and (1.1), we have

$$2 - 2(p+q) + \chi(\partial M) \geqslant \chi_4(\pi) \geqslant q^*(\pi).$$

To prove Theorem 1.5, we need only to prove that

$$q^*(\pi) \geqslant 2 - 2(p+q) + \chi(\partial M). \tag{4.6}$$

The condition where $\pi_1(M)$ contains no 2-torsion implies that all the prime factors of M with finite fundamental groups are lens spaces $L(p,q)$ with odd p. Moreover, M contains no 2-sided RP^2, so each prime factor with an infinite fundamental group is aspherical. By Convention $(*)$, we can rewrite M as

$$M = (\#_{i=1}^m M_i) \# (\#_{j=1}^n N_j) \# (\#_{l=1}^p Q_l)$$

such that each M_i is a closed aspherical 3-manifold, each N_j is a lens space with groups of odd orders, and each Q_l is an aspherical 3-manifold with $\partial Q_l \neq \emptyset$. In particular, $q = 0$ holds here.

Using the Poincaré duality and the Euler-Poincaré formula in $\mathbb{Z}_2$-coefficients and Proposition 2.5, and doing the same computation as in the proof of Theorem 1.4, we have

$$b(\pi_1(M_i); \mathbb{Z}_2) = 0 \tag{4.7}$$

and

$$b(\pi_1(Q_l); \mathbb{Z}_2) = \chi(\partial Q_l)/2 - 1. \tag{4.8}$$

Since each $\pi_1(N_j) = Z_{k_j}$ with odd k_j, we have $\tilde{H}^*(K(\pi_1(N_j), 1); \mathbb{Z}_2) = 0$, and it follows immediately that

$$b(\pi_1(N_j); \mathbb{Z}_2) = 0. \tag{4.9}$$

Since $b(*; \mathbb{Z}_2)$ is additive, by (4.7)–(4.9) we have

$$\begin{aligned} b(\pi; \mathbb{Z}_2) &= \sum_{i=1}^{m} b(\pi_1(M_i); \mathbb{Z}_2) + \sum_{j=1}^{n} b(\pi_1(N_j); \mathbb{Z}_2) + \sum_{l=1}^{p} b(\pi_1(Q_l); \mathbb{Z}_2) \\ &= \sum_{i=1}^{m} 0 + \sum_{j=1}^{n} 0 + \sum_{l=1}^{p} (\chi(\partial Q_l)/2 - 1) = \chi(\partial M)/2 - p. \end{aligned}$$

So

$$\beta_2(\pi;\mathbb{Z}_2) = b(\pi;\mathbb{Z}_2) + \beta_1(\pi;\mathbb{Z}_2) = \chi(\partial M)/2 - p + \beta_1(\pi;\mathbb{Z}_2).$$

Suppose that X is a closed 4-manifold such that $\pi_1(X) \cong \pi$. By Proposition 3.5 and Proposition 2.3(3), we have

$$\beta_2(X;\mathbb{Z}_2) \geqslant 2\beta_2(\pi;\mathbb{Z}_2) = \chi(\partial M) - 2p + 2\beta_1(\pi;\mathbb{Z}_2).$$

Then

$$\chi(X) = 2 - 2\beta_1(X;\mathbb{Z}_2) + \beta_2(X;\mathbb{Z}_2) = 2 - 2\beta_1(\pi;\mathbb{Z}_2) + \beta_2(X;\mathbb{Z}_2) \geqslant 2 - 2p + \chi(\partial M).$$

So $q^*(\pi) \geqslant 2 - 2p + \chi(\partial M)$. □

4.3 The proof of Theorem 1.3

We may assume that M is non-orientable, and otherwise it is proved in Theorem 1.4. Suppose that $M = P\#N$, where P is orientable and each prime factor W of N is non-orientable and contains no 2-sided RP^2 unless $W = RP^2 \times S^1$.

Let $\tilde{M}$ be the orientable double cover of M, $G = \pi_1(M)$ and $\tilde{G} = \pi_1(\tilde{M})$. By Proposition 3.1 and Proposition 2.3(1), we have

$$\begin{aligned}\chi_4(G) &\geqslant 1 - \beta_1(\tilde{G}) + \beta_2(\tilde{G}) + |2(\beta_1(G) - \beta_2(G)) - (\beta_1(\tilde{G}) - \beta_2(\tilde{G})) - 1| \\ &= 1 + b(\tilde{G}) + |-2b(G) + b(\tilde{G}) - 1|.\end{aligned} \tag{4.10}$$

Let $\tilde{N}$ be the orientable double cover of N, and $\bar{P}$ be the orientation reversal of P. Then

$$\tilde{M} = P\#\tilde{N}\#\bar{P}$$

and

$$G = \pi_1(P) * \pi_1(N), \quad \tilde{G} = \pi_1(P) * \pi_1(\tilde{N}) * \pi_1(P).$$

By the additivity of $b(*)$, we have

$$b(G) = b(\pi_1(P)) + b(\pi_1(N)), \quad b(\tilde{G}) = 2b(\pi_1(P)) + b(\pi_1(\tilde{N})).$$

Substituting them into (4.10), we have

$$\chi_4(G) \geqslant 1 + 2b(\pi_1(P)) + b(\pi_1(\tilde{N})) + |-2b(\pi_1(N)) + b(\pi_1(\tilde{N})) - 1|. \tag{4.11}$$

Suppose that P has p_1 prime factors with a boundary. Since P is orientable, by Proposition 4.2, we have

$$2b(\pi_1(P)) = -2p_1 + \chi(\partial P). \tag{4.12}$$

Suppose that the prime decomposition of N is

$$N = (\#_{i=1}^{v} V_i)\,\#\,(\#_{j=1}^{p_2} Q_j)\,\#\,(\#_{e=1}^{q} RP^2 \times S^1),$$

where each V_i is closed and aspherical, and each Q_j is aspherical with a boundary. Let $\tilde{V}_i$ and $\tilde{Q}_j$ be the orientable double covers of V_i and Q_j, respectively. Then

$$\tilde{N} = (\#_{i=1}^{v} \tilde{V}_i)\,\#\,(\#_{j=1}^{p_2} \tilde{Q}_j)\,\#\,(\#_{e=1}^{v+p_2+2q-1} S^2 \times S^1).$$

Since $\tilde{N}$ is orientable and $\chi(\partial\tilde{N})/2 = \chi(\partial N)$, by Proposition 4.2, we have

$$b(\pi_1(\tilde{N})) = -p_2 - (v + p_2 + 2q - 1) + \chi(\partial\tilde{N})/2 = -(v + 2p_2 + 2q - 1) + \chi(\partial N). \tag{4.13}$$

By (4.3)–(4.5), we have

$$b(\pi_1(N)) = \sum_{i=1}^{v} b(V_i) + \sum_{j=1}^{p_2} b(Q_j) + \sum_{e=1}^{q} b(RP^2 \times S^1) = -(v + p_2 + q) + \chi(\partial N)/2. \tag{4.14}$$

By (4.13) and (4.14), we have

$$\begin{aligned} &|-2b(\pi_1(N)) + b(\pi_1(\tilde{N})) - 1| \\ &\quad = |2(v + p_2 + q) - \chi(\partial N) - (v + 2p_2 + 2q - 1) + \chi(\partial N) - 1| = v. \end{aligned} \tag{4.15}$$

Substituting (4.12), (4.13) and (4.15) into (4.10), we have

$$\begin{aligned} \chi_4(G) &\geqslant 1 + (-2p_1 + \chi(\partial P)) + (-(v + 2p_2 + 2q - 1) + \chi(\partial N)) + v \\ &= 2 - 2(p_1 + p_2) - 2q + \chi(\partial P) + \chi(\partial N) = 2 - 2(p + q) + \chi(\partial M). \end{aligned} \tag{4.16}$$

Then by Proposition 2.2, we have $\chi_4(G) = 2 - (p + q) + \chi(\partial M)$.

Acknowledgements The first author was supported by Simons Collaborations in Mathematics and the Physical Sciences (Grant No. 615229). The referees provided insightful and efficient suggestions which greatly simplify the proofs and computations. The authors thank the referees for the advice. The authors also thank Professors Jonathan Hillman, Dieter Kotschick, Yang Su and Shicheng Wang for their comments on the literature and known results on this topic.

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