

# Doubled conformal compactification

SUN ZhaoYong<sup>1,2\*</sup> & TIAN Yu<sup>1,2\*</sup>

<sup>1</sup>*School of Physics, University of Chinese Academy of Sciences, Beijing 100049, China;*

<sup>2</sup>*State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China;*

Received December 3, 2013; accepted March 6, 2014; published online July 3, 2014

We use Weyl transformations between the Minkowski spacetime and dS/AdS spacetime to show that one cannot well define the electrodynamics globally on the ordinary conformal compactification of the Minkowski spacetime (or dS/AdS spacetime), where the electromagnetic field has a sign factor (and thus is discontinuous) at the light cone. This problem is intuitively and clearly shown by the Penrose diagrams, from which one may find the remedy without too much difficulty. We use the Minkowski and dS spacetimes together to cover the compactified space, which in fact leads to the doubled conformal compactification. On this doubled conformal compactification, we obtain the globally well-defined electrodynamics.

**conformal transformations, electrodynamics, doubled Penrose diagrams, conformal compactification**

**PACS number(s):** 03.50.-z, 03.50.De, 03.50.Kk

**Citation:** Sun Z Y, Tian Y. Doubled conformal compactification. *Sci China-Phys Mech Astron*, 2014, 57: 1630–1636, doi: 10.1007/s11433-014-5487-0

## 1 Introduction

Conformal transformations play an interesting role in physics and mathematics [1]. Conventionally, there are two meanings of the term “conformal transformation” in the literature. One is what we call the conformal coordinate transformation, which is a special kind of coordinate transformations and whose infinitesimal generators are the conformal Killing vectors (CKVs). The other is the local rescaling (more commonly called the Weyl transformation), which acts directly on the metric and can be regarded as a kind of gauge transformation. These two meanings certainly have some relationship, but it is more appropriate to clearly distinguish them. Unless otherwise specified, we mean in this paper by the term “conformal transformation” as the conformal coordinate transformation.

It has been shown that Maxwell’s equations are invariant under the larger conformal group [2–5]. Codirila and Osborn

pointed out that one can easily obtain the electromagnetic field associated with uniformly accelerated charged particles by the conformal invariance of the Maxwell equations with point-like charges [6]. And conformal transformations in 2-dimension also play an important role in string theory (see e.g. ref. [7]). However, as what we will introduce in the next section, conformal transformations can not be globally defined on the Minkowski spacetime, which means that the conformal invariance of electrodynamics can only be regarded as local. In fact, conformal transformations can always globally act on a  $d$ -dimensional compactification spacetime ( $d > 2$ ), which is called the conformally compactified spacetime. This idea inspires us to compactify the Minkowski, even dS/AdS, spacetime. In fact, electrodynamics even cannot be well defined globally on the ordinary conformal compactification of the Minkowski spacetime. Instead, electrodynamics globally defined on the double covering of that conformal compactification is possible, which has long been noticed as mentioned in ref. [6]. We call this double covering the doubled conformal compactification. Interestingly, Penrose had shown the

\*Corresponding author (SUN ZhaoYong, email: sunzhaoyong11@mailsucas.ac.cn; TIAN Yu, email: ytian@ucas.ac.cn)

doubled Penrose diagrams in ref. [8], which is closely related to the doubled conformal compactification here. In this paper, we try to construct the doubled conformal compactification to make the electrodynamics globally defined.

Considering that the zero-radius pseudo-sphere in  $(4+2)$ -dimensional Minkowski space can always simplify the problem and give an obvious map about conformal compactification (see for example refs. [6,9–12]).

## 2 General theory and the projective cone

The conformal transformations form a group, named the conformal group, under certain conditions. It is well known that the conformal group on the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  is of  $(d+1)(d+2)/2$  dimensions for  $d > 2$  and of infinite dimensions for  $d = 2$ . But general conformal transformations cannot be globally defined on such a flat space, so this well-known result has only considered the local aspects of conformal transformations. In fact, nontrivial conformal transformations can be globally defined on the  $d$ -sphere  $S^d$  instead, where the conformal group is of  $(d+1)(d+2)/2$  dimensions whether  $d > 2$  or  $d = 2$ . There exists a conformal mapping between  $\mathbb{R}^d$  and  $S^d$ , which smoothly extends the conventional conformal transformations on the former to the globally defined ones on the latter. For  $d = 2$  only a  $(d+1)(d+2)/2 = 6$  dimensional subgroup of the infinite-dimensional “conformal group” on  $\mathbb{R}^2$  can be so extended, which can be regarded as the “global” conformal group on  $\mathbb{R}^2$ . These two distinct conformal groups for the  $d = 2$  case can also be identified as the “angle-preserving” one and the “circle-preserving” one, respectively, see ref. [10]. For another elucidation of this problem, see ref. [1], Chapter 5.

There is exactly one point on  $S^d$  that has no image on  $\mathbb{R}^d$  under the conformal mapping. That point, though actually not on  $\mathbb{R}^d$ , is called the infinity point (also known as the conformal boundary for the case that the space has a non-positive-definite signature) of  $\mathbb{R}^d$ . The procedure that adds the infinity point to  $\mathbb{R}^d$ , so that the conformal transformations can be globally defined, is known as the conformal compactification of  $\mathbb{R}^d$ . The resulting compactified space ( $S^d$  here) is also called the conformal compactification (of  $\mathbb{R}^d$ ).  $\mathbb{R}^d$  and  $S^d$  are both constant curvature (or maximal symmetry) spaces. In differential geometry it is known that only constant curvature spaces have  $(d+1)(d+2)/2$  independent CKVs ( $d > 2$ ). In fact, all the constant curvature spaces, with any metric signatures, have  $(d+1)(d+2)/2$ -dimensional “global” conformal groups and the corresponding conformal compactifications for  $d \geq 2$ <sup>1)</sup>.

To see how the doubly conformal compactification arises, let us first have a simple review of the ordinary conformal compactification of the Minkowski spacetime. As constant curvature spaces with the same signature, de Sitter (dS) and

anti-de Sitter (AdS) space times also have that conformal compactification, which can be all treated from the viewpoint of the projective cone<sup>2)</sup>. From now on, we concentrate on the 4-dimensional case.

The projective cone  $[\mathcal{N}]$  is defined as a zero-radius pseudo-sphere  $\mathcal{N}$  in a  $(4+2)$ -dimensional Minkowski space:

$$\eta_{AB}\zeta^A\zeta^B = 0, \quad (\eta_{AB}) = \text{diag}(-1, 1, 1, 1, 1, -1), \quad (1)$$

modulo the projective equivalence relation<sup>3)</sup>

$$(\zeta^A) \sim \lambda(\zeta^A), \quad \lambda \neq 0. \quad (2)$$

The equivalence class corresponding to the point  $(\zeta^A)$  is denoted by  $[\zeta^A]$ . The whole  $(4+2)$ -dimensional Minkowski space  $\mathbb{R}^6$  modulo the projective equivalence relation (2) is the 5-dimensional projective space  $\mathbb{R}P^5$ , so  $[\mathcal{N}]$  is a submanifold of  $\mathbb{R}P^5$ . It is obvious that the pseudo-sphere (1) and the equivalence relation (2) are both invariant under general  $O(2, 4)$  transformations, among which a  $\mathbb{Z}_2$  antipodal reflection

$$(\zeta^A) \rightarrow -(\zeta^A), \quad (3)$$

acts trivially on  $[\mathcal{N}]$ .

Then it can be shown that  $[\mathcal{N}]$  is the conformal compactification of all the Minkowski, dS and AdS space times, where the  $O(2, 4)/\mathbb{Z}_2$  transformations act as conformal transformations on these space times. These different space times can be regarded as different choices of representatives in the equivalence classes on  $\mathcal{N}$ . In fact, these constant curvature space times correspond to choose representative points by intersecting  $\mathcal{N}$  with hyperplanes in  $\mathbb{R}^6$ , where the metrics on them are naturally induced from  $\eta_{AB}$ . For the hyperplane  $\mathcal{P}_a$ :

$$a_A\zeta^A = 1, \quad (4)$$

with  $(a_A) \neq 0$  the normal vector, it can be shown that the intersection manifold is characterized by

$$S = \eta^{AB}a_Aa_B, \quad (5)$$

as follows: (1)  $S < 0$ : dS spacetime, (2)  $S = 0$ : Minkowski spacetime, (3)  $S > 0$ : AdS spacetime, where in any case  $S$  is just the scalar curvature of the intersection manifold.

More concretely, to get the Minkowski spacetime we just choose a light-like normal vector

$$(a_A) = (0, 0, 0, 0, 1, 1). \quad (6)$$

Thus the intersection  $M$  of  $\mathcal{N}$  and the corresponding  $\mathcal{P}_a$  is flat with respect to the metric induced from  $\eta_{AB}$ , and can be parametrized by

$$x^\mu = L \frac{\zeta^\mu}{\zeta^+}, \quad \mu = 0, \dots, 3, \quad (7)$$

1)  $S^d$  is its own conformal compactification.

2) Also called the null cone or the Lie sphere.

3) In order for  $[\mathcal{N}]$  to be a (4-dimensional) manifold, the origin  $(\zeta^A) = 0$  must be excluded.

with  $L$  an arbitrary length scale parameter and  $\zeta^\pm = \zeta^5 \pm \zeta^4$  the lightcone coordinates. The induced metric is proportional to vect

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (8)$$

so  $x^\mu$  is just the Cartesian coordinates on  $M$ . Then it is straightforward to show that the  $O(2,4)/\mathbb{Z}_2$  transformations act as conformal transformations on  $M$  (for more details, see ref. [8]). Note that some equivalence classes on  $\mathcal{N}$ , which correspond to the infinity points of  $M$ , have no representatives on  $\mathcal{P}_a$ . They constitute precisely the intersection of  $\mathcal{N}$  and the hyperplane

$$a_A \zeta^A = 0, \quad (9)$$

which is parallel to  $\mathcal{P}_a$ . Adding those infinity points to  $M$  produces its conformal compactification  $[\mathcal{N}]$ , whose  $O(2,4)/\mathbb{Z}_2$  action is fully well-defined.

The dS and AdS space times can be obtained by choosing typically

$$(a_A) = (0, 0, 0, 0, 0, 1) \quad (10)$$

and

$$(a_A) = (0, 0, 0, 0, 1, 0), \quad (11)$$

respectively. However, they also have infinity points, or conformal boundary, to be included for a fully well-defined  $O(2,4)/\mathbb{Z}_2$  conformal action, since no single hyperplane can contain representatives of all the equivalence classes on  $\mathcal{N}$ . To remedy this problem, one may use general hypersurfaces (of antipodal symmetry) to intersect  $\mathcal{N}$ . It can be shown that these intersection manifolds are all conformally flat<sup>4)</sup>, and that the  $O(2,4)/\mathbb{Z}_2$  transformations induce conformal transformations on them. The simplest choice of this hypersurface is a 5-sphere

$$\delta_{AB} \zeta^A \zeta^B = 2L^2, \quad (12)$$

which intersects all the equivalence classes on  $\mathcal{N}$  precisely twice. The intersection of  $\mathcal{N}$  and the 5-sphere (12) is an  $S^1 \times S^3$ :

$$\begin{aligned} (\zeta^0)^2 + (\zeta^5)^2 &= L^2, \\ (\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2 + (\zeta^4)^2 &= L^2. \end{aligned} \quad (13)$$

Upon the antipodal identification (3), one sees that  $[\mathcal{N}]$  is of topology

$$S^1 \times S^3 / \mathbb{Z}_2, \quad (14)$$

which is actually homeomorphic to  $S^1 \times S^3$ .

Although there is no natural metric defined on  $[\mathcal{N}]$ , there is an induced metric on the intersection manifold (13) (modulo the antipodal identification), which is called  $N$  hereafter.  $N$  is conformally flat and has a globally defined  $O(2,4)/\mathbb{Z}_2$  conformal group, so it can be regarded as a metrical realization of  $[\mathcal{N}]$ , being the conformal compactification of all the Minkowski, dS and AdS space times.

The above construction of conformal compactification can be extended to any dimension and any metric signature. However, for even  $d$  it can be shown that  $S^1 \times S^d / \mathbb{Z}_2$  is not homeomorphic to  $S^1 \times S^d$  but is a non-orientable manifold, so the conformal compactification of  $(1+d)$ -dimensional spacetime is not simple and in some sense not suitable to be a spacetime. One immediately sees that this shortcoming can be simply overcome by discarding the  $\mathbb{Z}_2$  antipodal identification. That is the doubly conformal compactification, which can be realized by replacing the projective equivalence relation (2) with the pseudo-projective one:

$$(\zeta^A) \sim \lambda(\zeta^A), \quad \lambda > 0. \quad (15)$$

For the  $d = 3$  case, we use  $[\mathcal{N}]_+$  to denote  $\mathcal{N}$  modulo the above equivalence relation, i.e., the doubly conformal compactification of all the Minkowski, dS and AdS space times, whose metrical realization is exactly the intersection manifold (13), denoted by  $2N$ . Correspondingly, the conformal group on (13) is  $O(2,4)$  instead of  $O(2,4)/\mathbb{Z}_2$ .

### 3 The breakdown of the Maxwell equation on the compactification of Minkowski spacetime

We have shown the geometrical process of compactification. It is the most important that physical considerations support the introduction of the doubly conformal compactification. At this section we will introduce the discontinuity of electrodynamics caused by compactifying the Minkowski spacetime. The action functional of electrodynamics in general space times is

$$S_{\text{EM}} = \int d^4x \sqrt{-g} \left( -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + g^{\mu\nu} J_\mu A_\nu \right), \quad (16)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

In this paper the metrics always have a  $(-, +, +, +)$  signature. For the Minkowski spacetime with Cartesian coordinates  $(g_{\mu\nu} = \eta_{\mu\nu})$ , the above action functional is invariant (up to a boundary term) under the conformal transformations provided that  $A_\mu$  and  $J_\mu$  transform as:

$$A_\mu(x) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{A}_\nu(\tilde{x}), \quad J_\mu(x) = \Omega^2 \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{J}_\nu(\tilde{x}), \quad (17)$$

respectively, where the transformation of  $A_\mu$  is just trivial and that of  $J_\mu$  contains a conformal factor  $\Omega^2$  defined by

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial \tilde{x}^\nu}{\partial x^\sigma} \eta_{\mu\nu} = \Omega^2 \eta_{\rho\sigma}. \quad (18)$$

In fact, the action functional (16) in general space times is invariant under, in addition to the diffeomorphism, the Weyl transformation

$$g_{\mu\nu}(x) = k^{-2}(x) g'_{\mu\nu}(x), \quad (19)$$

provided that  $A_\mu$  (and thus  $F_{\mu\nu}$ ) is invariant and  $J_\mu$  transforms as:

$$J_\mu(x) = k^2(x) J'_\mu(x). \quad (20)$$

4) So they are all related by Weyl transformations (also called conformal mappings), at least locally.

Since the dS, AdS and  $N$  space times can all be obtained by Weyl transformations from the Minkowski spacetime, where the Cartesian coordinates  $x^\mu$  become the conformally flat coordinates on these space times, we can easily map the electrodynamics from the Minkowski spacetime to them.

The conformally flat coordinates on the dS spacetime can be obtained by the stereographic projection [14], with the Weyl factor

$$k_{\text{dS}}^{-2}(x) = \left(1 + \frac{x^2}{4R^2}\right)^2, \quad x^2 = \eta_{\mu\nu}x^\mu x^\nu, \quad (21)$$

with  $R$  the dS radius. The AdS case is simply achieved by replacing  $R^2$  with  $-R^2$  for most of the dS expressions, so we only mention the dS case. The Weyl factor for certain conformally flat coordinates on  $N$  can be shown to be

$$k_N^{-2}(x) = 1 + \frac{x^0 x^0 + \sum_i x^i x^i}{2L^2} + \left(\frac{x^2}{4L^2}\right)^2, \quad (22)$$

which is positive definite reflecting the fact that  $N$  has no conformal boundary.

The action functional (16) with  $g_{\mu\nu} = \eta_{\mu\nu}$

$$S_{\text{EM}} = \int d^4x \left( -\frac{1}{4} \eta^{\mu\alpha} \eta^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \eta^{\mu\nu} J_\mu A_\nu \right) \quad (23)$$

leads to the familiar equation of motion

$$\eta^{\alpha\mu} \partial_\alpha F_{\mu\nu} = J_\nu. \quad (24)$$

The simplest solution to the above equation is the Coulomb field for a static point charge:  $\mathbf{x}$

$$\begin{aligned} E_i &= F_{i0} = \frac{e}{4\pi} \frac{x^i}{|\mathbf{x}|^3}, \quad B_i = \frac{1}{2} \epsilon^{ijk} F_{jk} = 0, \\ J_0 &= e\delta^3(\mathbf{x}), \quad J_i = 0. \end{aligned} \quad (25)$$

General solutions to eq. (24) are just linear combinations of this fundamental solution. Of special interest is the electromagnetic field associated with uniformly accelerated point charge, which can be obtained from the solution (25) by conformal transformations [6]. There one sees that, however, an additional sign factor  $\epsilon(\Omega)$  has to be inserted (or equivalently, discarded) to attain a globally defined solution. That, in fact, already indicates the doubly conformal compactification. In the following, we will use Weyl transformations to map the solution (25) onto the dS and  $N$  space times, where the doubly conformal compactification is shown to be necessary. For simplicity, we take  $R = 1/2$  in eq. (21) and massless  $L = 1/2$  in eq. (22).

First we consider the dS case. In this case we have from eqs. (20) and (21)

$$J'_0 = e(1 + x^2)^2 \delta^3(\mathbf{x}), \quad J'_i = 0, \quad (26)$$

while  $E'_i$  and  $B'_i$  are the same as  $E_i$  and  $B_i$  in equation (25), respectively. Note that there are actually two antipodal point

charges here, due to the conformal boundary  $1 + x^2 = 0$  separating the world line  $\mathbf{x} = 0$  into two parts. Properly speaking, the line  $\mathbf{x} = 0$  is separated into three segments, but the outer two of them are joined through the Minkowski conformal infinity, as shown in Figure 1(a), where the dash line KN can be regarded as the world line of the charges in dS spacetime. For convenience, we omit the prime in these notations in the following equations. Since one patch of conformally flat coordinates cannot cover the whole dS spacetime<sup>5)</sup>, other coordinate patches must also be used to check whether the above solution is globally defined on the dS spacetime. It is easy to see that one more (conformally-flat-coordinate) patch is sufficient, which can be viewed as the inversion

$$x^\mu = \frac{\tilde{x}^\mu}{\tilde{x}^2} \quad (27)$$

of the original conformally flat coordinates. One can define special conformal transformations by combining an inversion with a translation and then another inversion [6]. Under the coordinate transformation (27), we have

$$\begin{aligned} \tilde{J}_\mu(\tilde{x}) &= \frac{\partial x^\nu}{\partial \tilde{x}^\mu} J_\nu(x) = \left[ \frac{\delta_\mu^\nu}{\tilde{x}^2} - 2 \frac{\tilde{x}^\nu \tilde{x}_\mu}{(\tilde{x}^2)^2} \right] J_\nu(x), \\ \tilde{x}_\mu &= \eta_{\mu\nu} \tilde{x}^\nu, \end{aligned} \quad (28)$$

which means

$$\begin{aligned} \tilde{J}_0(\tilde{x}) &= \frac{J_0(x)}{\tilde{x}^2} - 2\tilde{x}_0 \frac{\tilde{x}^0 J_0(x) + \tilde{x}^i J_i(x)}{(\tilde{x}^2)^2} \\ &= e[1 + (\tilde{x}^2)^{-1}]^2 \left[ \frac{1}{\tilde{x}^2} + \frac{2\tilde{t}^2}{(\tilde{x}^2)^2} \right] \delta^3\left(\frac{\tilde{\mathbf{x}}}{\tilde{x}^2}\right) \\ &= e(1 - \tilde{t}^2)^2 \delta^3(\tilde{\mathbf{x}}), \end{aligned} \quad (29)$$

with  $\tilde{t} = \tilde{x}^0$ , and

$$\tilde{J}_i(\tilde{x}) = \frac{J_i(x)}{\tilde{x}^2} - 2\tilde{x}_i \frac{\tilde{x}^0 J_0(x) + \tilde{x}^j J_j(x)}{(\tilde{x}^2)^2} = 0. \quad (30)$$

At the same time, the electromagnetic field transforms as:

$$\begin{aligned} \tilde{F}_{\mu\nu} &= \left[ \frac{\delta_\mu^\rho}{\tilde{x}^2} - 2 \frac{\tilde{x}^\rho \tilde{x}_\mu}{(\tilde{x}^2)^2} \right] \left[ \frac{\delta_\nu^\sigma}{\tilde{x}^2} - 2 \frac{\tilde{x}^\sigma \tilde{x}_\nu}{(\tilde{x}^2)^2} \right] F_{\rho\sigma} \\ &= \frac{F_{\mu\nu}}{(\tilde{x}^2)^2} - 2 \frac{\tilde{x}^\rho \tilde{x}_\mu}{(\tilde{x}^2)^3} F_{\rho\nu} - 2 \frac{\tilde{x}^\sigma \tilde{x}_\nu}{(\tilde{x}^2)^3} F_{\mu\sigma}, \end{aligned} \quad (31)$$

which means

$$\begin{aligned} \tilde{F}_{i0} &= (\tilde{x}^2)^{-3} [\tilde{x}^2 F_{i0} - 2\tilde{x}_i \tilde{x}^j F_{j0} + 2\tilde{t}(\tilde{t} F_{i0} + \tilde{x}^j F_{ij})] \\ &= \frac{e}{4\pi} (\tilde{x}^2)^{-3} \left[ (\tilde{t}^2 + \tilde{\mathbf{x}}^2) \frac{\tilde{x}^i / \tilde{x}^2}{|\tilde{\mathbf{x}} / \tilde{x}^2|^3} - 2\tilde{x}^i \frac{\tilde{\mathbf{x}}^2 / \tilde{x}^2}{|\tilde{\mathbf{x}} / \tilde{x}^2|^3} \right] \\ &= -\epsilon(\tilde{x}^2) \frac{e}{4\pi} \frac{\tilde{x}^i}{\tilde{\mathbf{x}}} \end{aligned} \quad (32)$$

with  $\epsilon(\tilde{x}^2)$  the sign of  $\tilde{x}^2$  and

$$\tilde{F}_{ij} = -2(\tilde{x}^2)^{-3} \tilde{t}(\tilde{x}_i F_{0j} + \tilde{x}_j F_{i0}) = 0. \quad (33)$$

<sup>5)</sup> The uncovered part corresponds to (part of) the conformal boundary of the Minkowski spacetime.

The discontinuity of  $\tilde{F}_{i0}$  from the  $\epsilon(\tilde{x}^2)$  factor in eq. (32) indicates the breakdown of the Maxwell equation at  $\tilde{x}^2 = 0$ . In fact, one may roughly understand this breakdown by the usual argument that there can be no net charge on compact spaces, since the dS spacetime can be viewed as an expanding  $S^3$  and it can be shown from eqs. (26) and (29) that there are two charges of the same signature on any one of the  $S^3$  simultaneity hypersurfaces.

Also, it's interesting to consider about Maxwell equation with the magnetic and electric charges. The solution (25) is changed to be

$$E_i = F_{i0} = \frac{e}{4\pi} \frac{x^i}{|\mathbf{x}|^3}, \quad B_i = \frac{1}{2} \epsilon^{ijk} F_{jk} = \frac{e_m}{4\pi} \frac{x^i}{|\mathbf{x}|^3},$$

$$J = e\delta^3(\mathbf{x}), \quad J_0^m = e_m\delta^3(\mathbf{x}),$$

where the  $e_m$  and  $J^m$  are the magnetic charge and current. With the second equation, one can easily see that the field

$$F_{ij} = \epsilon_{kij} B_k = \frac{e_m}{4\pi} \frac{\epsilon_{kij} x^k}{|\mathbf{x}|^3} \neq 0.$$

So the transformation of the field seems different with eqs. (32) and (33). In fact, after adding the magnetic charge, there will be a none zero term in  $\tilde{F}_{i0}$ , which means that the magnetic charges will contribute to the electric field in the conformal transformation. But the discontinuity of the  $\tilde{F}_{i0}$  at  $\mathbf{x} = 0$  still exists since the discontinuity is actually caused by the  $|\mathbf{x}|^3$  term. Evidently, the magnetic field is not zero, but a function containing the  $|\mathbf{x}|^3$  term also has a  $\epsilon(\tilde{x}^2)$  factor. This indicates the discontinuity of the magnetic field at  $\tilde{x}^2 = 0$ . One can roughly understand this by thinking about the symmetry of electric and the magnetic charges.

#### 4 Penrose diagrams and the doubled covering

The above discussion can be illustrated with Penrose diagrams, as in Figure 1. The lightcone  $\tilde{x}^2 = 0$  corresponds

to the dashed line segments  $JG$  and  $IH$  in Figure 1(b). Then it is easy to see that the discontinuity of  $\tilde{F}_{i0}$  at  $\tilde{x}^2 = 0$  can be removed by reversing the sign of the electromagnetic field and electric current:

$$A_\mu \rightarrow -A_\mu, \quad J_\mu \rightarrow -J_\mu, \quad (34)$$

in the regions “I” to “IV”, so that the Maxwell equation

$$\eta^{\alpha\mu} \partial_\alpha F_{\mu\nu} = k_{\text{dS}}^2(x) J_\nu \quad (35)$$

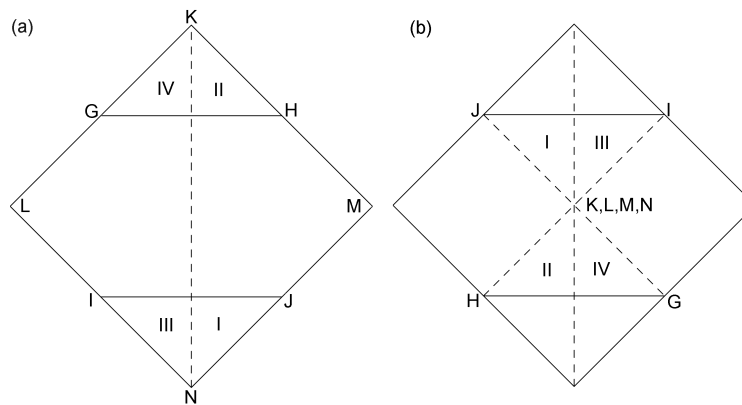
can be satisfied on the whole dS spacetime. We explicitly write this global solution as:

$$E_i = \epsilon(1+x^2) \frac{e}{4\pi} \frac{x^i}{|\mathbf{x}|^3}, \quad B_i = 0, \quad (36)$$

$$J_0 = \epsilon(1+x^2)e(1+x^2)^2\delta^3(\mathbf{x}), \quad J_i = 0.$$

For this solution the two antipodal point charges on the dS spacetime have opposite signs<sup>6)</sup>.

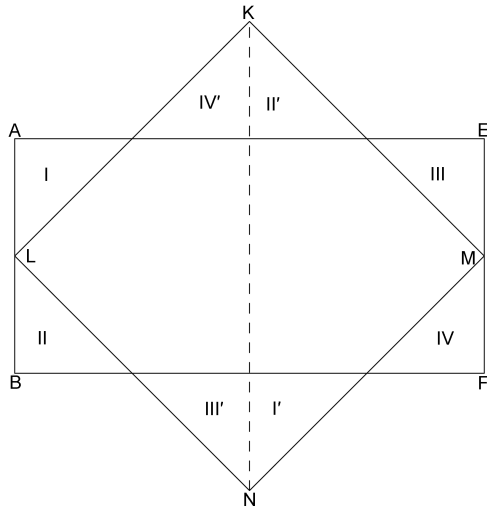
The Penrose diagram of dS spacetime as Figure 1(a) is not the familiar one, but has the shape of that of the Minkowski spacetime. It turns out to be interesting that we superpose the familiar Penrose diagrams of dS and Minkowski space times, as in Figure 2. There the cylinder AEFB (with identification AB=EF) is the Penrose diagram of dS spacetime, while the diamond KLMN is that of Minkowski spacetime. If we identify the regions “I” to “IV” with “I” to “IV”, respectively, we obtain the ordinary conformal compactification of dS and Minkowski space times (see also [14]). However, we have seen that for globally defined solutions on dS and Minkowski space times (25) and (36) the electromagnetic field and electric current in the regions “I” to “IV” differ from that in “I” to “IV” by a sign, up to positive Weyl factors. In order to find a conformal compactification of the Minkowski and dS/AdS space times where the electrodynamics can be globally defined, then, instead of identifying the regions “I” to “IV” with “I” to “IV” immediately, one should take them as, actually,



**Figure 1** An illustration of the point charge(s) in the dS spacetime with Penrose diagrams. (a) Penrose diagram of the dS spacetime in conformally flat coordinates, with identification  $KL=MN$  and  $KM=LN$ , which can be compared with Figure 1 in ref. [14]. The solid line segments  $GH$  and  $IJ$  are its conformal boundary. The dashed line segment stands for the world lines of the point charges. (b) Inversion (27) of (a). The points  $K, L, M$  and  $N$  are all transformed to the origin, and the triangles (numbered “I” to “IV”) in (a) to the corresponding positions in this Figure, respectively.

<sup>6)</sup> It is interesting if this fact has something to do with the arguments given in ref. [15], which is from a completely different point of view.

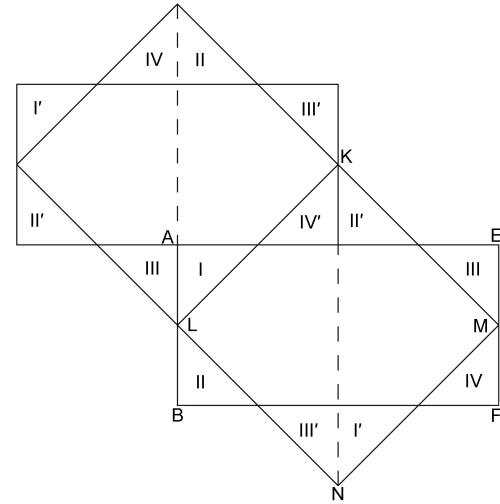




**Figure 2** Superposition of the familiar Penrose diagrams of dS and Minkowski space times, where extension of the ordinary conformal compactification arises.

antipodal regions<sup>7)</sup> on the doubly conformal compactification of these space times. In other words, we use Minkowski spacetime and dS spacetime to cover different parts of this doubly conformal compactification.

Although we have not seen in Figure 2 the whole of the doubly conformal compactification of Minkowski and dS/AdS space times, it is rather straightforward to construct it based on the above analysis. From Figure 2 it is clear that antipodal points have relative coordinates  $(\pm 1, \pm 1)$  on the Penrose diagram, where we have taken  $AB$  as the length unit. The antipode of antipode, with relative coordinate  $(\pm 2, 0)$  or  $(0, \pm 2)$ , should be itself. In fact, the electromagnetic field and electric current are of the same value, up to positive Weyl factors, at these points, so it is safe to identify these points in the conformal sense. A thus extended version of Figure 2 is Figure 3. The doubly conformal compactification of dS and Minkowski space times is shown, more clearly, in Figure 4. Note also that any pair of points related by an inversion-like



**Figure 3** Double extension of Figure 2.

transformation

$$x^\mu \rightarrow -\eta_{\mu\nu} \frac{x^\nu}{x^2} = -\frac{x_\mu}{x^2} \quad (37)$$

can be viewed as to have relative coordinates  $(\pm 1, 0)$  or  $(0, \pm 1)$  on these diagrams.

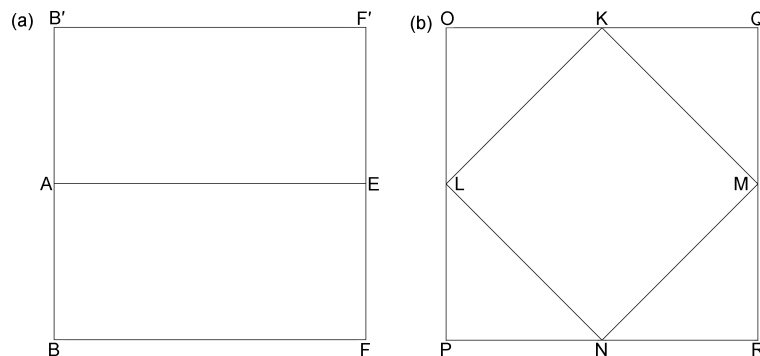
Then we consider the  $N$  case. Similarly, we have from eqs. (20) and (22)

$$J_0 = e[1 + 2(t^2 + x^2) + (x^2)^2]\delta^3(x) = e(1 + t^2)^2\delta^3(x), \quad J_i = 0, \quad (38)$$

with  $E_i$  and  $B_i$  still given by eq. (25), where we have omitted the prime in these notations. The region of  $N$  uncovered by the conformally flat coordinates corresponds to the ordinary conformal boundary of the Minkowski spacetime. By the inversion (27), we can examine the uncovered region (actually a “compactified” light cone). For the electric current, we have

$$\tilde{J}_0(\tilde{x}) = e(1 + \tilde{t}^2)^2\delta^3(\tilde{x}), \quad \tilde{J}_i(\tilde{x}) = 0. \quad (39)$$

For the electromagnetic field, we have also eqs. (32) and (33). So we can see the breakdown of the Maxwell equation at  $\tilde{x}^2 = 0$ , similar to the dS case. Unlike the dS case, however,



**Figure 4** An illustration of the doubly conformal compactification of dS and Minkowski space times with Penrose diagrams. (a) Double extension of the Penrose diagram of dS spacetime, with the usual cylindrical identification  $B'B=F'F$ , which is conformally compactified by identifying  $B'F'=BF$ . (b) Double extension of the Penrose diagram of Minkowski spacetime, which is conformally compactified by identifying  $OP=QR$  and  $OQ=PR$ .

7) This “antipodal” refers to the 6-dimensional one in eq. (3), as can be seen more clearly in the following discussion.

since  $N$  has no (conformal) boundary, this breakdown cannot be remedied by the sign reversion (34) of  $A_\mu$  and  $J_\mu$  in certain regions of  $N$ . Although one can see eqs. (38) and (39) as the only correct form of a point-like source that satisfies the continuity equation, there is no corresponding electromagnetic field that globally satisfies the Maxwell equation. In other words, one cannot find fundamental solutions to the Maxwell equation on  $N$ . This problem can be resolved by cutting open  $N$  along  $\tilde{x}^2 = 0$  and sewing another  $N$  (also cut open) onto it, which yields the double covering  $2N$ , as expected.

## 5 Concluding remarks

In sect. 2, we review the ordinary conformal compactification of the Minkowski, dS and AdS space times, where one can see that the pseudo-sphere really can help him to understand the conformal compactification. First, we get the intersection  $\mathcal{M}$ , which in fact is a Minkowski spacetime, of the hyperplane  $\mathcal{P}_a$  and the zero radius pseudo-sphere  $\mathcal{N}$  in a  $(4+2)$ -dimensional Minkowski spacetime. It's not difficult to see that the infinity points of  $\mathcal{M}$  lie on the hyperplane parallel to  $\mathcal{P}_a$ . The compactification  $[N]$ , in which the  $O(2,4)/\mathbb{Z}_2$  action can be well defined, is generated by adding those infinity points to  $\mathcal{M}$ . That is the compactification of Minkowski spacetime. For the dS and AdS case, we use a general hypersurface (of antipodal symmetry) to intersect  $N$  to get some conformally flat manifolds.

In sect. 3, we map the solution to Maxwell equation on Minkowski spacetime (25) to the dS spacetime to get eq. (26), from which one can find that there are really two antipodal point charges. This is caused by the fact that the conformal boundary  $1+x^2=0$  separates the world line  $x=0$  into two parts. With the inversion (27), one can find that the field (32) is discontinuous at  $\tilde{x}^2 = 0$ , which shows the breakdown of the Maxwell equation on the compactification of Minkowski spacetime. It is not hard to see that the breakdown is caused by the modulus term in equation (25). One should take care that in eq. (29) there is not a simple  $\tilde{x}^2$  factor but a Jacobian factor when one writes  $\delta^3(\tilde{x}/\tilde{x}^2)$  into  $\delta^3(\tilde{x})$ . And the condition for magnetic charge is not different.

In sect. 4, the discussion in sect. 3 is illustrated with Penrose diagrams. Reversing the sign of the electromagnetic field and current (see eq. (34)) in "I"–"IV" in Figure 1(b), one can see the Maxwell equations can be well defined on the whole dS spacetime. Eq. (36) is the global solution, and one can see that there are two antipodal point charges. We also show that the ordinary compactification, identifying "I"–"IV" with "I'"–"IV'", can not give a global defined electrodynamics. This leads us to consider the doubled conformal compactification, and we reveal that superposing the familiar Penrose diagrams of dS and Minkowski space times see Figure 2, and identifying the regions "I"–"IV" with "I'"–"IV'" as antipodal regions, respectively will give the doubled compactification. Figures 3 and 4 show this process clearly. And the doubled conformal compactification for the electrodynamics

with a magnetic charge is very similar to the electron, which we don't analysis here.

Since our attention is solely paid to the classical case here, one should further consider the conformal invariant quantum field theory containing electrodynamics correspondingly [16,17]. And zero-mass systems [18–20] can also be considered, including the Lienard-Wiechert field of massless charges [21]. These problems should be left for future works. The CPT invariance, and causality are also some interesting aspects for our further studying.

*We would like to thank HUANG C G, WU X N, XU Z and ZHOU B for helpful discussions. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11075206 and 11175245).*

- 1 Francesco P D, Mathieu P, Sncal D. Conformal Field Theory. New York: Springer-Verlag Press, 1997. 95–146
- 2 Cunningham E. The Principle of relativity in electrodynamics and an extension thereof. Proc London Math Soc, 1910, 2-8(1): 77–98
- 3 Bateman H. The Transformation of the electrodynamical equations. Proc London Math Soc, 1910, 2-8(1): 223–264
- 4 Rosen G. Conformal invariance of Maxwell's equations. Am J Phys, 1972, 40(7): 1023
- 5 Fushchich W I, Nikitin A G. Symmetries of Maxwell's Equations. Dordrecht: D.Reidel, 1987. 53–67
- 6 Codirla C, Osborn H. Conformal invariance and electrodynamics: Applications and general formalism. Annals Phys, 1997, 260(1): 91–116
- 7 Polchinski J. String Theory. Cambridge: Cambridge University Press, 1995. 32–74
- 8 Penrose R. Relativistic Symmetry Groups. In: Barut A O, eds. Group Theory in Non-linear Problems. Dordrecht: D.Reidel, 1974. 1–58
- 9 Kuzenko S M. Conformally compactified Minkowski superspaces revisited. J High Energy Phys, 1210(2012): 135
- 10 Weinberg S. Six-dimensional methods for four-dimensional conformal field theories. Phys Rev D, 2010, 82(4): 045031–045042
- 11 Dirac P A M. Wave equations in conformal space. Ann Math, 1936, 37(2): 429–442
- 12 Costa M S, Penedones J, Poland D, et al. Spinning conformal blocks. J High Energy Phys, 2011, 1111: 154
- 13 Herranz F J, Santander M. Conformal compactification and cycle-preserving symmetries of spacetimes. J Phys A Math Gen, 2002, 35(31): 6601–6629
- 14 Tian Y. De Sitter thermodynamics from diamonds's temperature. J High Energy Phys, 2005, 0506: 045
- 15 Parikh M K, Savonije I, Verlinde E. Elliptic de Sitter space: dS/Z<sub>2</sub>. Phys Rev D, 2003, 67(6): 064005–064020
- 16 Baker M, Johnson K. Applications of conformal symmetry in quantum electrodynamics. Physica A, 1979, 96(1-2): 120–130
- 17 Osborn H. N=1 Superconformal symmetry in four-dimensional quantum field theory. Ann Phys, 1999, 272(2): 243–294
- 18 McLennan J A. Conformal invariance and conservation laws for relativistic wave equations for zero rest mass. Nuovo Cimento, 1956, 3(6): 1360–1379
- 19 Mickelsson J, Niederle J. Conformally covariant field equations. Ann Inst H Poincaré, 1975, 23(2): 277–295
- 20 Jacobsen H P, Vergne M. Wave and Dirac operators, and representations of the conformal group. J Funct Anal, 1977, 24(1): 52–106
- 21 Azzurli F, Lechner K. The Lienard-Wiechert field of accelerated massless charges. Phys Let A, 2013, 377(14): 1025–029