SCIENTIA SINICA
Vol. VII, No. 8, 1958

MATHEMATICS

SOME COVERING PROPERTIES OF CONVEX DOMAINS IN THE THEORY OF CONFORMAL MAPPING*

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1. Let G be a convex domain in the w-plane. If for a boundary point w of G, there exists a circumference which contains G in its interior and passes through w, then we say that this circle is a supporting circle of G at w. Suppose that for every boundary point of G there is a supporting circle with radius $\rho\,(\rho>0)$ and that at a certain boundary point of G there exists no supporting circle with radius less than ρ, in these circumstances we say that G is a convex domain with the supporting radius ρ. Obviously, any convex domain is supported by a halfplane which may be regarded as a circle with the radius $\rho=\infty$. Denote by C_ρ the family of all the functions

$$w=f(z)=z+a_2z^2+\cdots$$

such that it maps the unit circle $|z|<1$ onto a convex domain D_f with the supporting radius ρ. The mapping radius of D_f at $w=0$ is evidently unity. Let K_ρ be the set of all those images D_f for which $f\in C_\rho$. We see that $\rho\geqslant 1$, that K_1 contains only the unit circle, and that $K_\rho \neq k_{\rho'}$ if $\rho\neq\rho'$. We have little knowledge about K_ρ $(\rho>1)$, although many properties concerning $\sum\limits_{\rho>1} K_\rho$ are known. The object of the present paper is to investigate the covering properties of K_ρ by using the method of extremal length.

2. The Szegö-problem[2] in the family C_ρ.

Let $G\in K_\rho$. Let n rays r_k issued from the origin $w=0$ make equal angles. Let γ_k be the length of the segment of r_k lying in $G(k=1,\cdots,n)$. Denote max $(r_1,\cdots,r_n)$ by $R^{(n)}(G)$ and

$$T_n(\rho)=\min_{G\in K_\rho} R^{(n)}(G)\,.$$

Theorem 1. *For* $1<\rho<\infty$, $n>1$, *we have*

*First published in Chinese in *Acta Mathematica Sinica*, Vol. VII, No. 3, pp. 421—432, 1957.

$$T_n(\rho) = \rho\left(1 - \frac{\sin x\pi}{\sin \frac{\pi}{n}}\right), \tag{1}$$

where x being the root, $0 < x < \frac{1}{n}$, *of the equation*

$$(1 - xn)\,\rho \cdot \Gamma\left(\frac{n-1}{2}\right)^2 = \Gamma\left(\frac{n-1}{n} - x\right)\Gamma\left(\frac{n+1}{n} + x\right). \tag{2}$$

Proof. Let $\rho > a > T_n(\rho)$. There is a domain $G \in K_\rho$ such that $R^{(n)}(G) < a$, and that $b_\nu = ae^{i\frac{2\pi\nu}{n}} \overline{\in} G$ $(\nu = 1, 2, \cdots, n)$. Let $\triangle_n$ be the regular circular polygon such that the middle points of the n sides are $b_1, b_2, \cdots, b_n$. Denote by $a_1, \cdots, a_n$ the n vertices of $\triangle_n$. Let U_ν be the domain bounded by the segments $\overline{Oa_\nu}$, $\overline{Oa_{\nu+1}}$ and the circular arc $\widehat{a_\nu a_{\nu+1}}$. Let V_ν be the inverse of U_ν with respect to $\widehat{a_\nu a_{\nu+1}}$, $D_\nu = U_\nu + \widehat{a_\nu a_{\nu+1}} + V_\nu$. On the segment $\overline{Ob_\nu}$ there exists a boundary point ξ of G. Let C be the supporting circumference of G at ξ with radius ρ. Let b'_ν be a point such that $|b_\nu - b'_\nu| = 2\rho$ and $\arg b_\nu = \arg b'_\nu + \pi$. Let η be the intersecting point either of C and $\overline{Oa_\nu}$ or of C and $\overline{Oa_{\nu+1}}$. Let C_ν be a circumference passing through η and b_ν, containing O in the interior. Then b'_ν is exterior to C_ν and GD_ν is in the interior of C_ν.

Let r be a positive number, sufficiently small. Let D be a doubly-connected domain which is the complementary domain of $|z| \leqslant r$ in G. Let Γ_1 and Γ_2 be the boundary curves of D. Let $\{\gamma\}$ be the set of all the Jordan arcs each of which is contained in D and connects Γ_1 and Γ_2. Denote by $d_D(\Gamma_1, \Gamma_2)$ the extremal length[3] of $\{\gamma\}$ in D. It is also the extremal distance[4] of Γ_1 and Γ_2 with respect to D. Let $M_D(\Gamma_1, \Gamma_2)$ be the reciprocal of $d_D(\Gamma_1, \Gamma_2)$. Then there is a continuous function $\rho(z)$ $(\geqslant 0)$ in D such that

$$M_D(\Gamma_1, \Gamma_2) = \iint_D \rho(z)^2\,dx\,dy^{[5]}, \quad \min_{\gamma \in \{\gamma\}} \int_\gamma \rho\,|dz| = 1.$$

Hence

$$M_D(\Gamma_1, \Gamma_2) \geqslant \sum_{\nu=1}^{n} \iint_{DD_\nu} \rho(z)^2\,dx\,dy. \tag{3}$$

Evidently, DD_ν is in the interior of C_ν. Let B_ν be the simply connected domain formed from D_ν cut by $|z| = r$ and C_ν. Let the boundary components of B_ν in $|z| = r$ and in C_ν be S_1 and S_2 respectively. On $B_\nu - DD_\nu$ we define $\rho(z) = 0$. If γ is a rectifiable Jordan arc in B_ν connecting S_1 and S_2, then there is a subarc γ' of γ, connecting Γ_1 and Γ_2. Hence

$$\int_{\gamma} \rho |dz| \geqslant \int_{\gamma'} \rho |dz| \geqslant 1 .$$

Let $d_{B_\nu}(S_1, S_2)$ be the extremal distance of S_1 and S_2 with respect to B_ν, then by definition, we have

$$\iint_{DD_\nu} \rho(z)^2 \, dx \, dy = \iint_{D_\nu} \rho(z)^2 \, dx \, dy \geqslant d_{B_\nu}(S_1, S_2)^{-1} . \tag{4}$$

We have to estimate $d_{B_\nu}(S_1, S_2)$. By a suitable linear transformation b'_ν is transformed into ∞, $\widehat{a_\nu a_{\nu+1}}$ into a segment l, b_ν into the middle point δ of l, S_1, S_2 into S, S' and O into a point O' on the line m which is orthogonal to l and bisects l. By this transformation, D_ν is transformed into D_ν^* which is symmetric with respect to δ. In virtue of b'_ν being exterior to C_ν and O being in the interior of C_ν, we see that the image of C_ν is a finite circle C_ν^* and O' is in the interior of C_ν^*, so that the image B_ν^* of B_ν is also in the interior of C_ν^* which passes through δ. Let the images of S_1 and S_2 be S and S' respectively. Since D_ν^* is symmetric with respect to δ, the tangent t of C_ν^* at the point δ cuts D_ν^* into two congruent domains E and $\hat{E}$, $B_\nu^* \subset E$. Rotate the whole configuration around δ through an angle π; E, B_ν^*, S and S' become $\hat{E}$, $\hat{B}_\nu^*$, $\hat{S}$ and $\hat{S}'$ respectively. Let $B \subset D_\nu^*$ be bounded by S and $\hat{S}$. Since the extremal distance is invariant under conformal mapping, we have

$$d_{B_\nu}(S_1, S_2) = d_{B_\nu^*}(S, S') = d_{\hat{B}_\nu^*}(\hat{S}, \hat{S}') . \tag{5}$$

Let $\{\gamma_1\}$ be the set of all rectifiable Jordan arcs in B_ν^*, each γ_1 connecting S and S', then there is a continuous positive function $\rho_1(z)$ such that

$$d_{B_\nu^*}(S, S')^{-1} = \iint_{B_\nu^*} \rho_1(z)^2 \, dx \, dy^{[5]}, \quad \min_{\{\gamma_1\}} \int_{\gamma_1} \rho_1 |dz| = 1 . \tag{6}$$

On $\hat{B}_\nu^*$ we define $\rho_1(z)$ by $\rho_1(z^*)$ where z^* and z are symmetric with respect to δ. On $B - B_\nu^* - \hat{B}_\nu^*$, we define $\rho_1(z) = 0$. If γ is a rectifiable arc connecting S and $\hat{S}$, then there are two subarcs γ' and γ'' connecting S, S' and $\hat{S}$, $\hat{S}'$ respectively. Hence

$$\int_{\gamma} \frac{\rho_1(z)}{2} |dz| \geqslant \int_{\gamma'} \frac{\rho_1}{2} |dz| + \int_{\gamma''} \frac{\rho_1}{2} |dz| \geqslant \frac{1}{2} + \frac{1}{2} = 1 .$$

Observing (6), we have

$$d_B(S, \hat{S})^{-1} \leqslant \iint_B \left(\frac{\rho_1}{2}\right)^2 dx \, dy = \frac{1}{4}\left(\iint_{B_\nu^*} \rho_1^2 \, dx \, dy + \iint_{\hat{B}^*} \rho_1^2 \, dx \, dy\right) = \\ = \frac{1}{2} d_{B_\nu^*}(S, S')^{-1} . \tag{7}$$

Let A^* be the subregion of B, bounded by l and S. Since B is symmetric with respect to l, we have[5]

$$2d_B(S,\hat{S}) = d_{A^*}(l,S)\,. \tag{8}$$

Let A_ν be the subregion of D_ν, bounded by $\widehat{a_\nu a_{\nu+1}}$ and S_1. Since the images of A_ν, S_1 and $\widehat{a_\nu a_{\nu+1}}$ are respectively A^*, l and S, we have

$$d_{A^*}(l,S) = d_{A_\nu}(S_1, \widehat{a_\nu a_{\nu+1}})\,, \tag{9}$$

and combining the relations (3)—(9), we have

$$M_D(\Gamma_1,\Gamma_2) \geqslant \sum d_{A_\nu}(S_1, \widehat{a_\nu a_{\nu+1}})^{-1}\,. \tag{10}$$

Let the function $f^*(\zeta)$ be regular in the unit circle $|\zeta|<1$ and map the unit circle into Δ_n with $f^*(0)=0$, $f^{*\prime}(0)>0$. The radius $\zeta = re^{i\frac{2\pi\nu}{n}}$ $(0\leqslant r\leqslant 1)$ is represented on the segment $\overline{Oa_\nu}$ $(\nu=1,2,\cdots,n)$. The inverse image of the circumference $|z|=r$ is nearly the circumference $|\zeta| = \frac{r}{f^{*\prime}(0)}$. It is easy to see that $\sum d_{A_\nu}(S_1, \widehat{a_\nu a_{\nu+1}})^{-1}$ is approximately the reciprocal of the extremal distance between $|\zeta| = \frac{r}{f^{*\prime}(0)}$ and $|\zeta|=1$ with respect to the circular ring

$$\frac{r}{f^{*\prime}(0)} < |\zeta| < 1\,,$$

i.e.

$$\sum d_{A_\nu}(S_1, \widehat{a_\nu a_{\nu+1}})^{-1} = \frac{2\pi}{\log\dfrac{f^{*\prime}(0)}{r} + o(1)}\,. \tag{11}$$

We may regard $f(\zeta)$ as a function which maps the ring $r+o(1)<|\zeta|<1$ onto D. Thus

$$d_D(\Gamma_1,\Gamma_2) = \frac{\log\dfrac{1}{r} + o(1)}{2\pi}\,. \tag{12}$$

It follows from (10), (11) and (12) that

$$\log\frac{1}{r} \leqslant \log\frac{f^{*\prime}(0)}{r} + o(1)\,.$$

so that

$$f^{*\prime}(0) \geqslant 1\,, \tag{13}$$

for $a > T_n(\rho)$.

3. We are in a position to compute $f^{*\prime}(0)$. Let the angle between $\widehat{a_\nu a_{\nu+1}}$ and segment $\overline{Oa_\nu}$ be $\frac{q}{2}\pi$. Let $\overline{ba_\nu}$ be a radius of the circular arc $\widehat{a_\nu a_{\nu+1}}$. Let the angle between $\overline{ba_\nu}$ and $\overline{Oa_\nu}$ be $x\pi$, then $x=\frac{1}{2}(1-q)$. Evidently the angle between $\overline{Ob}$ and $\overline{Oa_\nu}$ is $\left(1-\frac{1}{n}\right)\pi$ and that between $\overline{Ob}$ and $\overline{ba_\nu}$ is $\left(\frac{1}{n}-x\right)\pi$. Hence $0<x<\frac{1}{n}$. Writing $|a_\nu|=p$, $|b|=d$, we obtain

$$\frac{\rho}{\sin\frac{\pi}{n}}=\frac{p}{\sin\left(\frac{1}{n}-x\right)\pi}=\frac{d}{\sin x\pi}. \tag{14}$$

By Schwartz-Christoffel's formula, we have

$$f^*(\zeta)=p\zeta\frac{\int_0^1 t^{x-1}(1-t)^{\frac{1}{n}-x}(1-t\zeta^n)^{-x-\frac{1}{n}}dt}{\int_0^1 t^{x-1}(1-t)^{\frac{1}{n}-x}(1-t\zeta^n)^{-x+\frac{1}{n}}dt}.$$

Hence

$$f^{*\prime}(0)=p\frac{\int_0^1 t^{x-1}(1-t)^{\frac{1}{n}-x}dt}{\int_0^1 t^{x-1}(1-t)^{-\frac{1}{n}-x}dt}=p\frac{B\left(x,1+\frac{1}{n}-x\right)}{B\left(x,1-\frac{1}{n}-x\right)}=$$

$$=p\frac{(1-nx)\sin\frac{1}{n}\pi\Gamma\left(\frac{n-1}{n}\right)^2}{\sin\left(\frac{1}{n}-x\right)\pi\Gamma\left(\frac{n-1}{n}-x\right)\Gamma\left(\frac{n-1}{n}+x\right)}.$$

Using (14), we obtain

$$f^{*\prime}(0)=\frac{\rho(1-xn)\Gamma\left(\frac{n-1}{n}\right)^2}{\Gamma\left(\frac{n-1}{n}-x\right)\Gamma\left(\frac{n-1}{n}+x\right)}. \tag{15}$$

On the other hand, we have

$$a=\rho-d=\rho\left(1-\frac{\sin x\pi}{\sin\pi/n}\right).$$

If a decreases from ρ to 0, Δ_n decreases. If we regard $f^*(0)$ as a function of x, it is decreasing in $x\in\left(0,\frac{1}{n}\right)$. By (13), $f^{*\prime}(0)\geqslant 1$ for $a>T_n(\rho)$. Let $x_0\in\left(0,\frac{1}{n}\right)$ be such that

$$T_n(\rho)=\rho\left(1-\frac{\sin x_0\pi}{\sin\frac{1}{n}\pi}\right).$$

Let $x\to x_0-0$, it follows from $f^{*\prime}(0)\geqslant 1$ that

$$\frac{\rho\,(1-x_0\,n)\,\Gamma\left(\frac{n-1}{n}\right)^2}{\Gamma\left(\frac{n-1}{n}-x_0\right)\Gamma\left(\frac{n-1}{n}+x_0\right)}\geqslant 1.$$

The quantity on the right-hand side in the above formula is not greater than unity. Otherwise, there would be an $x_1>x_0$ such that

$$\rho\,(1-x_1n)\,\Gamma\left(\frac{n-1}{n}\right)^2=\Gamma\left(\frac{n-1}{n}-x_1\right)\Gamma\left(\frac{n-1}{n}+x_1\right).$$

But Δ_n corresponding to x_1 belongs to K_ρ and

$$a=\rho\left(1-\frac{\sin x_1\pi}{\sin\frac{1}{n}\pi}\right)<T_n(\rho).$$

This contradicts the definition of $T_n(\rho)$. Hence for $x=x_0$, (1) and (2) must hold. This completes the proof of Theorem 1.

4. The extremal domain Δ_n in § 2 is essentially unique, i.e. let G be any extremal domain such that $R^{(n)}(G)=T_n(\rho)$, then it can be shown that by a rotation, G coincides with Δ_n.

Suppose that there is another $G\in K_\rho$, different from Δ_n. Let $a=T_n(\rho)$. We may suppose that $ae^{i\frac{2\pi\nu}{n}}$, $\nu=1, 2, \cdots, n$ do not belong to G. Let D_ν, D, B, B_ν^*, $\hat{B}_\nu^*$, etc., be the domains as in § 2. There exists a ν such that $B-B_\nu^*-\hat{B}_\nu^*$ possesses interior points, and accordingly the closure of $B_\nu^*+B_\nu^*$ does not contain B. The straight line t divides D_ν^* into two subdomains, and let E be the subdomain which has O' as a boundary point. There is a conformal mapping such that E maps onto a domain lying in the upper half-plane, O' maps into the origin, and the boundary points of E which do not belong to t map into the segment L on the real axis. Let $\hat{E}_1$ be the symmetric domain of E_1 with respect to the real axis. Denote by G_1 the simply connected domain $E_1+\hat{E}_1+L$. The image S^* of S is nearly a semi-circumference. Let $C_\nu^{(1)}$ be the image of C_ν^*. Let $\hat{C}_\nu^{(1)}$ be a curve lying in the lower half-plane such that $C_\nu^{(1)}+\hat{C}_\nu^{(1)}$ is symmetric with respect to the real axis. Since C_ν^* and O' lie on the same side of t, the domain G_1^* bounded by $C_\nu^{(1)}$ and $\hat{C}_\nu^{(1)}$ is simply-connected and is contained in G_1. By Lindelöf's principle, the mapping radius

R_1 of G_1 with respect to O is greater than the mapping radius R_1^* of G_1^* with respect to O. We have, as $x \to o$,

$$d_{B^*}(S, t) = \frac{\log \frac{R_1}{\frac{n}{2} r} + o(1)}{\pi}$$

and

$$d_{B_\nu^*}(S, S') = \frac{\log \frac{R_1^*}{\frac{n}{2} r} + o(1)}{\pi}.$$

Hence, for sufficiently small r,

$$d_{B^*}(S, t) - d_{B_\nu^*}(S, S') \geqslant k, \tag{16}$$

k being a positive number independent of r. By (7), (8), (9), we have

$$d_{A_\mu}(S_1, \widehat{a_\nu a_{\nu+1}}) \geqslant d_{B_\mu^*}(S, S'), \tag{17}$$

if $\mu \neq \nu$. Using the method of proof for (7), we can establish

$$d_B(S, \hat{S}) \geqslant 2 d_{B^*}(S, t).$$

By (8) and (9), we have

$$d_{B^*}(S, t) \leqslant d_{A_\nu}(S_1, \widehat{a_\nu a_{\nu+1}}). \tag{18}$$

Since A_μ and $A_{\mu'}$ are congruent, we obtain

$$d_{A_\mu}(S_1, \widehat{a_\mu a_{\mu+1}}) = d_{A_{\mu'}}(S_1, \widehat{a_{\mu'} a_{\mu'+1}})$$

for $\mu \neq \mu'$. It follows from (11) that

$$d_{A_\mu}(S_1, \widehat{a_\mu a_{\mu+1}}) = \frac{n \log \frac{f^{*\prime}(0)}{r} + o(1)}{2\pi}. \tag{19}$$

Combining (3), (4), (16), (17), (18), (19), we obtain

$$M_D(\Gamma_1, \Gamma_2) \geqslant \frac{n-1}{n} \left(\frac{2\pi}{\log \frac{1}{r} + o(1)} \right) + \frac{1}{n} \frac{2\pi}{\log \frac{e^{-2\pi \frac{k}{n}}}{r} + o(1)}.$$

But the equation $M_D(\Gamma_1, \Gamma_2) = \dfrac{2\pi}{\log\frac{1}{r}}$ implies the contradiction that $k \leqslant 0$. We see that the extremal domain must be Δ_n up to a rotation.

As an application of Theorem 1, we see from the equation

$$\sqrt{T_2(\rho)\,(2\rho - T_2(\rho))}\ \sin^{-1}\sqrt{\frac{T_2(\rho)}{2\rho}} = \frac{\pi}{4} \tag{20}$$

that $T_2(\rho)$ is the Bloch constant for the family C_ρ.

5. In this section we shall determine $T_1(\rho)$. Evidently there exists a domain G satisfying $R^{(1)}(G)=T_1(\rho)$. Without loss of generality, we may suppose that the point $a = T_1(\rho)$ lies on the boundary of G. Let C be the supporting circumference of G at a with radius ρ. Denote by $\hat{G}$ the interior of C. Let Δ_1 be a circle passing through a with radius ρ and centre $b = a - \rho$. Let b' be the centre of $\hat{G}$, then $|b'| \geqslant |b|$, the equality holds good when and only when $\hat{G} = \Delta_1$. Since $\hat{G} \supseteq G$, by Lindelöf's principle the mapping radius of $\hat{G}$ at O is greater than 1, i.e.

$$\frac{\rho^2 - |b'|^2}{\rho^2} \geqslant 1 \qquad \text{or} \qquad |b'| \leqslant \sqrt{\rho^2 - \rho}\,.$$

Hence $T_1(\rho) = a \geqslant \rho - |b| \geqslant \rho - \sqrt{\rho^2 - \rho}$. If $|\varepsilon| = 1$, then the function

$$f(z) = \frac{z}{1 + \sqrt{1 - \frac{1}{\rho}\varepsilon z}} \tag{21}$$

belongs to C_ρ. And $\min\limits_{|z|=\rho} |f(z)| \geqslant \dfrac{1}{1+\sqrt{1-\frac{1}{\rho}}} = \rho - \sqrt{\rho^2 - \rho}$. Hence

$$T_1(\rho) = \rho - \sqrt{\rho^2 - \rho}\,.$$

It can also be shown that the extremal function is (21), i.e. the extremal domain is a circle with the centre $\sqrt{\rho^2 - \rho}\, e^{i\theta}$ $(0 \leqslant \theta < 2\pi)$ and radius ρ.

Theorem 2. *For* $1 \leqslant \rho < \infty$, $T_1(\rho) = \rho - \sqrt{\rho^2 - \rho}$, *the extremal function must be* (21).

Let us now consider the case $\rho \to \infty$. Let $T_n = \lim\limits_{\rho\to\infty} T_n(\rho)$, we have

$$T_1 = \frac{1}{2}\,.$$

Since the right-hand side of (2) is bounded for $x \in \left(0, \frac{1}{n}\right)$, the root x of (2) must tend to $\frac{1}{n}$, for $n > 2$, $\rho \to \infty$, and we obtain

$$\lim_{\rho \to \infty} (1 - nx)\rho = \frac{\Gamma\left(1 - \frac{2}{n}\right)}{\Gamma\left(\frac{n-1}{n}\right)^2}.$$

Thus for $n > 2$,

$$T_n = \lim_{\rho \to \infty} \rho\left(1 - \frac{\sin x\pi}{\sin \frac{\pi}{n}}\right) = \lim_{\rho \to \infty} \rho(1 - xn) = \frac{\Gamma\left(1 - \frac{2}{n}\right)}{\Gamma\left(\frac{n-1}{n}\right)^2} =$$

$$= \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1 - \frac{2}{n}\right)}{\Gamma\left(1 - \frac{1}{n}\right)} \cos \frac{\pi}{n} =$$

$$= \int_0^1 \frac{dt}{(1 - t^n)^{2/n}} \cos \frac{\pi}{n} = \int_0^1 \frac{dt}{(1 + t^n)^{2/n}}.$$

The formula (20) gives $T_2(\rho) = \frac{\pi}{4}$. In general, we have

$$T_n = \int_0^1 \frac{dt}{(1 + t^n)^{2/n}}, \qquad n = 1, 2, \cdots.$$

6. Let T_ρ be the Bloch contant of C_ρ, then the corresponding Bloch function $w = f(z)$ maps the unit circle onto a domain which contains the circle $|w| < T_\rho$. This theorem is due to M. Y. Chang[1]. But he leaves out the problem of the determination of all the Bloch functions concerning C_ρ. We can now establish the following

Theorem 3. *Let $w = f(z) \in C_\rho$ be a Bloch function of C_ρ, then the image of the unit circle by the mapping $w=f(z)$ is a domain bounded by two circular arcs with radius ρ and symmetric with respect to $w = 0$. And furthermore, $w=f(z)$ satisfies*

$$\left(\frac{a + \varepsilon w}{a - \varepsilon w}\right)^\theta = \left(\frac{1 + \varepsilon z}{1 - \varepsilon z}\right)^\pi, \tag{22}$$

where $|\varepsilon| = 1$, $\rho\,\theta \sin \frac{\theta}{2} = \pi$, $a = \rho \sin \frac{\pi}{2a}$.

Proof. Let Δ_2 be a domain in the w-plane, bounded by two circular arcs with radius ρ. The two arcs intersect at $-a$ and a. At a the angle of intersection is θ, thus $a=\rho \sin \frac{\theta}{2}$. The function

$$u = \frac{a + w}{a - w} = h(w)$$

maps $\triangle_2$ onto the angular domain $|\arg u| < \frac{\theta}{2}$ which is transformed into the half-plane $\Re(v) > 0$ by the transformation $v = u^{\frac{\pi}{\theta}}$. Accordingly the inverse function of $v = \frac{1+\zeta}{1-\zeta}$ maps $\Re(v) > 0$ onto $|\zeta| < 1$.

Combining these mappings, we obtain a function $\zeta = g(w)$ which maps $\triangle_2$ onto $|\zeta| < 1$. Evidently the inverse function $w = f^*(\zeta)$ satisfies the equation

$$\left(\frac{a + w}{a - w}\right)^{\pi/\theta} = \left(\frac{1+\zeta}{1-\zeta}\right).$$

Hence $f^{*\prime}(0) = \frac{a\theta}{\pi}$. Select $a\theta$ such that $f^{*\prime}(0) = 1$, then $f^*(\zeta) \in C_\rho$, $\triangle_2 \in K_\rho$. Moreover we have $\rho\,\theta \sin\frac{\theta}{2} = \pi$, $a = \rho \sin\frac{\pi}{2a}$.

We proceed to prove that any Bloch function of C_ρ takes the form $f(\zeta) = \frac{1}{\varepsilon} f^*(\varepsilon\zeta)$ where $|\varepsilon| = 1$. In other words, the image G of the unit circle by $w = f(\zeta)$ is congruent to $\triangle_2$ after a suitable rotation. If this is not true, then on the circle[1] there exists either (i) a pair of boundary points C and $-C$ of G or (ii) three boundary points A, B, C (not on a semi-circumference) of G. In the case (i), the supporting circles of G at C and $-C$ contact $|w| = T_\rho$, since $|w| < T_\rho$ is contained in G. The common part of these two circles contains G. In fact, G^* is of the type $\triangle_2$ and the mapping radius of G^* at $w = 0$ is 1. Hence by Lindelöf's principle, $G^* = G$.

Now we consider the case (ii). Let G^* be the common part of the three supporting circles of G at A, B and C. The domain G^* contains the circle $|w| < T_\rho$, and is contained in G. If $G^* \neq G$, then by Lindelöf's principle the mapping radius d of G^* at O is greater than unity. Hence K_ρ would have a domain which would contain a circle with the radius $\frac{T_\rho}{d}$. This is impossible.

It leaves us to consider the case where $G^* = G$. The boundary of G is a circular triangle whose vertices shall be denoted by A', B', C'. The segments $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OA'}$, $\overline{OB'}$, $\overline{OC'}$ divide G into six subdomains $D_1, \cdots, D_6$. Suppose that the common boundary of D_4 and D_5 (D_6 and D_1, D_2 and D_3) is $\overline{OC}$ ($\overline{OA}$, $\overline{OB}$). Denote the circle $|w| \leqslant r$ by E_r and write $D = G - E_r$. Let the boundary of D be Γ_1 and Γ_2. Let $\rho(z)$ be a positive function continuous on D such that

$$M_D(\Gamma_1,\Gamma_2)=\iint_D \rho(z)^2\,dx\,dy\,,\qquad \min_{\gamma\in\{\gamma\}}\int_\gamma \rho|dz|=1\,.$$

Then

$$M_D(\Gamma_1,\Gamma_2)=\sum_{\nu=1}^{6}\iint_{DD_\nu}\rho(z)^2\,dx\,dy\,.$$

Denote respectively by S_ν and t_ν the partial boundary of DD_ν in Γ_1 and Γ_2. Denote by $d_{D_\nu}(S_\nu, t_\nu)$ the extremal distance of S_ν and t_ν with respect to D_ν. As in § 2, we have

$$M_D(\Gamma_1,\Gamma_2)\geqslant\sum_{\nu=1}^{6} d_{D_\nu}(S_\nu,t_\nu)^{-1}\,. \tag{23}$$

Now we have to estimate $d_{D_1}(S_1, t_1)$. By a rotation, we may suppose that D_1 lies in $\triangle_2$, the boundary OA of D on the imaginary axis and D_2 within the second quadrant. Let the angle of D at O be θ_1. The function $u=h(z)$ maps the segment $\arg z=\frac{\pi}{2}+\theta_1=\varphi_1$ onto a circular arc γ with the end points 1 and $u_0\left(\arg u_0=\frac{\theta}{2}\right)$ of a circle K passing through -1 and 1. The angle between γ and the positive real axis is φ_1. The function $w=u^{\frac{\pi}{\theta}}$ maps γ onto γ_1 lying in the first quadrant of the v-plane. Consider K in the v-plane, and we see that γ_1 lies in the interior of K. The angle between γ_1 and the positive axis is φ_1. One end of γ_1 is $v=1$ and the other lies on the imaginary axis. The function $v=\dfrac{1+\zeta}{1-\zeta}$ maps γ_1 onto a curve γ_2 lying in the circle $|\zeta|<1$ and outside the angular domain $\varphi_1>\arg\zeta>\frac{\pi}{2}$. Hence $\zeta=g(w)$ maps D_1 onto a domain D_1^* lying in $|\zeta|<1$ and bounded by γ_2, $\arg\zeta=\frac{\pi}{2}$ and $|\zeta|=1$. Let the images of S_1 and t_1 be S_1^* and t_1^* respectively.

Denote by $\hat{D}_1$, $\hat{S}_1$ and $\hat{t}_1$ respectively the components of D_1^*, S_1^* and t_1^* lying in $|\zeta|<1$. As in § 4 (cf. [8]) we can prove that there is a positive number k_1 such that

$$-d_{D_1^*}(S_1^*,t_1^*)+d_{\hat{D}_1}(\hat{S}_1,\hat{t}_1)\geqslant k_1$$

for sufficiently small r. On the other hand, we have $d_{\hat{D}_1}(\hat{S}_1,\hat{t}_1)=$ $=\dfrac{\log\frac{1}{r}+o(1)}{\theta_1}$. Hence we obtain

$$d_{D_\nu}(S_\nu, t_\nu) = d_{D_\nu^*}(S_\nu^*, t_\nu^*) \leqslant \frac{\log \frac{1}{r} - k\theta_\nu + o(1)}{\theta_\nu},$$

for $\nu = 1$. Similarly, we can prove this formula for $\nu = 2, \cdots, 6$. Hence, by (23),

$$M_D(\Gamma_1, \Gamma_2) \geqslant \sum_{\nu=1}^{6} \frac{\theta_\nu}{\log \frac{1}{r} - k_\nu \theta_\nu + o(1)}.$$

On account of the relations $d_D(\Gamma_1, \Gamma_2) = \dfrac{2\pi}{\log \frac{1}{r} + o(1)}$ and $\sum_{\nu=1}^{6} \theta_\nu = 2\pi$, we arrive at the result

$$\frac{2\pi}{\log \frac{1}{r} + o(1)} \geqslant \sum_{\nu=1}^{6} \frac{\theta_\nu}{\log \frac{1}{r} - k_\nu \theta_\nu + o(1)} \qquad (k_\nu > 0).$$

This is absurd. Therefore (ii) is impossible. This completes the proof.

7. The Bloch constant of convex domains.

Let the convex domain G belong to $\sum_{\rho \geqslant 1} c_\rho$, and $B(G)$ be the radius of maximal circle contained in G. The minimum B of $B(G)$ for $G \in \sum c_\rho$ is called the Bloch constant of convex domains. M. Y. Chang discovered $B = \frac{\pi}{4}$. Here we shall give a new proof of $B = \frac{\pi}{4}$ through the principle of symmetrization. Furthermore, we can determine the extremal domain simultaneously.

Theorem 4. *The extremal domain G, (i.e. $B(G) = B$) of $\sum_{\rho \geqslant 1} c_\rho$ is a strip symmetric with respect to the origin, with width $\frac{\pi}{2}$.*

Proof. Let $G \in \sum_{\rho \geqslant 1} c_\rho$, $B(G) = B$. Using Lindelöf's principle we can prove that G is either a strip or a triangle. Suppose that G is a triangle with vertices A, B and C, and further that $AD \perp BC$ and $D \in BC$, and BC is parallel to the real axis. Let G change into G^* by the Steiner symmetrization with respect to the real axis. Let the length of AD be $h = 2d$, and write $DB = a_1$, $DC = a_2$. Denote by ρ the distance from O to AD. Then G^* is the quadrilateral with the vertices

$$-(a_1 + \rho), \quad a_2 - \rho, \quad -\rho + di, \quad -\rho - di.$$

The radius of a maximal circle contained in G^* is

$$k_1 = \frac{(a_1 + a_2)\, d}{\sqrt{a_1^2 + d^2} + \sqrt{a_2^2 + d^2}}\,.$$

The radius of a maximal circle contained in G is

$$k_2 = \frac{2\,(a_1 + a_2)\, d}{a_1 + a_2 + \sqrt{a_1^2 + d^2} + \sqrt{a_2^2 + d^2}}\,.$$

Evidently $k_2 > k_1$, i.e. $k_1 < B$. Since the mapping radius R of G^* at O is greater than unity, hence by $w = \frac{\zeta}{R}$, G^* is represented onto $G_1 \in \sum_{\rho} c_\rho$ with $R(G_1) = \frac{k_1}{R} < B$. This is a contradiction. Therefore G must be a strip. If the strip G is not symmetric to the origin, then the mapping function for $|z| < 1$ onto G must be

$$\frac{\varepsilon}{2(1 - |z_0|^2)} \left\{ \log \frac{1 - \dfrac{z - z_0}{1 - \bar{z}_0 z}}{1 + \dfrac{z - z_0}{1 - \bar{z}_0 z}} - \log \frac{1 + z_0}{1 - z_0} \right\}, \qquad |\varepsilon| = 1\,.$$

Then $B(G) = \frac{\pi}{4} \frac{1}{1 - |z_0|^2} > \frac{\pi}{4}$. If follows that G must be symmetric with respect to the origin, and $B(G) = \frac{\pi}{4}$, $B = \frac{\pi}{4}$.

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