Rosenthal type inequalities for B-valued strong mixing random fields and their applications*

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Abstract Some inequalities for moments of partial sums of a B-valued strong mixing field are established and their applications to the weak and strong laws of large numbers and the complete convergences are discussed.

Keywords: Rosenthal type inequality, law of large numbers, ρ^*-mixing ϕ^*-mixing random fields.

Suppose that $(B, \|\cdot\|)$ is a separable Banach space with its dual B^*, and $(\Omega, \mathscr{U}, P)$ is a probability space. For any $f\in B^*$ and $x\in B$, by fx we mean $f(x)$. For a B-valued random variable X on $(\Omega, \mathscr{U}, \mathrm{P})$, define $\|X\|_p=(\mathrm{E}\|X\|^p)^{1/p}$. Given two σ-fields $\mathscr{F}, \mathscr{S}$ in $\mathscr{U}$, let

$$\phi(\mathscr{F},\mathscr{S}):=\sup\{|\mathrm{P}(B\mid A)-\mathrm{P}(B)|;\ A\in\mathscr{F},\ B\in\mathscr{S},\ \mathrm{P}(A)\neq 0\},$$

$$\rho_B(\mathscr{F},\mathscr{S}):=\sup\left\{\frac{|\mathrm{E}fX-\mathrm{E}f\,\mathrm{E}X|}{\|f\|_2\|X\|_2};\ f\in L_2(\mathscr{F},B^*),\ X\in L_2(\mathscr{S},B)\right\},$$

where $L_p(\mathscr{F},B)$ denotes L_p space of $\mathscr{F}$-measurable B-valued random variables, and $L_p(\mathscr{U},B)$ is denoted simply by $L_p(B)$ or just L_p. Here and in the sequel measurability and integrability mean the strong measurability and strong integrability, respectively. Clearly, $\rho_{\mathbb{R}}(\mathscr{F},\mathscr{S})\leqslant\rho_B(\mathscr{F},\mathscr{S})$ where $\mathbb{R}$ is the real space, and if $B=\mathbb{R}$, then $\rho_B(\mathscr{F},\mathscr{S})$ is the usual ρ-mixing coefficients. It is well known that $\rho_{\mathbb{R}}(\mathscr{F},\mathscr{S})\leqslant 2\phi^{1/2}(\mathscr{F},\mathscr{S})$. The following Property 1 tells us that it is also true for general Banach space B.

Property 1. *Given two σ-fields $\mathscr{F}, \mathscr{S}$ in $\mathscr{U}$ and $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, for any $f\in L_p(\mathscr{F},B^*)$ and $X\in L_q(\mathscr{F},B)$, we have*

$$|\mathrm{E}fX-\mathrm{E}f\,\mathrm{E}X|\leqslant 2\phi^{1/p}(\mathscr{F},\mathscr{S})\|f\|_p\|X\|_q.$$

Property 2. *Suppose that $f\in L_p(\mathscr{F},B^*)$ and $X\in L_q(\mathscr{S},B)$, where $p\cdot q>1$ with $1/p+1/q=1$. Then*

$$|\mathrm{E}fX-\mathrm{E}f\,\mathrm{E}X|\leqslant 4\rho_B^{\frac{2}{p}\wedge\frac{2}{q}}(\mathscr{F},\mathscr{S})\|f\|_p\|X\|_q.$$

The proofs of Properties 1 and 2 are similar to those of Lemmas 1.2.7 and 1.2.8 of ref. [1], so they are omitted here.

Let d be a positive integer and, let N^d be the d-dimensional lattice equipped with the coordinatewise partial order, $\leqslant$. Let $\{X_k;\ k\in N^d\}$ be a d-dimensional discrete field of B-valued random variables on $(\Omega, \mathscr{U}, P)$. It will be called "centered" if $EX_k=0$. For any $A\subset N^d$, set

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$S_A = \sum_{n \in A} X_n$, $|A|$ = the cardinal number of A. For any $n \in N^d$, let $(n) = \{m \in N^d, m \leqslant n\}$, $S_n = S_{(n)}$, $|n| = |(n)| = n_1 n_2 \cdots n_d$ and $\| n \|$ denotes the Euclidean norm. Occasionally, n, k, n, etc. will also denote positive integers, the reader will not be confused from their context.

For two nonempty disjoint sets $S, T \subset N^d$, we define $\text{dist}(S, T)$ as $\min\{\| j - k \|; j \in S, k \in T\}$. Let $\sigma(S)$ be the σ-field generated by $\{X_k; k \in S\}$, and define $\sigma(T)$ similarly.

We define two mixing coefficients of $\{X_k; k \in N^d\}$. For any real number $s \geqslant 1$, define $\rho_B^*(s) = \sup \rho_B(\sigma(S), \sigma(T))$, where the supremum is taken over all pairs of sets S, T such that $\text{dist}(S, T) \geqslant s$, and define $\varphi^*(s)$ similarly.

Many limit results were obtained for real mixing random sequences in the past twenty years (cf. ref. [1] and references therein) and for some real mixing random fields recently (cf. refs. [2—4]). But, to the best of our knowledge, little is known of B-valued mixing random sequences or fields. The main purpose of this paper is to establish some moment inequalities just as the Rosenthal -type inequalities for block sums of ρ^*-mixing or ϕ^*-mixing fields which are useful tools in studying the limit properties of these mixing random fields. The main results are stated in sec. 1. In sec. 2, we give some applications to weak laws of large numbers, Marcinkiewicz-Zygmund strong laws of large numbers and complete convergences.

1 Inequalities for moments of partial sums

The first inequality seems simple but will be useful in proving our main inequalities.

Theorem 1. *Let $\mathscr{F}, \mathscr{G}$ be two σ-fields in $\mathscr{U}$, and let $p > 1$ be a real number and* $\rho := \max\{\rho_B(\mathscr{F}, \mathscr{G}), \rho_B(\mathscr{G}, \mathscr{F})\}$. *Suppose that one of the following two conditons is satisfied*:

(a) *B^* has the Radon-Nikodym (R-N) property and $\rho^{\frac{2}{p} \wedge \frac{2}{q}} < 1/4$, where $\frac{1}{p} + \frac{1}{q} = 1$,*

(b) *B is reflexive (i.e., $B^{**} = B$) and $\rho < 1$.*

Then there exists a constant $C_p = C(\rho, p)$ depending only on ρ and p such that

$$\mathrm{E} \| X \|^p \leqslant C_p \mathrm{E} \| X + Y \|^p \tag{1}$$

holds for any $X \in L_p(\mathscr{F}, B)$, $Y \in L_p(\mathscr{G}, B)$ with $\mathrm{E}X = \mathrm{E}Y = 0$.

To prove Theorem 1, we need some lemmas. The first one is well known.

Lemma 1. *Given $1 \leqslant p < \infty$, for any probability space $(\Omega, \mathscr{F}, \mathrm{P})$ $L_p(\mathscr{F}, B)^* = L_q(\mathscr{F}, B^*)$ if and only if B^* has the R-N property, where $\frac{1}{p} + \frac{1}{q} = 1$ and $L_p(\mathscr{F}, B)^*$ denotes the daul space of $L_p(\mathscr{F}, B)$.*

The following is the interpolation theorem[5].

Lemma 2. *Let $1 \leqslant p_i, q_i \leqslant \infty$, $i = 0, 1$, and let T be a linear operator mapping from $L_{p_0} + L_{q_0}$ into $L_{p_1} + L_{q_1}$ with its restriction to $L_{p_0}(L_{p_1})$ being a continuous mapping from $L_{p_0}(L_{p_1})$ into $L_{q_0}(L_{q_1})$. If $0 \leqslant \theta \leqslant 1$ and*

$$\frac{1}{r_i} = \frac{1-\theta}{p_i} + \frac{\theta}{q_i}, \quad i = 0, 1.$$

Then T is a continuous mapping from L_{r_0} into L_{r_1} and

$$\| T \|_{L_{r_0} \to L_{r_1}} \leqslant \| T \|^{1-\theta}_{L_{p_0} \to L_{p_1}} \| T \|^{\theta}_{L_{q_0} \to L_{q_1}}.$$

The following lemma is an extension of Lemma 1 of reference [6].

Lemma 3 *Suppose B^* has the R-N property. Given $1 < p < \infty$ and two σ-fields $\mathscr{F}$, $\mathscr{F}^*$ in $\mathscr{U}$, if $\rho := \max\{\rho_B(\mathscr{F}, \mathscr{F}_*), \rho_B(\mathscr{F}_*, \mathscr{F})\} < 1$, then there exists $C = C(p, \rho)$ such that the following implication holds. If $X \in L_p(\mathscr{F}, B)$, $Y \in L_p(\mathscr{F}_*, B)$ with $\mathrm{E}X = \mathrm{E}Y = 0$, then there is a $Z \in L_p(\mathscr{U}, B)$ such that*

$$X = \mathrm{E}\{Z \mid \mathscr{F}\} \text{ and } Y = \mathrm{E}\{Z \mid \mathscr{F}_*\}, \quad \| Z \|_p \leqslant C(\| X \|_p + \| Y \|_p).$$

Proof. Let A be a linear operator defined by $A(\cdot) = \mathrm{E}\{\mathrm{E}\{\cdot \mid \mathscr{F}_*\} \mid \mathscr{F}\}$ and let $E(\cdot)$ denote the linear operator of taking the expected value $\mathrm{E}\{\cdot\}$. Clearly, $\| A - E \|_{L_1(B) \to L_1(B)} \leqslant 2$, $\| A - E \|_{L_\infty(B) \to L_\infty(B)} \leqslant 2$ and $(A - E)^k = A^k - E$ for each $k \geqslant 1$. First, we show that

$$\| A - E \|_{L_2(B) \to L_2(B)} \leqslant \rho. \tag{2}$$

For any $\xi \in L_2(\mathscr{F}_*, B)$, by a well-known consequence of the Hahn-Banach theorem and Lemma 1, there exists a $\xi^* \in L_2(\mathscr{F}, B^*)$ with $\| \xi^* \|_2 = 1$ such that

$$\begin{aligned}\| \mathrm{E}\{\xi \mid \mathscr{F}\} - \mathrm{E}\xi \|_2 &= \| \mathrm{E}[\xi - \mathrm{E}\xi \mid \mathscr{F}] \|_2 = \mathrm{E}\{\xi^*(\mathrm{E}[\xi - \mathrm{E}\xi \mid \mathscr{F}])\} \\ &= \mathrm{E}\{\mathrm{E}[\xi^*(\xi - \mathrm{E}\xi) \mid \mathscr{F}]\} = \mathrm{E}\{\xi^*(\xi - \mathrm{E}\xi)\} \\ &= \mathrm{E}\xi^*\xi - \mathrm{E}\xi^*\mathrm{E}\xi \leqslant \rho \| \xi^* \|_2 \| \xi \|_2 = \rho \| \xi \|_2.\end{aligned}$$

It follows that for any $\xi \in L_2(B)$, $\| (A - E)(\xi) \|_2 \leqslant \rho \| \mathrm{E}\{\xi \mid \mathscr{F}_*\} \|_2 \leqslant \rho \| \xi \|_2$, which implies (2) immediately. Now, for $1 < p < \infty$, Lemma 2.2 implies

$$\begin{aligned}\| A^k - E \|_{L_p \to L_p} &\leqslant (\| (A - E)^k \|_{L_2 \to L_2})^{2 - \frac{2}{p}} (\| A^k - E \|_{L_1 \to L_1})^{\frac{2}{p} - 1} \\ &\leqslant \rho^{2k\left(1 - \frac{1}{p}\right)} 2^{\frac{2}{p} - 1}, \quad 1 < p \leqslant 2; \\ \| A^k - E \|_{L_p \to L_p} &\leqslant (\| (A - E)^k \|_{L_2 \to L_2})^{\frac{2}{p}} (\| A^k - E \|_{L_\infty \to L_\infty})^{1 - \frac{2}{p}} \\ &\leqslant \rho^{\frac{2}{p}k} 2^{1 - \frac{2}{p}}, \quad 2 \leqslant p < \infty.\end{aligned}$$

In particular, for centered ξ we have $\| A^k \xi \|_p \leqslant 2^{\frac{2}{p} - 1} \rho^{2k\left(1 - \frac{1}{p}\right)} \| \xi \|_p$ for $1 < p \leqslant 2$ and $\| A^k \xi \|_p \leqslant 2^{1 - \frac{2}{p}} \rho^{\frac{2}{p}k} \| \xi \|_p$ for $2 \leqslant p < \infty$. Since the estimate also holds for $k = 0$ and, by symmetry, also for $A^*(\xi) = \mathrm{E}\{\mathrm{E}\{\xi \mid \mathscr{F}\} \mid \mathscr{F}_*\}$,

$$Z = \sum_{k=0}^{\infty} A^k(X - \mathrm{E}\{Y \mid \mathscr{F}\}) + \sum_{k=0}^{\infty} (A^*)^k(Y - \mathrm{E}\{X \mid \mathscr{F}_*\}).$$

is well defined and satisfies the asserted property by the triangle inequality and a simple computation.

Proof of Theorem 1. Suppose that (a) is satisfied. By a well-known consequence of Hahn-Banach theorem and Lemma 1, we get

$$\| X \|_p \leqslant \| X \|_p + \| Y \|_p = \mathrm{E}fX + \mathrm{E}gY, \tag{3}$$

where $f \in L_q(\mathscr{F}, B^*)$, $g \in L_q(\mathscr{G}, B^*)$ with $\| f \|_q = \| g \|_q = 1$. It follows from Property 2 that

$$\begin{aligned}\| X \|_p + \| Y \|_p &= \mathrm{E}(f + g)(X + Y) - \mathrm{E}fY - \mathrm{E}gX \\ &\leqslant \| f + g \|_q \| X + Y \|_p + | \mathrm{E}fY | + | \mathrm{E}gX |\end{aligned}$$

$$\leqslant 2\|X+Y\|_p + 4\rho^{\frac{2}{p}\wedge\frac{2}{q}}\|f\|_q\|Y\|_p + 4\rho^{\frac{2}{p}\wedge\frac{2}{q}}\|g\|_q\|X\|_p$$
$$\leqslant 2\|X+Y\|_p + 4\rho^{\frac{2}{p}\wedge\frac{2}{q}}(\|X\|_p + \|Y\|_p).$$

Putting $C_p = 2(1-4\rho^{\frac{2}{p}\wedge\frac{2}{q}})^{-1}$ yields (1). Now suppose that (b) is satisfied. Since B is reflexive, B and B^* have the R-N property, and (3) remains true. By assumption $EX = EY = 0$, we can replace f and g in (3) by centered variables $f_1 = f - Ef$ and $g_1 = g - Eg$, respectively. Noting $B^{**} = B$, by the definition of $\rho_B(\cdot,\cdot)$, we have $\rho_{B^*}(\mathscr{F},\mathscr{S}) = \rho_B(\mathscr{S},\mathscr{F})$ and $\rho_{B^*}(\mathscr{S},\mathscr{F}) = \rho_B(\mathscr{F},\mathscr{S}) < 1$. By Lemma 3, there exists $h \in L_q(\mathscr{U}, B^*)$, such that

$$\|h\|_q \leqslant 4C(q,\rho), \mathrm{E}\{h \mid \mathscr{F}\} = f_1, \ \mathrm{E}\{h \mid \mathscr{S}\} = g_1.$$

Theorefore

$$\mathrm{E}fX + \mathrm{E}gY = \mathrm{E}\{(\mathrm{E}[h \mid \mathscr{F}])X\} + \mathrm{E}\{(\mathrm{E}[h \mid \mathscr{F}_*])Y\} = \mathrm{E}h(X+Y)$$
$$\leqslant \mathrm{E}\{\|h\| \cdot \|X+Y\|\} \leqslant \|h\|_q\|X+Y\|_p \leqslant 4C(q,\rho)\|X+Y\|_p.$$

The proof is completed.

The following is the Rosenthal type inequality for ρ^*-mixing random fields.

Theorem 2. *Let B be of type p and let $\{X_k; k \in N^d\}$ be a B-valued centered random field. Suppose that one of the following conditions is satisfied:*

(a) *B^* has the R-N property and* $\lim \rho_B^*(\tau) = 0$,

(b) *B is reflexive and* $\lim \rho_B^*(\tau) < 1$.

Then for any $r \geqslant p$, there exists a positive constant B_r depending only on r, p and $\rho_B^*(\cdot)$ such that for any finite set $S \subset N^d$,

$$\mathrm{E}\left\|\sum_{k\in S}X_k\right\|^r \leqslant B_r\left(\sum_{k\in S}\mathrm{E}\|X_k\|^r + \left(\sum_{k\in S}\mathrm{E}\|X_k\|^p\right)^{r/p}\right). \tag{4}$$

Remark. If B is a p-uniformly smooth space, then it is reflexive and of type p.

Proof. First, we prove that for any $q > 1$ there exists a positive constant $D_q = D(q, p, \rho^*(\cdot))$ such that

$$\mathrm{E}\left\|\sum_{j\in S}X_j\right\|^q \leqslant D_q\mathrm{E}\left(\sum_{j\in S}\|X_j\|^p\right)^{q/p}. \tag{5}$$

If (a) is satisfied, we can assume that $(\rho_B^*(1))^{\frac{2}{q}\wedge\frac{2}{q'}} < \frac{1}{4}$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Otherwise, we define J as the smallest integer for which $(\rho_B^*(J))^{\frac{2}{q}\wedge\frac{2}{q'}} < \frac{1}{4}$. For each $l \in \{1, \cdots, J\}^d$, let $T(l) = \{k \in S;\ k_u \equiv l_u \bmod J$ for $u = 1, \cdots, d\}$. X_j's in $T(l)$ are at least J apart from each other, and then we can consider each $T(l)$ separately. Similarly, if (b) is satisfied, we can assume that $\rho_B^*(1) < 1$. Now, suppose that $\{\varepsilon_j; j \in N^d\}$ is a sequence of independent and identically distributed random variables such that $P(\varepsilon_j = 1) = P(\varepsilon_j = -1) = \frac{1}{2}$, all j. $\{\varepsilon_j\}$ will also be independent of the X_j. Let $Y = \sum_{j\in S,\varepsilon_j=1}X_j$, $Z = \sum_{j\in S,\varepsilon_j=-1}X_j$. Then $Y + Z = \sum_{j\in S}X_j$, $Y - Z = \sum_{j\in S}\varepsilon_j\{X_j\}$. Noting that for fixed $\{\varepsilon_j\}$ the distance between the two sets $\{j; \varepsilon_j = 1\}$ and $\{j; \varepsilon_j = -1\}$ is one, by Theorem 1 we have

$$\mathrm{E}_X\|Y+Z\|^q \leqslant C_q\{\mathrm{E}_X\|Y\|^q + \mathrm{E}_X\|Z\|^q\} \leqslant C_q\mathrm{E}_X\|Y-Z\|^q.$$

It follows that $\mathrm{E}\|Y+Z\|^q \leqslant C_q \mathrm{E}\|Y-Z\|^q$, i.e. $\mathrm{E}\left\|\sum_{j\in S} X_j\right\|^q \leqslant C_q \mathrm{E}\left\|\sum_{j\in S}\varepsilon_j X_j\right\|^q$. Noting that there exists a positive constant c'_1 depending only on p and the space B such that

$$\mathrm{E}_\varepsilon\left\|\sum_{j\in S}\varepsilon_j X_j\right\|^p \leqslant c'_2\left(\sum_{j\in S}\|X_j\|^p\right)^{q/p},$$

we finish the proof of (5).

We first show (4) for $B=\mathbb{R}$ and $p=2$. For any $r\geqslant 2$, there exists $k\in\mathbb{N}$ and $1\leqslant q<2$ such that $r=2^k q$, so (4) is equivalent to

$$\mathrm{E}\left|\sum_{j\in S}X_j\right|^{2^k q} \leqslant C_{k,q}\left\{\sum_{j\in S}\mathrm{E}\,|X_j|^{2^k q}+\left(\sum_{j\in S}\mathrm{E}\,|X_j|^2\right)^{2^{k-1}q}\right\}. \tag{6}$$

When $k=1$, noting that $q/2<1$, from (5) ($p=2$) we have

$$\begin{aligned}
\mathrm{E}\left|\sum_{j\in S}X_j\right|^{2q} &\leqslant D_{2q}\mathrm{E}\left(\sum_{j\in S}X_j^2\right)^q \\
&\leqslant 2^{q-1}D_{2q}\left\{\mathrm{E}\left(\sum_{j\in S}(X_j^2-\mathrm{E}X_j^2)\right)^q+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^q\right\} \\
&\leqslant 2^{q-1}D_{2q}\left\{D_q\mathrm{E}\left(\sum_{j\in S}(X_j^2-\mathrm{E}X_j^2)^2\right)^{q/2}+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^q\right\} \\
&\leqslant 2^{q-1}D_{2q}\left\{D_q\mathrm{E}\left(\sum_{j\in S}(X_j^2-\mathrm{E}X_j^2)^{2\times q/2}\right)+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^q\right\} \\
&\leqslant C_{1,q}\left\{\sum_{j\in S}\mathrm{E}\,|X_j|^{2q}+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^q\right\}.
\end{aligned}$$

Hence (6) holds when $k=1$. When $k\geqslant 2$, assuming that (6) holds for any integer less than k, we will prove that (6) remains valid for k itself. From (5) ($p=2$) and the hypothesis of induction, it follows that

$$\begin{aligned}
\mathrm{E}\left|\sum_{j\in S}X_j\right|^{2^k q} &\leqslant D_{2^k q}\mathrm{E}\left(\sum_{j\in S}X_j^2\right)^{2^{k-1}q} \\
&\leqslant 2^{(2^{k-1}q)}D_{2^k q}\left\{\mathrm{E}\left(\sum_{j\in S}(X_j^2-\mathrm{E}X_j^2)\right)^{2^{k-1}q}+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^{2^{k-1}q}\right\} \\
&\leqslant 2^{(2^{k-1}q)}D_{2^k q}\left\{C_{k-1,q}\sum_{j\in S}\mathrm{E}\,|X_j^2-\mathrm{E}X_j^2|^{2^{k-1}q}\right. \\
&\quad\left.+C_{k-1,q}\left(\sum_{j\in S}\mathrm{E}(X_j^2-\mathrm{E}X_j^2)^2\right)^{2^{k-2}q}+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^{2^{k-1}q}\right\} \\
&\leqslant A_{k,q}\left\{\sum_{j\in S}\mathrm{E}\,|X_j|^{2^k q}+\left(\sum_{j\in S}\mathrm{E}(X_j^2-\mathrm{E}X_j^2)^2\right)^{2^{k-2}q}+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^{2^{k-1}q}\right\}.
\end{aligned} \tag{7}$$

Suppose that $\{\bar{X}_j; j\in N^d\}$ is a field of independent random variables and $\bar{X}_j$ has the same distribution as X_j for each j. Then by the Rosenthal inequality and the Marcinkiewicz-Zygmund inequality for independent random variables, it follows that

$$\begin{aligned}
&\left(\sum_{j\in S}\mathrm{E}(X_j^2-\mathrm{E}X_j^2)^2\right)^{2^{k-2}q} = \left(\mathrm{E}\left(\sum_{j\in S}(\bar{X}_j^2-\mathrm{E}\bar{X}_j^2)\right)^2\right)^{2^{k-2}q} \\
&\leqslant \mathrm{E}\left(\sum_{j\in S}(\bar{X}_j^2-\mathrm{E}\bar{X}_j^2)\right)^{2^{k-1}q} \leqslant 2^{(2^{k-1}q-1)}\left\{\mathrm{E}\left(\sum_{j\in S}\bar{X}_j^2\right)^{2^{k-1}q}+\left(\sum_{j\in S}\mathrm{E}\bar{X}_j^2\right)^{2^{k-1}q}\right\}
\end{aligned}$$

$$\leqslant B_{k,q}\left\{\mathrm{E}\left|\sum_{j\in S}\bar{X}_j\right|^{2^k q}+\left(\sum_{j\in S}\mathrm{E}\bar{X}_j^2\right)^{2^{k-1}q}\right\}$$

$$\leqslant B'_{k,q}\left\{\sum_{j\in S}\mathrm{E}\mid \bar{X}_j\mid^{2^k q}+\left(\sum_{j\in S}\mathrm{E}\bar{X}_j^2\right)^{2^{k-1}q}\right\}$$

$$=B'_{k,q}\left\{\sum_{j\in S}\mathrm{E}\mid X_j\mid^{2^k q}+\left(\sum_{j\in S}\mathrm{E}X_j^2\right)^{2^{k-1}q}\right\}. \tag{8}$$

Putting (8) into (7), we know that (6) remains valid for k. This proves (4) for $B=R$ and $p=2$. Now, suppose that $r\geqslant p$. Then

$$\mathrm{E}\left\|\sum_{n\in S}X_n\right\|^r\leqslant C_r\mathrm{E}\left(\sum_{n\in S}\|X_n\|^p\right)^{\frac{r}{p}}$$

$$\leqslant C_r\left\{\mathrm{E}\left|\sum_{n\in S}\xi_n\right|^{\frac{r}{p}}+\left(\sum_{n\in S}\mathrm{E}\|X_n\|^p\right)^{\frac{r}{p}}\right\},$$

where $\xi_n=\|X_n\|^p-\mathrm{E}\|X_n\|^p$. It is easily seen that $\{\xi_n; n\in N^d\}$ is a real ρ^*-mixing field with $\rho_{\mathbb{R}}^*(\tau)\leqslant\rho_B^*(\tau)$. If $p<r\leqslant 2p$, by (5) (where $p=2$), it follows that

$$\mathrm{E}\left|\sum_{n\in S}\xi_n\right|^{\frac{r}{p}}\leqslant C_r\mathrm{E}\left(\sum_{n\in S}\xi_n^2\right)^{\frac{r}{2p}}$$

$$\leqslant C_r\sum_{n\in S}\mathrm{E}\mid\xi_n\mid^{2\times\frac{r}{2p}}\leqslant C_r\sum_{n\in S}\mathrm{E}\|X_n\|^r.$$

The above inequality is obviously true for $r=p$. Now, suppose $r\geqslant 2p$. Since (4) is true for real fields (where $p=2$), it follows that

$$\mathrm{E}\left|\sum_{n\in S}\xi_n\right|^{\frac{r}{p}}\leqslant C_r\left\{\left(\sum_{n\in S}\mathrm{E}\xi_n^2\right)^{\frac{r}{2p}}+\sum_{n\in S}\mathrm{E}\mid\xi_n\mid^{\frac{r}{p}}\right\}.$$

Let $\{\tilde{\xi}; n\in N^d\}$ be independent random variables such that for each n, $\tilde{\xi}_n$ has the same distribution as ξ_n. Then

$$\left(\sum_{n\in S}\mathrm{E}\xi_n^2\right)^{\frac{r}{2p}}=\left(\sum_{n\in S}\mathrm{E}\tilde{\xi}_n^2\right)^{\frac{r}{2p}}=\left(\mathrm{E}\left(\sum_{n\in S}\tilde{\xi}_n\right)^2\right)^{\frac{r}{2p}}$$

$$\leqslant\mathrm{E}\left|\sum_{n\in S}\tilde{\xi}_n\right|^{\frac{r}{p}}\leqslant C_{r,p}\left\{\left(\mathrm{E}\left|\sum_{n\in S}\tilde{\xi}_n\right|\right)^{\frac{r}{p}}+\sum_{n\in S}\mathrm{E}\mid\tilde{\xi}_n\mid^{\frac{r}{p}}\right\},$$

where the last inequality is by the Rosenthal-type inequality for independent random variables[7]. It follows that

$$\mathrm{E}\left|\sum_{n\in S}\xi_n\right|^{\frac{r}{p}}\leqslant C_{r,p}\left\{\left(\sum_{n\in S}\mathrm{E}\mid\xi_n\mid\right)^{\frac{r}{p}}+\sum_{n\in S}\mathrm{E}\mid\xi_n\mid^{\frac{r}{p}}\right\}$$

$$\leqslant C_{r,p}\left\{\left(\sum_{n\in S}\mathrm{E}\|X_n\|^p\right)^{\frac{r}{p}}+\sum_{n\in S}\mathrm{E}\|X_n\|^r\right\}.$$

The proof is completed.

The Rosenthal-type inequality remains true for the maximal partial sums, if the random field is ϕ^*-mixing.

Theorem 3. *Let B be of type p and let* $\{X_k;\ k\in N^d\}$ *be a B-valued centered random field. For each* $m, n\in N^d$, *set* $S_n(m)=\sum_{k\leqslant n}X_{m+k}$. *Suppose that one of the following conditions is satisfied:*

(a) B^* *has the R-N property and* $\lim\phi^*(\tau)=0$;

(b) *B is reflexive and* $\lim\phi^*(\tau)<\frac{1}{4}$.

Then for any $r\geqslant p$, *there exists a positive constant* B_r *depending only on r, p and* $\phi^*(\cdot)$ *such that for any* $m, n\in N^d$,

$$\mathrm{E}\max_{k\leqslant n}\|S_k(m)\|^r\leqslant B_r\left(\sum_{k\leqslant n}\mathrm{E}\|X_{m+k}\|^r+\left(\sum_{k\leqslant n}\mathrm{E}\|X_{m+k}\|^p\right)^{r/p}\right).$$

Proof. By Theorem 2 and the fact $\rho_B^*(\tau)\leqslant 2\phi^{*1/2}(\tau)$, it is enough to prove the following lemma.

Lemma 4. *Let* $\{X_n;n\in N^d\}$ *be a B-valued random field with* $\phi^*(1)<1$. *Then there exists a constant* $C=C(\phi^*(1),q)$ *such that for any* $n\in N^d$,

$$\mathrm{E}\max_{k\leqslant n}\|S_k\|^q\leqslant C\max_{k\leqslant n}\mathrm{E}\|S_k\|^q.$$

Proof. Let $S_k=0$ if $k_l=0$ for some $1\leqslant l\leqslant d$. Given n, we may assume that $X_k=0$ for $k\nleqslant n$. Let $0<\varepsilon<1-\phi^*(1)$ and

$$d_{n,i}(\varepsilon)=\varepsilon^{-\frac{1}{q}}\left(\mathrm{E}\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}-S_{i,k_2,\cdots,k_d}\|^q\right)^{\frac{1}{q}},$$

$$M_i^*=\max_{j\leqslant i}\left\{\max_{k_2,\cdots,k_d}\|S_{j,k_2,\cdots,k_d}\|-d_{n,j}(\varepsilon)\right\},$$

$$E_i=\left\{M_{i-1}^*<x,\ \max_{k_2,\cdots,k_d}\|S_{i,k_2,\cdots,k_d}\|-d_{n,i}(\varepsilon)\geqslant x\right\},$$

$$B_i=\left\{\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}-S_{i,k_2,\cdots,k_d}\|\leqslant d_{n,i}(\varepsilon)\right\}.$$

Then $P(B_i)\geqslant 1-\varepsilon$, and

$$\mathrm{P}\left(\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}\|\geqslant x\right)\geqslant\sum_{i=1}^{n_1}\mathrm{P}\left(E_i,\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}\|\geqslant x\right).$$

On E_iB_i, we have $\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}\|\geqslant x$. It follows that

$$\mathrm{P}\left(\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}\|\geqslant x\right)\geqslant\sum_{i=1}^{n_1}\mathrm{P}(E_i,B_i)\geqslant\sum_{i=1}^{n_1}\{\mathrm{P}(B_i)-\phi^*(1)\}\mathrm{P}(E_i)$$
$$\geqslant\left\{\min_{1\leqslant i\leqslant n_1}\mathrm{P}(B_i)-\phi^*(1)\right\}\mathrm{P}(M_n^*\geqslant x),$$

which implies $\mathrm{P}(M_n^*\geqslant x)\leqslant\{1-\varepsilon-\phi^*(1)\}^{-1}\mathrm{P}(\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}\|\geqslant x)$. Noting that $d_{n,i}\leqslant 2\varepsilon^{-\frac{1}{q}}\max_{1\leqslant i\leqslant n_1}(\mathrm{E}\max_{k_2,\cdots,k_d}\|S_{i,k_2,\cdots,k_d}\|^q)^{\frac{1}{q}}$, we conclude that

$$\mathrm{P}\left\{\max_k\|S_k\|\geqslant x+2\varepsilon^{-\frac{1}{q}}\max_{1\leqslant i\leqslant n_1}\left(\mathrm{E}\max_{k_2,\cdots,k_d}\|S_{i,k_2,\cdots,k_d}\|^q\right)^{\frac{1}{q}}\right\}$$
$$\leqslant\{1-\varepsilon-\phi^*(1)\}^{-1}\mathrm{P}\left(\max_{k_2,\cdots,k_d}\|S_{n_1,k_2,\cdots,k_d}\|\geqslant x\right).$$

It follows that

$$\mathrm{E}\max_{k}\|S_k\|^q\leqslant C\max_{1\leqslant k_1\leqslant n_1}\mathrm{E}\max_{k_2,\cdots,k_d}\|S_k\|^q.$$

By induction on d, we have proved the lemma.

For ρ^*-mixing random fields we also have the following inequalities for the maximal partial sums.

Theorem 4. *Let B be of type p and let $\{X_k; k\in N^d\}$ be a B-valued centered random field. For each $m, n\in N^d$, set $S_n(m)=\sum_{k\leqslant n}X_{m+k}$. Suppose that one of the following conditions is satidfied:*

(a) *B^* has the R-N property and* $\lim\rho_B^*(\tau)=0$,

(b) *B is reflexive and* $\lim\rho_B^*(\tau)<1$.

$k, n\in N^d$. Then for any $r>p$, There exists a positive constant B_r depending only on r, p and $\rho_B^*(\cdot)$ such that for any k, $n\in N^d$,

$$\mathrm{E}\max_{k\leqslant n}\|S_k(m)\|^r\leqslant B_r\left\{\left(|n|\max_{k\leqslant n}\mathrm{E}\|X_{m+k}\|^p\right)^{r/p}\right.$$
$$\left.+|n|([\log n_1]\cdots[\log n_d])^r\max_{k\leqslant n}\mathrm{E}\|X_{m+k}\|^r\right\}.$$

Proof. By induction on d and using the arguments in Corollary 3 of ref.[8], the proof is easy. One can refer to the author's another paper[1)].

2 Some applications

Using Theorem 2, we can get the following theorem on the weak laws of large numbers.

Theorem 5. *Let $1\leqslant p<2$ and let B be a Banach space with B^* having the R-N property. Then B being of stable type p is equivalent to the statement that:*

(1) *for every sequence of identically distributed B-valued* X, *variables $\{X, X_n;\ n\geqslant 1\}$ with* $\lim\rho_B^*(\tau)=0$ *and* $\lim_{t\to\infty}t^p\mathrm{P}(\|X\|>t)=0$, *there exists a sequence $\{M_n\}$ in B such that $(S_n-M_n)/n^{1/p}\longrightarrow 0$ in probability, where* $\mathrm{M_n}$ *can be chosen to be $n\mathrm{E}(XI\{\|X\|<n^{1/p}\})$;*

B being of type p is equivalent to the statement that

(2) *for every sequence of identically distributed B-valued random variables $\{X, X_n; n\geqslant 1\}$ with* $\lim\rho_B^*(\tau)=0$, *$(S_n-\mathrm{E}S_n)/n^{1/p}\longrightarrow 0$ in probability if* $\mathrm{E}\|X\|^p<\infty$.

Furthermore, if B is reflexive, the condition $\lim\rho_B^*(\tau)=0$ *can be replaced by* $\lim\rho_B^*(\tau)<1$.

The following is the result on strong laws of large numbers and complete convergences. Let $X_n\prec X$ denote $\sup_n\mathrm{P}(\|X_n\|>t)\leqslant C\mathrm{P}(\|X\|>t)$ for some $C>0$ and all $t>0$.

Theorem 6. *Let $1\leqslant p<2$ and B be a Banach space with B^* having the R-N property. The following are equivalent.*

(i) *B is of stable type p.*

(ii) *For $r>1$ and every B-valued centered random field $\{X_n; n\in N^d\}$ with* $\lim\rho_B^*(\tau)=0$ *and*

1) Zhang, L. X., Convergence rates in the strong laws of nonstationary ρ^*-mixing random fields, Preprint, 1996.

$$X_n \prec X,\ \mathrm{E}\,\| X \|^{rp} \log^{d-1}(\| X \|) < \infty, \tag{9}$$

we have

$$\sum_n |n|^{r-2} \mathrm{P}\left(\max_{k \le n} \| S_k \| > \varepsilon |n|^{1/p} \right) < \infty \text{ for any } \varepsilon > 0. \tag{10}$$

(iii) *For every B-valued centered random field* $\{X_n;\ n \in N^d\}$ *with* $\lim \rho_B^*(\tau) = 0$ *and*

$$X_n \prec X,\ \mathrm{E}\,\| X \|^{p} \log^{\beta \vee (d-1)}(\| X \|) < \infty \text{ for some } \beta > d(p-1), \tag{11}$$

we have

$$\sum_n |n|^{-1} \mathrm{P}\left(\max_{k \le n} \| S_k \| > \varepsilon |n|^{1/p} \right) < \infty \text{ for any } \varepsilon > 0. \tag{12}$$

(iv) *For* $r \geq 1$ *and every B-valued centered random field* $\{X_n; n \in N^d\}$ *with* $\lim \phi^*(\tau) = 0$ *and* (9), *we have* (10).

(v) *For every B-valued centered random field* $\{X_n;\ n \in N^d\}$ *with* $\lim \rho_B^*(\tau) = 0$ *and* (11), *we have*

$$\lim_n \frac{S_n}{|n|^{1/p}} = 0,\ a.s. \tag{13}$$

(vi) *For every B-valued centered random field* $\{X_n; n \in N^d\}$ *with* $\lim \phi^*(\tau) = 0$ *and*

$$X_n \prec X,\ \mathrm{E}\,\| X \|^{p} \log^{d-1}(\| X \|) < \infty, \tag{14}$$

we have (13). *Furthermore, if B is reflexive, then the conditions* $\lim \rho_B^*(\tau) = 0, \lim \phi^*(\tau) = 0$ *can be replaced by* $\lim \rho_B^*(\tau) < 1, \lim \phi^*(\tau) < 1/4$ *respectively in each statement.*

Proof. (iii)⇒(v), (iv)⇒(vi) are obvious and, it is easily seen that each of (ii)—(vi) implies the following statement.

(vii) For every bounded sequence $\{x_k; k \geq 1\}$ in B, we have

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \varepsilon_k x_k}{n^{1/p}} = 0,\ \text{a.s.},$$

where $\{\varepsilon_k\}$ is a Rademacher sequence, which implies (i). On the other hand, by using Theorems 3 and 4 we can prove (i)⇒(iv) and (i)⇒(ii), (iii). The proof is omitted here. One can refer to the author's another paper and sec. 8.3 of reference [1].

Open Problems. (a) *Can condition that* B^* *has the R-N property (or B is reflexive) be removed or not?*

(b) *Are* (12) *and* (13) *also true with condition* (11) *being replaced by* (14), *if necessary, with some more conditions on the convergent rates of* $\rho_B^*(\tau)$?

(c) *If condition* $X_n \prec X$ *is replaced by the saying that* $\{X, X_n\}$ *are identically distributed, is each of statements* (ii)—(vi) *in Theorem* 6 *equivalent to the statement that B is of type p? It is known that it is true in the case of independent fields.*

References

1 Lin, Z. Y., Lu, C. R., *Limit Theory on Mixing Random Variables*, Dordrecht-Beijing: Kluwer Academic Publishers and Science Press, 1997.

2 Bradkey, R. C., On the spectral density and asymptotic normality of dependence between random variables, *J. Theoret. Probab.*, 1992, 5: 355.

3 Miller, C., Three theorems on random fields with ρ^*-mixing, *J. Theoret. Probab.*, 1994, 7: 867.

4 Miller, C., A CLT for periodograms of a ρ^*-mixing random field, *Stoch. Processes Appl.*, 1995, 60: 313.

5 Triebel, H., *Interpolation Theory, Function Space, Differential Operators*, Amsterdam-New York-Oxeford: North-Holland Publishing Company, 1978.

6 Bryc, W., Smolenski, W., Moment conditions for almost sure convergence of weakly correlated random variables, *Proc. Amer. Math. Soc.*, 1993, 119: 629.

7 Ledoux, M., Talagrand, M., *Probability in Banach Spaces: Isoperimetry and Processes*, Berlin-New York: Springer-Verlag, 1991.

8 Móricz, F., A general moment inequality for the maximum of partial sums of single series, *Acta. Sci. Math.*, 1982, 44: 67.