A TRANSFORMATION METHOD OF NON-HAMILTONIAN SYSTEMS AND ITS APPLICATION

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Abstract

In a previous paper[1] a transformation method of non-Hamiltonian systems has been presented. In this paper, this method is concretized, and the principle, process and forms of two kinds (the forms of the implicit function and the explicit function) of the method are discussed in detail, with the motion solution of a small celestial body under the oblateness perturbation of a principal celestial body given as an example.

1. The Principle of the Method

If we take coordinates and velocities (or other similar variables) as elementary variables, the corresponding motion equations of a nonlinear perturbation system with n freedoms are just a system of $2n$ first order differential equations. Let x and X represent the coordinate vector and velocity vector respectively:

$$x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad X=\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}. \tag{1}$$

Note

$$\sigma=\begin{pmatrix} X \\ x \end{pmatrix}, \tag{2}$$

and the corresponding motion equation is

$$\frac{d\sigma}{dt}=f(\sigma,t;\varepsilon), \tag{3}$$

where ε is a small parameter, f is a vector function of $2n$ dimensions.

Usually x is chosen as the angle coordinate and X as the moment, and they need not be canonical conjugate variables. The variables of every component of x are divided into fast and slow variables; the terms which contain fast variables (or at the same time contain slow variables, but their character depends on the fast variables) are called short-period terms; the terms which only contain slow variables are called long-period terms. If the solutions of Eq. (3) can be constructed in these period terms and the secular terms which are linearly variables with the time t, then Eq. (3) can be solved by the present transformation method and the corresponding transformation may be taken as,

$$\sigma = \sigma^* + \sum_{m\geqslant 1} \sigma_s^{(m)}(Y, x, t; \varepsilon^m), \tag{4}$$

$$\sigma^* = \begin{pmatrix} Y \\ y \end{pmatrix}, \tag{5}$$

where the new variables y and Y all are n-vectors, $\sigma_s^{(m)}$ is the mth order period term that needs to be defined and can be defined in the solving process. The function $\sigma_s^{(m)}$ plays the same role as the generation function in the canonical transformation and the transformation is the form of the implicit function which is constructed by mixed variables. It may also take the form of the explicit function, namely,

$$\sigma = \sigma^* + \sum_{m\geqslant 1} \sigma_s^{(m)}(\sigma^*, t; \varepsilon^m). \tag{6}$$

According to the transformation (4), the equation of new variable σ^* can be obtained as,

$$\begin{aligned} \frac{d\sigma^*}{dt} &= \frac{\partial\sigma^*}{\partial\sigma} f(\sigma, t; \varepsilon) + R, \\ R &= \frac{\partial\sigma^*}{\partial t}, \end{aligned} \tag{7}$$

where R is named the remainder function of the transformation; if the transformation does not contain t explicitly, then $R = 0$. The transformation matrix $\partial\sigma^*/\partial\sigma$ and the vector function $f(\sigma, t; \varepsilon)$ and $\partial\sigma^*/\partial t$ all can be expanded as the power series of the mixed variables Y and x. Let the operator

$$D = \frac{\partial}{\partial(Y, x)} \tag{8}$$

denote the partial derivative with the vector $\begin{pmatrix} Y \\ x \end{pmatrix}$, then

$$\frac{\partial\sigma^*}{\partial\sigma} = E - \frac{\partial}{\partial\sigma} \sum_m \sigma_s^{(m)} = E + D \sum_m \sigma_s^{(m)} \sum_{k\geqslant 1} (-1)^k \left[D \sum_m Y_s^{(m)}\right]^{k-1}, \tag{9}$$

$$f(\sigma, t; \varepsilon) = f(Y, x, t; \varepsilon) + \sum_{k\geqslant 1} \frac{1}{k!} \frac{\partial^k f}{\partial Y^k} \left[\sum_m Y_s^{(m)}\right]^k, \tag{10}$$

$$\frac{\partial\sigma^*}{\partial t} = -\sum_m \frac{\partial\sigma_s^{(m)}}{\partial t} + \left(\sum_{k\geqslant 0} (-1)^k \frac{\partial}{\partial Y} \sum_m \sigma_s^{(m)} \left[\frac{\partial}{\partial Y} \sum_m Y_s^{(m)}\right]^k\right) \sum_m \frac{\partial Y_s^{(m)}}{\partial t}. \tag{11}$$

In the right side of (9), E is a unity matrix of $2n$ dimensions, other terms are all matrices of $2n$ dimensions. $[D\sum_m Y_s^{(m)}]^{k-1}$ is the power of matrices, with the same symbol as in (11). In addition, $\partial\sigma_s^{(m)}/\partial t$ and $\partial Y_s^{(m)}/\partial t$ denote the partial derivative with the explicit t in $\sigma_s^{(m)}$ and $Y_s^{(m)}$.

In the right side of (10), all terms are $2n$-vector functions. For every formula, the arrangement of elements of matrices and vectors is the same as the original. The matrices and vectors with dimensions under $2n$ must make up the deficiencies with zero elements. The similar cases in the following will be treated in the same way; the $Y_s^{(m)}$ represents the Y component of the vector $\sigma_s^{(m)}$.

The function on the right side of (3) can be expanded to convergent power series of the small parameter ε (usually the series is finite), namely,

$$f(\sigma, t; \varepsilon) = f_0(X) + \sum_{k\geq 1} f_k(X, x, t; \varepsilon^k), \tag{12}$$

where $f_k = 0(\varepsilon^k)$. Introducting (9)—(11) into (7), we can obtain the motion equation for the new variable σ^* as,

$$\frac{d\sigma^*}{dt} = f^*(Y, x, t; \varepsilon) = \sum_{k\geq 0} f_k^*(Y, x, t; \varepsilon^k). \tag{13}$$

The functions on the right side are formed by the mixed variables Y, x. Their particular forms are:

$$f_0^* = f_0(Y), \tag{14}$$

$$f_1^* = f_1(Y, x, t; \varepsilon) + \frac{\partial f_0}{\partial Y} Y_s^{(1)} - D\sigma_s^{(1)} f_0 - \frac{\partial \sigma_s^{(1)}}{\partial t}, \tag{15}$$

$$\begin{aligned} f_2^* = {} & f_2(Y, x, t; \varepsilon^2) + \frac{\partial f_1}{\partial Y} Y_s^{(1)} + \frac{1}{2}\frac{\partial^2 f_0}{\partial Y^2} Y_s^{(1)^2} + \frac{\partial f_0}{\partial Y} Y_s^{(2)} \\ & + [-D\sigma_s^{(2)} + D\sigma_s^{(1)} D Y_s^{(1)}] f_0 - D\sigma_s^{(1)}\left[f_1 + \frac{\partial f_0}{\partial Y} Y_s^{(1)}\right] - \frac{\partial \sigma_s^{(2)}}{\partial t} + \frac{\partial \sigma_s^{(1)}}{\partial Y}\frac{\partial Y_s^{(1)}}{\partial t}, \end{aligned} \tag{16}$$

$$\begin{aligned} f_3^* = {} & f_3(Y, x, t; \varepsilon^3) + \frac{\partial f_2}{\partial Y} Y_s^{(1)} + \frac{\partial f_1}{\partial Y} Y_s^{(2)} + \frac{1}{2}\frac{\partial^2 f_1}{\partial Y^2} Y_s^{(1)^2} \\ & + \frac{1}{6}\frac{\partial^3 f_0}{\partial Y^3} Y_s^{(1)^3} + \frac{\partial^2 f_0}{\partial Y^2} Y_s^{(1)} Y_s^{(2)} + \frac{\partial f_0}{\partial Y} Y_s^{(3)} \\ & + [-D\sigma_s^{(3)} + D\sigma_s^{(1)} D Y_s^{(2)} + D\sigma_s^{(2)} D Y_s^{(1)} - D\sigma_s^{(1)} (D Y_s^{(1)})^2] f_0 \\ & + [-D\sigma_s^{(2)} + D\sigma_s^{(1)} D Y_s^{(1)}] \cdot \left[f_1 + \frac{\partial f_0}{\partial Y} Y_s^{(1)}\right] \\ & + [-D\sigma_s^{(1)}] \cdot \left[f_2 + \frac{\partial f_1}{\partial Y} Y_s^{(1)} + \frac{\partial f_0}{\partial Y} Y_s^{(2)} + \frac{1}{2}\frac{\partial^2 f_0}{\partial Y^2} Y_s^{(1)^2}\right] - \frac{\partial \sigma_s^{(3)}}{\partial t} \\ & + \left(\frac{\partial \sigma_s^{(2)}}{\partial Y}\frac{\partial Y_s^{(1)}}{\partial t} + \frac{\partial \sigma_s^{(1)}}{\partial Y}\frac{\partial Y_s^{(2)}}{\partial t}\right) - \frac{\partial \sigma_s^{(1)}}{\partial Y}\frac{\partial Y_s^{(1)}}{\partial Y}\frac{\partial Y_s^{(1)}}{\partial t}. \end{aligned} \tag{17}$$

...

We may go on like that if necessary. According to the specifically required accuracy it is easy to substitute the new variables y for old variables x in the function f^*, which will be further discussed later.

According to the transformation form of the explicit function (6), the equation of new variable σ^* can be obtained by

$$\frac{d\sigma^*}{dt} = \frac{\partial \sigma^*}{\partial \sigma} f(\sigma, t; \varepsilon) + R, \quad R = \frac{\partial \sigma^*}{\partial t}. \tag{18}$$

The transformation matrix $\partial\sigma^*/\partial\sigma$, vector function f and $\partial\sigma^*/\partial t$ can be written as,

$$\frac{\partial\sigma^*}{\partial\sigma} = E + \sum_{k\geqslant 1}(-1)^k\left[\frac{\partial}{\partial\sigma^*}\sum_m \sigma_s^{(m)}\right]^k \tag{19}$$

$$f(\sigma, t;\, \varepsilon) = f(\sigma^*, t;\, \varepsilon) + \sum_{k\geqslant 1}\frac{1}{k!}\frac{\partial^k f}{\partial\sigma^{*k}}\left[\sum_m \sigma_s^{(m)}\right]^k, \tag{20}$$

$$\frac{\partial\sigma^*}{\partial t} = -\left(\sum_{k\geqslant 0}(-1)^k\left[\frac{\partial}{\partial\sigma^*}\sum_m \sigma_s^{(m)}\right]^k\right)\sum_m \frac{\partial\sigma_s^{(m)}}{\partial t}. \tag{21}$$

If Condition (12) is satisfied, Eq. (18) becomes

$$\frac{d\sigma^*}{dt} = f^*(\sigma^*, t;\varepsilon) = \sum_{k\geqslant 0} f_k^*(\sigma^*, t;\varepsilon^k). \tag{22}$$

We still use D to express the new operator:

$$D = \frac{\partial}{\partial\sigma^*}.$$

Then the particular forms of f^* are:

$$f_0^* = f_0(Y), \tag{23}$$

$$f_1^* = f_1(\sigma^*,\, t;\, \varepsilon) + \frac{\partial f_0}{\partial Y}Y_s^{(1)} - D\sigma_s^{(1)}f_0 - \frac{\partial\sigma_s^{(1)}}{\partial t}, \tag{24}$$

$$\begin{aligned} f_2^* = {} & f_2(\sigma^*, t;\, \varepsilon^2) + Df_1\sigma_s^{(1)} + \frac{1}{2}\frac{\partial^2 f_0}{\partial Y^2}Y_s^{(1)^2} + \frac{\partial f_0}{\partial Y}Y_s^{(2)} \\ & + [-D\sigma_s^{(2)} + (D\sigma_s^{(1)})^2]f_0 - D\sigma_s^{(1)}\left[f_1 + \frac{\partial f_0}{\partial Y}Y_s^{(1)}\right] - \frac{\partial\sigma_s^{(2)}}{\partial t} + D\sigma_s^{(1)}\frac{\partial\sigma_s^{(1)}}{\partial t}, \end{aligned} \tag{25}$$

$$\begin{aligned} f_3^* = {} & f_3(\sigma^*, t;\, \varepsilon^3) + Df_2\sigma_s^{(1)} + Df_1\sigma_s^{(2)} + \frac{1}{2}D^2f_1\sigma_s^{(1)^2} \\ & + \frac{1}{6}\frac{\partial^3 f_0}{\partial Y^3}Y_s^{(1)^3} + \frac{\partial^2 f_0}{\partial Y^2}Y_s^{(1)}Y_s^{(2)} + \frac{\partial f_0}{\partial Y}Y_s^{(3)} \\ & + [-D\sigma_s^{(3)} + D\sigma_s^{(2)}D\sigma_s^{(1)} + D\sigma_s^{(1)}D\sigma_s^{(2)} - (D\sigma_s^{(1)})^3]f_0 \\ & + [-D\sigma_s^{(2)} + (D\sigma_s^{(1)})^2]\left[f_1 + \frac{\partial f_0}{\partial Y}Y_s^{(1)}\right] \\ & + [-D\sigma_s^{(1)}]\left[f_2 + Df_1\sigma_s^{(1)} + \frac{\partial f_0}{\partial Y}Y_s^{(2)} + \frac{1}{2}\frac{\partial^2 f_0}{\partial Y^2}Y_s^{(1)^2}\right] - \frac{\partial\sigma_s^{(3)}}{\partial t} \\ & + \left(D\sigma_s^{(1)}\frac{\partial\sigma_s^{(2)}}{\partial t} + D\sigma_s^{(2)}\frac{\partial\sigma_s^{(1)}}{\partial t}\right) - (D\sigma_s^{(1)})^2\frac{\partial\sigma_s^{(1)}}{\partial t} \end{aligned} \tag{26}$$

$\cdots$

The purpose of transformation (4) is, under a certain accuracy, to lead to

$$f^* = \sum_{k=0}^{l} f_k^*(Y;\varepsilon^k) + \sum_{k\geqslant l+1} f_k^*(Y, x, t;\, \varepsilon^k), \tag{27}$$

$$f_k^*(Y,\; \varepsilon^k) = \begin{pmatrix}(f_k^*)_Y \\ (f_k^*)_y\end{pmatrix},$$

$$(f_k^*)_Y = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (f_k^*)_y = \begin{pmatrix} (f_k^*)_{y_1} \\ (f_k^*)_{y_2} \\ \vdots \\ (f_k^*)_{y_n} \end{pmatrix}, \quad (k = 0, 1, \cdots, l) \tag{28}$$

where $(f_k^*)_y$ expresses the functions on the right side of the equations about y. In the following paragraphs analogous symbols are not explained any longer. It is in the similar case with the transformation form (6) of the explicit function, but on the right side of (27), $k \geqslant l+1$ corresponding f_k^* are also the explicit functions, that is, $f_k^*(\sigma^*, t; \varepsilon^k)$. Therefore the new equation (13) or (22) becomes the integrable form. Under the accuracy $O(\varepsilon^l)$, its solution is

$$Y = Y_0, \quad y = y_0 + \left[\sum_{k=0}^{l} f_k^*(Y_0; \varepsilon^k)\right](t - t_0), \tag{29}$$

where Y_0 and y_0 are integral constants defined by the initial values at t_0. The solution for the old variable σ can be given through the transformation relation (4) or (6). In the transformation form of the implicit function $\sigma_s^{(m)}$ contains the mixed variables and the problem of substitution variables can be dealt with in the same way as in the von-Zeipel canonical transformation.

As the $\sigma_s^{(m)}$ in the transformation (4) or (6) needs to be defined, so the above-mentioned purpose may be achieved. For example, we can take these terms which only contain the variable Y on the right side of (14)—(17) or (23)—(26) as f_0^*, f_1^*, f_2^*, $f_3^* \cdots$ We can use all other terms which contain fast and slow angle variables x (or y) to lead to $\sigma_s^{(m)}$. But some condition must be satisfied. For that according to character of variables that they contain we must divide the function f on the right side of (3) or the function f given by (12) into the short-period terms, long-period terms and the terms that only contain X (this will become the secular terms of variables after integrating), and we write them as f_s, f_l, and f_c respectively. In what follows the related terms that appear are divided in the same way.

The necessary conditions for the transformation are:

(1) The function f does not contain the first order long-period term, namely,

$$f_{1l} = 0, \tag{30}$$

and

$$f_{kc} = \begin{pmatrix} (f_{kc})_X \\ (f_{kc})_x \end{pmatrix}, \quad (f_{kc})_X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (f_{kc})_x = \begin{pmatrix} (f_{kc})_{x_1} \\ (f_{kc})_{x_2} \\ \vdots \\ (f_{kc})_{x_n} \end{pmatrix}. \tag{31}$$

$$(k = 0, 1 \cdots)$$

If $(f_{kc})_X$ is a non-zero vector, the $(f_k^*)_Y$, obtained after transforming, is generally a non-zero vector, too. Consequently the integrable form can not be constructed directly. Neither can the solution (29) be given. But the transformation can make the equation reduce to n-order. Hence, we can first discuss the n-order equation:

$$\frac{dY}{dt} = \sum_{k=0}^{l} (f_k^*(Y; \varepsilon^k))_Y.$$

Suppose its solution is

$$Y = Y(t, t_0, Y_0; \varepsilon),$$

thereupon the n integrals about y can be immediately constructed, namely,

$$y = y_0 + \int^{t} \sum_{k=0}^{l} (f_k^*(Y(t); \varepsilon^k))_y dt.$$

The Van der Pol equation, a typical nonlinear vibration equation, and some non-attractive perturbation systems (such as the atmospheric resistance for the artificial earth satellite, etc.) belong to this type, which do not satisfy Condition (31). Another typical nonlinear vibration equation, the Lindstedt equation, and the attractive perturbation system can satisfy Condition (31) (such as the oblateness perturbation which is to be discussed in Sec. III of this paper).

(2) In the courses of the transformation there is not a small divisor appearing due to commensurability. If the small divisor appears, for example, in the mth order, the corresponding characteristic value μ must satisfy this inequality,

$$\mu > \max(\varepsilon^{\frac{m}{2}}, \varepsilon). \tag{32}$$

Under the above-mentioned conditions an effective transformation series (4) or (6) can be obtained, but to attain the above purpose the transformation has to be done time and again. We can use the method of accelerated convergence[2] for the transformation. In the first transformation all the functions are expanded as a power series of ε, by taking $\sigma_s^{(1)}$ for $\sigma_s^{(m)}$, and then making the functions f_0^* and f_1^* become integrable forms, and the lowest power of remainder terms should not be less than two orders. In the second transformation the functions are expanded as a power series of $\nu = \varepsilon^N (N \geqslant 2)$, thereby the present transformation method of non-Hamiltonian systems is very simple not only in principle but in its course for perturbation solution of a problem. This will be discussed in detail in the following two sections.

II. Construction of Form Solution

For the sake of convenience, let us suppose that there is just one fast variable (if there are several fast variables, in this method of transformation there is not any substantive difference) which is denoted by x_n. When $\varepsilon = 0$, the angle velocity (or angle frequency) corresponding to x_n is denoted by θ, and θ is only related to X_1 (usually x_n corresponds to X_n, the notation we use is just for simplicity). All the remaining angle variables $x_1, x_2, \cdots, x_{n-1}$ all are slow variables. In addition the case of containing t explicitly can be reduced to the case of not containing t explicitly, if we can add a variable or take an appropriate variable transformation. When a problem needs to be treated as a case of containing t explicitly it would not cause any trouble, for the general method is already stated in the previous section. Therefore we will limit ourselves to the discussion of the case not containing t explicitly. The equation of motion can be written as follows:

$$\frac{d\sigma}{dt} = f(\sigma, \varepsilon) = f_0(X_1) + \sum_{k \geqslant 1} f_k(\sigma, \varepsilon^k), \tag{33}$$

$$f_0(X_1)=\delta\theta(X_1),\quad \delta=\begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix},\tag{34}$$

where δ is an vector and has only the $2n$th component.

The solution of the equation can be obtained by the transformation method of two kinds, namely, the form of the explicit function and the form of the implicit function. They are given as follows.

(1) *The transformation of the form of the implicit function*

The first transformation which eliminates the fast variable (x_n) is $\sigma\Rightarrow\sigma^*$, namely, Eq. (4). From (14)—(17) we can obtain f_k^* and $\sigma_s^{(m)}$ by using

$$f_0^*=f_0(Y_1)=\delta\theta^*(Y_1),\tag{35}$$

$$f_1^*=f_{1c}(Y,\varepsilon),\quad \theta^*\frac{\partial\sigma_s^{(1)}}{\partial x_n}=f_{1s}(Y,x;\varepsilon)+\delta\left[\frac{\partial\theta^*}{\partial Y_1}Y_{1s}^{(1)}\right],\tag{36}$$

$$f_2^*=[\Phi_2]_{c,l},\quad \theta^*\frac{\partial\sigma_s^{(2)}}{\partial x_n}=[\Phi_2]_s+\delta\left[\frac{\partial\theta^*}{\partial Y_1}Y_{1s}^{(2)}\right],\tag{37}$$

$$f_3^*=[\Phi_3]_{c,l},\quad \theta^*\frac{\partial\sigma_s^{(3)}}{\partial x_n}=[\Phi_3]_s+\delta\left[\frac{\partial\theta^*}{\partial Y_1}Y_{1s}^{(3)}\right],\tag{38}$$

$$\Phi_2=f_2+\left(\frac{\partial f_1}{\partial Y}Y_s^{(1)}-\frac{\partial\sigma_s^{(1)}}{\partial x}f_1\right)-\frac{1}{\theta^*}\frac{\partial\theta^*}{\partial Y_1}Y_{1s}^{(1)}f_{1s}+\delta\left[\frac{1}{2}\frac{\partial^2\theta^*}{\partial Y_1^2}-\frac{1}{\theta^*}\left(\frac{\partial\theta^*}{\partial Y_1}\right)^2\right]Y_{1s}^{(1)^2},\tag{39}$$

$$\begin{aligned}\Phi_3=&f_3+\left(\frac{\partial f_2}{\partial Y}Y_s^{(1)}-\frac{\partial\sigma_s^{(1)}}{\partial x}f_2\right)+\left(\frac{\partial f_1}{\partial Y}Y_s^{(2)}-\frac{\partial\sigma_s^{(2)}}{\partial x}f_1\right)+\left(\frac{1}{2}\frac{\partial^2 f_1}{\partial Y^2}Y_s^{(1)}\right.\\&\left.-\frac{\partial\sigma_s^{(1)}}{\partial x}\frac{\partial f_1}{\partial Y}\right)Y_s^{(1)}-\left(\frac{\partial\theta^*}{\partial Y_1}Y_{1s}^{(1)}\right)\frac{\partial\sigma_s^{(2)}}{\partial x_n}-\left(\frac{\partial\theta^*}{\partial Y_1}Y_{1s}^{(2)}+\frac{1}{2}\frac{\partial^2\theta^*}{\partial Y_1^2}Y_{1s}^{(1)^2}\right)\frac{\partial\sigma_s^{(1)}}{\partial x_n}\\&-\frac{\partial\sigma_s^{(1)}}{\partial Y}f_2^*+\delta\left[\frac{\partial^2\theta^*}{\partial Y_1^2}Y_{1s}^{(1)}Y_{1s}^{(2)}+\frac{1}{6}\frac{\partial^3\theta^*}{\partial Y_1^3}Y_{1s}^{(1)^3}\right].\end{aligned}\tag{40}$$

If the computation continues to $O(\varepsilon^3)$, the x in f_3^* can be replaced by y directly, but for the f_2^*, when x is replaced by y directly, $(\partial f_2^*/\partial y)y_s^{(1)}$ will be introduced and then put into Φ_3. Therefore after eliminating x_n, the function on the right side of the new equation becomes

$$\begin{aligned}f^*(\sigma^*,\varepsilon)=&f_0^*(Y_1)+f_1^*(Y,\varepsilon)+f_2^*(Y,y_1,y_2,\cdots,y_{n-1},\varepsilon^2)\\&+f_3^*(Y,y_1,y_2,\cdots,y_{n-1},\varepsilon^3)+\cdots\end{aligned}\tag{41}$$

The second transformation which eliminates the slow variables $(y_1, y_2, \cdots, y_{n-1})$ is $\sigma^*\Rightarrow\sigma^{**}$, that is,

$$\sigma^*=\sigma^{**}+\sum_{m\geqslant1}\sigma_l^{(m)}(Z,y,\varepsilon^m),\tag{42}$$

$$\sigma^{**} = \begin{pmatrix} Z \\ z \end{pmatrix}. \tag{43}$$

From (14)—(17), after transforming, the functions f_k^* on the right side of the new equation and $\sigma_l^{(m)}$ can be obtained by

$$f_0^{**} = f_0^*(Z_1) = \delta\theta^{**}(Z_1), \tag{44}$$

$$f_1^{**} = f_1^*(Z, \varepsilon), \tag{45}$$

$$f_2^{**} = f_{2c}^*(Z, \varepsilon^2), \qquad \frac{\partial\sigma_l^{(1)}}{\partial y} f_1^* = f_{2l}^* + \frac{\partial f_1}{\partial Z} Z_l^{(1)} + \delta\left[\frac{\partial\theta^{**}}{\partial Z_1} Z_{1l}^{(2)}\right], \tag{46}$$

$$f_3^{**} = [\Phi_3^*]_c, \qquad \frac{\partial\sigma_l^{(2)}}{\partial y} f_1^* = [\Phi_3^*]_l + \frac{\partial f_1^*}{\partial Z} Z_l^{(2)} + \delta\left[\frac{\partial\theta^{**}}{\partial Z_1} Z_{1l}^{(3)}\right], \tag{47}$$

$$\Phi_3^* = f_3^* + \left(\frac{\partial f_2^*}{\partial Z} Z_l^{(1)} - \frac{\partial\sigma_l^{(1)}}{\partial y} f_2^*\right) + \left(\frac{1}{2}\frac{\partial^2 f_1^*}{\partial Z^2} Z_l^{(1)} - \frac{\partial\sigma_l^{(1)}}{\partial y}\frac{\partial f_1^*}{\partial Z}\right) Z_l^{(1)}. \tag{48}$$

It can be shown that $Z_{1l}^{(1)} = 0$ because of the condition that $f_{1l} = 0$. After eliminating all the fast and slow angle variables, the function on the right side of the equation of the new variables is

$$f^{**}(\sigma^{**}, \varepsilon) = f_0^{**}(Z_1) + f_1^{**}(Z, \varepsilon) + f_2^{**}(Z, \varepsilon^2) + f_3^{**}(Z, \varepsilon^3) + \cdots \tag{49}$$

In this vector function, the frontal n components $(f^{**})_{Z_1}, (f^{**})_{Z_2}, \cdots, (f^{**})_{Z_n}$ all are zero. Thereby the solution for σ^{**} is

$$Z = Z_0, \quad z = z_0 + \left[\sum_{k=0}^{3} f_k^{**}(Z_0, \varepsilon^k)\right](t - t_0). \tag{50}$$

After transforming twice the solution for the old variable σ can be obtainded from

$$\sigma = \sigma^{**} + \sigma_l^{(1)} + \sigma_l^{(2)} + \cdots + \sigma_s^{(1)} + \sigma_s^{(2)} + \cdots \tag{51}$$

(2) *The transformation of the form of the explicit function*

The first transformation which eliminates fast variable (y_n) yields

$$f_0^* = f_0(Y_1) + \delta\theta^*(Y_1), \tag{52}$$

$$f_1^* = f_{1c}(Y, \varepsilon), \quad \theta^*\frac{\partial\sigma_s^{(1)}}{\partial y_n} = f_{1s}(\sigma^*, \varepsilon) + \delta\left[\frac{\partial\theta^*}{\partial Y_1} Y_{1s}^{(1)}\right], \tag{53}$$

$$f_2^* = [\Psi_2]_{c,l}, \quad \theta^*\frac{\partial\sigma_s^{(2)}}{\partial y_n} = [\Psi_2]_s + \delta\left[\frac{\partial\theta^*}{\partial Y_1} Y_{1s}^{(2)}\right], \tag{54}$$

$$f_3^* = [\Psi_3]_{c,l}, \quad \theta^*\frac{\partial\sigma_s^{(3)}}{\partial y_n} = [\Psi_3]_s + \delta\left[\frac{\partial\theta^*}{\partial Y_1} Y_{1s}^{(3)}\right], \tag{55}$$

$$\Psi_2 = f_2 + \left(\frac{\partial f_1}{\partial\sigma^*}\sigma_s^{(1)} - \frac{\partial\sigma_s^{(1)}}{\partial y} f_{1c}\right) + \delta\left[\frac{1}{2}\frac{\partial^2\theta^*}{\partial Y_1^2} Y_{1s}^{(1)^2}\right], \tag{56}$$

$$\Psi_3 = f_3 + \left(\frac{\partial f_2}{\partial\sigma^*}\sigma_s^{(1)} - \frac{\partial\sigma_s^{(1)}}{\partial\sigma^*} f_2^*\right) + \left(\frac{\partial f_1}{\partial\sigma^*}\sigma_s^{(2)} - \frac{\partial\sigma_s^{(2)}}{\partial y} f_{1c}\right) + \frac{1}{2}\frac{\partial^2 f_1}{\partial\sigma^{*2}}\sigma_s^{(1)^2}$$
$$+ \delta\left[\frac{\partial^2\theta^*}{\partial Y_1^2} Y_{1s}^{(1)} Y_{1s}^{(2)} + \frac{1}{6}\frac{\partial^3\theta^*}{\partial Y_1^3} Y_{1s}^{(1)^3}\right]. \tag{57}$$

The second transformation which eliminates the slow variables $(z_1, z_2, \cdots, z_{n-1})$

yields

$$f_0^{**} = f_0^*(Z_1) = \delta\theta_\cdot^{**}(Z_1), \tag{58}$$

$$f_1^{**} = f_1^*(Z, \varepsilon), \tag{59}$$

$$f_2^{**} = f_{2c}^*(Z, \varepsilon^2), \qquad \frac{\partial \sigma_l^{(1)}}{\partial z} f_1^* = f_{2l}^* + \frac{\partial f_1^*}{\partial Z} Z_l^{(1)} + \delta \left[\frac{\partial \theta^{**}}{\partial Z_1} Z_{1l}^{(2)} \right], \tag{60}$$

$$f_3^{**} = [\Psi_3^*]_c, \qquad \frac{\partial \sigma_l^{(2)}}{\partial z} f_1^* = [\Psi_3^*]_l + \frac{\partial f_1^*}{\partial Z} Z_l^{(2)} + \delta \left[\frac{\partial \theta^{**}}{\partial Z_1} Z_{1l}^{(3)} \right], \tag{61}$$

$$\Psi_3^* = f_3^* + \left(\frac{\partial f_7^*}{\partial \sigma^{**}} \sigma_l^{(1)} - \frac{\partial \sigma_l^{(1)}}{\partial z} f_{2c}^* \right) + \frac{1}{2} \frac{\partial^2 f_1^*}{\partial Z^2} Z_l^{(1)^2}. \tag{62}$$

After transforming twice the solution of (33) can be obtained. As the case is similar with the transformation of the implicit function form, it is not necessary to repeat.

According to the course of obtaining the perturbation solution by using two transformation methods, the explicit function form is clearer than the implicit one, but the latter is a bit simpler and has some special advantages for solving certain problem as will be shown in the next section. It must be mentioned both are simpler than the Lie transformation of non-Hamiltonian systems.

III. Solution of Oblateness Perturbation

To see the advantage of our method, we may use it to obtain the motion solution of a small celestial body under oblateness perturbation. We may take the six elements of an elliptical orbit: a, e, i, Ω, ω, M, as the elementary variables because of their very simple and definite geometric meaning, namely,

$$X = \begin{pmatrix} a \\ e \\ i \end{pmatrix}, \quad x = \begin{pmatrix} \Omega \\ \omega \\ M \end{pmatrix}. \tag{63}$$

The equator coordinate system of the principal celestial body is taken to be the coordinate system, while the centroid of the principal body is taken to be the origin of the coordinate system. In the computing unit of the system, as is often used in studying an artificial earth satellite, the perturbing potential function of oblateness J_2 is

$$\Delta V = -\frac{J_2}{r^3} \left(\frac{3}{2} \sin^2\phi - \frac{1}{2} \right), \tag{64}$$

where r and ϕ are respectively the radial vector and latitude of the small body. So the relation between ϕ and orbit elements is

$$\sin\phi = \sin i \sin(v + \omega), \tag{65}$$

where v is the true anomaly.

Then substitution of the potential function V into the Lagrangian motion equation[4] yields the system of the first order ordinary differential equations of the six elements:

$$\frac{d\sigma}{dt} = f(a, e, i, \Omega, \omega, M; J_2). \tag{66}$$

The function on the right side of this system is independent of t and the angle variable $x_1 = \Omega$, while the small parameter ε is J_2. The function f on the right side can be expanded to the following finite terms:

$$f = f_0(a) + f_1(a, e, i, \omega, M; J_2), \tag{67}$$

where

$$f_0 = \delta\theta, \qquad \theta = n = a^{-\frac{3}{2}}. \tag{68}$$

$$f_1 = f_{1c}(a, e, i; J_2) + f_{1s}(a, e, i, \omega, M; J_2), \quad f_{1l} = 0. \tag{69}$$

Notice that the vector function f_{1c} just contains the last three components. If only the critical inclination is removed ($i = i_c = 63°26'$ or $116°34'$) then the two conditions given in the first section are fully satisfied, and the solution can be obtained by the transformation method. We now use the transformation of the form of the implicit function, in addition to the method of the accelerated convergence.

The first transformation which separates the first order short-period term is $(X, x) \Rightarrow (Y, y)$, namely,

$$\sigma = \sigma^* + \sigma_s^{(1)}(Y, x; J_2), \tag{70}$$

$$\sigma^* = \begin{pmatrix} Y \\ y \end{pmatrix}, \quad Y = \begin{pmatrix} a^* \\ e^* \\ i^* \end{pmatrix}, \quad y = \begin{pmatrix} \Omega^* \\ \omega^* \\ M^* \end{pmatrix}. \tag{71}$$

Then Eqs. (35)—(38) become

$$f_0^* = f_0(a^*) = \delta n^*(a^*), \tag{72}$$

$$f_1^* = f_{1c}(a^*, e^*, i^*; J_2), \quad n^* \frac{\partial \sigma_s^{(1)}}{\partial M} = f_{1s}(a^*, e^*, i^*, \omega, M; J_2) + \delta\left[-\frac{3}{2} a^{*-\frac{5}{2}} a_s^{(1)}\right], \tag{73}$$

$$f_2^* = \left[\left(\frac{\partial f_1}{\partial a^*} + \frac{3}{2a^*} f_{1s}\right) a_s^{(1)} + \frac{\partial f_1}{\partial e^*} e_s^{(1)} + \frac{\partial f_1}{\partial i^*} i_s^{(1)} - \frac{\partial \sigma_s^{(1)}}{\partial \omega} (f_1)_\omega - \frac{\partial \sigma_s^{(1)}}{\partial M} (f_1)_M\right] + \delta\left[-\frac{3}{8} a^{*-\frac{7}{2}} a_s^{(1)^2}\right], \tag{74}$$

$$\begin{aligned} f_3^* = {} & \left[\left(\frac{1}{2}\frac{\partial^2 f_1}{\partial a^{*2}} - \frac{15}{8a^{*2}} f_{1s}\right) a_s^{(1)^2} + \frac{1}{2}\frac{\partial^2 f_1}{\partial e^{*2}} e_s^{(1)^2} + \frac{1}{2}\frac{\partial^2 f_1}{\partial i^{*2}} i_s^{(1)^2} + \frac{\partial^2 f_1}{\partial a^* \partial e^*} a_s^{(1)} e_s^{(1)} \right. \\ & \left. + \frac{\partial^2 f_1}{\partial a^* \partial i^*} a_s^{(1)} i_s^{(1)} + \frac{\partial^2 f_1}{\partial e^* \partial i^*} e_s^{(1)} i_s^{(1)}\right] - \left[\frac{\partial \sigma_s^{(1)}}{\partial \omega}\left(\frac{\partial (f_1)_\omega}{\partial a^*} a_s^{(1)} + \frac{\partial (f_1)_\omega}{\partial e^*} e_s^{(1)} \right.\right. \\ & \left.\left. + \frac{\partial (f_1)_\omega}{\partial i^*} i_s^{(1)}\right) + \frac{\partial \sigma_s^{(1)}}{\partial M}\left(\frac{\partial (f_1)_M}{\partial a^*} a_s^{(1)} + \frac{\partial (f_1)_M}{\partial e^*} e_s^{(1)} + \frac{\partial (f_1)_M}{\partial i^*} i_s^{(1)}\right)\right] \\ & - \left[\frac{\partial \sigma_s^{(1)}}{\partial a^*} (f_2^*)_a + \frac{\partial \sigma_s^{(1)}}{\partial e^*} (f_2^*)_e + \frac{\partial \sigma_s^{(1)}}{\partial i^*} (f_2^*)_i\right] + \delta\left[\frac{5}{8} a^{*-\frac{9}{2}} a_s^{(1)^3}\right]. \end{aligned} \tag{75}$$

If the computation just continues to J_2^3, the ω, M on the right side of f_3^* can be replaced by ω^*, M^* directly, for f_2^* on the right side after being replaced in the same way, the two terms

$$\frac{\partial f_2}{\partial \omega^*}\omega_s^{(1)} + \frac{\partial f_2^*}{\partial M^*} M_s^{(1)}$$

are introduced into f_3^*. Then the first transformation gives

$$f^* = f_0^*(a^*) + f_1^*(a^*, e^*, i^*; J_2) + f_2^*(a^*, e^*, i^*, \omega^*, M^*; J_2^2) + f_3^*(a^*, e^*, i^*, \omega^*, M^*; J_2^3). \tag{76}$$

The second transformation is $(Y, y) \Rightarrow (Z, z)$, at this moment the small parameter $\varepsilon = J_2^2, f^*$ can be written as,

$$\begin{aligned} f^* &= F_0^*(a^*, e^*, i^*; J_2) + F_1^*(a^*, e^*, i^*; J_2, \varepsilon), \\ F_0^* &= f_0^* + f_1^*, \qquad F_1^* = f_2^* + f_3^*. \end{aligned} \tag{77}$$

Its corresponding transformation is

$$\sigma^* = \sigma^{**} + \sigma_s^{*(1)}(Z, y; \varepsilon), \tag{78}$$

$$\sigma^{**} = \begin{pmatrix} Z \\ z \end{pmatrix}, \quad Z = \begin{pmatrix} a^{**} \\ e^{**} \\ i^{**} \end{pmatrix}, \quad z = \begin{pmatrix} \Omega^{**} \\ \omega^{**} \\ M^{**} \end{pmatrix}, \tag{79}$$

where $\sigma_s^{*(1)}$ can be divided into four parts, namely,

$$\sigma_s^{*(1)} = \sigma_s^{(2)} + \sigma_s^{(3)} + \sigma_l^{(1)} + \sigma_l^{(2)}. \tag{80}$$

The result of the transformation is

$$f_0^{**} = f_0^*(a^{**}) + f_1^*(a^{**}, e^{**}, i^{**}; J_2), \tag{81}$$

$$f_1^{**} = f_{2c}^*(a^{**}, e^{**}, i^{**}; J_2^2) = f_{3c}^*(a^{**}, e^{**}, i^{**}; J_2^3) + [\Phi]_c, \tag{82}$$

$$n^{**}\frac{\partial \sigma_s^{(2)}}{\partial M^*} = f_{2s}^* + \delta\left[-\frac{3}{2}a^{**-\frac{5}{2}}a_s^{(2)}\right], \tag{83}$$

$$(f_1^*)_\omega \frac{\partial \sigma_l^{(1)}}{\partial \omega^*} = f_{2l}^* + \left[\frac{\partial f_1^*}{\partial e^{**}}e_l^{(1)} + \frac{\partial f_1^*}{\partial i^{**}}i_l^{(1)}\right] + \delta\left[-\frac{3}{2}a^{**-\frac{5}{2}}a_l^{(2)}\right], \tag{84}$$

$$\begin{aligned} (f_1^*)_\omega \frac{\partial \sigma_l^{(2)}}{\partial \omega^*} &= f_{3l}^* + \left[\frac{\partial f_1^*}{\partial a^{**}}a_l^{(2)} + \frac{\partial f_1^*}{\partial e^{**}}e_l^{(2)} + \frac{\partial f_1^*}{\partial i^{**}}i_l^{(2)}\right] + [\Phi]_l \\ &\quad + \delta\left[-\frac{3}{2}a^{**-\frac{5}{2}}a_l^{(3)}\right], \end{aligned} \tag{85}$$

$$\begin{aligned} \Phi = &\left[\frac{\partial f_{2l}^*}{\partial e^{**}}e_l^{(1)} + \frac{\partial f_{2l}^*}{\partial i^{**}}i_l^{(1)}\right] + \left[\frac{1}{2}\frac{\partial^2 f_1^*}{\partial e^{**2}}e_l^{(1)^2} + \frac{1}{2}\frac{\partial^2 f_1^*}{\partial i^{**2}}i_l^{(1)^2} \right. \\ &\left. + \frac{\partial^2 f_1^*}{\partial e^{**}\partial i^{**}}e_l^{(1)}i_l^{(1)}\right] - \frac{\partial \sigma_l^{(1)}}{\partial \omega^*}\left[(f_{2l}^*)_\omega + \frac{\partial (f_1^*)_\omega}{\partial e^{**}}e_l^{(1)} + \frac{\partial (f_1^*)_\omega}{\partial i^{**}}i_l^{(1)}\right]. \end{aligned} \tag{86}$$

$\sigma_s^{(3)}$ is the third order short-period term which can be neglected.

After transforming twice under the accuracy $O(\varepsilon^3)$, the new equations have become an integrable form. By computing the form solution of $f^{**}, \sigma_s^{(1)}, \sigma_l^{(1)}, \cdots,$ it can be shown that the method is simpler than Lie transformation of non-Hamiltonian systems and the mean elements method. In addition it also has two characteristics:

(1) It avoids the difficulty which we come across in using the mean elements method to compute $\sigma_s^{(1)}$, $\sigma_s^{(2)}$, $\cdots$[4] The closed solution about e can be obtained directly.

(2) As we use $\frac{\partial \sigma_s^{(1)}}{\partial \omega}(f_1)_\omega$, $\frac{\partial \sigma_s^{(1)}}{\partial M}(f_1)_M, \cdots$ instead of $\frac{\partial f_1}{\partial \omega}\omega_s^{(1)}$, $\frac{\partial f_1}{\partial M}M_s^{(1)}, \cdots$ the difficulty is avoided in integrating these terms like $\left(\frac{a}{r}\right)^p (f-M)\binom{\sin qf}{\cos qf}$, $(p, q = 0, 1, \cdots)$ which appear in the mean elements method and the canonical transformation in calculating the higher order terms. It simplifies the calculation remarkably and is convenient to use the computer to calculate the closed solution about e.

The above-mentioned transformation method can also be employed to solve perturbation problems of other types. Therefore, to some extent the transformation method of non-Hamiltonian systems presented in this paper can certainly replace the mean elements method.

Because of the limited space, the specific result will appear elsewhere. The important short periodic term $a_s^{(2)}$ to $O(\varepsilon^2)$ is as follows:

$$
\begin{aligned}
a_s^{(2)} = & \frac{J_2^2}{p^3}(1-e^2)^{-3}\left\{(1-e^2)\sin^2 i\left(9-\frac{45}{4}\sin^2 i\right)(v-M)\right. \\
& \times \sum_{k=-1}^{5} A_{1k} e^{|k-2|}\sin(kv+2\omega) + \sum_{k=1}^{6}\left[\left(1-\frac{3}{2}\sin^2 i\right)^2\left(A_{2k}\right.\right. \\
& \left.+\frac{1-e^2}{1+\sqrt{1-e^2}}A_{3k}\right) + \frac{3}{8}(1-e^2)\sin^2 i A_{4k} + \left.\frac{3}{64}\sin^4 i A_{5k}\right] e^k \cos kv \\
& + \sum_{k=-4}^{8}\left[\frac{3}{8}\sin^2 i\left(1-\frac{3}{2}\sin^2 i\right)\left(A_{6k}+\frac{1-e^2}{1+\sqrt{1-e^2}}A_{7k}\right)\right. \\
& \left.+\frac{3}{64}(1-e^2)\sin^4 i A_{8k}\right] e^{|k-2|}\cos(kv+2\omega) \\
& \left.+\sum_{k=-2}^{10}\left[\frac{3}{128}\sin^4 i A_{9k}\right] e^{|k-4|}\cos(kv+4\omega)\right\}, \qquad (87)
\end{aligned}
$$

where $p = a(1-e^2)$. On the right side of the formula σ should be σ^* and we have already considered the added two terms because of transformation variables,

$$
\frac{\partial a_s^{(1)}}{\partial \omega}\omega_s^{(1)} + \frac{\partial a_s^{(1)}}{\partial M}M_s^{(1)}.
$$

The coefficients of every term are:

$$
\begin{aligned}
A_{1k} = & -\frac{1}{8},\ -\frac{3}{4},\ -\left(\frac{3}{2}+\frac{3}{8}e^2\right),\ -\left(1+\frac{3}{2}e^2\right),\ -\left(\frac{3}{2}+\frac{3}{8}e^2\right), \\
& -\frac{3}{4},\ -\frac{1}{8}; \\
A_{2k} = & \left(\frac{27}{2}+\frac{15}{2}e^2-\frac{27}{16}e^4\right),\ \left(\frac{21}{2}+\frac{3}{4}e^2-\frac{9}{32}e^4\right),\ \left(\frac{9}{2}-\frac{3}{32}e^2\right),\ \frac{21}{16},\ \frac{9}{32},\ \frac{1}{32};
\end{aligned}
$$

$$A_{3k} = \left(\frac{3}{2} + \frac{15}{4}e^2 + \frac{3}{16}e^4\right),\ \left(3 + \frac{3}{2}e^2\right),\ \left(\frac{9}{4} + \frac{7}{32}e^2\right),\ \frac{3}{4},\ \frac{3}{32},\ 0;$$

$$A_{4k} = \left(26 + \frac{45}{2}e^2\right),\ (21 + 4e^2),\ \frac{15}{2}, 1,\ 0,\ 0;$$

$$A_{5k} = (-140 + 389e^2 + 345e^4),\ (-54 + 368e^2 + 109e^4),\ (35 + 130e^2),\ \left(\frac{67}{2} + 16e^2\right),\ -9,\ -\frac{3}{4};$$

$$A_{6k} = \frac{1}{8},\ \frac{23}{16},\ \left(\frac{11}{2} + \frac{5}{2}e^2\right),\ \left(\frac{39}{4} + \frac{283}{16}e^2\right),\ 0,\ \left(14 + \frac{221}{4}e^2 + \frac{221}{8}e^4\right),\ \left(10 + 47e^2 + \frac{87}{2}e^4 + 4e^6\right),\ \left(\frac{40}{3} + \frac{161}{4}e^2 + \frac{73}{8}e^4\right),\ \left(\frac{67}{2} + 10e^2 - \frac{1}{8}e^4\right),\ \left(\frac{75}{4} - \frac{17}{16}e^2\right),\ \left(6 - \frac{1}{2}e^2\right),\ \frac{19}{16},\ \frac{1}{8};$$

$$A_{7k} = 0,\ \frac{1}{16},\ \frac{1}{4},\ \frac{1}{16}e^2,\ 0,\ \left(-1 + 3e^2 + \frac{1}{8}e^4\right),\ (8 + 3e^2),\ \left(7 + 12e^2 + \frac{5}{8}e^4\right),\ \left(13 + \frac{11}{2}e^2\right),\ \left(9 + \frac{13}{16}e^2\right),\ \frac{11}{4},\ \frac{5}{16},\ 0;$$

$$A_{8k} = 0,\ 0, -6,\ -39,\ 0,\ (-84 - 69e^2),\ (-24 - 84e^2 - 12e^4),\ (-20 - 21e^2),\ 6,\ 9,\ 2,\ 0,\ 0;$$

$$A_{9k} = \frac{3}{4},\ 9,\ 0,\ (133 + 32e^2),\ \left(198 + 168e^2 + \frac{21}{4}e^4\right),\ (140 + 379e^2 + 75e^4),\ (32 + 364e^2 + 285e^4 + 12e^6),\ (108 + 387e^2 + 99e^4),\ \left(150 + 208e^2 + \frac{53}{4}e^4\right),\ (109 + 56e^2),\ \left(\frac{87}{2} + 6e^2\right),\ 9,\ \frac{3}{4}.$$

The above coefficients are arranged according to the values of k in summation of the original formula (87).

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