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A geometric characterization of planar Sobolev extension domains

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Abstract We characterize bounded simply connected planar $W^{1,p}$-extension domains for $1 < p < 2$ as those bounded simply connected domains $\Omega \subset \mathbb{R}^2$ for which any two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be connected with a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} ds(z) \leqslant C(\Omega, p)|z_1 - z_2|^{2-p}.$$

By combining earlier results, we obtain the following duality result: a Jordan domain $\Omega \subset \mathbb{R}^2$ is a $W^{1,p}$-extension domain, $1 < p < \infty$ if and only if the complementary domain $\mathbb{R}^2 \setminus \overline{\Omega}$ is a $W^{1,p/(p-1)}$-extension domain.

Keywords Sobolev extension, quasiconvexity, John domain

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1 Introduction

In this paper, we study those planar domains $\Omega \subset \mathbb{R}^2$ for which there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. Here, the Sobolev space $W^{1,p}$, $1 \leqslant p \leqslant \infty$, is

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega, \mathbb{R}^2)\},$$

where ∇u denotes the distributional gradient of u. The usual norm in $W^{1,p}(\Omega)$ is

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

More precisely, $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ is an extension operator if there exists a constant $C \geqslant 1$ so that for every $u \in W^{1,p}(\Omega)$, we have

$$\|Eu\|_{W^{1,p}(\mathbb{R}^2)} \leqslant C\|u\|_{W^{1,p}(\Omega)}$$

and $Eu|_\Omega = u$. Notice that we are not assuming the operator E to be linear. However, for $p > 1$, there also always exists a *linear* extension operator provided that there exists an extension operator

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(see [16, 38]). Finally, a domain $\Omega \subset \mathbb{R}^2$ is called a $W^{1,p}$-extension domain if there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. For example, each Lipschitz domain is a $W^{1,p}$-extension domain for each $1 \leqslant p \leqslant \infty$ by the results of Calderón [6] and Stein [40]. However, as proven by Jones [23], the class of extension domains is much larger. The boundary of a $W^{1,p}$-extension domain can be of full Hausdorff dimension and it can include fractal parts.

In this paper, we prefer to use the homogeneous seminorm $\|u\|_{L^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$. This makes no difference because we only consider domains Ω with a bounded (and hence compact) boundary; for such domains, one has a bounded (linear) extension operator for the homogeneous seminorms if and only if there is one for the non-homogeneous ones (see [21]). In what follows, the norm of the extension operator is usually with respect to the homogeneous seminorms.

The main result of our paper is the following geometric characterization of simply connected bounded planar $W^{1,p}$-extension domains.

Theorem 1.1. *Let $1 < p < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then Ω is a $W^{1,p}$-extension domain if and only if for all $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining z_1 and z_2 such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} ds(z) \leqslant C(\Omega, p)|z_1 - z_2|^{2-p}. \tag{1.1}$$

Both the necessity and sufficiency in Theorem 1.1 are new. Notice that the curve γ above is allowed to touch the boundary of Ω even if the points in question lie outside the closure of Ω. This is crucial: there exist bounded simply connected $W^{1,p}$-extension domains for which $\mathbb{R}^2 \setminus \overline{\Omega}$ has multiple components (see, e.g., [7, 25]).

When combined with earlier results, Theorem 1.1 essentially completes the search for a geometric characterization of bounded simply connected planar $W^{1,p}$-extension domains. The unbounded case requires extra technical work and it will be discussed elsewhere.

The condition (1.1) on the complement in Theorem 1.1 appears also in the characterization of $W^{1,q}$-extension domains when $2 < q < \infty$. For such domains, a characterization using the condition (1.1) in the domain itself with the Hölder dual exponent $q/(q-1)$ of q was proved in [39, Theorem 1.2] (see also earlier partial results in [5, 26]).

Theorem 1.2 (See [39])**.** *Let $2 < q < \infty$ and let Ω be a bounded simply connected planar domain. Then Ω is a $W^{1,q}$ -extension domain if and only if for all $z_1, z_2 \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining z_1 and z_2 such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-q}} ds(z) \leqslant C(\Omega, q)|z_1 - z_2|^{\frac{q-2}{q-1}}. \tag{1.2}$$

The above two theorems leave out the case $p = 2$. This is settled by earlier results [13–15, 23], according to which a bounded simply connected domain is a $W^{1,2}$-extension domain if and only if it is a quasidisk (equivalently, a uniform Jordan domain). Thus, Ω is a bounded simply connected $W^{1,2}$-extension domain if and only if Ω is a uniform (bounded) Jordan domain which in turn is true if and only if Ω is a Jordan domain and $\mathbb{R}^2 \setminus \overline{\Omega}$ is uniform or, equivalently, if and only if Ω is bounded and simply connected with $\mathbb{R}^2 \setminus \overline{\Omega}$ a $W^{1,2}$-extension domain.

By combining (the proof of) our characterization in Theorem 1.1 with Shvartsman's characterization stated in Theorem 1.2, we verify the following duality result between the extendability of Sobolev functions from a Jordan domain and from its complementary domain in Subsection 4.7.

Corollary 1.3. *Let $1 < p, q < \infty$ be Hölder dual exponents and let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. Then Ω is a $W^{1,p}$-extension domain if and only if $\mathbb{R}^2 \setminus \bar{\Omega}$ is a $W^{1,q}$-extension domain.*

Corollary 1.3 was hinted by the example in [28] (see also [32, 37]) that exhibits such a duality.

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $W^{1,p}$-extension domain, where $1 < p \leqslant 2$. Then there is $q > p$ so that Ω is a $W^{1,s}$-extension domain for all $1 < s < q$.*

The case $1 < p < 2$ follows from Theorem 1.1 together with the fact that (1.1) implies the analogous inequality for all $1 < s < p + \epsilon$. The case of smaller s is essentially just Hölder's inequality (see [30]), while the improvement to larger exponents follows from Lemma 2.3 that relies on ideas in the proof of [39, Proposition 2.6]. Again, the case $p = 2$ of Corollary 1.4 was already known to hold: one then has extendability for all $1 < s < \infty$.

By combining Corollary 1.4 with results from [26, 39], we obtain an open-ended property.

Corollary 1.5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $W^{1,p}$-extension domain, where $1 < p < \infty$. Then the set of all $1 < s < \infty$ for which Ω is a $W^{1,s}$-extension domain is an open interval.*

Actually, the open interval above can only be one of $1 < s < \infty$, $1 < s < q$ with $q \leqslant 2$, or $q < s < \infty$ with $q \geqslant 2$.

Let us comment on some earlier partial results related to Theorem 1.1. First of all, bounded simply connected $W^{1,p}$-extension domains are John domains when $1 \leqslant p < 2$ (see, e.g., [25, Theorem 6.4], [14, Theorem 3.4], [33, Theorem 4.5] and the references therein). The definition of a John domain is given in Definition 2.18 below. However, there exist John domains that fail to be extension domains and, even after Theorem 1.1, there is no interior geometric characterization available for this range of exponents. Secondly, in [27], it was shown that the complement of a bounded simply connected $W^{1,1}$-extension domain is quasiconvex. This was obtained as a corollary to a characterization of bounded simply connected BV-extension domains. Recall that a set $E \subset \mathbb{R}^2$ is called *quasiconvex* if there exists a constant $C \geqslant 1$ such that any pair of points $z_1, z_2 \in E$ can be connected to each other with a rectifiable curve $\gamma \subset E$ whose length satisfies $\mathrm{le}(\gamma) \leqslant C|z_1 - z_2|$. In [27], it was conjectured that the quasiconvexity of the complement holds for every bounded simply connected $W^{1,p}$-extension planar domain when $1 < p < 2$. This conjecture follows from Theorem 1.1 (see Lemma 2.2), but again, quasiconvexity is a weaker condition than our geometric characterization.

Next, we describe the idea of the proof as follows. We show the necessity of (1.1) in Section 3 by first verifying this condition under the additional requirement that the domain in question is a Jordan domain. This additional assumption together with the extension property allows us to construct suitable test functions that are employed to verify (1.1) (see Lemma 3.3). These are motivated by the function $u(x, y) = \frac{y}{x}$ on $\Omega = \{(x, y) : 0 < y < x, 0 < x < 1\}$. In the complex notation, we have $|\nabla u(z)| \leqslant C|z|^{-1}$; a variant of this property (see (3.2)) holds for the function ϕ constructed in the proof of Lemma 3.3. The curve γ in (1.1) in the case of a Jordan domain is given as the image under the exterior Riemann mapping function of a uniform curve (in the sense of Definition 2.27) that we construct by hand in the exterior of the unit disk (see Figure 3). For readers who are familiar with conformal geometry, it might be helpful to think of the constructed curve in the exterior of the unit disk as almost a quasihyperbolic geodesic for which many useful geometric properties are preserved under conformal maps.

The general case is then handled via an approximation argument, for which we fill Ω by an increasing sequence of Jordan $W^{1,p}$-extension domains with control on the norms of the respective extension operators (see Theorem 3.9). These domains Ω_n are the images of $B(0, 1 - \frac{1}{n})$, $n = 1, 2, \ldots$, under the Riemann mapping function from the unit disk onto the domain Ω. To prove the uniform $W^{1,p}$-extension property, we employ a variant of the technique used by Jones [23] to construct extensions from Ω_n to Ω. The main differences with the setting in [23] are that we only extend to an annular region and that the uniformity of the domain considered by Jones is with respect to the Euclidean metric, while in our case, it is with respect to the inner metric of the domain.

For sufficiency, we again first deal with the Jordan case and then use a compactness argument to pass to a limit. This is done in Section 4. The crucial point in the proof is the introduction of a new version of the Whitney extension technique in the case of Jordan domains. The extension operator is defined in Subsection 4.3. In order to build this extension, we assign a Whitney square of Ω to each complementary Whitney square of size at most the size of Ω. In [23], Jones chose a square of the comparable diameter, and the uniformity gives that each Whitney square of Ω gets assigned to at most uniformly finitely many Whitney squares of the exterior. Roughly speaking, this gives a bi-Lipschitz correspondence between Whitney squares. In our case, this kind of correspondence cannot be expected. To overcome this problem, we pick a Whitney square whose *shadow* along hyperbolic rays has the diameter comparable to that of the shadow of the complementary square. In a sense, we reflect with respect to harmonic measure. Hence, the diameter of a "reflected" square can be much larger than the diameter of the original one and we cannot uniformly bound the number of exterior squares that correspond to a single Whitney square of Ω. Nevertheless, (1.1) allows us to eventually establish appropriate bounds on our extension. Towards this, roughly speaking, we establish estimates on the many-to-one map from the collection of the exterior Whitney squares to Whitney squares of Ω (see Lemma 4.13). This is obtained through a delicate analysis of the locations and sizes of those complementary squares that share a "reflected" square.

The reader familiar with Sobolev extensions may wonder why do we not simply employ the existing extension operators such as those in [17, 38]. This is because we have not been able to directly show that these operators work under our assumptions. However, once we know by our main theorem that the domains are extension domains, we conclude that also these extension operators work under our assumptions.

The rest of this paper is organized as follows. In Section 2, we introduce notation and initial results. Theorem 1.1 gets proven in Sections 3 and 4. Finally, Corollary 1.3 is proven at the very end of this paper in Section 5.

2 Preliminaries

Let us fix some notation. When we make estimates, we often write the constants as positive real numbers $C(\cdot)$ with the parentheses including all the parameters which the constant depends on; we just simply write C if it is absolute. The constant $C(\cdot)$ may vary between appearances, even within a chain of inequalities. By $a \lesssim b$, we mean that $a \leqslant Cb$ for some constant $C \geqslant 2$. Then $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$ hold. If we need to stress the dependence of the respective constant C only on data A, we write $a \lesssim_A b$ and $a \sim_A b$, respectively. The Euclidean distance between two sets A, $B \subset \mathbb{R}^2$ is denoted by $\mathrm{dist}(A, B)$. By $\mathbb{D}$, we always mean the open unit disk in $\mathbb{R}^2$ and by $\mathbb{S}^1$ its boundary. The interior of a set A is denoted by A° and the closure by $\overline{A}$. Given a measurable set A of strictly positive area $|A|$ and a function $u \in L^1(A)$, we write

$$u_A = \fint_A u dz = \frac{1}{|A|} \int_A u dz.$$

2.1 Curves and integrals over curves

Let us next define the curves and line integrals that we use throughout this paper. A continuous map $\gamma\colon I \to \mathbb{R}^2$ is called a curve when I is a (possibly unbounded) interval. When there is no danger of confusion, we sometimes refer also to the image $\gamma(I) \subset \mathbb{R}^2$ by γ. Recall that the derivative $\gamma'(t)$ exists for almost every $t \in I$ for a locally Lipschitz γ. Then the Euclidean length of such a curve can be defined by

$$\mathrm{le}(\gamma) = \int_I |\gamma'(t)| dt.$$

In general,

$$\mathrm{le}(\gamma) = \sup \left\{ \sum_{j=1}^{k} |\gamma(t_{j+1}) - \gamma(t_j)| \right\},$$

where the supremum runs over all $k \geqslant 1$ and all $t_1 < t_2 < \cdots < t_{k+1} \in I$. If $\mathrm{le}(\gamma) < \infty$, which is the case for the Lipschitz curves defined on compact intervals, we call the curve γ rectifiable. In this case, after a reparametrization, we may assume that $\gamma\colon [0,1] \to \mathbb{R}^2$ and that $|\gamma'(t)| = \mathrm{le}(\gamma)$ almost everywhere. We call such a parametrization a constant speed parametrization. Alternatively, we may parametrize the curve γ by arc-length. In this way, we obtain $\tilde{\gamma}\colon [0, \mathrm{le}(\gamma)] \to \mathbb{R}^2$ with $|\tilde{\gamma}'(t)| = 1$ almost everywhere. A change of variable argument shows that reparametrization does not change the length of the curve. From now on, when using the term *rectifiable curve*, by default, we assume the constant speed parametrization on $[0, 1]$, unless otherwise stated.

Notice that we are not requiring our curves to be injective. However, if $E \subset \mathbb{R}^n$ is a continuum with $\mathcal{H}^1(E) < \infty$, then for any $x, y \in E$, there exists an injective curve $\gamma_{x,y}\colon [0,1] \to E$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\mathrm{le}(\gamma_{x,y}) \leqslant \mathcal{H}^1(E)$ (see [8, Lemma 3.12]).

The line integral of a measurable function $f\colon \mathbb{R}^2 \to \overline{\mathbb{R}}$ along a rectifiable curve γ is defined as

$$\int_\gamma f(z)ds(z) = \int_0^1 f(\gamma(t))|\gamma'(t)|dt = \int_0^1 f(\gamma(t))\,\mathrm{le}(\gamma)dt,$$

whenever the integral on the right-hand side exists. Alternatively, for the arc-length parametrization $\tilde{\gamma}$ for γ, we have

$$\int_\gamma f(z)ds(z) = \int_0^{\mathrm{le}(\gamma)} f(\tilde{\gamma}(t))dt.$$

We equip $[0,\infty]$ with the topology whose basis consists of restrictions of open sets of $\mathbb{R}$ to $[0,\infty)$ together with all the intervals $(M,\infty]$ with $M > 0$. A function $f\colon \mathbb{R}^2 \to [0,\infty]$ is then continuous at $x_0 \in \mathbb{R}^2$ where $f(x_0) = \infty$, if for every $M > 0$, there exists $\varepsilon > 0$ for which $f(x) > M$ for all $x \in B(x_0, \varepsilon)$. The continuity of f at each x_0 where $f(x_0) < \infty$ has the usual meaning.

By the Arzelá-Ascoli lemma, we have the following result.

Lemma 2.1. *Let $\gamma_i\colon [0,1] \to \mathbb{R}^2$, $i \in \mathbb{N}$ be a collection of rectifiable curves so that $\bigcup_i \gamma_i([0,1])$ is bounded and $\sup_i \mathrm{le}(\gamma_i) < \infty$. Then there exist a sequence $i_j \nearrow \infty$ and a rectifiable curve γ_∞ so that $\gamma_{i_j}(t) \to \gamma_\infty(t)$ for all $t \in [0,1]$ when $j \to \infty$. Moreover, for any continuous function $f\colon \mathbb{R}^2 \to [0,\infty]$, we have*

$$\int_{\gamma_\infty} f(z)ds(z) \leqslant \liminf_{j\to\infty} \int_{\gamma_{i_j}} f(z)ds(z).$$

In particular,

$$\mathrm{le}(\gamma_\infty) \leqslant \liminf_{j\to\infty} \mathrm{le}(\gamma_{i_j}).$$

If $\gamma_1, \gamma_2\colon [0,1] \to \mathbb{R}^2$ are two non-constant rectifiable curves with $\gamma_1(1) = \gamma_2(0)$, we define their concatenation $\gamma_1 * \gamma_2$ by

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1\left(t\dfrac{\mathrm{le}(\gamma_1)+\mathrm{le}(\gamma_2)}{\mathrm{le}(\gamma_1)}\right), & \text{if } 0 \leqslant t \leqslant \dfrac{\mathrm{le}(\gamma_1)}{\mathrm{le}(\gamma_1)+\mathrm{le}(\gamma_2)}, \\ \gamma_2\left(t\dfrac{\mathrm{le}(\gamma_1)+\mathrm{le}(\gamma_2)}{\mathrm{le}(\gamma_2)} - \dfrac{\mathrm{le}(\gamma_1)}{\mathrm{le}(\gamma_2)}\right), & \text{if } \dfrac{\mathrm{le}(\gamma_1)}{\mathrm{le}(\gamma_1)+\mathrm{le}(\gamma_2)} < t \leqslant 1. \end{cases}$$

For a curve γ, we define the reversed curve by $\overleftarrow{\gamma}$ by setting $\overleftarrow{\gamma}(t) = \gamma(1-t)$. Because of possible noninjectivity, a restriction of a curve to a subcurve between $x, y \in \gamma$ can be defined in many ways. We use the first times in the parameter space, where we hit x and y, i.e., the restriction $\gamma[x,y]$ is defined as

$$\gamma[x,y](t) = \gamma((1-t)t_x + tt_y),$$

where $t_x = \inf\{t \in [0,1] : \gamma(t) = x\}$ and $t_y = \inf\{t \in [0,1] : \gamma(t) = y\}$. Notice that with this definition, we have $\gamma[x,y](0) = x$ and $\gamma[x,y](1) = y$, and the subcurve might go in the reversed direction along γ. For $x, y \in \mathbb{R}^2$, we also use the notation $[x,y]\colon [0,1] \to \mathbb{R}^2\colon t \mapsto (1-t)x + ty$ to denote the line segment from x to y.

Let $\Omega \subset \mathbb{R}^2$ be a domain and $x, y \in \overline{\Omega}$. We say that a curve $\gamma\colon [0,1] \to \mathbb{R}^2$ joins x and y in Ω, if $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma([0,\,1]) \subset \Omega \cup \{x,y\}$. We then define the *inner distance with respect to* Ω between x, $y \in \overline{\Omega}$ by setting

$$\mathrm{dist}_\Omega(x,\,y) = \inf_{\gamma\subset\Omega} \mathrm{le}(\gamma),$$

where the infimum runs over all the curves joining x and y in Ω. We postpone the proof of the fact that dist_Ω is a distance on Jordan domains to Lemma 2.16. Note that any pair $x, y \in \Omega$ is rectifiably joinable in Ω, but the inner distance from $x \in \Omega$ to a point $y \in \partial\Omega$ might well be infinite. If $\mathrm{dist}_\Omega(x,\,y)$

$<\infty$, we say that x *and* y *are rectifiably joinable in* Ω. The inner diameter $\operatorname{diam}_\Omega(E)$ of a set $E\subset\overline{\Omega}$ is then defined to be the supremum of $\operatorname{dist}_\Omega(x,\,y)$ over pairs of points $x,y\in E$ and

$$B_\Omega(z,r)=\{y\in\Omega\mid \operatorname{dist}_\Omega(z,\,y)<r\}$$

denotes the open ball in Ω with respect to the inner distance.

2.2 Curve condition

We begin by recording a consequence of (1.1) that essentially follows from [39, Lemma 2.1] (see also the proof of [11, Theorem 2.15] and [30]). Since the results of [39, Lemma 2.1] are stated for curves contained in open sets, we check below that the arguments work in our setting.

Lemma 2.2. *Let $1<p<2$, $\Omega\subset\mathbb{R}^2$ be a bounded simply connected domain, and $z_1,z_2\in\mathbb{R}^2\setminus\Omega$.*

(1) *If*

$$\max\{\operatorname{dist}(z_1,\partial\Omega),\operatorname{dist}(z_2,\partial\Omega)\}\leqslant 2|z_1-z_2|, \tag{2.1}$$

and $\gamma\subset\mathbb{R}^2\setminus\Omega$ is a curve joining z_1 and z_2 so that

$$\int_\gamma \operatorname{dist}(z,\partial\Omega)^{1-p}ds(z)\leqslant C_1|z_1-z_2|^{2-p},$$

then we have

$$\operatorname{le}(\gamma)\leqslant C(p,C_1)|z_1-z_2|.$$

(2) *If*

$$\max\{\operatorname{dist}(z_1,\partial\Omega),\operatorname{dist}(z_2,\partial\Omega)\}> 2|z_1-z_2|, \tag{2.2}$$

then the line segment $[z_1,z_2]\subset\mathbb{R}^2\setminus\Omega$ joining z_1 and z_2 satisfies

$$\int_{[z_1,z_2]} \operatorname{dist}(z,\partial\Omega)^{1-p}ds(z)\leqslant C(p)|z_1-z_2|^{2-p}.$$

Especially, if the curve condition (1.1) *holds, then $\mathbb{R}^2\setminus\Omega$ is quasiconvex with a constant depending only on p and C_1.*

Proof. Let us first verify the part (1). We claim that

$$\gamma\subset B(z_1,c|z_1-z_2|)\setminus\Omega \tag{2.3}$$

with $c=((2-p)(C_1+1)+3^{2-p})^{1/(2-p)}-2$. If (2.3) holds, then for any $z\in\gamma$, according to (2.1), we have

$$\operatorname{dist}(z,\partial\Omega)\leqslant \operatorname{dist}(z_1,\partial\Omega)+c|z_1-z|\leqslant(2+c)|z_1-z_2|,$$

and by $1<p<2$,

$$(2+c)^{1-p}|z_1-z_2|^{1-p}\operatorname{le}(\gamma)\leqslant\int_\gamma \operatorname{dist}(z,\partial\Omega)^{1-p}ds(z)\leqslant C_1|z_1-z_2|^{2-p}.$$

Hence,

$$\operatorname{le}(\gamma)\leqslant C(p,C_1)|z_1-z_2|,$$

and we conclude that we only need to establish (2.3).

Let us verify (2.3). By the curve condition on γ, the triangle inequality and (2.1),

$$\begin{aligned}C_1|z_1-z_2|^{2-p}&\geqslant\int_\gamma \operatorname{dist}(z,\partial\Omega)^{1-p}ds(z)\\&\geqslant\int_\gamma(\operatorname{dist}(z_1,\partial\Omega)+|z-z_1|)^{1-p}ds(z)\end{aligned}$$

$$\geqslant \int_\gamma (2|z_1 - z_2| + |z - z_1|)^{1-p} ds(z). \tag{2.4}$$

Suppose that $\gamma \subset \mathbb{R}^2 \setminus \Omega$ is not contained in $B(z_1, c|z_1 - z_2|) \setminus \Omega$. Then by restricting the curve to the part contained in the disk $B(z_1, c|z_1 - z_2|)$, we further have

$$\begin{aligned}\int_\gamma (2|z_1 - z_2| + |z - z_1|)^{1-p} ds(z) &\geqslant \int_{|z_1-z_2|}^{c|z_1-z_2|} (2|z_1 - z_2| + t)^{1-p} dt \\ &= \frac{|z_1 - z_2|^{2-p}}{2-p}((c+2)^{2-p} - 3^{2-p}). \end{aligned}\tag{2.5}$$

By combining (2.4) and (2.5), we arrive at

$$\frac{1}{2-p}((c+2)^{2-p} - 3^{2-p}) \leqslant C_1,$$

which is impossible for our choice of c. Thus, we conclude (2.3), and we have proven the part (1) of our claim.

Towards (2), clearly (2.2) implies $[z_1, z_2] \subset \mathbb{R}^2 \setminus \Omega$. With possibly changing the roles of z_1 and z_2, we may assume that

$$\operatorname{dist}(z_1, \partial\Omega) > 2|z_1 - z_2|.$$

Thus, we have

$$\int_{[z_1,z_2]} \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \leqslant |z_1 - z_2| 2^{p-1} \operatorname{dist}(z_1, \partial\Omega)^{1-p} \leqslant C(p)|z_1 - z_2|^{2-p},$$

where we used the facts that $1 < p < 2$, and that by (2.2) together with the triangle inequality, we have that for each $z \in [z_1, z_2]$,

$$\operatorname{dist}(z, \partial\Omega) \geqslant \operatorname{dist}(z_1, \partial\Omega) - |z_1 - z| \geqslant \operatorname{dist}(z_1, \partial\Omega) - |z_1 - z_2| \geqslant \frac{1}{2}\operatorname{dist}(z_1, \partial\Omega).$$

This gives the claim of the second part. □

We establish the following self-improving property of (1.1) via ideas from the proof of [39, Proposition 2.6].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain for which* (1.1) *holds for $\mathbb{R}^2 \setminus \Omega$. Then there exists $\epsilon > 0$ that depends only on p and the constant $C_1 = C(\Omega, p)$ in* (1.1) *so that for every $1 < \hat{p} < p + \epsilon$ and all $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining z_1 and z_2 such that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-\hat{p}} ds(z) \leqslant C(p, C_1)|z_1 - z_2|^{2-\hat{p}}.$$

Proof. We begin by showing that under the assumption of the lemma, for any pair of points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, there exists a rectifiable curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining them with $\operatorname{le}(\gamma) \leqslant C|z_1 - z_2|$ such that for all $w_1, w_2 \in \gamma$, any subcurve $\gamma[w_1, w_2] \subset \gamma$ joining w_1 and w_2 satisfies

$$\int_{\gamma[w_1,w_2]} \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \leqslant c|w_1 - w_2|^{2-p}, \tag{2.6}$$

where the constants C andc depend only on p and C_1. In the case where z_1 and z_2 satisfy (2.2), we claim that we may take $\gamma = [z_1, z_2]$, the line segment joining z_1 to z_2. Towards this, we may clearly assume that

$$\operatorname{dist}(z_1, \partial\Omega) > 2|z_1 - z_2|.$$

Then since every subcurve of our line segment γ is still a line segment, we have

$$\int_{[w_1,w_2]} \operatorname{dist}(z, \partial\Omega)^{1-p}\, ds(z) \leqslant C(p)|w_1 - w_2| \operatorname{dist}(z_1, \partial\Omega)^{1-p}$$

$$\leqslant C(p)|w_1-w_2||z_1-z_2|^{1-p} \leqslant C(p)|w_1-w_2|^{2-p},$$

where we used the facts that $1<p<2$, and that by (2.2) with the triangle inequality, we have that for each $z \in [z_1, z_2]$,

$$\mathrm{dist}(z,\partial\Omega) \geqslant \mathrm{dist}(z_1,\partial\Omega) - |z_1-z| \geqslant \mathrm{dist}(z_1,\partial\Omega) - |z_1-z_2| \geqslant \frac{1}{2}\mathrm{dist}(z_1,\partial\Omega).$$

Thus, (2.6) holds whenever (2.2) holds.

We are left with the case where (2.2) fails. Then (2.1) holds. We claim that there exists a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ that joins z_1 and z_2 and minimizes the integral in (1.1).

Let γ_j be a sequence of curves joining z_1 and z_2 such that

$$\int_{\gamma_j} \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) \leqslant c_j |z_1-z_2|^{2-p},$$

where $c_j \leqslant C_1$ converge to the infimum c of such constants c_j for the pair z_1, z_2. Then this condition ensures that

$$\mathrm{le}(\gamma_j) \leqslant C|z_1-z_2|$$

for all j by Lemma 2.2(1). Therefore, by Lemma 2.1, there exist a sequence $j_i \to \infty$ and a limit curve γ so that $\gamma_{j_i}(t) \to \gamma(t)$ for all t as $i \to \infty$ and

$$\int_{\gamma} \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) \leqslant \liminf_{i\to\infty} \int_{\gamma_{j_i}} \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) \leqslant c|z_1-z_2|^{2-p}. \tag{2.7}$$

Now, fix $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ satisfying (2.1), and let $\gamma \subset \mathbb{R}^2 \setminus \Omega$ be a minimizer for the integral in (1.1) for z_1 and z_2. We claim that any subcurve $\gamma[w_1, w_2]$ of γ is also a minimizer for w_1 and w_2. Otherwise, let $\gamma'[w_1, w_2]$ be a minimizer for w_1 and w_2. Because of symmetry, we may assume that γ passes z_1, w_1, w_2 and z_2 in this order. Then by the linearity of the integral, we have

$$\begin{aligned}\int_{\gamma} \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) &= \left(\int_{\gamma[z_1,w_1]} + \int_{\gamma[w_1,w_2]} + \int_{\gamma[w_2,z_2]}\right) \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) \\ &> \left(\int_{\gamma[z_1,w_1]} + \int_{\gamma'[w_1,w_2]} + \int_{\gamma[w_2,z_2]}\right) \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) \\ &= \int_{\gamma'} \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z),\end{aligned}$$

where

$$\gamma' = \gamma[z_1, w_1] * \gamma'[w_1, w_2] * \gamma[w_2, z_2]$$

joins z_1 and z_2. This contradicts the minimality assumption on γ. Thus, our claim follows, and hence (2.6) also holds for points satisfying (2.1).

To conclude, for any pair of points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$, there exists a rectifiable curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ joining them with $\mathrm{le}(\gamma) \leqslant C|z_1-z_2|$ so that (2.6) holds. In other words, the curve γ satisfies the so-called "strong α-hyperbolicity" in [39, Definition 2.4] with $\alpha = 2-p$. Thus, we can use the proof of [39, Proposition 2.6] to conclude the lemma. For the sake of completeness, let us give the details of this argument.

We first show that whenever a curve γ satisfies (2.6) and $w_1, w_2 \in \gamma$, we have

$$\frac{1}{\mathrm{le}(\gamma[w_1,w_2])} \int_{\gamma[w_1,w_2]} \mathrm{dist}(z,\partial\Omega)^{1-p} ds(z) \leqslant C(p,c) \min_{z\in\gamma[w_1,w_2]} \mathrm{dist}(z,\partial\Omega)^{1-p}. \tag{2.8}$$

We have two cases. If

$$\max_{z\in\gamma[w_1,w_2]} \mathrm{dist}(z,\partial\Omega) < 2\,\mathrm{le}(\gamma[w_1,w_2]),$$

then as $1<p<2$,

$$\min_{z\in\gamma[w_1,w_2]} \mathrm{dist}(z,\partial\Omega)^{1-p} > 2^{1-p}\,\mathrm{le}(\gamma[w_1,w_2])^{1-p}.$$

Therefore,

$$\int_{\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega)^{1-p} ds(z) \leqslant c|w_1-w_2|^{2-p} \leqslant c\,\mathrm{le}(\gamma[w_1,w_2])\,\mathrm{le}(\gamma[w_1,w_2])^{1-p}$$
$$\leqslant C(p,c)\,\mathrm{le}(\gamma[w_1,w_2]) \min_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega)^{1-p},$$

and (2.8) follows. If

$$\max_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega) \geqslant 2\,\mathrm{le}(\gamma[w_1,w_2]),$$

then by the triangle inequality,

$$\max_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega) \leqslant \min_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega) + \mathrm{le}(\gamma[w_1,w_2])$$
$$\leqslant \min_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega) + \frac{1}{2} \max_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega).$$

Thus,

$$\min_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega) \leqslant \max_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega) \leqslant 2 \min_{z\in\gamma[w_1,w_2]} \operatorname{dist}(z,\partial\Omega),$$

and (2.8) again follows from (2.6).

Now let us complete the proof by relying on $\mathrm{le}(\gamma) \leqslant C|z_1-z_2|$, (2.6) and (2.8). Parametrize γ by arc-length, $\gamma\colon [0,\mathrm{le}(\gamma)] \to \mathbb{R}^2 \setminus \Omega$. Then (2.8) gives the estimate

$$\frac{1}{|t_2-t_1|}\int_{t_1}^{t_2} \operatorname{dist}(\gamma(t),\partial\Omega)^{1-p} dt \leqslant C(p,c) \min_{t\in[t_1,t_2]} \operatorname{dist}(\gamma(t),\partial\Omega)^{1-p}$$

for all $0 \leqslant t_1 < t_2 \leqslant \mathrm{le}(\gamma)$. This implies that $\omega(t) = \operatorname{dist}(\gamma(t),\partial\Omega)^{1-p}$ is a Muckenhoupt $\mathcal{A}_1$-weight on $[0,\mathrm{le}(\gamma)]$. By the reverse Hölder's inequality (see, e.g., [18, 15.3]), there exists $\beta > 1$ that depends only on $C(p,c)$ such that

$$\left(\frac{1}{\mathrm{le}(\gamma)}\int_0^{\mathrm{le}(\gamma)} \omega(t)^\beta dt\right)^{\frac{1}{\beta}} \leqslant C(p,c)\frac{1}{\mathrm{le}(\gamma)}\int_0^{\mathrm{le}(\gamma)} \omega(t)dt.$$

This estimate together with (2.6) and the fact that $|z_1-z_2| \leqslant \mathrm{le}(\gamma) \leqslant C|z_1-z_2|$ implies the claim. □

We close this subsection with the following technical existence result that will be employed in Section 4.

Lemma 2.4. *Let $\frac{1}{2} < R < 1$ and $1 < \hat{p} < 2$. There are an absolute constant $\delta > 0$ and a constant $C(\hat{p})$ that depend only on $\hat{p}$ so that the following holds. Let $w_1 \in \mathbb{D} \setminus \overline{B}(0,R)$ and $w_1 \neq w_2 \in B(z_1, \delta(1-|w_1|)) \setminus \overline{B}(0,R)$. Then there is a curve $\gamma \subset B(w_1,(1-|w_1|)/2) \setminus \overline{B}(0,R)$ joining w_2 to w_1 so that*

$$\int_\gamma \operatorname{dist}(z, \partial B(0,R) \cup \partial B(w_1,(1-|w_1|)/2))^{1-\hat{p}} ds(z) \leqslant C(\hat{p})|w_1-w_2|^{2-\hat{p}}.$$

Proof. We prove the claim with $\delta = 1/30$. Fix R, w_1 and w_2 as in our assumptions. If w_2 lies on the radial segment through w_1, then we may clearly choose γ to be the radial segment between the points w_2 and w_1. Otherwise, consider the additional points $\xi_j = (|w_1| + |w_2-w_1|)\frac{w_j}{|w_j|}$, $j=1,2$. We let γ_j be the radial segment from w_j to ξ_j for $j=1,2$ and let γ_3 be the shorter arc on the circle $S(0,|w_1|+|w_2-w_1|)$ from ξ_1 to ξ_2. We can estimate the lengths of these curves by

$$\mathrm{le}(\gamma_2) = |w_2-\xi_2| = |w_1| - |w_2| + |w_2-w_1| \leqslant 2|w_1-w_1|$$

and

$$\mathrm{le}(\gamma_3) \leqslant \pi|\xi_1-\xi_2| \leqslant 2\pi|w_1-w_2|$$

since $w_1, w_2, \xi_1, \xi_2 \in \mathbb{D} \setminus \overline{B}(0,\frac{1}{2})$. We define γ as the concatenation $\gamma = \gamma_1 * \gamma_3 * \overleftarrow{\gamma}_2$. Then

$$\mathrm{le}(\gamma) \leqslant \mathrm{le}(\gamma_1) + \mathrm{le}(\gamma_2) + \mathrm{le}(\gamma_3) \leqslant |w_2-w_1| + 2|w_2-w_1| + 2\pi|w_2-w_1| < 10|w_2-w_1|. \tag{2.9}$$

Moreover, clearly $\gamma \cap \overline{B}(0,R) = \emptyset$. Since $\gamma_3 \subset S(0,|w_1|+|w_2-w_1|)$ and $|w_1| > R$, we have $\mathrm{dist}(\gamma_3, \partial B(0,R)) \geqslant |w_2-w_1|$. Furthermore,

$$\mathrm{dist}(\gamma, \partial B(w_1,(1-|w_1|)/2)) \geqslant (1-|w_1|)/2 - 10|w_2-w_1| \geqslant 5|w_2-w_1|$$

by (2.9) since $w_1 \in \gamma$ and $z_2 \in B(z_1,(1-|w_1|)/30$. This together with (2.9) yields

$$\begin{aligned}&\int_\gamma \mathrm{dist}(z, \partial B(0,R) \cup \partial B(w_1,(1-|w_1|)/2))^{1-\hat{p}} ds(z)\\ &\quad \leqslant \int_{\gamma_1} \mathrm{dist}(z,\partial B(0,R))^{1-\hat{p}} ds(z) + \int_{\gamma_2} \mathrm{dist}(z,\partial B(0,R))^{1-\hat{p}} ds + 20|w_2-w_1|^{1-\hat{p}}.\end{aligned}$$

The claim follows by integrating since γ_1 and γ_2 are radial segments and both are of the length no more than $2|w_2-w_1|$. □

2.3 Hyperbolic metric

Recall that the hyperbolic distance between $z_1, z_2 \in \mathbb{D}$ is defined to be

$$\mathrm{dist}_h(z_1,z_2) = \inf_\gamma \int_\gamma \frac{2}{1-|z|^2} ds(z),$$

where the infimum is taken over all rectifiable curves γ joining z_1 to z_2 in $\mathbb{D}$. Notice that the density above is comparable to $\frac{1}{1-|z|} = \mathrm{dist}(z,\partial\mathbb{D})^{-1}$. The infimum is achieved by a unique curve joining z_1 and z_2 that we call the (geodesic) hyperbolic segment between z_1 and z_2. It is an arc of a (generalized) circle that intersects the unit circle orthogonally. Especially, if the hyperbolic segment contains the origin, then it is a Euclidean segment. Conversely, each Euclidean segment that contains the origin and is contained in $\mathbb{D}$ is a hyperbolic segment. It is not obvious from the definition that the hyperbolic distance is preserved under conformal self maps of the disk, but this is indeed the case and follows from the fact that conformal self maps of the disk are Möbius transformations of $\mathbb{D}$ onto itself.

The hyperbolic distance in a simply connected domain is defined via a conformal map. Precisely, given a simply connected domain Ω, we pick a conformal map $\varphi\colon \mathbb{D} \to \Omega$ and define, for $x, y \in \Omega$,

$$\mathrm{dist}_h(x,y) = \mathrm{dist}_h(\varphi^{-1}(x), \varphi^{-1}(y)).$$

This is independent of the choice of φ since φ is unique modulo composition with a Möbius transformation that maps $\mathbb{D}$ onto $\mathbb{D}$. Equivalently,

$$\mathrm{dist}_h(x,y) = \inf_\gamma \int_\gamma \frac{2|g'(z)|}{1-|g(z)|^2} ds(z),$$

where $g = \varphi^{-1}$ and the infimum is taken over all rectifiable curves that join x to y in Ω. Hyperbolic segments in Ω are then both minimizers of this integral and images of hyperbolic segments in the unit disk. Even though the hyperbolic metric is defined via conformal maps, one can estimate it without knowing the map in question. For this, one uses the following Koebe distortion theorem.

Lemma 2.5 (See [1, Theorem 2.10.6]). *Suppose that φ is conformal in a domain $\Omega \subsetneqq \mathbb{C}$ with $\varphi(\Omega) = \Omega' \subsetneqq \mathbb{C}$. Let $z_0 \in \Omega$. Then*

$$\frac{1}{4}|\varphi'(z_0)|\mathrm{dist}(z_0,\partial\Omega) \leqslant \mathrm{dist}(\varphi(z_0),\partial\Omega') \leqslant |\varphi'(z_0)|\mathrm{dist}(z_0,\partial\Omega).$$

By the Koebe distortion theorem, the density in the definition of the hyperbolic distance is comparable to $\mathrm{dist}(z,\partial\Omega)^{-1}$ with absolute constants. For example, in the upper half-plane $\mathbb{H}$, the hyperbolic metric has the density y^{-1} at the point $(x,y) \in \mathbb{H}$, and the hyperbolic geodesics are circular arcs perpendicular to the real axis (contained in half-circles with the center on the real axis) and segments of vertical lines ending at the real axis. See [1, Chapter 2] for more information on the hyperbolic metric.

We also need the hyperbolic distance in the complement of the closed unit disk and in complementary Jordan domains. Towards this, we recall that the hyperbolic distance in the punctured disk $\mathbb{D}\setminus\{0\}$ is defined via the density $\rho(z)=\frac{1}{|z|\log(1/|z|)}$ and this time taking the infimum over curves in $\mathbb{D}\setminus\{0\}$. For the exterior of the closed unit disk, we transform this density and the hyperbolic distance via the (conformal) Möbius transformation $\varphi(z)=\frac{1}{z}$. Then the density of the hyperbolic distance is still controlled from above by an absolute constant multiple of $\frac{1}{|z|-1}=\operatorname{dist}(z,\partial\mathbb{D})^{-1}$ (and also from below when $z\in B(0,10)$).

Recall that a Jordan curve divides the plane into two domains, the boundary of each of which equals this curve; we refer to the bounded one as a Jordan domain Ω. Then the Jordan domain Ω is conformally equivalent to the unit disk and the corresponding unbounded domain $\widetilde{\Omega}=\mathbb{R}^2\setminus\overline{\Omega}$ is conformally equivalent to $\mathbb{R}^2\setminus\overline{\mathbb{D}}$. We define the hyperbolic distance and the corresponding density in $\widetilde{\Omega}$ via the conformal map and our hyperbolic distance and density in the exterior of the closed unit disk. This does not depend on the choice of the conformal map in question since any two conformal maps from the exterior domain of the unit circle onto our Jordan domain can only differ by a precomposition with a rotation. This follows since the composition of the inverse of the second map with the first one would be a conformal self map of the exterior domain of the unit circle. Each such map is a rotation. This can be seen by pre- and postcomposing with the Möbius transformation $\varphi(z)=\frac{1}{z}$ so as to obtain a conformal self map of the punctured disk, noticing that the origin is a removable singularity and mapped to the origin by the extension. Thus, we obtain a conformal self map of the unit disk that maps 0 to 0. Such maps are rotations.

Given a Jordan domain Ω and a conformal map $\varphi\colon\mathbb{D}\to\Omega$ or $\varphi\colon\mathbb{R}^2\setminus\overline{\mathbb{D}}\to\mathbb{R}^2\setminus\overline{\Omega}$, our map φ extends homeomorphically up to the boundary by the Carathéodory-Osgood theorem [34, Theorem 4.9, p. 445]. Then the *hyperbolic ray in* Ω, ending at $z\in\partial\Omega$, is the image under φ of the radial ray from the origin to $\varphi^{-1}(z)$, or in $\mathbb{R}^2\setminus\overline{\Omega}$, is the image under φ of the radial half-line starting from $\varphi^{-1}(z)$. Most of the hyperbolic rays in a Jordan domain Ω have finite length in the sense that

$$\operatorname{le}(\varphi([0,w)))<\infty\quad\text{for a.e. } w\in S^1=\partial\mathbb{D}. \tag{2.10}$$

This follows since

$$\int_0^{2\pi}\int_0^1 r|\varphi'(re^{i\theta})|drd\theta=\int_{\mathbb{D}}|\varphi'(z)|\leqslant\pi^{1/2}\bigg(\int_{\mathbb{D}}|\varphi'(z)|^2\bigg)^{1/2}=\pi^{1/2}\bigg(\int_{\mathbb{D}}J_\varphi(z)\bigg)^{1/2}$$

by Hölder's inequality and the Cauchy-Riemann equations; the integral of J_φ over $\mathbb{D}$ is the area of Ω and hence finite and $|\varphi'(z)|$ is bounded in $\overline{B}(0,1/2)$ since φ is smooth.

The following lemma provides us with estimates on the oscillation of $|\varphi'(z)|$ in terms of the hyperbolic metric.

Lemma 2.6 (See [1, Theorem 2.10.8])**.** *Suppose that φ is conformal in U, where U is the unit disk $\mathbb{D}$ or $U=\mathbb{R}^2\setminus\overline{\mathbb{D}}$, and let $z,w\in U$. Then*

$$\exp(-3\operatorname{dist}_h(z,w))|\varphi'(w)|\leqslant|\varphi'(z)|\leqslant\exp(3\operatorname{dist}_h(z,w))|\varphi'(w)|.$$

We also record the following estimates, referred to as the Gehring-Hayman inequalities, e.g., in [35, Theorem 4.20, p. 88]. They show the significance of hyperbolic segments.

Lemma 2.7 (See [10])**.** *Let $\varphi\colon\mathbb{D}\to\Omega$ be a conformal map. Given a pair of points $x,y\in\overline{\mathbb{D}}$, denoting the corresponding hyperbolic segment in $\mathbb{D}$ by $\Gamma_{x,y}$, and by $\gamma_{x,y}$ any curve connecting x and y in $\mathbb{D}$, we have*

$$\operatorname{le}(\varphi(\Gamma_{x,y}))\leqslant C\operatorname{le}(\varphi(\gamma_{x,y}))$$

and

$$\operatorname{diam}(\varphi(\Gamma_{x,y}))\leqslant C\operatorname{diam}(\varphi(\gamma_{x,y})),$$

where C is an absolute constant.

We close this subsection with the following lemma that employs hyperbolic segments. In general, the internal distance between two boundary points of a Jordan domain can well be infinite. However, such points cannot be obtained as Euclidean limits of sequences of points with uniformly bounded internal distances.

Lemma 2.8. *Let Ω be a Jordan domain and $x_j, y_j \in \Omega$, $j \in \mathbb{N}$ be points so that $x_j \to x \in \overline{\Omega}$ and $y_j \to y \in \overline{\Omega}$ with $y \neq x$. Then*

$$\mathrm{le}(\Gamma_{x,y}) \leqslant C \liminf_{j\to\infty} \mathrm{dist}_\Omega(x_j, y_j)$$

for the hyperbolic segment $\Gamma_{x,y}$ between x and y in Ω, where C is an absolute constant. Especially, if $\mathrm{dist}_\Omega(x_j, y_j) \leqslant M < \infty$ for all j, then $\mathrm{dist}_\Omega(x, y) \leqslant CM$.

Proof. Let $x_j, y_j \in \Omega$ be points so that $x_j \to x \in \overline{\Omega}$ and $y_j \to y \in \overline{\Omega}$ with $y \neq x$. We may assume that $x_j \neq y_j$. Pick rectifiable curves γ_j joining x_j to y_j in Ω so that

$$\mathrm{le}(\gamma_j) \leqslant 2\mathrm{dist}_\Omega(x_j, y_j). \tag{2.11}$$

Fix a conformal map $\varphi\colon \mathbb{D} \to \Omega$. By the Carathéodory-Osgood theorem, we may extend φ homeomorphically up to the boundary. We refer also to this extension by φ. Write $z_j = \varphi^{-1}(x_j)$ and $w_j = \varphi^{-1}(y_j)$. By Lemma 2.7, we conclude that

$$\mathrm{le}(\varphi(\Gamma_j)) \leqslant C\,\mathrm{le}(\gamma_j) \tag{2.12}$$

for the hyperbolic segment $\Gamma_j := \Gamma_{z_j,w_j}$ between z_j and w_j in $\mathbb{D}$ with an absolute constant C. Since φ is uniformly continuous, $x_j \neq y_j$ and $\lim_{j\to\infty} x_j = x \neq y = \lim_{j\to\infty} y_j$, there exists $\delta > 0$ so that $|z_j - w_j| \geqslant \delta$ for every $j \geqslant 1$. Because each Γ_j is a hyperbolic segment and hence an arc of a (generalized) circle that intersects the unit circle orthogonally, we deduce from this the existence of a $\delta' > 0$ and points $\xi_j \in \Gamma_j$ so that $|\xi_j| \geqslant 1 - \delta'$ for all $j \geqslant 1$. Define

$$T_j(z) = \frac{z - \xi_j}{1 + \overline{\xi}_j z},$$

where $\overline{\xi}_j$ is the complex conjugate of ξ_j. Then T_j is a conformal (Möbius) self map of $\mathbb{D}$ and maps ξ_j to the origin. Now, $T_j \circ \Gamma_j$ is a hyperbolic segment that contains 0 and hence a Euclidean line segment. On the other hand, a subsequence of the points ξ_j converges to some ξ with $|\xi| \geqslant 1 - \delta'$ and a subsequence of the corresponding $T_j \circ \Gamma_j$ converges to some curve α by Lemma 2.1. Clearly, α is a Euclidean line segment that contains the origin and hence also a hyperbolic segment. Define

$$T(z) = \frac{z - \xi}{1 + \overline{\xi} z}.$$

Then T is a conformal self map of the disk and hence $T^{-1} \circ \alpha$ is a hyperbolic segment between $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$. Consequently, $\Gamma := \varphi \circ T^{-1} \circ \alpha$ is a hyperbolic segment in Ω with endpoints x and y.

We are left to estimate the length of Γ. By switching to a subsequence in the beginning of our proof, we may assume by (2.11) and (2.12) that

$$\liminf_{j\to\infty} \mathrm{le}(\varphi(\Gamma_j)) \leqslant 2C \liminf_{j\to\infty} \mathrm{dist}_\Omega(x_j, y_j).$$

Since φ is uniformly continuous, the above subsequence $T_j \circ \Gamma_j$ converges to α, and T_j^{-1} converges to T^{-1}, we have that $\varphi \circ \Gamma_j$ converges to $\varphi \circ T^{-1} \circ \alpha = \Gamma$. Hence, we may deduce from Lemma 2.1 that

$$\mathrm{le}(\Gamma) \leqslant 2C \liminf_{j\to\infty} \mathrm{dist}_\Omega(x_j, y_j).$$

This completes the proof. $\square$

2.4 Whitney-type set

A *dyadic square* in $\mathbb{R}^2$ refers to any set

$$[m_i 2^{-k}, (m_i+1)2^{-k}] \times [m_j 2^{-k}, (m_j+1)2^{-k}],$$

where $m_i, m_j, k \in \mathbb{Z}$. We denote by $\ell(Q)$ the side length of the given square Q. The barycenter of a set $A \subset \mathbb{R}^2$ with positive and finite Lebesgue-measure will be denoted by x_A, and for $c > 0$, the dilation of A by factor c will be denoted by

$$cA = \{c(x - x_A) + x_A : x \in A\}.$$

We use these concepts in particular for dyadic squares Q.

Recall that any open set in $\mathbb{R}^2$, different from the entire $\mathbb{R}^2$, admits a Whitney decomposition (see, e.g., [40, Chapter VI]).

Lemma 2.9 (Whitney decomposition). *For any open set $U \neq \mathbb{R}^2$, there exists a collection $W = \{Q_j\}_{j\in\mathbb{N}}$ of countably many closed dyadic squares such that*

(i) $U = \bigcup_{j\in\mathbb{N}} Q_j$ *and* $(Q_k)^\circ \cap (Q_j)^\circ = \emptyset$ *for all* $j, k \in \mathbb{N}$ *with* $j \neq k$;

(ii) $\ell(Q_k) \leqslant \mathrm{dist}(Q_k, \partial U) \leqslant 4\sqrt{2}\ell(Q_k)$ *for all* $k \in \mathbb{N}$;

(iii) $\frac{1}{4}\ell(Q_k) \leqslant \ell(Q_j) \leqslant 4\ell(Q_k)$ *whenever* $k, j \in \mathbb{N}$ *and* $Q_k \cap Q_j \neq \emptyset$.

The above squares Q_j are called Whitney squares of U. We also need the following more general concept since the image of a Whitney square under a conformal map needs not be a Whitney square.

Definition 2.10. A bounded connected set $A \subset U \neq \mathbb{R}^2$ is said to be of λ-Whitney type in U (with constant $\lambda \geqslant 1$) if the following hold:

(i) There exists a disk with radius $\frac{1}{\lambda}\mathrm{diam}(A)$ contained in A.

(ii) $\frac{1}{\lambda}\mathrm{diam}(A) \leqslant \mathrm{dist}(A, \partial U) \leqslant \lambda \mathrm{diam}(A)$.

For example, the Whitney squares in Lemma 2.9 are $4\sqrt{2}$-Whitney-type sets. Conversely, each λ-Whitney-type set $A \subset U$ intersects at most $N(\lambda)$ Whitney squares of U. By Lemma 2.9(ii) and Definition 2.10(ii), we have

$$Q \subset B(x, C(\lambda)\mathrm{dist}(x, \partial U))$$

with $C(\lambda) = \sqrt{2}(\lambda + 1) + \lambda$ for any $x \in A$ and any Whitney square Q of U that intersects A, and that

$$\ell(Q) \geqslant (5\sqrt{2})^{-1}\mathrm{dist}(A, \partial U)$$

for any such Q.

Observe that for a λ-Whitney-type set A in U and any $x \in A$, by the triangle inequality and Definition 2.10(ii), we have

$$\mathrm{dist}(A, \partial U) \leqslant \mathrm{dist}(x, \partial U) \leqslant (1 + \lambda)\mathrm{dist}(A, \partial U). \tag{2.13}$$

Thus, if a pair A_1, A_2 of λ-Whitney-type sets has non-empty intersection, then

$$\mathrm{diam}(A_1) \sim \mathrm{diam}(A_2) \tag{2.14}$$

with the constant depending only on λ.

In terms of hyperbolic metric, Whitney-type sets have a uniformly bounded diameter in the following sense.

Lemma 2.11. *Let Ω be a Jordan (or exterior Jordan) domain in $\mathbb{R}^2$ and $A \subset \Omega$ be a λ-Whitney-type set with $\lambda \geqslant 1$. Then*

$$\mathrm{dist}_h(x, y) \leqslant C(\lambda) \tag{2.15}$$

for all $x, y \in A$.

Proof. Let $x, y \in A$ be fixed. Let $r = \frac{1}{10}\mathrm{dist}(A, \partial\Omega)$ and consider the cover $\{B(z,r)\}_{z\in A}$ of the set A. By the $5r$-covering lemma, there exists a pairwise disjoint subcollection $\{B(z_i,r)\}_{i=1}^N$ so that $A \subset \bigcup_{i=1}^N B(z_i, 5r)$. For every i, we have

$$B(z_i, r) \subset B(x, \mathrm{diam}(A) + r) \subset B(x, 11r).$$

Hence, since $B(z_i, r)$ are pairwise disjoint, we have $N \leqslant (11\lambda)^2$. By the fact that A is connected, there exists a sequence $\{i(k)\}_{k=1}^M$ with $M \leqslant N$ so that $x \in B(z_{i(1)}, 5r)$, $y \in B(z_{i(M)}, 5r)$ and $B(z_{i(k)}, 5r) \cap B(z_{i(k+1)}, 5r) \neq \emptyset$ for all $k = 1, \ldots, M-1$. Then the curve

$$\gamma = [x, z_{i(1)}] * [z_{i(1)}, z_{i(2)}] * \cdots * [z_{i(M-1)}, z_{i(M)}] * [z_{i(M)}, y]$$

satisfies $\mathrm{le}(\gamma) \leqslant 10(N+2)r$ and $\mathrm{dist}(\gamma, \partial\Omega) \geqslant \mathrm{dist}(A, \partial\Omega) - 5r = 5r$.

Since the density of the hyperbolic metric is bounded from above by $\mathrm{dist}(z, \partial\Omega)^{-1}$ up to a multiplicative constant, we have

$$\mathrm{dist}_h(x,y) \leqslant C\int_\gamma \mathrm{dist}(z,\partial\Omega)^{-1} ds(z) \leqslant C\,\mathrm{le}(\gamma)\mathrm{dist}(\gamma,\partial\Omega)^{-1} \leqslant 2C(N+2),$$

concluding the proof. □

Given a λ-Whitney-type set $A \subset \mathbb{D}$, one has $\mathrm{dist}_h(z,w) \leqslant C(\lambda)$ for all $z, w \in A$ by (2.15). Hence, Lemma 2.6 implies $|\varphi'(z)| \sim |\varphi'(w)|$ with a constant depending only on λ. By this (applied to suitable disks) in combination with Lemma 2.5, one can prove that the images of Whitney squares of a simply connected domain Ω under a conformal map of Ω onto Ω' get mapped to Whitney-type sets. Following the idea from [9, Theorem 11], we see that a more general version of this can be proven with the help of Lemma 2.5 and [41, Theorem 18.1]. Since we use it later on, let us recall that [41, Theorem 18.1] gives the following: there exists a universal increasing (continuous) function $\Theta\colon (0,1) \to \mathbb{R}$ such that $\lim_{x\to 0^+}\Theta(x) = 0$, $\lim_{x\to 1^-}\Theta(x) = \infty$ and for every conformal map $\varphi\colon \Omega \to \Omega'$ with domains $\Omega, \Omega' \subsetneqq \mathbb{R}^2$ and each point $x \in \Omega$, we have

$$\frac{|\varphi(x) - \varphi(y)|}{\mathrm{dist}(\varphi(x), \partial\Omega')} \leqslant \Theta\left(\frac{|x-y|}{\mathrm{dist}(x,\partial\Omega)}\right) \tag{2.16}$$

for every y with $0 < |x-y| < \mathrm{dist}(x, \partial\Omega)$.

Lemma 2.12. *Suppose that $\varphi\colon \Omega \to \Omega'$ is conformal, where $\Omega, \Omega' \subsetneqq \mathbb{R}^2$ are domains and $A \subset \Omega$ is a λ_1-Whitney-type set. Then $\varphi(A) \subset \Omega'$ is a λ_2-Whitney-type set with $\lambda_2 = \lambda_2(\lambda_1)$.*

Our next estimate shows that a conformal map from the unit disk (or from the exterior domain of the disk) onto a simply connected domain (to an exterior domain) is C_λ-bi-Lipschitz modulo a scaling factor on each λ-Whitney-type set.

Lemma 2.13. *Let $\varphi\colon U \to \Omega$ be conformal, where $U = \mathbb{D}$ or $U = \mathbb{R}^2 \setminus \overline{\mathbb{D}}$. If $A \subset U$ is of λ-Whitney type and $z_0, z_1, z_2 \in A$, then*

$$C_\lambda^{-1}|\varphi'(z_0)||z_2 - z_1| \leqslant |\varphi(z_2) - \varphi(z_1)| \leqslant C_\lambda|\varphi'(z_0)||z_2 - z_1|,$$

where C_λ depends only on λ.

Proof. Fix $z_0, z_1, z_2 \in A$, where $A \subset U$ is of λ-Whitney type. As in the proof of Lemma 2.11, let $r = \frac{1}{10}\mathrm{dist}(A, \partial U)$. Then $\mathrm{dist}_h(z, z_1) \leqslant C$ for an absolute constant when $z_2 \in B(z_1, 5r)$. Hence, Lemma 2.6 gives us the estimate

$$|\varphi(z_2) - \varphi(z_1)| \leqslant \exp(C)|\varphi'(z_1)||z_2 - z_1| \tag{2.17}$$

if $z_2 \in B(z_1, 5r)$. Let us assume that $z_2 \notin B(z_1, 5r)$. Then $|z_2 - z_1| \geqslant 5r$. Let γ be the curve from the proof of Lemma 2.11 for the pair z_1, z_2. Then $\mathrm{le}(\gamma) \leqslant (11\lambda)^2 r$ and $\mathrm{dist}_h(z, z_1) \leqslant 2C(2 + (11\lambda)^2) =: C_1$ for each $z \in \gamma$. Hence, Lemma 2.6 gives

$$|\varphi(z_2) - \varphi(z_1)| \leqslant \int_\gamma |\varphi'(z)| ds(z) \leqslant \exp(C_1)|\varphi'(z_1)|\,\mathrm{le}(\gamma) \leqslant \frac{1}{5}(11\lambda^2)\exp(C_1)|\varphi'(z_1)||z_2 - z_1|. \tag{2.18}$$

By combining (2.17) and (2.18), we see that Lemma 2.11 together with Lemma 2.6 allows us to further deduce that

$$|\varphi(z_2) - \varphi(z_1)| \leqslant C(\lambda)|\varphi'(z_1)||z_2 - z_1| \leqslant C'(\lambda)|\varphi'(z_0)||z_2 - z_1|.$$

Towards the opposite inequality, notice first that $\mathrm{dist}_h(\hat{w}, \hat{z}) \leqslant C$ with an absolute constant whenever $\hat{z} \in \Omega$ and $\hat{w} \in B(\hat{z}, \mathrm{dist}(\hat{z}, \partial\Omega)/2)$. This holds since the density of the hyperbolic distance in this disk is bounded from above by a fixed multiple of $\mathrm{dist}(\hat{z}, \partial\Omega)^{-1}$. Especially, $\mathrm{dist}_h(\varphi^{-1}(\hat{\xi}), \varphi^{-1}(\hat{z})) \leqslant C$ for each $\hat{\xi} \in B(\hat{z}, \mathrm{dist}(\hat{z}, \partial\Omega)/2)$. By Lemma 2.6, we conclude that

$$|\varphi'(\varphi^{-1}(\hat{z}))| \leqslant C_3|\varphi'(\varphi^{-1}(\hat{\xi}))| \tag{2.19}$$

for all $\hat{\xi} \in B(\hat{z}, \partial\Omega)/2)$. Since $\varphi^{-1}(\hat{\xi}) = \frac{1}{\varphi'(\varphi^{-1}(\hat{\xi}))}$, from (2.19), we deduce the estimate

$$|(\varphi^{-1})'(\hat{\xi})| \leqslant C_3|(\varphi^{-1})'(\hat{z})|. \tag{2.20}$$

Let I be the Euclidean line segment between $\hat{z} \in \varphi(A)$ and given $\hat{w} \in B(\hat{z}, \mathrm{dist}(\hat{z}, \partial\Omega)/2)$. Then $I \subset B(\hat{z}, \mathrm{dist}(\hat{z}, \partial\Omega)/2)$. By integrating the estimate (2.20) over I, we conclude that

$$|\varphi^{-1}(\hat{w}) - \varphi^{-1}(\hat{z})| \leqslant \int_I |(\varphi^{-1})'(\hat{\xi})|ds(\hat{\xi}) \leqslant C_3|(\varphi^{-1})'(\hat{z})||\hat{w} - \hat{z}| \leqslant C_1|\varphi'(\varphi^{-1}(\hat{z}))|^{-1}|\hat{w} - \hat{z}|. \tag{2.21}$$

Especially, if $\hat{w}, \hat{z} \in \varphi(A)$ satisfy

$$|\hat{w} - \hat{z}| \leqslant \frac{1}{2}\max\{\mathrm{dist}(\hat{z}, \partial\Omega), \mathrm{dist}(\hat{w}, \partial\Omega)\}, \tag{2.22}$$

then by (2.21), Lemmas 2.6 and 2.11, we get

$$|\varphi^{-1}(\hat{w}) - \varphi^{-1}(\hat{z})| \leqslant C_3 \max\{|\varphi'(\varphi^{-1}(\hat{\xi}))|^{-1} : \hat{\xi} \in \varphi(A)\}|\hat{w} - \hat{z}| \leqslant C_1(\lambda)C_3|\varphi'(z_0)|^{-1}|\hat{w} - \hat{z}|.$$

We are left to consider the case where (2.22) fails. By Lemma 2.12, we know that $\varphi(A)$ is of $C_4(\lambda)$-Whitney type. Hence,

$$\mathrm{diam}\varphi(A) \leqslant C_5(\lambda)\mathrm{dist}(\varphi(A), \partial\Omega) \leqslant C_5\mathrm{dist}(\hat{z}, \partial\Omega) \tag{2.23}$$

for each $\hat{z} \in A$. If (2.22) fails, then $\mathrm{dist}(\hat{z}, \partial\Omega) \leqslant 2|\hat{w} - \hat{z}|$ and we conclude that

$$\mathrm{diam}(\varphi(A)) \leqslant 2C_5(\lambda)|\hat{w} - \hat{z}|$$

and further that

$$\begin{aligned}|\varphi^{-1}(\hat{w}) - \varphi^{-1}(\hat{z})| &\leqslant \mathrm{diam}(A) = \mathrm{diam}(A)|\hat{w} - \hat{z}|^{-1}|\hat{w} - \hat{z}| \\ &\leqslant 2C_5(\lambda)\mathrm{diam}(A)\mathrm{diam}(\varphi(A))^{-1}|\hat{w} - \hat{z}|.\end{aligned}$$

It only remains to be noticed that $|\varphi'(z_0)|$ is comparable to $\mathrm{dist}(\varphi(z_0), \partial\Omega)/\mathrm{dist}(z_0, \partial U)$ with absolute constants by Lemma 2.5, that $C_5(\lambda)\mathrm{dist}(\varphi(z_0), \partial\Omega) \geqslant \mathrm{diam}(\varphi(A))$ by (2.23) and that $\mathrm{diam}(A) \leqslant C_6(\lambda)\mathrm{dist}(z_0, \partial U)$ since A is of λ-Whitney type in U with $z_0 \in A$. □

2.5 Conformal capacity

Let $\Omega \subset \mathbb{R}^2$ be a domain. For a given pair of disjoint continua $E, F \subset \overline{\Omega}$, define the *conformal capacity between E and F in Ω* as

$$\mathrm{Cap}(E, F, \Omega) = \inf\{\|\nabla u\|^2_{L^2(\Omega)} : u \in \Delta(E, F, \Omega)\},$$

where $\Delta(E, F, \Omega)$ denotes the class of all $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap C(\Omega \cup E \cup F)$ that satisfy $u = 1$ on E, and $u = 0$ on F. We remark that in general, one has to be careful when defining the capacity of continua that are allowed to intersect the boundary of the domain. However, for our purposes, the definition above is enough, so we can avoid considering the prime-end compactification of Ω and the more subtle definitions of capacity. The conformal capacity is by definition increasing both with respect to the sets E and F and with respect to Ω. We refer to this by the monotonicity of conformal capacity. Also notice that $\mathrm{Cap}(E,F,\Omega)=\mathrm{Cap}(F,E,\Omega)$, as seen by switching a given test function u to $v=1-u$.

Let us introduce the properties of conformal capacity, which will be used in the rest of the paper (see, e.g., [41, Chapter 1] for more properties). We remark that even though [41] (as well as some other references below) states estimates for "modulus", "modulus" is equivalent with conformal capacity in our setting below (see, e.g., [20, Theorem 2.6] and [36, Proposition 10.2, p. 54]).

Lemma 2.14. *The conformal capacity is conformally invariant, i.e., for domains Ω and Ω' in $\mathbb{R}^2$, a conformal (onto) map $\varphi\colon \Omega\to\Omega'$ and disjoint continua E and F in Ω, we have*

$$\mathrm{Cap}(\varphi(E),\varphi(F),\Omega')=\mathrm{Cap}(E,F,\Omega). \tag{2.24}$$

Moreover, if φ has a homeomorphic extension, still denoted by φ, $\varphi\colon\overline{\Omega}\to\overline{\Omega'}$, then (2.24) also holds for disjoint continua in $\overline{\Omega}$. Especially, this is the case if both Ω and Ω' are Jordan.

In what follows, whenever we mention the conformal invariance of conformal capacity, we always refer to the above lemma.

We have the following estimate for the conformal capacity in the unit disk $\mathbb{D}$ (and in its exterior domain $\mathbb{R}^2\setminus\overline{\mathbb{D}}$). Let E and F be disjoint continua in $\overline{\mathbb{D}}$. Then

$$\mathrm{Cap}(E,F,\mathbb{D})\geqslant c\log\left(1+\frac{\min\{\mathrm{diam}(E),\mathrm{diam}(F)\}}{\mathrm{dist}(E,F)}\right), \tag{2.25}$$

where $c>0$ is a universal constant. Moreover, the analogous inequality holds for $E,F\subset\mathbb{R}^2\setminus\mathbb{D}$. For these results, see [42, Lemma 7.38] that gives (2.25) for the entire plane instead of $\mathbb{D}$, and see [12, Remark 2.12] and [20, Theorems 2.6 and 2.8] that allow us to deduce the desired estimates from the global one.

We call a domain $A\subset\mathbb{R}^2$ a *ring domain* if its complement has exactly two components and at least one of the components is compact. If the exterior components of A are U_0 and U_1, then we write $A=R(U_0,U_1)$. It follows from topology that also ∂A has two components, $V_0=U_0\cap\overline{A}$ and $V_1=U_1\cap\overline{A}$. If U_0, V_0 and V_1 are compact, we have

$$\mathrm{Cap}(V_0,V_1,A)=\mathrm{Cap}(U_0,V_1,A\cup U_0); \tag{2.26}$$

indeed, "$\leqslant$" directly follows from the definition and "$\geqslant$" follows by extending each $u\in\Delta(V_0,V_1,A)$ as constant 1 to $U_0\setminus V_0$ (see also [41, Theorem 11.3] and its proof). Furthermore, we have the following estimate for the capacity of the boundary components of a ring domain.

Lemma 2.15. *Let $A=R(U_0,U_1)\subset\mathbb{R}^2$ be a ring domain with U_1 unbounded. Assume that $V_0=U_0\cap\overline{A}$ and $V_1=U_1\cap\overline{A}$ are compact. There exist two universal increasing functions $\phi_i\colon(0,\infty)\to(0,\infty)$, $i=1,2$ so that $\lim_{t\to0+}\phi_i(t)=0$ and $\lim_{t\to\infty}\phi_i(t)=\infty$, and so that*

$$\phi_1\left(\frac{\mathrm{diam}(U_0)}{\mathrm{dist}(U_0,U_1)}\right)\leqslant\mathrm{Cap}(V_0,V_1,A)\leqslant\phi_2\left(\frac{\mathrm{diam}(U_0)}{\mathrm{dist}(U_0,U_1)}\right). \tag{2.27}$$

We need the fact that the inner distance satisfies the triangle inequality [4, Lemma 2.3].

Lemma 2.16. *Let Ω be a Jordan domain and $z_1,z_2,z_3\in\overline{\Omega}$ be three distinct points. Then*

$$\mathrm{dist}_\Omega(z_1,z_3)\leqslant\mathrm{dist}_\Omega(z_1,z_2)+\mathrm{dist}_\Omega(z_2,z_3).$$

We record the following estimate, which states a kind of converse to (2.25). It builds on [29, Lemma 2.2]. (Recall from Subsection 2.1 that diam_Ω and $B_\Omega(z,r)$ refer to the diameter and ball in the inner distance with respect to Ω.)

Lemma 2.17. *Let Ω be a domain and $E,F\subset\Omega$ be a pair of disjoint continua. Then if $\mathrm{Cap}(E,F,\Omega)\geqslant\delta_0>0$, we have*

$$\min\{\mathrm{diam}_\Omega(E),\mathrm{diam}_\Omega(F)\}\gtrsim\mathrm{dist}_\Omega(E,F), \tag{2.28}$$

where the constant depends only on δ_0. Especially,

$$\min\{\mathrm{diam}_\Omega(E), \mathrm{diam}_\Omega(F)\} \gtrsim \mathrm{dist}(E, F),$$

and if $\Omega = \mathbb{R}^2$,

$$\min\{\mathrm{diam}(E), \mathrm{diam}(F)\} \gtrsim \mathrm{dist}(E, F). \tag{2.29}$$

If we further assume that Ω is Jordan, then (2.28) *also holds if $E \subset \overline{\Omega}$ and $F \subset \overline{\Omega}$ are disjoint continua with* $\mathrm{Cap}(E, F, \Omega) \geqslant \delta_0$.

Proof. **Step 1.** We begin with the case where $E, F \subset \Omega$. By switching E and F, we may assume that $\mathrm{diam}_\Omega(E) \leqslant \mathrm{diam}_\Omega(F)$. We may also assume that $2\mathrm{diam}_\Omega(E) \leqslant \mathrm{dist}_\Omega(E, F)$; otherwise, the claim holds trivially. Fix $z \in E$, and write $\frac{\mathrm{dist}_\Omega(E,F)}{\mathrm{diam}_\Omega(E)} = M$. We define

$$u(x) = \begin{cases} 1, & \text{if } \mathrm{dist}_\Omega(x, z) \leqslant \mathrm{diam}_\Omega(E), \\ 0, & \text{if } \mathrm{dist}_\Omega(x, z) \geqslant \mathrm{dist}_\Omega(E, F), \\ \dfrac{\log(\mathrm{dist}_\Omega(E, F)) - \log(\mathrm{dist}_\Omega(x, z))}{\log(M)}, & \text{otherwise.} \end{cases}$$

Then u is locally Lipschitz and

$$|\nabla u(x)| \leqslant (\log M)^{-1} \mathrm{dist}_\Omega(x, z)^{-1}$$

for all $x \in \Omega$ with $\mathrm{diam}_\Omega(E) \leqslant \mathrm{dist}_\Omega(x, z) \leqslant \mathrm{dist}_\Omega(E, F)$, and $|\nabla u(x)| = 0$ elsewhere. Write

$$R = B_\Omega(z, \mathrm{dist}_\Omega(E, F)) \setminus B_\Omega(z, \mathrm{diam}_\Omega(E)),$$

and for $i \geqslant 1$,

$$A_i = B_\Omega(z, 2^i \mathrm{diam}_\Omega(E)) \setminus B_\Omega(z, 2^{i-1} \mathrm{diam}_\Omega(E)),$$

where $B_\Omega(z, r)$ is the disk centered at z with radius r with respect to the inner distance. The assumption $\mathrm{Cap}(E, F, \Omega) \geqslant \delta_0 > 0$ and a direct calculation via our dyadic annular decomposition with respect to the inner distance give

$$\begin{aligned} \delta_0 &\leqslant \int_\Omega |\nabla u|^2 dx \leqslant (\log M)^{-2} \int_R \mathrm{dist}_\Omega(x, z)^{-2} dx \\ &\leqslant (\log M)^{-2} \sum_{i=1}^{\infty} \int_{R \cap A_i} 2^{2-2i} \mathrm{diam}_\Omega(E)^{-2} dx \\ &\leqslant 2(\log M)^{-2} \sum_{i=1}^{[\log M]+1} 4\pi \\ &\lesssim (\log M)^{-2} \log M \lesssim (\log M)^{-1}, \end{aligned}$$

where $[\log M]$ denotes the integer part of $\log M$, and in the third inequality we used the fact that $B_\Omega(z, r) \subset B(z, r)$. Hence, $M \leqslant C(\delta_0)$, which means that $\mathrm{dist}_\Omega(E, F) \lesssim \mathrm{diam}_\Omega(E)$.

Step 2. We continue with the case where at least one of the sets E and F intersects $\partial\Omega$. We cannot directly use a test-function defined like the function u from the previous step since it would not necessarily be continuous in $\Omega \cup E \cup F$.

To begin, let $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism, conformal in $\mathbb{D}$, given by the Riemann mapping and Carathéodory-Osgood theorems. Since

$$\mathrm{Cap}(E, F, \mathbb{R}^2) \geqslant \mathrm{Cap}(E, F, \Omega) \geqslant \delta_0$$

by monotonicity, we conclude by (2.29), which is given by Step 1, that neither E nor F is a singleton.

Suppose that $E \subset \partial\Omega$. Then $\varphi^{-1}(E)$ is a closed nondegenerate arc contained in the unit circle and hence (2.10) provides us with $w \in \varphi^{-1}(E)$ for which $\mathrm{le}(\varphi([0,w])) < \infty$. We choose $0 < t_E < 1$ so that

$$\mathrm{diam}_\Omega(\varphi([t_E w, w]) \cup E) \leqslant 2\mathrm{diam}_\Omega(E)$$

and

$$\mathrm{dist}_\Omega(\varphi([t_E w, w]) \cup E, F) \geqslant \frac{1}{2}\mathrm{dist}_\Omega(E,F).$$

Set $E' := \varphi([t_E w, w]) \cup E = \varphi([t_E w, w] \cup \varphi^{-1}(E))$. Then E' is compact and connected. Notice that E' intersects Ω and $E \subset E'$.

If E intersects Ω, we simply let $E' = E$. We construct F' in an analogous manner, considering now the internal distance to E'. Then $E', F' \subset \overline{\Omega}$ are continua,

$$\mathrm{diam}_\Omega(E') \leqslant 2\mathrm{diam}_\Omega(E), \quad \mathrm{diam}_\Omega(F') \leqslant 2\mathrm{diam}_\Omega(F)$$

and

$$\mathrm{dist}_\Omega(E,F) \leqslant 4\mathrm{dist}_\Omega(E',F').$$

Moreover, $E \subset E'$ and $F \subset F'$, and hence monotonicity of capacity together with our capacity assumption ensures that

$$\mathrm{Cap}(E', F,' \Omega) \geqslant \delta_0.$$

Hence, it suffices to prove (2.28) for E' and F' instead of E and F. For simplicity of notation, we refer to E' and F' by E and F in what follows.

By switching the roles of E and F if necessary, we show that

$$C\mathrm{diam}_\Omega(E) \geqslant \mathrm{dist}_\Omega(E,F) \tag{2.30}$$

for some constant that may depend only on δ_0. At this point, it is perhaps worth pointing out that the right-hand side is finite since both E and F intersect Ω.

Towards (2.30), we first pick $z_0 \in E \cap \Omega$ and choose a new conformal map $\psi\colon \Omega \to \mathbb{D}$ so that $\psi(z_0) = 0$. This can be done by post-composing φ^{-1} with a suitable conformal self (Möbius) map of the disk. Given $j \geqslant 1$, we define $E_j = \psi^{-1}((1-\frac{1}{j})\psi(E))$ and $F_j = \psi^{-1}((1-\frac{1}{j})\psi(F))$. Here, tA for a subset of the unit disk is the image of A under the map $f(z) = tz$. Then $E_j, F_j \subset \Omega$ are disjoint continua. By conformal invariance and monotonicity,

$$\mathrm{Cap}(E_j, F_j, \Omega) = \mathrm{Cap}(\psi(E_j), \psi(F_j), \mathbb{D}) \geqslant \mathrm{Cap}\left(\psi(E_j), \psi(F_j), B\left(0, 1-\frac{1}{j}\right)\right). \tag{2.31}$$

Notice that $\psi(E_j) = (1-\frac{1}{j})\psi(E)$ and $\psi(F_j) = (1-\frac{1}{j})\psi(F)$ are the images of $\psi(E)$ and $\psi(F)$, respectively, under the conformal map $f(z) = (1-\frac{1}{j})z$, and $B(0, 1-\frac{1}{j})$ is the image of the unit disk under this map. Hence, conformal invariance gives

$$\mathrm{Cap}\left(\psi(E_j), \psi(F_j), B\left(0, 1-\frac{1}{j}\right)\right) = \mathrm{Cap}(\psi(E), \psi(F), \mathbb{D}) = \mathrm{Cap}(E, F, \Omega). \tag{2.32}$$

By combining (2.31) and (2.32), we conclude from our capacity assumption on E and F that

$$\mathrm{Cap}(E_j, F_j, \Omega) \geqslant \delta_0$$

for all $j \geqslant 1$. Hence, we are allowed to infer from Step 1 that

$$\mathrm{dist}_\Omega(E_j, F_j) \leqslant C\mathrm{diam}_\Omega(E_j) \tag{2.33}$$

with $C = C(\delta_0)$ and all $j \geqslant 1$.

We proceed with a limiting process that relies on (2.33). First, notice that $0 \in \psi(E_j)$ for all j. Hence, by Lemma 2.16, there exists z_j in E_j so that $\mathrm{diam}_\Omega(E_j) \leqslant 3\mathrm{dist}_\Omega(\psi^{-1}(0), z_j)$. Thus,

$$\mathrm{diam}_\Omega(E_j) \leqslant 3\,\mathrm{le}(\psi^{-1}([0, \psi(z_j)])), \tag{2.34}$$

where $[0, \psi(z_j)]$ is the radial segment between the points 0 and $\psi(z_j)$. Since $\psi(E_j) = (1 - \frac{1}{j})\psi(E)$, the point $\xi_j = (1 - \frac{1}{j})^{-1}\psi(z_j)$ belongs to $\psi(E)$. Since also $0 \in \psi(E)$, we deduce that

$$\mathrm{diam}_\Omega(E) \geqslant \mathrm{dist}_\Omega(\psi^{-1}(0), \psi^{-1}(\xi_j)).$$

By Lemma 2.7, we conclude that

$$\mathrm{le}(\psi^{-1}([0, \xi_j])) \leqslant C_1 \mathrm{diam}_\Omega(E) \tag{2.35}$$

with an absolute constant C_1. By the definition of ξ_j, we have $[0, \psi(z_j)] \subset [0, \xi_j]$, and hence (2.34) together with (2.35) gives the uniform estimate

$$\mathrm{diam}_\Omega(E_j) \leqslant 3C_1 \mathrm{diam}_\Omega(E). \tag{2.36}$$

Next, (2.33) and Lemma 2.7 provide us with points $x_j \in E_j$, $y_j \in F_j$ and corresponding hyperbolic segments Γ_j between x_j and y_j in Ω so that

$$\mathrm{le}(\Gamma_j) \leqslant C_2 \mathrm{diam}_\Omega(E_j), \tag{2.37}$$

where C_2 depends only on δ_0. Since Ω is a Jordan domain, it is especially bounded, and hence by switching to a subsequence if necessary, we may assume that $x_j \to x$ and $y_j \to y$, where $x, y \in \overline{\Omega}$. This together with (2.36) and (2.37) allows us to employ Lemma 2.8 so as to conclude that

$$\mathrm{dist}_\Omega(x, y) \leqslant C_3 \mathrm{diam}_\Omega(E).$$

By construction, $\psi(x_j) \in (1-\frac{1}{j})E$ and $\psi(y_j) \in (1-\frac{1}{j})F$, and since ψ is homeomorphic up to the boundary, we deduce that $x \in E$ and $y \in F$. Thus, we have established (2.30) and the proof is complete. □

2.6 John domains

Let us recall the definition of a John domain.

Definition 2.18 (John domain). An open bounded subset $\Omega \subset \mathbb{R}^2$ is called a John domain provided that it satisfies the following condition: there exist a distinguished point $x_0 \in \Omega$ and a constant $J > 0$ such that for every $x \in \Omega$, there is a rectifiable curve γ joining x and x_0 in Ω and satisfying

$$\mathrm{dist}(y, \mathbb{R}^2 \setminus \Omega) \geqslant J\,\mathrm{le}(\gamma[x, y])$$

for all $y \in \gamma$. Such a curve γ is called a J-John curve, J is called a John constant, and we refer to a John domain with a John constant J by a J-John domain and to x_0 by a John center of Ω.

We continue with results related to John domains. For the convenience of the reader, we refer to [33] whenever possible, even when the result in question has a longer history.

As an example, every disk is a 1-John domain with the center as the John center and radial segments as John curves. In fact, one can always choose hyperbolic segments for the John curves in the simply connected situation.

Lemma 2.19 (See [33, Theorem 5.4]). *If Ω is a simply connected J-John domain, then hyperbolic segments from the John center x_0 to points in Ω are J'-John curves, where J' depends only on J.*

Remark 2.20. Actually, also the hyperbolic segment Γ connecting x_0 and $y \in \partial\Omega$ is a J'-John curve for a simply connected planar J-John domain Ω with the base point x_0. This follows from the preceding lemma and the definition of a hyperbolic segment: the hyperbolic segment between x_0 and a point x on this segment is the part of Γ between x_0 and x. Consequently, any two points $x, y \in \overline{\Omega}$ are rectifiably joinable and the diameter of a simply connected John domain with respect to the inner distance is finite.

For further reference, let us record the following consequence that deals with integrals as in (1.1). Let $1 < p < 2$.

Parameterizing Γ via arc-length, the John condition and integration gives

$$\int_\Gamma \operatorname{dist}(z, \partial\Omega)^{1-p} ds(z) \leqslant (J')^{1-p} \operatorname{le}(\gamma)^{2-p} \leqslant C(p, J') \operatorname{dist}(x_0, \partial\Omega)^{2-p}.$$

We move towards explaining the role of John domains in our work.

Definition 2.21. A set E is of *bounded turning if there is a constant C such that any pair of points z_1 and z_2 can be joined by a curve $\gamma \subset E$ whose diameter satisfies* $\operatorname{diam}(\gamma) \leqslant C|z_1 - z_2|$. *We then say that E is of C-bounded turning.*

Recall that quasiconvexity was defined analogously but with the length instead of the diameter. Hence, being of bounded turning is a weaker condition than being quasiconvex.

Lemma 2.22 (See [33, Theorem 4.5])**.** *Let Ω be a bounded simply connected planar domain. Then Ω is John if and only if $\mathbb{R}^2 \setminus \Omega$ is of bounded turning. This equivalence is quantitative in the sense that the John constant and the constant in bounded turning depend only on each other.*

By Lemma 2.2, the complement of a bounded simply connected domain whose complement satisfies (1.1) is C'-quasiconvex and hence also of C'-bounded turning with a constant that depends only on the exponent p and the constant C in (1.1). Hence, we obtain the following important corollary to the preceding lemma.

Corollary 2.23. *Let Ω be a bounded simply connected domain whose complement satisfies* (1.1). *Then Ω is J-John with a constant J that depends only on the exponent p and the constant C in* (1.1).

We need the fact that the boundaries of bounded simply connected John domains are of area zero.

Lemma 2.24. *If Ω is a bounded simply connected planar John domain, then the Lebesgue area of $\partial\Omega$ is zero.*

Conformal maps from the unit disk onto a John domain behave nicely with respect to the inner distance. In order to state this quantitatively, we need a definition.

We say that a homeomorphism $\varphi\colon \mathbb{D} \to \Omega$ is *quasisymmetric with respect to the inner distance* if there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ so that

$$|z - x| \leqslant t|y - x| \quad \text{implies } \operatorname{dist}_\Omega(\varphi(z), \varphi(x)) \leqslant \eta(t) \operatorname{dist}_\Omega(\varphi(y), \varphi(x)) \tag{2.38}$$

for each triple z, x and y of points in $\mathbb{D}$.

It follows from the definition that the inverse of a quasisymmetric map is also quasisymmetric, i.e.,

$$\operatorname{dist}_\Omega(z, x) \leqslant t \operatorname{dist}_\Omega(y, x) \quad \text{implies } |\varphi^{-1}(z) - \varphi^{-1}(x)| \leqslant \eta(1/t)^{-1} |\varphi^{-1}(y) - \varphi^{-1}(x)|$$

when $z, x, y \in \Omega$.

Roughly speaking, the definition means that the homeomorphism φ maps round objects to essentially round objects (with respect to the inner distance). The following result will be an important technical tool for us.

Lemma 2.25 (See [33, Theorem 7.2])**.** *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map. Then Ω is John if and only if φ is quasisymmetric with respect to the inner distance. This statement is quantitative in the sense that the John constant and the function η in quasisymmetry depend only on each other and on* $\operatorname{diam}(\Omega)/\operatorname{dist}(\varphi(0), \partial\Omega)$.

Remark 2.26. Notice that quasisymmetry is a strong version of uniform continuity of the conformal map from the unit disk onto Ω equipped with the inner distance. Hence, the quasisymmetry condition extends up to the boundary: one is allowed to use (2.38), when correctly interpreted, for triples of points in $\overline{\mathbb{D}}$. For example, when Ω is a slit disk, every other point of the slit than the tip corresponds to two different points with respect to the completion of Ω in the inner metric. In what follows, we only use the quasisymmetry condition up to the boundary in situations, where Ω is a Jordan John domain where this is not an issue. However, we will later employ the fact that a conformal map from the unit disk onto a bounded simply connected John domain extends continuously to $\partial\mathbb{D}$, with respect to the Euclidean distances. This follows since quasisymmetry of a quasisymmetric map from the unit disk implies uniform continuity with respect to the Euclidean distances.

Recall that hyperbolic segments in the unit disk are arcs of (generalized) circles perpendicular to the unit circle. Hence, they are essentially the shortest connecting curves and stay "optimally away from the boundary". The following definition gives an analog of this property.

Definition 2.27 (Inner uniform domain). A domain Ω is called *inner uniform* if there exists a positive constant ϵ_0 such that for any pair of points $x, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x and y and satisfying

$$\mathrm{le}(\gamma) \leqslant \frac{1}{\epsilon_0}\mathrm{dist}_\Omega(x,y) \quad \text{and} \quad \mathrm{dist}(z,\partial\Omega) \geqslant \epsilon_0 \min\{\mathrm{le}(\gamma[x,z]), \mathrm{le}(\gamma[z,y])\} \quad \text{for all } z \in \gamma. \tag{2.39}$$

Since a conformal map of the unit disk onto a John domain is quasisymmetric, the definition of hyperbolic segments and Lemma 2.7 suggest that each bounded simply connected John domain should be inner uniform. This is indeed the case.

Lemma 2.28 (See [33, Theorem 2.29 and Example 2.18(2)]). *Let Ω be a bounded simply connected J-John domain. Then Ω is inner uniform with an associated constant ϵ_0 that depends only on J. Moreover, the curves in the definition may be chosen to be hyperbolic segments.*

We continue by relating the inner and Euclidean diameters of boundary arcs of a Jordan John domain.

Lemma 2.29. *Let Ω be a Jordan J-John domain and $\gamma \subset \partial\Omega$ be a subarc containing its endpoints. Then we have*

$$\mathrm{diam}(\gamma) \leqslant \mathrm{diam}_\Omega(\gamma) \leqslant C\mathrm{diam}(\gamma),$$

where C depends only on J.

Proof. We only need to show that $\mathrm{diam}_\Omega(\gamma) \leqslant C\mathrm{diam}(\gamma)$ since the first inequality is trivial. Pick $x, y \in \gamma$ such that $\mathrm{diam}_\Omega(\gamma) \leqslant 2\mathrm{dist}_\Omega(x,y)$. By the definition of the inner distance, the hyperbolic segment Γ joining x and y satisfies $\mathrm{dist}_\Omega(x,y) \leqslant \mathrm{le}(\Gamma)$. Let z be the midpoint (in the sense of the length) of Γ. Then since Ω is a John domain and Γ is a hyperbolic segment, by applying Lemma 2.28 to pairs of points on γ converging to the endpoints of γ, we deduce that

$$\mathrm{le}(\Gamma) \leqslant C(J)\mathrm{dist}(z,\partial\Omega).$$

Hence, we have

$$\mathrm{diam}_\Omega(\gamma) \leqslant C(J)\mathrm{dist}(z,\partial\Omega). \tag{2.40}$$

Fix a conformal map $\varphi\colon \mathbb{D} \to \Omega$. Since Ω is Jordan, φ extends to a homeomorphism (still denoted φ) of $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. Let B be the closed disk of radius $\frac{1}{8}|1-\varphi^{-1}(z)|$, tangent to the circular arc $\varphi^{-1}(\Gamma)$ at $\varphi^{-1}(z)$, and contained in the Jordan domain enclosed by $\varphi^{-1}(\Gamma)$ and $\varphi^{-1}(\gamma)$; recall that $\varphi^{-1}(\Gamma)$ is a hyperbolic segment in $\mathbb{D}$ and hence a circle that meets the unit circle orthogonally at two endpoints of the arc $\varphi^{-1}(\gamma)$ of the unit circle. Since B is a 3-Whitney-type set, by Lemma 2.12, $Q' = \varphi(B)$ is a λ'-Whitney-type set. Here, λ' is an absolute constant. Let α be the radial projection of B to $\partial\mathbb{D}$. Then $\alpha \subset \varphi^{-1}(\gamma)$ and $\mathrm{diam}(B) = \frac{1}{4}|1-\varphi^{-1}(z)| \leqslant \mathrm{diam}(\alpha)$. Hence,

$$\mathrm{diam}(\varphi^{-1}(\gamma)) \geqslant \mathrm{diam}(B) \geqslant \frac{1}{4}\mathrm{dist}(B,\alpha) \geqslant \frac{1}{4}\mathrm{dist}(B,\varphi^{-1}(\gamma)).$$

Consequently, by (2.25),

$$\mathrm{Cap}(B,\varphi^{-1}(\gamma),\mathbb{D}) \geqslant \delta > 0$$

for an absolute constant δ. By the conformal invariance of capacity and monotonicity,

$$\delta \leqslant \mathrm{Cap}(Q', \gamma, \Omega) \leqslant \mathrm{Cap}(Q', \gamma, \mathbb{R}^2),$$

which with Lemma 2.17 implies

$$\mathrm{dist}(Q', \gamma) \leqslant C(\delta)\mathrm{diam}(\gamma). \tag{2.41}$$

Since Q' is of λ'-Whitney type and $z \in Q'$, we conclude via (2.13) and (2.41) that

$$\mathrm{dist}(z, \partial\Omega) \sim \mathrm{diam}(Q') \lesssim \mathrm{dist}(Q', \gamma) \lesssim \mathrm{diam}(\gamma),$$

where all the constants are absolute. This together with (2.40) gives

$$\mathrm{diam}_\Omega(\gamma) \lesssim \mathrm{dist}(z, \partial\Omega) \lesssim \mathrm{diam}(\gamma)$$

with constants depending only on J as desired. □

Based on the above lemma, one would expect $\partial\Omega$ to be compact with respect to the inner metric for each Jordan John domain. This is indeed the case by [4, Remark 3.14] (see also [19]).

We close this subsection with a subinvariance property.

Lemma 2.30 (See [22, Theorem 1]). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain and $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map. Suppose that Ω is J-John with the John center $\varphi(0)$. Then φ maps every J'-John domain $G \subset \mathbb{D}$ with the John center z_0 to a $c(J, J')$-John domain $G' = \varphi(G)$ with the John center $\varphi(z_0)$.*

2.7 Conformal geometry of the exterior domain

Let us fix our notation for this subsection. Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and a homeomorphism $\varphi\colon \mathbb{R}^2 \setminus \mathbb{D} \to \mathbb{R}^2 \setminus \Omega$ be conformal in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. For $z_1 \in \partial\Omega$, define

$$A(z_1, k) := \{x \in \mathbb{R}^2 \setminus \overline{\mathbb{D}} \mid 2^{k-1} < |x - \varphi^{-1}(z_1)| \leqslant 2^k\} \tag{2.42}$$

for $k \in \mathbb{Z}$. Furthermore, let $\Gamma(z_1) \subset \mathbb{R}^2 \setminus \overline{\Omega}$ be the hyperbolic ray corresponding to z_1, and set

$$\Gamma_k := \varphi(A(z_1, k)) \cap \Gamma(z_1).$$

We call $\varphi(\{x \in \mathbb{R}^2 \setminus \mathbb{D} \mid 2^{k-1} = |x - \varphi^{-1}(z_1)|\})$ the inner boundary of $\varphi(A(z_1, k))$ and $\varphi(\{x \in \mathbb{R}^2 \setminus \mathbb{D} \mid 2^k = |x - \varphi^{-1}(z_1)|\})$ the outer boundary of $\varphi(A(z_1, k))$. See Figure 1 for an illustration of our notation.

The following technical lemma is a version of a step in the proof of an analog of [3, Lemma 2.7].

Lemma 2.31. *With the notation introduced in the beginning of this subsection, let $z_2 \in \Gamma(z_1)$, and let $\gamma \subset \mathbb{R}^2 \setminus \Omega$ be any curve connecting z_1 and z_2. Let $k \in \mathbb{Z}$ be such that $2^k \leqslant |\varphi^{-1}(z_1) - \varphi^{-1}(z_2)|$ and let γ_k be any subcurve of γ in $\varphi(A(z_1, k))$ joining the inner boundary and the outer boundary of $\varphi(A(z_1, k))$. Then*

$$\mathrm{le}(\Gamma_k) \sim \mathrm{dist}(\Gamma_k, \partial\Omega) \sim \mathrm{diam}(\Gamma_k) \tag{2.43}$$

and

$$\mathrm{le}(\gamma_k) \gtrsim \mathrm{le}(\Gamma_k).$$

Here, all the constants are absolute and especially independent of Ω and the choice of φ, z_1, γ, z_2 and k.

Proof. The fact that $\mathrm{le}(\Gamma_k) \sim \mathrm{dist}(\Gamma_k, \partial\Omega) \sim \mathrm{diam}(\Gamma_k)$ immediately follows from Lemmas 2.12 and 2.13 since by definition, $\varphi^{-1}(\Gamma_k)$ is contained in a 2-Whitney-type disk in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$.

Hence, we only need to prove that $\mathrm{le}(\gamma_k) \gtrsim \mathrm{le}(\Gamma_k)$. Observe that since γ_k by definition joins the inner and outer boundaries of $\varphi(A(z_1, k))$, we have

$$\mathrm{le}(\varphi^{-1}(\gamma_k)) \geqslant \mathrm{diam}(\varphi^{-1}(\gamma_k)) \geqslant \mathrm{diam}(\varphi^{-1}(\Gamma_k)) = \mathrm{le}(\varphi^{-1}(\Gamma_k)) = \mathrm{dist}(\varphi^{-1}(\Gamma_k), \partial\mathbb{D}). \tag{2.44}$$

We next argue by case study.

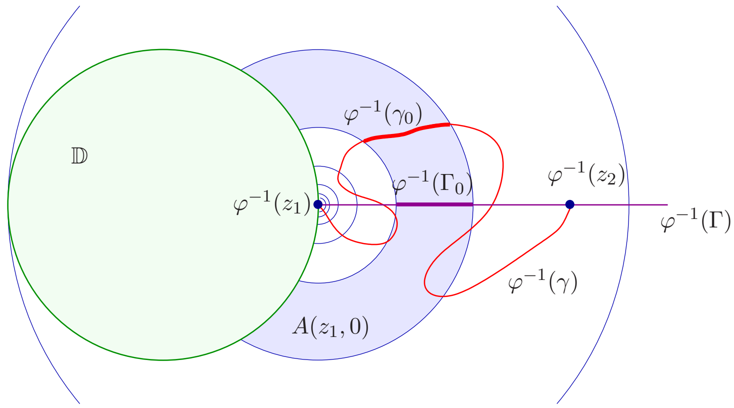

Figure 1 (Color online) An illustration of the annular parts $\varphi^{-1}(\Gamma_k)$ and $\varphi^{-1}(\gamma_k)$ for $k=0$, which are considered in Lemma 2.31

Case 1. $\operatorname{dist}(\varphi^{-1}(\gamma_k),\varphi^{-1}(\Gamma_k)) < \frac{1}{3}\operatorname{dist}(\varphi^{-1}(\Gamma_k),\partial\mathbb{D})$. Write $r=\operatorname{dist}(\varphi^{-1}(\Gamma_k),\partial\mathbb{D})=2^{k-1}$ and pick $w\in\varphi^{-1}(\Gamma_k)$ so that $\operatorname{dist}(w,\varphi^{-1}(\gamma_k))<\frac{r}{3}$. Then $B(w,\frac{r}{2})$ contains a subcurve α of $\varphi^{-1}(\gamma_k)$ of the length at least $r/6$. Since $\varphi^{-1}(\Gamma_k)\cup\alpha$ is contained in the 3-Whitney-type set $B(w,\frac{r}{2})\cup\varphi^{-1}(\Gamma_k)$ and $6\operatorname{le}(\alpha)\geqslant\operatorname{le}(\varphi^{-1}(\Gamma_k))$, Lemma 2.13 gives

$$\begin{aligned}\operatorname{le}(\gamma_k)\geqslant\operatorname{le}(\varphi(\alpha))&=\int_\alpha|\varphi'(z)|ds(z)\geqslant\frac{1}{C}\operatorname{le}(\alpha)|\varphi'(w)|\\&\geqslant\frac{1}{C}\operatorname{le}(\varphi^{-1}(\Gamma_k))|\varphi'(w)|\geqslant\frac{1}{C}\int_{\varphi^{-1}(\Gamma_k)}|\varphi'(z)|ds(z)=\frac{1}{C}\operatorname{le}(\Gamma_k)\end{aligned}$$

for an absolute constant C.

Case 2. $\operatorname{dist}(\varphi^{-1}(\gamma_k),\varphi^{-1}(\Gamma_k))\geqslant\frac{1}{3}\operatorname{dist}(\varphi^{-1}(\Gamma_k),\partial\mathbb{D})$. Let $\alpha'\subset\mathbb{R}^2\setminus\overline{\Omega}$ be a curve that joins γ_k and Γ_k. Since $\varphi^{-1}(\Gamma_k)$ is contained in a (2-Whitney-type) disk B, $\varphi^{-1}(\alpha')$ contains a subcurve $\alpha\subset\frac{3}{2}B$ of the length at least $\frac{1}{6}\operatorname{dist}(\varphi^{-1}(\Gamma_k),\partial\mathbb{D})$. Since $\frac{3}{2}B$ is of 6-Whitney type, we may again apply Lemma 2.13 to conclude that

$$\operatorname{le}(\alpha')\geqslant\operatorname{le}(\varphi(\alpha))\geqslant\frac{1}{C}\operatorname{le}(\Gamma_k)$$

with an absolute constant. Hence,

$$C\operatorname{dist}_\Omega(\gamma_k,\Gamma_k)\geqslant\operatorname{diam}(\Gamma_k). \tag{2.45}$$

Next, by (2.25) for the exterior of the unit disk, (2.44), the fact that

$$\operatorname{dist}(\varphi^{-1}(\gamma_k),\varphi^{-1}(\Gamma_k))\leqslant 2\operatorname{dist}(\varphi^{-1}(\Gamma_k),\partial\mathbb{D})$$

and the monotonicity of the capacity, we obtain

$$c\log\left(\frac{3}{2}\right)\leqslant\operatorname{Cap}(\varphi^{-1}(\overline{\gamma}_k),\varphi^{-1}(\overline{\Gamma}_k),\mathbb{R}^2\setminus\overline{\mathbb{D}})=\operatorname{Cap}(\overline{\gamma}_k,\overline{\Gamma}_k,\mathbb{R}^2\setminus\overline{\Omega})\leqslant\operatorname{Cap}(\overline{\gamma}_k,\overline{\Gamma}_k,\mathbb{R}^2).$$

Hence, by (2.45) and Lemma 2.17 for $\mathbb{R}^2$, we conclude that

$$\operatorname{le}(\gamma_k)\geqslant\operatorname{diam}(\gamma_k)\gtrsim\operatorname{dist}(\gamma_k,\Gamma_k)\gtrsim\operatorname{diam}(\Gamma_k)\sim\operatorname{le}(\Gamma_k)$$

with absolute constants. □

We record another technical result (see [35, Corollary 4.18] and [3, Proof of Theorem 3.1, p. 645]).

Lemma 2.32. *Let Ω and φ be as in the beginning of this subsection and $0<\sigma\leqslant 1$. Let $z_0\in\mathbb{R}^2\setminus\overline{\mathbb{D}}$ and let I be an arc of $\partial\mathbb{D}$ with*

$$\operatorname{le}(I)\geqslant\sigma(|z_0|-1)$$

and

$$\mathrm{dist}(I, z_0) \leqslant \frac{|z_0|-1}{\sigma}.$$

Then there is a rectifiable curve $\alpha \subset \mathbb{R}^2 \setminus \mathbb{D}$ *joining* z_0 *to* I *so that*

$$\mathrm{le}(\varphi(\alpha)) \leqslant C(\sigma)\mathrm{dist}(\varphi(z_0), \partial\Omega),$$

where $C(\sigma)$ *is independent of* φ, z_0, Ω.

Proof. Notice first that it is enough to prove the claim for a subarc of I. Let $w \in I$ be such that $|w-z_0| \leqslant 2\mathrm{dist}(I, z_0)$. Then by taking a subarc of I that contains w, we may assume that $\mathrm{le}(I) \leqslant |z_0|-1$, that I is closed and that $\mathrm{dist}(I, z_0) \leqslant 2\frac{|z_0|-1}{\sigma}$. Define $I_t = \{t\xi : \xi \in I\}$ for all $1 < t \leqslant |z_0|$. Then I_t is also a continuum and $\mathrm{le}(I_t) \geqslant \mathrm{le}(I)$. According to (2.25) (for $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$), the assumptions on I and the conformal invariance of capacity, we have the estimate

$$0 < \delta(\sigma) \leqslant \mathrm{Cap}(I_{|z_0|}, I_t, \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \mathrm{Cap}(\varphi(I_{|z_0|}), \varphi(I_t), \mathbb{R}^2 \setminus \overline{\Omega}) \tag{2.46}$$

for all $1 < t < |z_0|$. Then, by Lemma 2.17, we conclude that

$$\mathrm{dist}_{\mathbb{R}^2 \setminus \overline{\Omega}}(\varphi(I_{|z_0|}), \varphi(I_t)) \leqslant C(\sigma)\mathrm{diam}_{\mathbb{R}^2 \setminus \overline{\Omega}}(\varphi(I_{|z_0|})) \leqslant C(\sigma)\,\mathrm{le}(\varphi(I_{|z_0|})).$$

Hence, we can connect $I_{|z_0|}$ to I_t with a curve β_t for which $\mathrm{le}(\varphi(\beta_t)) \leqslant C(\sigma)\,\mathrm{le}(\varphi(I_{|z_0|}))$.

By Lemma 2.1, there exists a sequence $t_i \searrow 1$ so that $\varphi \circ \beta_{t_i}$ converges to a rectifiable curve $\hat{\beta} \subset \mathbb{R}^2 \setminus \Omega$ joining $\varphi(I)$ to $\varphi(I_{|z_0|})$ with

$$\mathrm{le}(\hat{\beta}) \leqslant \liminf_{i \to \infty} \mathrm{le}(\varphi(\beta_{t_i})) \leqslant C(\sigma)\,\mathrm{le}(\varphi(I_{|z_0|})).$$

Next, take J to be a shortest closed subarc of $\partial B(0, |z_0|)$ containing both $I_{|z_0|}$ and z_0. Then

$$\begin{aligned}\mathrm{le}(J) &\leqslant \mathrm{le}(I_{|z_0|}) + \pi\mathrm{dist}(z_0, I_{|z_0|}) \leqslant |z_0|\,\mathrm{le}(I) + \pi|z_0 - w| + \pi\mathrm{dist}(w, I_{|z_0|})\\ &\leqslant |z_0|\min((|z_0|-1), 2\pi) + 2\pi\frac{|z_0|-1}{\sigma} + \pi(|z_0|-1)\\ &\leqslant C(\sigma)(|z_0|-1) = C(\sigma)\mathrm{dist}(z_0, \mathbb{D}).\end{aligned}$$

Since J is contained in a $\lambda(\sigma)$-Whitney-type set $B = J \cup B(z_0, \frac{|z_0|-1}{2})$, by Lemmas 2.13 and 2.12,

$$\mathrm{le}(\varphi(I_{|z_0|})) \leqslant \mathrm{le}(\varphi(J)) \leqslant C(\sigma)\mathrm{diam}(\varphi(B)) \leqslant C(\sigma)\mathrm{dist}(\varphi(B), \partial\Omega).$$

Now, take $z \in I_{|z_0|} \cap \varphi^{-1}(\hat{\beta})$ and define $\alpha = \varphi^{-1}(\hat{\beta}) * J[z, z_0]$. Then α connects I to z_0 in $\mathbb{R}^2 \setminus \mathbb{D}$ and

$$\begin{aligned}\mathrm{le}(\varphi(\alpha)) &\leqslant \mathrm{le}(\hat{\beta}) + \mathrm{le}(\varphi(J)) \leqslant C(\sigma)\,\mathrm{le}(\varphi(I_{|z_0|})) + \mathrm{le}(\varphi(J))\\ &\leqslant C(\sigma)\mathrm{dist}(\varphi(B), \partial\Omega) \leqslant C(\sigma)\mathrm{dist}(\varphi(z_0), \partial\Omega).\end{aligned}$$

This completes the proof. □

3 Proof of necessity

In this section, we prove that a bounded simply connected planar $W^{1,p}$-extension domain with $1 < p < 2$ necessarily has the property that any two points $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$ can be connected with a curve $\gamma \subset \mathbb{R}^2 \setminus \Omega$ satisfying

$$\int_\gamma \mathrm{dist}(z, \partial\Omega)^{1-p} ds(z) \leqslant C(\|E\|, p)|z_1 - z_2|^{2-p}.$$

We first consider the case where Ω is additionally assumed to be Jordan. Under this assumption, we usually denote the complementary domain of Ω by $\widetilde{\Omega}$.

Theorem 3.1. *Let $1 < p < 2$ and let Ω be a Jordan domain. Suppose that there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$. Then, given $z_1, z_2 \in \widetilde{\Omega} \cup \partial\Omega$, there is a curve $\gamma \subset \widetilde{\Omega} \cup \partial\Omega$ so that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} ds(z) \leqslant C(\|E\|, p)|z_1 - z_2|^{2-p}, \tag{3.1}$$

where $C(\|E\|, p)$ depends only on p and the norm of the extension operator.

After this, based on inner uniformity (see Definition 2.27 below), we prove that if Ω is a bounded simply connected $W^{1,p}$-extension domain, then for $n \geqslant 2$, the Jordan domains $\Omega_n = \varphi(B(0, 1 - \frac{1}{n}))$ are also $W^{1,p}$-extension domains with extension operator norms depending only on p and the norm of the extension operator for Ω. Here, $\varphi\colon \mathbb{D} \to \Omega$ is a suitable conformal map. Finally, by a limiting argument, we obtain the result in the general case.

Actually, we remark that when $z_1, z_2 \in \widetilde{\Omega}$, one can require that the curve γ in Theorem 3.1 is contained in $\widetilde{\Omega}$. For this, see Remark 3.7.

3.1 Necessity in the Jordan case

In this subsection, we prove Theorem 3.1. Recall that the existence of our extension operator guarantees that Ω is a John domain with a constant J depending only on p and the norm of E (see, e.g., [25, Theorem 6.4], [14, Theorem 3.4], [33, Theorem 4.5] and the references therein). In what follows, J refers to this constant. Because of technical issues, we first consider the case $z_1, z_2 \in \partial\widetilde{\Omega} = \partial\Omega$ with $z_2 \neq z_1$.

Since Ω is Jordan, $\partial\Omega \setminus \{z_1, z_2\}$ consists of two open arcs P_1 and P_2. Without loss of generality, we assume that $\operatorname{diam}(P_1) \leqslant \operatorname{diam}(P_2)$. For the following four lemmas, let Ω, z_1, z_2, P_1 and P_2 be fixed.

We need the following general lower bound on the Sobolev norm.

Lemma 3.2. *Let Q be a square with sides parallel to the coordinate axes and fix $1 \leqslant p < 2$. Let $u \in W^{1,1}(Q)$ be absolutely continuous on almost all lines parallel to the coordinate axes. Write*

$$A_0 = \{x \in Q \mid u(x) \leqslant 0\} \quad \text{and} \quad A_1 = \{x \in Q \mid u(x) \geqslant 1\}.$$

Suppose further that

$$\max\{\mathscr{H}^1(\pi_1(A_0)), \mathscr{H}^1(\pi_2(A_0))\} \geqslant \delta\ell(Q)$$

and

$$\max\{\mathscr{H}^1(\pi_1(A_1)), \mathscr{H}^1(\pi_2(A_1))\} \geqslant \delta\ell(Q)$$

for some $\delta > 0$, where $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure, and π_i stands for the projection to the x_i-axis for each $i = 1, 2$. Then

$$\ell(Q)^{2-p} \leqslant C(\delta, p) \int_Q |\nabla u|^p dx.$$

Proof. We may assume that

$$\int_Q |\nabla u|^p dx < \infty;$$

otherwise the claim is trivial.

Suppose first that $\mathscr{H}^1(\pi_1(A_0)) \geqslant \delta\ell(Q)$ and $\mathscr{H}^1(\pi_1(A_1)) \geqslant \delta\ell(Q)$. If for $\mathscr{H}^1$-almost every $x_1 \in \pi_1(A_0)$, there exists some $x_2 \in \pi_2(Q)$ such that $u(x_1, x_2) \geqslant \frac{1}{3}$, then by our absolute continuity assumption and Hölder's inequality,

$$\frac{1}{3} \leqslant \int_{\pi_2(Q)} |\nabla u(x_1, t)| dt \leqslant \ell(Q)^{\frac{p-1}{p}} \left(\int_{\pi_2(Q)} |\nabla u(x_1, t)|^p dt \right)^{\frac{1}{p}}$$

for $\mathscr{H}^1$-almost every $x_1 \in \pi_1(A_0)$, and our claim follows from Fubini's theorem that

$$\int_Q |\nabla u|^p dx \geqslant \int_{\pi_1(A_0)} \int_{\pi_2(Q)} |\nabla u(x_1, t)|^p dt dx_1 \geqslant \mathcal{H}^1(\pi_1(A_0)) \frac{1}{3^p} \ell(Q)^{1-p} \geqslant \frac{\delta}{3^p} \ell(Q)^{2-p}.$$

Similarly, the claim holds if for $\mathscr{H}^1$-almost every $x_1 \in \pi_1(A_1)$, there exists $x_2 \in \pi_2(Q)$ such that $u(x_1, x_2) \leqslant \frac{2}{3}$. If both of the above two conditions fail, we find $x_1 \in \pi_1(A_0)$ and $\hat{x}_1 \in \pi_1(A_1)$ such that for all $x_2 \in \pi_2(Q)$, $u(x_1, x_2) \leqslant \frac{1}{3}$ and $u(\hat{x}_1, x_2) \geqslant \frac{2}{3}$. Then by absolute continuity and Hölder's inequality, for $\mathscr{H}^1$-almost every $x_2 \in \pi_2(Q)$, we have

$$\frac{1}{3} \leqslant u(\hat{x}_1, x_2) - u(x_1, x_2) \leqslant \int_{\pi_1(Q)} |\nabla u(t, x_2)| dt \leqslant \ell(Q)^{\frac{p-1}{p}} \left(\int_{\pi_1(Q)} |\nabla u(t, x_2)|^p dt \right)^{\frac{1}{p}},$$

and we again conclude by Fubini's theorem that

$$\int_Q |\nabla u|^p dx \geqslant \int_{\pi_2(Q)} \int_{\pi_1(Q)} |\nabla u(t, x_2)|^p dt dx_2 \geqslant \mathcal{H}^1(\pi_2(Q)) \frac{1}{3^p} \ell(Q)^{1-p} \geqslant \frac{1}{3^p} \ell(Q)^{2-p}.$$

If $\mathscr{H}^1(\pi_2(A_0)) \geqslant \delta\ell(Q)$ and $\mathscr{H}^1(\pi_2(A_1)) \geqslant \delta\ell(Q)$, the argument for the previous case gives the asserted estimate after switching the roles of the coordinates x_1 and x_2. We are left with the cases where

$$\mathscr{H}^1(\pi_1(A_0)) \geqslant \delta\ell(Q) \quad \text{and} \quad \mathscr{H}^1(\pi_2(A_1)) \geqslant \delta\ell(Q)$$

and

$$\mathscr{H}^1(\pi_2(A_0)) \geqslant \delta\ell(Q) \quad \text{and} \quad \mathscr{H}^1(\pi_1(A_1)) \geqslant \delta\ell(Q).$$

By symmetry, it suffices to consider the first one. As above, if for $\mathscr{H}^1$-almost every $x_1 \in \pi_1(A_0)$, there exists some $x_2 \in \pi_2(Q)$ such that $u(x_1, x_2) \geqslant \frac{1}{3}$, then we get

$$\int_Q |\nabla u|^p dx \geqslant \frac{\delta}{3^p} \ell(Q)^{2-p}.$$

Similarly, if for $\mathscr{H}^1$-almost every $x_2 \in \pi_2(A_1)$, there exists some $x_1 \in \pi_2(Q)$ such that $u(x_1, x_2) \leqslant \frac{1}{3}$, then we get

$$\int_Q |\nabla u|^p dx \geqslant \frac{\delta}{3^p} \ell(Q)^{2-p}.$$

Thus, the only case remaining is the one in which there exist $x_1 \in \pi_1(A_0)$ and $x_2 \in \pi_2(A_1)$ such that for all $t \in \pi_2(Q)$ and $s \in \pi_1(Q)$, $u(x_1, t) \leqslant \frac{1}{3}$ and $u(s, x_2) \geqslant \frac{2}{3}$, and so that u is absolutely continuous along these two line segments. This is impossible as these segments intersect. □

We continue with the existence of suitable test functions. Recall that the curves P_1 and P_2 are open.

Lemma 3.3. *Let $c_1 \geqslant 1$ and $1 < p < 2$. With the above notation, there exists a function $\Phi \in W^{1,p}(\Omega)$ such that for any $0 < \epsilon < \frac{1}{9}$, we have $\Phi \geqslant 1 - \epsilon$ in some neighborhood of $P_1 \cap B(z_1, c_1|z_2 - z_1|)$, $\Phi \leqslant \epsilon$ in some neighborhood of $P_2 \cap B(z_1, c_1|z_2 - z_1|)$, and $\|\nabla\Phi\|^p_{L^p(\Omega)} \leqslant C(p, c_1, J)|z_1 - z_2|^{2-p}$. Here, the neighborhoods are defined with respect to the topology of $\overline{\Omega}$.*

Proof. Let $x_0 \in \Omega$ be a distinguished point as in Definition 2.18. Denote by γ_1 the hyperbolic segment from x_0 to z_1 and by γ_2 the hyperbolic segment from x_0 to z_2. By Lemma 2.19(4), the curves γ_1 and γ_2 are John curves. We define $\gamma_0 = \gamma_1 \cup \gamma_2$. The existence of John curves is actually only guaranteed by the definition for points inside the domain, but the general case follows easily from this (see Remark 2.20). Let $\varphi\colon \overline{\mathbb{D}} \to \overline{\Omega}$ be a homeomorphism, which is conformal inside and satisfies $\varphi(0) = x_0$. Then it is clear that the preimages of γ_1 and γ_2 under φ are radial line segments, and $\varphi^{-1}(P_1 \cup \gamma_0)$ is a Jordan curve. Hence, $P_1 \cup \gamma_0$ is also Jordan as φ is a homeomorphism. It follows that $P_1 \cup \gamma_0$ bounds a Jordan subdomain $\Omega_1 \subset \Omega$.

Define a function $\phi\colon \Omega \to \mathbb{R}$ by setting

$$\phi(x) = \max\left\{ \inf_{\gamma(x,P_2)} \int_{\gamma(x,P_2)} \frac{1}{|\hat{z} - z_1|} ds(\hat{z}), \inf_{\gamma(x,P_2)} \int_{\gamma(x,P_2)} \frac{1}{|\hat{z} - z_2|} ds(\hat{z}) \right\}$$

for $x \in \Omega$, where the infima are taken over all the rectifiable curves $\gamma(x, P_2) \subset \Omega$ joining x to P_2.

Since Ω is a Jordan domain, γ_0, P_1 and P_2 are pairwise disjoint. By the John condition, we have $\operatorname{dist}(w,\partial\Omega) \geqslant J|w-z_1|$ for every $w \in \gamma_1$. Therefore, for $w \in \gamma_1$, we get

$$\phi(w) \geqslant \inf_{\gamma(w,P_2)} \int_{\gamma(w,P_2)} \frac{1}{|\hat{z}-z_1|} ds(\hat{z}) \geqslant \frac{\operatorname{dist}(w,\partial\Omega)}{\operatorname{dist}(w,\partial\Omega)+|w-z_1|} \geqslant \frac{J}{J+1} =: c_0,$$

where we have used the fact that $\gamma(w,P_2)$ necessarily exits $B(w,\operatorname{dist}(w,\partial\Omega))$ and that for points $\hat{z} \in B(w,\operatorname{dist}(w,\partial\Omega))$, we have by the triangle inequality that

$$|\hat{z}-z_1| \leqslant |\hat{z}-w| + |w-z_1| \leqslant \operatorname{dist}(w,\partial\Omega) + |w-z_1|.$$

The same estimate follows similarly for $w \in \gamma_2$. Hence, for any point $w \in \Omega_1$, we have $\phi(w) \geqslant c_0$ as Ω_1 is Jordan and P_2 is outside Ω_1; any curve $\gamma(w,P_2) \subset \Omega$ must cross γ_0 by the Jordan curve theorem (see Figure 2).

Fix $0 < \epsilon < \frac{1}{9}$. We claim that we have $\phi \leqslant \epsilon$ in some neighborhood of P_2. Indeed, for any $x \in P_2$, there is the radius $R_x > 0$ such that $B(x,2R_x) \cap P_1 = \emptyset$. Let $0 < r_x \leqslant R_x$. Then for any $y \in B(x,r_x) \cap \Omega$, there is a point $z \in P_2 \cup \{z_1,z_2\}$ such that $|y-z| = \operatorname{dist}(y,P_2) = \operatorname{dist}(y,\partial\Omega) \leqslant r_x$ while $\operatorname{dist}(y,P_1) \geqslant R_x$ via the triangle inequality. By choosing r_x sufficiently small, we conclude, via letting $\gamma(w,P_2)$ be the line segment joining y and z in the definition of ϕ, that

$$\phi(y) \leqslant r_x R_x^{-1} \leqslant \epsilon.$$

Hence, by taking the union of $B(x,r_x) \cap \overline{\Omega}$ over $x \in P_2$, we obtain a neighborhood of P_2 in which $\phi \leqslant \epsilon$.

Recall that $c_1 \geqslant 1$. We define a cut-off function by setting

$$\alpha(z) = \begin{cases} 1, & \text{if } |z-z_1| < c_1|z_1-z_2|, \\ \log_2 \dfrac{2c_1|z_1-z_2|}{|z-z_1|}, & \text{if } c_1|z_1-z_2| \leqslant |z-z_1| \leqslant 2c_1|z_1-z_2|, \\ 0, & \text{otherwise} \end{cases}$$

for $z \in \Omega$. Using this cut-off function, we define

$$\Phi(z) = \alpha(z) \min\left\{\frac{1}{c_0}\phi(z), 1\right\}$$

when $z \in \Omega$. We also define $\Phi(x) = 1$ for $x \in P_1 \cap B(z_1, c_1|z_1-z_2|)$, and $\Phi(x) = 0$ when $x \in P_2$. Then by the properties of ϕ, we know that for any $0 < \epsilon < \frac{1}{9}$, $\Phi \geqslant 1-\epsilon$ in some neighborhood of the set $P_1 \cap B(z_1, c_1|z_1-z_2|)$, and $\Phi \leqslant \epsilon$ in some neighborhood of $P_2 \cap B(z_1, c_1|z_1-z_2|)$.

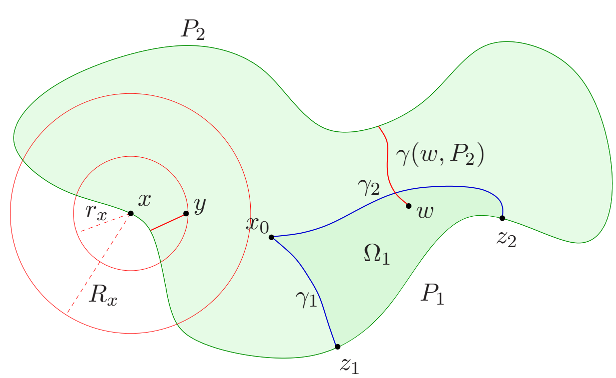

Figure 2 (Color online) The function ϕ is seen to have large value in Ω_1 by observing that any curve $\gamma(w,P_2)$ connecting a point $w \in \Omega_1$ to P_2 in Ω must intersect $\gamma_0 = \gamma_1 \cup \gamma_2$. In order to see that ϕ has small value near P_2, one observes that ϕ near $x \in P_2$ can be estimated by integrating $\frac{1}{R_x}$ along a curve with the length at most $r_x \leqslant \epsilon R_x$

We claim that ϕ is locally Lipschitz in Ω with

$$|\nabla\phi(z)| \leqslant \frac{3}{2}\max\{|z-z_1|^{-1}, |z-z_2|^{-1}\} \tag{3.2}$$

for almost every $z \in \Omega$. Indeed, for any $y \in B(z, 3^{-1}\mathrm{dist}(z, \partial\Omega))$, we have that by the definition of ϕ and the fact that $z_1, z_2 \in \partial\Omega$,

$$\begin{aligned}|\phi(y)-\phi(z)| &\leqslant \max\left\{\int_{[y,z]}|\hat{z}-z_1|^{-1}ds(\hat{z}), \int_{[y,z]}|\hat{z}-z_2|^{-1}ds(\hat{z})\right\}\\ &\leqslant \frac{3}{2}\max\{|z-z_1|^{-1}, |z-z_2|^{-1}\}|y-z|,\end{aligned}$$

where $[y,z]$ is the line segment joining y and z. Thus, our claim follows. Furthermore, by applying the Leibniz rule, we obtain

$$\begin{aligned}\|\nabla\Phi\|^p_{L^p(\Omega)} &\lesssim \|\nabla\alpha\|^p_{L^p(\Omega)} + \|\nabla\phi\|^p_{L^p(\Omega\cap B(z_1, 2c_1|z_1-z_2|))}\\ &\lesssim \int_{B(z_1,2c_1|z_1-z_2|)\setminus B(z_1,|z_1-z_2|)}|\hat{z}-z_1|^{-p}d\hat{z} + \int_{B(z_1,2c_1|z_1-z_2|)}|\hat{z}-z_1|^{-p}+|\hat{z}-z_2|^{-p}d\hat{z}\\ &\leqslant C(p,c_1,J)|z_1-z_2|^{2-p},\end{aligned}$$

by calculating in polar coordinates with $1<p<2$. Thus, $\Phi \in W^{1,p}(\Omega)$ with the desired properties since $\|\Phi\|_{L^\infty(\Omega)} \leqslant 1$ and Ω is bounded. □

Let $\widetilde{\varphi}: \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ be a conformal map. Since Ω is Jordan, $\widetilde{\varphi}$ extends homeomorphically up to the boundary by the Carathéodory-Osgood theorem. We refer to this extension also by $\widetilde{\varphi}$. For our fixed $z_1, z_2 \in \partial\widetilde{\Omega}$, let Γ_k be the hyperbolic ray starting at $\widetilde{\varphi}^{-1}(z_k)$, where $k=1,2$. Pick $y_k \in \Gamma_k$ with

$$|\widetilde{\varphi}^{-1}(z_k) - y_k| = |\widetilde{\varphi}^{-1}(z_2) - \widetilde{\varphi}^{-1}(z_1)|,$$

and let α_c be a shorter one of the two circular arcs from y_1 to y_2. Define

$$\alpha = [\widetilde{\varphi}^{-1}(z_1), y_1] * \alpha_c * [y_2, \widetilde{\varphi}^{-1}(z_2)] \quad \text{and} \quad \gamma = \widetilde{\varphi}(\alpha) \tag{3.3}$$

(see Figure 3). We establish the curve condition (3.1) for γ. The reason for using γ instead of the corresponding hyperbolic segment is partially that this is technically easier.

Let $\widetilde{W}$ be a Whitney decomposition of $\widetilde{\Omega}$ given by Lemma 2.9 and set

$$\widetilde{W}_\gamma = \{\widetilde{Q}_i \in \widetilde{W} \mid \widetilde{Q}_i \cap \gamma \neq \emptyset\}.$$

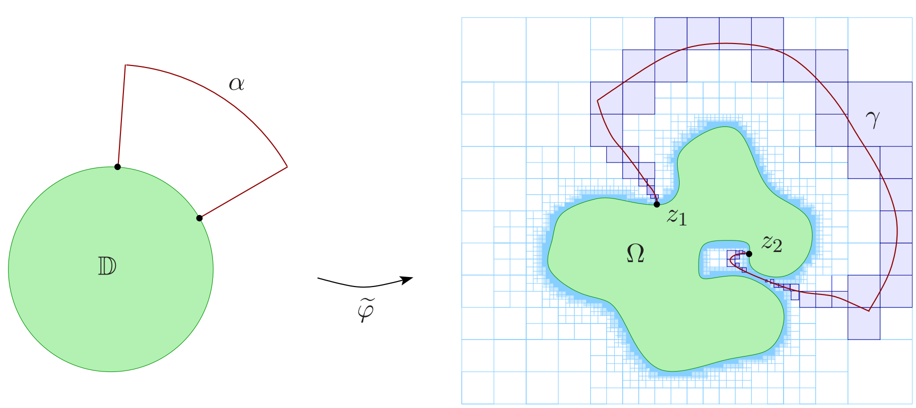

Figure 3 (Color online) The curve γ is obtained as the image of the curve α under the conformal map $\widetilde{\varphi}$: $\mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$. In the illustration, the Whitney squares in $\widetilde{W}_\gamma$ are highlighted

We index the squares in W_γ according to the side length: $\widetilde{Q}_{i1},\dots,\widetilde{Q}_{in_i}$ are those with the side length 2^i when $i\in\mathbb{Z}$, if there are such squares. Notice that since $\partial\widetilde{\Omega}$ is bounded, each n_i is necessarily finite.

We start with a simple observation on the Whitney squares that intersect the circular part of the curve α.

Lemma 3.4. *The number of squares $\widetilde{Q}_{ij}\in\widetilde{W}$ for which $\varphi^{-1}(\widetilde{Q}_{ij})\cap\alpha[y_1,y_2]\neq\emptyset$ is bounded from above by a universal constant.*

Proof. By Lemma 2.12, we have that $\widetilde{\varphi}^{-1}(\widetilde{Q}_{i,j})$ is a λ-Whitney-type set with a universal constant λ. If $\varphi^{-1}(\widetilde{Q}_{ij})\cap\alpha[y_1,y_2]\neq\emptyset$, we have

$$\begin{aligned}|\widetilde{\varphi}^{-1}(z_2)-\widetilde{\varphi}^{-1}(z_1)|&=\operatorname{dist}(\partial\mathbb{D},\alpha[y_1,y_2])\\&\leqslant\operatorname{dist}(\partial\mathbb{D},\widetilde{\varphi}^{-1}(\widetilde{Q}_{ij}))+\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_{ij}))\\&\leqslant(\lambda+1)\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_{ij})),\end{aligned}$$

so $\varphi^{-1}(\widetilde{Q}_{ij})$ contains a disk of radius $\frac{1}{\lambda(\lambda+1)}|\widetilde{\varphi}^{-1}(z_2)-\widetilde{\varphi}^{-1}(z_1)|$. Since for different $\widetilde{Q}_{ij}$, these disks are disjoint, and

$$\operatorname{dist}(\widetilde{\varphi}^{-1}(z_1),\widetilde{\varphi}^{-1}(\widetilde{Q}_{ij}))\leqslant\operatorname{diam}(\alpha)\leqslant 3|\widetilde{\varphi}^{-1}(z_2)-\widetilde{\varphi}^{-1}(z_1)|,$$

the claim follows. □

Lemma 3.5. *For the curve γ defined in* (3.3) *and each Whitney square $\widetilde{Q}\in\widetilde{W}_\gamma$, we have $\widetilde{Q}\subset B(z_1,C|z_1-z_2|)$, where $C=C(J)$ is independent of z_1, z_2 and $\widetilde{\varphi}$.*

Proof. Since Ω is John, by Lemma 2.22, the set $\mathbb{R}^2\setminus\Omega$ is of $C(J)$-bounded turning, where $C(J)$ depends only on J. Thus, there is a (closed) curve $\beta\subset\mathbb{R}^2\setminus\Omega$ that joins z_1 and z_2 and so that $\beta\subset\overline{B}(z_1,C(J)|z_1-z_2|)$.

Now, if $\widetilde{Q}\cap\beta\neq\emptyset$, we have

$$\widetilde{Q}\subset B(z_1,C(J)|z_1-z_2|+\operatorname{diam}(\widetilde{Q}))\subset B(z_1,(C(J)+\sqrt{2}C(J))|z_1-z_2|),$$

as $z_1\in\partial\Omega$.

Suppose then that $\widetilde{Q}\cap\beta=\emptyset$. We have

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(\beta))\geqslant|\widetilde{\varphi}^{-1}(z_1)-\widetilde{\varphi}^{-1}(z_2)|\tag{3.4}$$

since $z_1,z_2\in\widetilde{\varphi}^{-1}(\beta)$. Next, $\widetilde{\varphi}^{-1}(\widetilde{Q})$ is a λ-Whitney-type set by Lemma 2.12 with a universal constant λ and $\widetilde{\varphi}^{-1}(\widetilde{Q})\cap\alpha\neq\emptyset$. Hence, the definition of α (α is an inner uniform curve for the exterior domain of the unit disk) gives

$$\begin{aligned}\operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}),\widetilde{\varphi}^{-1}(\beta))&\leqslant\min\{\operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}),\widetilde{\varphi}^{-1}(z_1)),\operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}),\widetilde{\varphi}^{-1}(z_2))\}\\&\leqslant C\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}))\leqslant C|\widetilde{\varphi}^{-1}(z_1)-\widetilde{\varphi}^{-1}(z_2)|.\end{aligned}$$

This together with (3.4) shows that

$$C\min\{\operatorname{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})),\operatorname{diam}(\widetilde{\varphi}^{-1}(\beta))\}\geqslant\operatorname{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}),\varphi^{-1}(\beta)).$$

Then the version of (2.25) for $\mathbb{R}^2\setminus\overline{\mathbb{D}}$ and conformal invariance of capacity give

$$0<\delta(C)\leqslant\operatorname{Cap}(\widetilde{\varphi}^{-1}(\widetilde{Q}),\widetilde{\varphi}^{-1}(\beta),\mathbb{R}^2\setminus\overline{\mathbb{D}})=\operatorname{Cap}(\widetilde{Q},\beta,\widetilde{\Omega})\leqslant\operatorname{Cap}(\widetilde{Q},\beta,\mathbb{R}^2),$$

where in the last inequality, we used the monotonicity of capacity.

Hence, Lemma 2.17 shows that $\operatorname{dist}(\widetilde{Q},\beta)\leqslant C\operatorname{diam}(\beta)$, and since $z_1\in\beta$ and $\operatorname{diam}(\beta)\leqslant C(J)|z_1-z_2|$, we conclude that

$$\begin{aligned}\widetilde{Q}&\subset B(z_1,\operatorname{dist}(z_1,\widetilde{Q})+\operatorname{diam}(\widetilde{Q}))\subset B(z_1,(1+\sqrt{2})\operatorname{dist}(z_1,\widetilde{Q}))\\&\subset B(z_1,(1+\sqrt{2})(\operatorname{diam}(\beta)+\operatorname{dist}(\beta,\widetilde{Q})))\\&\subset B(z_1,C\operatorname{diam}(\beta))\subset B(z_1,C(J)|z_1-z_2|).\end{aligned}$$

This completes the proof. □

We apply the preceding four lemmas to prove the following estimate for $\widetilde{W}_\gamma$. Recall that n_i stands for the number (if any) of $\widetilde{Q}_{ij} \in \widetilde{W}_\gamma$ of the side length 2^i and $\|E\|$ stands for the norm of the homogeneous extension operator.

Lemma 3.6. *We have*

$$\sum_i n_i 2^{i(2-p)} \leqslant C(\|E\|, p)|z_1 - z_2|^{2-p}.$$

Proof. We claim that there exists a constant c_0 such that for every $\widetilde{Q}_{ij} \in \widetilde{W}_\gamma$,

$$c_0\widetilde{Q}_{ij} \cap P_1 \neq \emptyset \neq c_0\widetilde{Q}_{ij} \cap P_2. \tag{3.5}$$

Towards this, suppose first that $\widetilde{\varphi}^{-1}(Q_{ij}) \cap [\widetilde{\varphi}^{-1}(z_k), y_k] \neq \emptyset$ for $k = 1$ or for $k = 2$, where the points $y_k \in \alpha$ are from the definition of α and γ. Pick $z_0 \in \widetilde{\varphi}^{-1}(\widetilde{Q}_{ij}) \cap [\widetilde{\varphi}^{-1}(z_k), y_k]$. Then $\mathrm{le}(\widetilde{\varphi}^{-1}(P_k)) \geqslant |z_0| - 1$ and $\mathrm{dist}(\varphi^{-1}(P_k), z_0) \leqslant |z_0 - \widetilde{\varphi}^{-1}(z_k)| \leqslant |z_0| - 1$ for $k = 1, 2$. Hence, Lemma 2.32, applied to both $\widetilde{\varphi}^{-1}(P_1)$ and $\widetilde{\varphi}^{-1}(P_2)$, gives a curve α' connecting P_1 and P_2 and passing through z_0 such that

$$\mathrm{le}(\widetilde{\varphi}(\alpha')) \leqslant C_0 \mathrm{dist}(\widetilde{\varphi}(z_0), \partial\widetilde{\Omega}).$$

Since $\widetilde{Q}_{ij}$ is a Whitney square, it follows that $\widetilde{\varphi}(\alpha') \subset c_0'\widetilde{Q}_{ij}$, where $c_0' = c_0'(C_0) \geqslant 1$, and we conclude (3.5) for our $\widetilde{Q}_{ij}$.

We are left with the case where $\widetilde{Q}_{ij}$ only intersects the image of the circular part of α. By Lemma 3.4, there are only uniformly finitely many such $\widetilde{Q}_{ij}$, so there exists a constant c_0'' such that

$$\ell(\widetilde{Q}') \leqslant c_0''\ell(\widetilde{Q}_{ij}) \quad \text{and} \quad \mathrm{dist}(\widetilde{Q}_{ij}, \widetilde{Q}') \leqslant c_0''\ell(\widetilde{Q}_{ij})$$

for each such $\widetilde{Q}_{ij}$ and some $\widetilde{Q}'$ from our first case. By setting $c_0 = c_0'c_0''$, we obtain (3.5) also in this case.

Next, Lemma 3.5 allows us to infer that

$$2c_0\widetilde{Q}_{ij} \subset B(z_1, 2c_0C|z_1 - z_2|) \tag{3.6}$$

for each $\widetilde{Q}_{ij} \in \widetilde{W}_\gamma$. Here, $C = C(J) = C(p, \|E\|)$.

Let Φ be defined as in Lemma 3.3 for the choice $c_1 = 2c_0C$, where c_0C is from (3.6). Let $s = \frac{1+p}{2}$. Then $1 < s < p$.

Since Ω is a $W^{1,p}$-extension domain, we have $E\Phi \in W^{1,p}(\mathbb{R}^2)$, where E is the corresponding extension operator. Therefore, denoting the Hardy-Littlewood maximal operator by M and using the boundedness of $M\colon L^{p/s}(\mathbb{R}^2) \to L^{p/s}(\mathbb{R}^2)$ applied to the function $|\nabla E\Phi|^s$, we obtain

$$\begin{aligned}
&\sum_i \sum_{j=1}^{n_i} |\widetilde{Q}_{ij}|^{1-\frac{p}{s}} \bigg(\int_{2c_0\widetilde{Q}_{ij}} |\nabla E\Phi(x)|^s dx \bigg)^{\frac{p}{s}} \\
&\quad\leqslant C(c_0, p) \sum_i \sum_{j=1}^{n_i} |\widetilde{Q}_{ij}| \bigg(\fint_{2c_0\widetilde{Q}_{ij}} |\nabla E\Phi(x)|^s dx \bigg)^{\frac{p}{s}} \\
&\quad\leqslant C(c_0, p) \sum_i \sum_{j=1}^{n_i} \int_{\widetilde{Q}_{ij}} |M(|\nabla E\Phi|^s)(x)|^{\frac{p}{s}} dx \\
&\quad\leqslant C(c_0, p) \int_{\widetilde{\Omega}} |M(|\nabla E\Phi|^s)(x)|^{\frac{p}{s}} dx \\
&\quad\leqslant C(c_0, p) \int_{\mathbb{R}^2} |\nabla E\Phi(x)|^p dx \\
&\quad\leqslant C(c_0, \|E\|, p) \int_{\Omega} |\nabla \Phi(x)|^p dx \leqslant C(c_0, c_1, \|E\|, p)|z_1 - z_2|^{2-p}.
\end{aligned} \tag{3.7}$$

Notice that for any $\widetilde{Q}_{ij} \in \widetilde{W}_\gamma$,

$$\mathrm{diam}(\gamma_1) \sim_{c_0} \ell(\widetilde{Q}_{ij}) \sim_{c_0} \mathrm{diam}(\gamma_2)$$

for subcurves $\gamma_1 \subset 2c_0\widetilde{Q}_{ij}$ of P_1 and $\gamma_2 \subset 2c_0\widetilde{Q}_{ij}$ of P_2 by (3.5), (3.6) and the definition of c_1. Then by Lemma 3.2 (with $p = s$ there) applied to a representative of $E\Phi$ that is absolutely continuous on almost every line segment parallel to the coordinate axes, relying on the values of Φ on P_1 and P_2 from Lemma 3.3, we have

$$\ell(2c_0\widetilde{Q}_{i,j})^{2\frac{p}{s}-p} \leqslant C(c_0,p)\left(\int_{2c_0\widetilde{Q}_{ij}} |\nabla E\Phi(x)|^s dx\right)^{\frac{p}{s}},$$

which, by summing over all the squares $\widetilde{Q}_{ij}$, gives the estimate

$$C(c_0,p)\sum_i\sum_{j=1}^{n_i} |2c_0\widetilde{Q}_{ij}|^{1-\frac{p}{s}}\left(\int_{2c_0\widetilde{Q}_{ij}} |\nabla E\Phi(x)|^s dx\right)^{\frac{p}{s}} \geqslant \sum_i n_i 2^{i(2-p)}.$$

Therefore, (3.7) yields the asserted inequality. □

Proof of Theorem 3.1. We establish the result via a case study.

Case 1. $z_1, z_2 \in \partial\Omega$. Let γ be the curve constructed in (3.3) for the pair z_1, z_2. Then $\widetilde{\varphi}^{-1}(\gamma) = \alpha$. Since each $\widetilde{Q}_{ij} \in \widetilde{W}_\gamma$ is a Whitney square, its diameter is comparable to $\operatorname{dist}(\widetilde{Q}_{ij}, \partial\Omega)$, which means for the points $w \in \gamma \cap \widetilde{Q}_{ij}$ that

$$\operatorname{dist}(w, \partial\Omega) \sim \operatorname{diam}(\widetilde{Q}_{ij}) \tag{3.8}$$

with absolute constants.

Since α consists of two line segments and a circular arc, we have

$$\mathcal{H}^1(\varphi^{-1}(\widetilde{Q}_{ij}) \cap \alpha) \leqslant (2+\pi)\operatorname{diam}(\varphi^{-1}(\widetilde{Q}_{ij})),$$

and by Lemma 2.12, the set $\widetilde{\varphi}^{-1}(\widetilde{Q}_{ij})$ is of λ-Whitney type with an absolute constant λ. Thus, by Lemma 2.13, we get

$$\mathcal{H}^1(\widetilde{Q}_{ij} \cap \gamma) \leqslant C\ell(\widetilde{Q}_{ij}) \tag{3.9}$$

for some absolute constant C.

By combining the claim of Lemma 3.6 with (3.8) and (3.9), we arrive at

$$\begin{aligned}\int_\gamma \operatorname{dist}(z,\partial\Omega)^{1-p} ds &\leqslant \sum_{\widetilde{Q}_{ij}\in\widetilde{W}_\gamma} \int_{\gamma\cap\widetilde{Q}_{ij}} \operatorname{dist}(z,\partial\Omega)^{1-p} ds \\ &\leqslant C(p)\sum_{\widetilde{Q}_{ij}\in\widetilde{W}_\gamma} \operatorname{dist}(\widetilde{Q}_{ij},\partial\Omega)^{2-p} \leqslant C(\|E\|,p)|z_1-z_2|^{2-p}.\end{aligned}$$

Hence, we have proven the existence of the desired curve when $z_1, z_2 \in \partial\Omega$.

Case 2. $z_1, z_2 \in \widetilde{\Omega} \cup \partial\Omega$ and at least one of the points belongs to $\widetilde{\Omega}$. By swapping z_1 and z_2, if needed, we may assume that $z_2 \in \widetilde{\Omega}$ and $\operatorname{dist}(z_1,\partial\Omega) \leqslant \operatorname{dist}(z_2,\partial\Omega)$.

Suppose first that

$$|z_1 - z_2| \leqslant \operatorname{dist}(z_2, \partial\Omega). \tag{3.10}$$

Then we may choose γ to be the line segment $[z_1, z_2]$ between z_1 and z_2, and the curve condition (3.1) is satisfied as $1 < p < 2$:

$$\int_{[z_1,z_2]} \operatorname{dist}(z,\partial\Omega)^{1-p} ds(z) \leqslant \int_{[z_1,z_2]} \operatorname{dist}(z, \partial B(z_2, \operatorname{dist}(z_2,\partial\Omega)))^{1-p} ds \leqslant C(p)|z_1-z_2|^{2-p}. \tag{3.11}$$

Assume now that (3.10) fails. Choose $z_3, z_4 \in \partial\Omega$ so that

$$|z_i - z_{i+2}| = \operatorname{dist}(z_i, \partial\Omega)$$

for $i = 1, 2$. Then

$$|z_1 - z_3| + |z_2 - z_4| < 2|z_1 - z_2|$$

and

$$|z_3 - z_4| \leqslant 3|z_1 - z_2|. \tag{3.12}$$

Let γ' be the curve connecting z_3 and z_4 obtained from Case 1, and define $\gamma = [z_1, z_3] * \gamma' * [z_4, z_2]$. Then, by (3.11) for $[z_1, z_3]$ and $[z_4, z_2]$, Case 1 and (3.12), we get

$$\begin{aligned}\int_\gamma \mathrm{dist}(z, \partial\Omega)^{1-p} ds(z) &\leqslant C(\|E\|, p)(|z_1 - z_3|^{2-p} + |z_3 - z_4|^{2-p} + |z_4 - z_2|^{2-p}) \\ &\leqslant C(\|E\|, p)|z_1 - z_2|^{2-p},\end{aligned}$$

concluding the proof also in this case. □

Remark 3.7. Let $z_1, z_2 \in \widetilde{\Omega}$. Even though the curve joining z_1 and z_2 which we constructed in the proof above may touch the boundary $\partial\Omega$, it can be modified so as to be contained in $\widetilde{\Omega}$.

To begin, we may again assume that

$$\mathrm{dist}(z_1, \partial\Omega) \leqslant \mathrm{dist}(z_2, \partial\Omega)$$

and (3.10) fails. Consider the points $z_3, z_4 \in \partial\Omega$ from the proof above and let $w_i = \widetilde{\varphi}^{-1}(z_i)$ for $i = 3, 4$. Since $\widetilde{\varphi}$ is continuous up to the boundary and (3.12) holds, we find $\epsilon > 0$ so that

$$|\widetilde{\varphi}(w) - \widetilde{\varphi}(w')| < 4|z_1 - z_2|$$

whenever $w, w' \in \partial\mathbb{D}$ satisfy $|w - w_3| + |w' - w_4| < \epsilon$. Recall that the curve γ in the above proof in Case 1 is the image of the curve α that consists of two radial segments and a circular arc (see Figure 4). Suppose that $w_3 \neq w_4$. Then we may choose w and w' as above so that the corresponding curve α between w and w' intersects the preimages of the line segments between z_1 and z_3 and between z_2 and z_4. This allows us to reroute our curve so that it does not intersect the boundary. The case of $w_3 = w_4$ is similar; choose w and w' from "different sides" of w_3.

Remark 3.8. The inequality in Lemma 3.6 is actually equivalent to (3.1) for our γ. One of the directions was shown above. For the other direction, we first note that each Whitney square has at most 20 neighboring squares, which tells us that we can distribute the squares in $\widetilde{W}_\gamma$ into no more than 21 subcollections $\{\widetilde{W}_k\}_{k=1}^{21}$ so that each of them consists of pairwise disjoint squares. Then for any two distinct $\widetilde{Q}_i, \widetilde{Q}_j \in \widetilde{W}_k$, by Lemma 2.9, we have

$$\frac{11}{10}\widetilde{Q}_i \cap \frac{11}{10}\widetilde{Q}_j = \emptyset.$$

Notice that for each $\widetilde{Q}_{ij} \in \widetilde{W}_\gamma$, by definition, we have

$$\mathcal{H}^1\left(\frac{11}{10}\widetilde{Q}_{ij} \cap \gamma\right) \geqslant \frac{1}{10}\ell(\widetilde{Q}_{ij}).$$

Thus, by applying the estimate

$$\ell(\widetilde{Q}_{ij}) \leqslant \mathrm{dist}(\widetilde{Q}_{ij}, \partial\Omega) \leqslant 4\sqrt{2}\ell(\widetilde{Q}_{ij}),$$

we have

$$\begin{aligned}\sum_{\widetilde{Q}_{ij} \in \widetilde{W}_\gamma} \mathrm{dist}(\widetilde{Q}_{ij}, \partial\Omega)^{2-p} &\leqslant C(p) \sum_{k=1}^{21} \sum_{\widetilde{Q}_{ij} \in \widetilde{W}_k} \int_{\gamma \cap \widetilde{Q}_{ij}} \mathrm{dist}(z, \partial\Omega)^{1-p} ds \\ &\leqslant C(p) \int_\gamma \mathrm{dist}(z, \partial\Omega)^{1-p} ds \leqslant C(\|E\|, p)|z_1 - z_2|^{2-p},\end{aligned}$$

which gives the other direction.

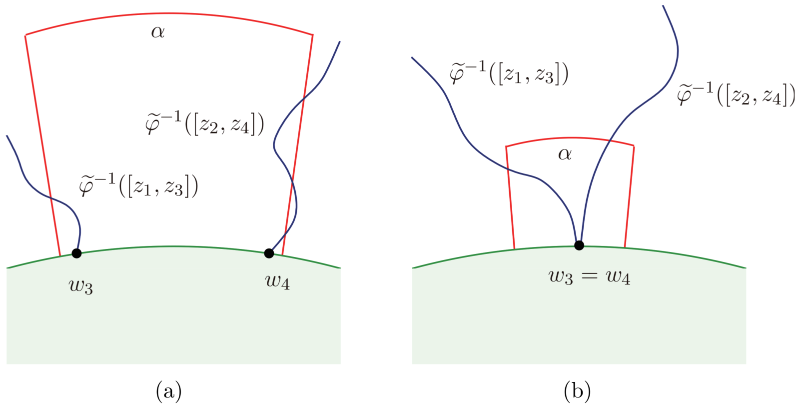

Figure 4 (Color online) The curve constructed in Theorem 3.1 can be modified so as to travel inside $\widetilde{\Omega}$ by perturbing slightly the starting point and the endpoint of the intermediate curve $\widetilde{\varphi}(\alpha)$ and by disregarding the unnecessary parts of the concatenated curves. In (a), we have the case where the selected points z_3 and z_4 differ, and in (b), we have the case where they agree

3.2 Inner extension

We prove the following inner extension theorem in this subsection.

Theorem 3.9. *Let $\varphi\colon \mathbb{D} \to \Omega$ be a conformal map, where $\Omega \subset \mathbb{R}^2$ is a simply connected John domain with John constant J. Suppose that $\varphi(0)$ is the distinguished point in the definition of a John domain. Set $\Omega_\epsilon = \varphi(B(0, 1-\epsilon))$ for $0 < \epsilon \leqslant \frac{1}{2}$ and let $1 < p < \infty$. Then there exists an extension operator $E_\epsilon\colon W^{1,p}(\Omega_\epsilon) \to W^{1,p}(\Omega)$ such that $\|E_\epsilon\| \leqslant C(p, J)$.*

Fix ϵ, and notice that Ω_ϵ is a Jordan domain. Let $\Omega'_\epsilon = \mathbb{R}^2 \setminus \overline{\Omega}_\epsilon$ and $\widetilde{\Omega}_\epsilon = \Omega'_\epsilon \cap \Omega$. Let φ be a conformal map as in Theorem 3.9, with $\varphi(0)$ a John center of Ω. This map will be fixed through this subsection. By Lemma 2.30, φ is η-quasisymmetric with respect to the inner distance, where η depends only on J. Moreover, by Remark 2.26, we may extend φ continuously to the boundary $\partial\mathbb{D}$; we denote the extended map still by φ.

We are going to modify the method of Jones from [23] to prove Theorem 3.9. We construct a suitable cover for $\widetilde{\Omega}_\epsilon$ inside Ω and an associated partition of unity. Towards this, recall that Ω is John and by Lemma 2.30, so is Ω_ϵ, with a constant depending only on J. Thus, by Lemma 2.28, we have that Ω_ϵ is inner uniform and we may use hyperbolic segments of Ω_ϵ as curves referred to in the definition of inner uniformity, with constant ϵ_0 depending only on J.

Fix $k_0 \in \mathbb{N}$ with $2^{-k_0-1} < \epsilon \leqslant 2^{-k_0}$. We begin by constructing a decomposition of the preimage $A = \mathbb{D} \setminus \overline{B(0, 1-\epsilon)}$, of $\widetilde{\Omega}_\epsilon$ under φ, and then obtain a decomposition of $\widetilde{\Omega}_\epsilon$ with the help of the map φ (see Figure 5).

For $k \in \mathbb{N}$, let

$$A_k = B(0, 1-\epsilon+2^{-k}\epsilon) \setminus B(0, 1-\epsilon+2^{-k-1}\epsilon).$$

For each $k \geqslant 0$, the collection of the 2^{k+k_0} radial rays is obtained by dividing the polar angle 2π evenly and starting with the zero angle subdivides A_k into closed (with respect to $\mathbb{D}$) sets. Run this process for all $k \in \mathbb{N}$. We refer to these closed sets by $\widetilde{Q}_i$. They satisfy the version

$$\frac{1}{\lambda}\mathrm{diam}(\widetilde{Q}_i) \leqslant \mathrm{dist}(\widetilde{Q}_i, \partial(B(0, 1-\epsilon))) \leqslant \lambda \mathrm{diam}(\widetilde{Q}_i) \tag{3.13}$$

of Definition 2.10(i) with $\lambda = 16\pi$.

Set $\widetilde{S}_i = \varphi(\widetilde{Q}_i)$ and let $\widetilde{W} = \{\widetilde{S}_i\}$. We claim that each $\widetilde{S}_i$ is a λ-Whitney-type set with respect to the inner distance of Ω and $\partial\Omega_\epsilon$ in the following sense.

Lemma 3.10. *There exists a constant $0 < c = c(J) < 1$ such that*

$$B_\Omega(w_i, c\,\mathrm{diam}_\Omega(\widetilde{S}_i)) \subset \widetilde{S}_i \tag{3.14}$$

for some $w_i \in \widetilde{S}_i$,

$$c\mathrm{diam}_\Omega(\widetilde{S}_i) \leqslant \mathrm{dist}_\Omega(\widetilde{S}_i, \partial\Omega_\epsilon) \leqslant \frac{1}{c}\mathrm{diam}_\Omega(\widetilde{S}_i) \tag{3.15}$$

and

$$c\mathrm{diam}_\Omega(\widetilde{S}_i) \leqslant \mathrm{diam}_\Omega(\widetilde{S}_j) \leqslant \frac{1}{c}\mathrm{diam}_\Omega(\widetilde{S}_i) \tag{3.16}$$

whenever $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$. *Here,* $B_\Omega(x, r)$ *denotes the open disk centered at* x *with radius* r *with respect to the inner distance.*

Proof. Let $\widetilde{S}_i$ be fixed. By the construction of $\widetilde{Q}_i = \varphi^{-1}(\widetilde{S}_i)$, there is a disk $B(z_0, c_0\mathrm{diam}(\widetilde{Q}_i))$ contained in $\widetilde{Q}_i$ for some absolute constant $c_0 \leqslant 1$. Let z_1 be an arbitrary point on the boundary of $B(z_0, c_0\mathrm{diam}(\widetilde{Q}_i))$ and let $z_2 \in \widetilde{Q}_i$ be such that

$$\mathrm{dist}_\Omega(\varphi(z_2), \varphi(z_0)) \geqslant \frac{1}{3}\mathrm{diam}_\Omega(\widetilde{S}_i); \tag{3.17}$$

the existence of such a point follows from the triangle inequality. Then

$$|z_2 - z_1| \leqslant c_0^{-1}|z_1 - z_0|$$

and hence (3.17) together with quasisymmetry gives

$$\mathrm{diam}_\Omega(\widetilde{S}_i) \leqslant 3\mathrm{dist}_\Omega(\varphi(z_2), \varphi(z_0)) \leqslant 3\eta(c_0^{-1})\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_0)). \tag{3.18}$$

By the arbitrariness of z_1 and the fact that φ is a homeomorphism, we conclude (3.14) for a constant $c = c(\eta) = c(J)$.

Towards (3.15), first choose points $z_3 \in \partial B(0, 1-\epsilon)$ and $z_4 \in \widetilde{Q}_i$ such that

$$\mathrm{dist}_\Omega(\varphi(z_4), \varphi(z_3)) \leqslant 2\mathrm{dist}_\Omega(\widetilde{S}_i, \partial\Omega_\epsilon). \tag{3.19}$$

Let $z \in \widetilde{Q}_i$ be such that

$$\mathrm{diam}(\widetilde{Q}_i) \leqslant 2|z_4 - z|. \tag{3.20}$$

By (3.13),

$$|z_4 - z| \leqslant \mathrm{diam}(\widetilde{Q}_i) \sim \mathrm{dist}(\widetilde{Q}_i, \partial B(0, 1-\epsilon)) \lesssim |z_4 - z_3| \tag{3.21}$$

with absolute constants. Now (3.21), quasisymmetry of φ and (3.19) give

$$\mathrm{dist}_\Omega(\varphi(z_4), \varphi(z)) \leqslant C(\eta)\mathrm{dist}_\Omega(\varphi(z_3), \varphi(z_4)) \leqslant C(\eta)\mathrm{dist}_\Omega(\widetilde{S}_i, \partial\Omega_\epsilon). \tag{3.22}$$

Let z_0 be as in the first paragraph of the proof. By the triangle inequality, $|z - z_0| \geqslant \frac{1}{4}\mathrm{diam}(\widetilde{Q}_i)$ or $|z_4 - z_0| \geqslant \frac{1}{4}\mathrm{diam}(\widetilde{Q}_i)$. Assume that the latter inequality holds; the other case is handled analogously. Clearly, $|z_4 - z_0| \leqslant \mathrm{diam}(\widetilde{Q}_i) \leqslant 2|z_4 - z|$ by (3.20). Hence, quasisymmetry gives

$$\mathrm{dist}_\Omega(\varphi(z_4), \varphi(z_0)) \leqslant \eta(2)\mathrm{dist}_\Omega(\varphi(z_4), \varphi(z)). \tag{3.23}$$

Let us argue as in the first paragraph of the proof with the help of the point $z_2 \in \widetilde{Q}_i$. Our assumption that $|z_4 - z_0| \geqslant \frac{1}{4}\mathrm{diam}(\widetilde{Q}_i) \geqslant \frac{1}{4}|z_2 - z_0|$ together with quasisymmetry further gives

$$\mathrm{diam}_\Omega(\widetilde{S}_i) \leqslant 3\mathrm{dist}_\Omega(\varphi(z_2), \varphi(z_0)) \leqslant 3\eta(4)\mathrm{dist}_\Omega(\varphi(z_4), \varphi(z_0)). \tag{3.24}$$

We obtain the lower bound of the distance in (3.15) by combining (3.22)–(3.24).

Towards the upper bound in (3.15), pick points $z_5 \in \partial B(0, 1-\epsilon)$ and $z_6 \in \widetilde{Q}_i$ such that

$$|z_5 - z_6| = \mathrm{dist}(\widetilde{Q}_i, \partial B(0, 1-\epsilon)).$$

Let z_0 and c_0 be as in the first paragraph of our proof. Then (3.13) gives

$$|z_5 - z_6| \leqslant c_0^{-1}\lambda|z_0 - z_6|,$$

and by quasisymmetry,

$$\mathrm{dist}_\Omega(\widetilde{S}_i, \partial\Omega_\epsilon) \leqslant \mathrm{dist}_\Omega(\varphi(z_5), \varphi(z_6)) \leqslant \eta(c_0^{-1}\lambda)\mathrm{dist}_\Omega(\varphi(z_6), \varphi(z_0)) \leqslant \eta(c_0^{-1}\lambda)\mathrm{diam}_\Omega(\widetilde{S}_i),$$

as desired.

We are left to prove (3.16). Since $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$, both $\mathrm{dist}_\Omega(\widetilde{S}_j, \partial\Omega_\epsilon) \leqslant \mathrm{dist}_\Omega(\widetilde{S}_i, \partial\Omega_\epsilon) + \mathrm{diam}_\Omega(\widetilde{S}_i)$ and the analogous inequality with the roles of i and j reversed hold. Hence, (3.16) follows from (3.15). □

Given $\widetilde{S}_i \in \widetilde{W}$ and $M > 1$ that will be selected soon, define

$$\widetilde{U}_i := \left\{ x \in \Omega \,\middle|\, \mathrm{dist}_\Omega(x, \widetilde{S}_i) < \frac{1}{M}\mathrm{diam}_\Omega(\widetilde{S}_i) \right\}.$$

We claim that we can choose $M > 1$ depending only on J such that these sets $\widetilde{U}_i$ have uniformly finite overlaps. Notice that $\widetilde{U}_i \subset \widetilde{\Omega}$ whenever $M \geqslant 2/c$ for the constant c in (3.15).

Lemma 3.11. *If $\widetilde{S}_i \cap \widetilde{S}_j = \emptyset$, then*

$$\max\{\mathrm{diam}_\Omega(\widetilde{S}_i), \mathrm{diam}_\Omega(\widetilde{S}_j)\} \leqslant C(J)\mathrm{dist}_\Omega(\widetilde{S}_i, \widetilde{S}_j). \tag{3.25}$$

Especially, for $M \geqslant 2C(J)$ in the definition of the sets $\widetilde{U}_i$, we have

$$1 \leqslant \sum_i \chi_{\widetilde{U}_i}(x) \leqslant 9 \tag{3.26}$$

for every $x \in \widetilde{\Omega}_\epsilon$, where $\chi_{\widetilde{U}_i}$ is the characteristic function of $\widetilde{U}_i$.

Proof. First, observe that $\widetilde{Q}_i \cap \widetilde{Q}_j = \emptyset$ gives

$$\mathrm{dist}(\widetilde{Q}_i, \widetilde{Q}_j) \geqslant C \max\{\mathrm{diam}(\widetilde{Q}_i), \mathrm{diam}(\widetilde{Q}_j)\},$$

where the constant is absolute. We apply quasisymmetry to show that $\widetilde{S}_i \cap \widetilde{S}_j = \emptyset$ implies

$$\mathrm{dist}_\Omega(\widetilde{S}_i, \widetilde{S}_j) \gtrsim \max\{\mathrm{diam}_\Omega(\widetilde{S}_i), \mathrm{diam}_\Omega(\widetilde{S}_j)\},$$

where the constant depends only on the John constant (see also [29, Formula (3.5)] for a version of this). Towards this, choose $w_1 \in \widetilde{S}_i$ and $w_2 \in \widetilde{S}_j$ such that

$$\mathrm{dist}_\Omega(w_1, w_2) \leqslant 2\mathrm{dist}_\Omega(\widetilde{S}_i, \widetilde{S}_j), \tag{3.27}$$

and let $w_3 \in \widetilde{S}_i$ be an arbitrary point. Then since

$$|\varphi^{-1}(w_1) - \varphi^{-1}(w_2)| \geqslant \mathrm{dist}(\widetilde{Q}_i, \widetilde{Q}_j) \gtrsim \mathrm{diam}(\widetilde{Q}_i) \geqslant |\varphi^{-1}(w_1) - \varphi^{-1}(w_3)|$$

with an absolute constant, the quasisymmetry of φ applied to $\varphi^{-1}(w_1), \varphi^{-1}(w_2)$ and $\varphi^{-1}(w_3)$ gives

$$\mathrm{dist}_\Omega(w_1, w_2) \gtrsim \mathrm{dist}_\Omega(w_1, w_3).$$

Thus, by the arbitrariness of w_3, (3.27) shows that

$$\mathrm{dist}_\Omega(\widetilde{S}_i, \widetilde{S}_j) \gtrsim \mathrm{diam}_\Omega(\widetilde{S}_i).$$

By replacing $w_3 \in \widetilde{S}_i$ with $w_3 \in \widetilde{S}_j$ and $\mathrm{diam}(\widetilde{Q}_i)$ with $\mathrm{diam}(\widetilde{Q}_j)$ above, respectively, we analogously obtain the inequality

$$\mathrm{dist}_\Omega(\widetilde{S}_i, \widetilde{S}_j) \gtrsim \mathrm{diam}_\Omega(\widetilde{S}_j),$$

and (3.25) follows.

Regarding (3.26), the lower bound is trivial since $\widetilde{W}$ forms a cover of $\widetilde{\Omega}_\epsilon$. Since each $\widetilde{S}_i$ has at most 8 neighboring sets, we obtain the upper bound in (3.26) from (3.25). □

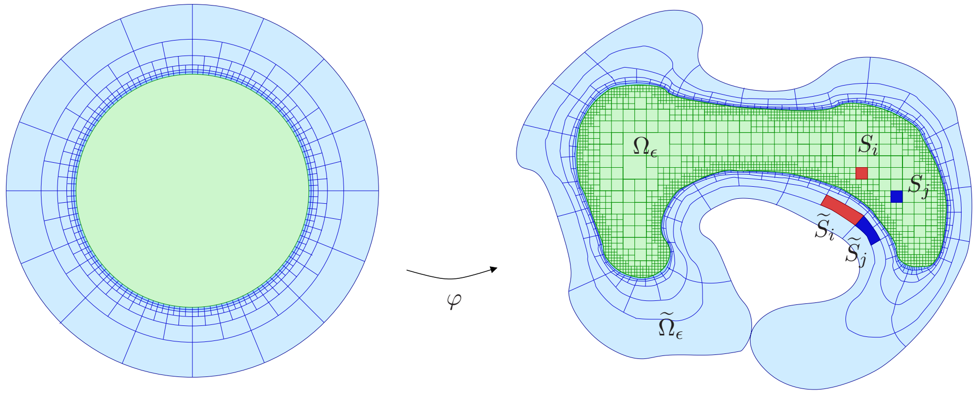

Figure 5 (Color online) In the inner extension, the annular region $\widetilde{\Omega}_\epsilon$ is divided into Whitney-type sets that are obtained by mapping a Whitney-type decomposition of the annulus inside the disk conformally. For the inner part Ω_ϵ, we use a standard Whitney decomposition. Two pairs of sets $(\widetilde{S}_i, S_i)$ and $(\widetilde{S}_j, S_j)$ are highlighted

We now fix $M = \max\{2C(J), 2/c\}$, where the constant $C(J)$ is from (3.25) and c is from (3.15). Given $\widetilde{S}_i \in \widetilde{W}$, set

$$\psi_i(x) = \max\{1 - 2M\operatorname{diam}_\Omega(\widetilde{S}_i)^{-1}\operatorname{dist}_\Omega(x, \widetilde{S}_i), 0\}$$

for $x \in \Omega$. Then ψ_i is locally Lipschitz with bounded and relatively closed support in Ω, $|\nabla\psi_i| \leqslant C(J)\operatorname{diam}_\Omega(\widetilde{S}_i)^{-1}$ and $\psi_i(x) = 1$ for any $x \in \widetilde{S}_i$. Moreover, the support of ψ_i is contained in $\widetilde{U}_i$. Define

$$\phi_j(x) = \frac{\psi_j(x)}{\sum_i \psi_i(x)}$$

for $x \in \widetilde{\Omega}_\epsilon$. Then our collection of the functions ϕ_j is a partition of unity in $\widetilde{\Omega}_\epsilon : \sum \phi_j(x) = 1$ in $\widetilde{\Omega}_\epsilon$. By (3.26), the functions ϕ_j are locally Lipschitz and have supports in U_j, and

$$|\nabla\phi_j| \leqslant C(J)\operatorname{diam}_\Omega(\widetilde{S}_j)^{-1}. \tag{3.28}$$

In order to construct our extension operator, we associate each $\widetilde{S}_i \in \widetilde{W}$ with a suitable square $Q \in W$, where $W = \{Q_1, Q_2, \dots\}$ is a fixed Whitney decomposition of Ω_ϵ (see Figure 5).

Lemma 3.12. *Given $\widetilde{S}_i \in \widetilde{W}$, there is $Q \in W$ such that*

$$\operatorname{diam}(Q) = \operatorname{diam}_\Omega(Q) \sim_J \operatorname{dist}_\Omega(\widetilde{S}_i, Q) \sim_J \operatorname{diam}_\Omega(\widetilde{S}_i). \tag{3.29}$$

Proof. To see that a Whitney square of the desired size can be chosen, trace back towards $\varphi(0)$ along any hyperbolic ray of Ω that intersects $\widetilde{S}_i$ and let Q be a first Whitney square of Ω_ϵ intersecting this hyperbolic ray such that

$$\operatorname{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i)) \geqslant \frac{1}{9\lambda}\operatorname{diam}(\varphi^{-1}(\widetilde{S}_i)), \tag{3.30}$$

where λ is an absolute constant given by Lemma 2.12 so that $\varphi^{-1}(Q)$ is of λ-Whitney type with respect to $B(0, 1-\epsilon)$. We show the existence of such a square via Definition 2.10 and the assumption that $0 < \epsilon \leqslant \frac{1}{2}$. Towards this, if such a square does not exist, then

$$\frac{\operatorname{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i))}{\operatorname{diam}(\varphi^{-1}(\widetilde{S}_i))} < \frac{1}{9\lambda}$$

for all the Whitney squares Q intersecting our fixed hyperbolic ray. However, the diameter of $\varphi^{-1}(\widetilde{S}_i) = \widetilde{Q}_i$ is at most 2, while a λ-Whitney-type set in $B(0, 1-\epsilon)$ containing the origin has the distance to $\partial B(0, 1-\epsilon)$ at least $\frac{1}{4\lambda}$ since $\epsilon \leqslant \frac{1}{2}$ and $\lambda \geqslant 1$. Therefore, we have

$$\frac{1}{8\lambda} \leqslant \frac{\operatorname{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i))}{\operatorname{diam}(\varphi^{-1}(\widetilde{S}_i))} < \frac{1}{9\lambda},$$

which leads to a contradiction. Then the facts that Q is a first square satisfying (3.30), $\varphi^{-1}(Q)$ is of λ-Whitney type, $\varphi^{-1}(\widetilde{S}_i) = \widetilde{Q}_i$ satisfies (3.13) and $\mathrm{dist}_\Omega(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i))$ is comparable to the length of the segment of our hyperbolic ray between $\varphi^{-1}(Q)$ and $\varphi^{-1}(\widetilde{S}_i)$ allow us to deduce that

$$\mathrm{diam}(\varphi^{-1}(\widetilde{S}_i)) \sim \mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i)) \sim \mathrm{diam}(\varphi^{-1}(S)). \tag{3.31}$$

Next, we apply the quasisymmetry of φ with respect to the inner distance to establish (3.29). First of all, choose $x_1 \in \widetilde{S}_i$ and $x_2 \in Q$ such that

$$\mathrm{dist}_\Omega(Q, \widetilde{S}_i) \leqslant \mathrm{dist}_\Omega(x_1, x_2) \leqslant 2\mathrm{dist}_\Omega(Q, \widetilde{S}_i), \tag{3.32}$$

and let $x_3 \in \widetilde{S}_i$ be an arbitrary point. Since $x_1, x_3 \in \widetilde{S}_i$ and $x_2 \in Q$, (3.31) implies that

$$|\varphi^{-1}(x_1) - \varphi^{-1}(x_2)| \geqslant \mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i)) \geqslant C^{-1}|\varphi^{-1}(x_1) - \varphi^{-1}(x_3)|$$

with an absolute constant C, and hence the quasisymmetry of φ gives

$$\mathrm{dist}_\Omega(x_1, x_3) \leqslant C(J)\mathrm{dist}_\Omega(x_1, x_2).$$

Thus, (3.32) gives

$$\mathrm{diam}_\Omega(\widetilde{S}_i) \leqslant C(J)\mathrm{dist}_\Omega(Q, \widetilde{S}_i) \tag{3.33}$$

according to the arbitrariness of x_3. For the other direction, choose $x_4 \in \varphi^{-1}(\widetilde{S}_i)$, and $x_5 \in \varphi^{-1}(Q)$ such that

$$\mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i)) \leqslant |x_4 - x_5| \leqslant 2\mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i)).$$

Pick $x_6 \in \varphi^{-1}(\widetilde{S}_i)$ such that

$$\mathrm{diam}(\varphi^{-1}(\widetilde{S}_i)) \leqslant 2|x_6 - x_4|;$$

the existence of such a point follows from the triangle inequality. By (3.31),

$$|x_6 - x_4| \geqslant \frac{1}{2}\mathrm{diam}(\varphi^{-1}(\widetilde{S}_i)) \geqslant C^{-1}\mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(\widetilde{S}_i)) \geqslant C^{-1}|x_5 - x_4|.$$

Then the quasisymmetry of φ gives

$$\mathrm{diam}_\Omega(\widetilde{S}_i) \geqslant \mathrm{dist}_\Omega(\varphi(x_6), \varphi(x_4)) \geqslant \eta(C)^{-1}\mathrm{dist}_\Omega(\varphi(x_5), \varphi(x_4)) \geqslant \eta(C)^{-1}\mathrm{dist}_\Omega(Q, \widetilde{S}_i),$$

which together with (3.33) gives the last equivalence in (3.29). The second equivalence follows by an analogous argument by changing the roles of $\widetilde{S}_i$ and Q, and the first equality is obvious. □

We now use Lemma 3.12 to pick for each $\widetilde{S}_i \in \widetilde{W}$ a square $Q \in W$. By Lemma 3.12, the collection of those Q that satisfy (3.29) is a nonempty subcollection of W. Recalling that $W = \{Q_1, Q_2, \dots\}$, we simply pick the one of them that has the smallest index amongst those that belong to our subcollection. For simplicity of notation, we refer to this Q associated with $\widetilde{S}_i$ by S_i.

Then, by (3.29), we know that the inner distance between $\widetilde{S}_i$ and S_i with respect to Ω is no more than a constant times $\mathrm{diam}_\Omega(\widetilde{S}_i)$. By the triangle inequality and (3.16), it follows that

$$\mathrm{dist}_\Omega(S_i, S_j) \lesssim \mathrm{diam}_\Omega(\widetilde{S}_i)$$

if $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$. Given $\widetilde{S}_i$ and $\widetilde{S}_j$ with $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$, we consider the hyperbolic segment Γ in Ω joining the barycenters x_{S_i} and x_{S_j} of S_i and S_j, respectively. From Lemma 2.7 and (3.29), we conclude that the Euclidean length of the hyperbolic segment Γ is no more than constant times $\mathrm{diam}_\Omega(\widetilde{S}_i)$. Since $\Omega_\epsilon = \varphi(B(0, 1-\epsilon))$ and $S_i, S_j \subset \Omega_\epsilon$, it follows that the hyperbolic segment Γ joining x_{S_i} and x_{S_j} is contained in Ω_ϵ. We use Lemma 2.7 a second time to conclude that the Euclidean length of a hyperbolic segment $\Gamma_{i,j}$ joining x_{S_i} and x_{S_j} that is hyperbolic with respect to Ω_ϵ is also bounded from above by a constant times $\mathrm{diam}_\Omega(\widetilde{S}_i)$:

$$\mathrm{le}(\Gamma_{i,j}) \lesssim \mathrm{diam}_\Omega(\widetilde{S}_i). \tag{3.34}$$

Let us define $G(\widetilde{S}_i, \widetilde{S}_j)$ to be the union of all squares in our fixed Whitney decomposition W of Ω_ϵ that intersect this fixed geodesic $\Gamma_{i,j}$.

Next, we show that the inner uniformity of Ω_ϵ allows us to conclude that there are uniformly finitely many Whitney squares in every $G(\widetilde{S}_i, \widetilde{S}_j)$ with $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$. This is a counterpart of [23, Lemma 2.8] with a similar proof.

Lemma 3.13. *Let i, j be such that $\widetilde{S}_i \cap \widetilde{S}_j \neq \emptyset$. Then*

$$\#\{S_k \in W \mid S_k \in G(\widetilde{S}_i, \widetilde{S}_j)\} \leqslant C(J), \tag{3.35}$$

where # denotes the counting measure.

Proof. Since $\mathrm{diam}(S_i) \lesssim \mathrm{diam}_\Omega(\widetilde{S}_i)$ by (3.29) and the curve $\Gamma_{i,j}$ intersects the Whitney square S_i, we conclude by (3.34) that the diameter of each Whitney square of Ω_ϵ that intersects $\Gamma_{i,j}$ is bounded from above by a fixed multiple of $\mathrm{diam}_\Omega(\widetilde{S}_i)$.

On the other hand, by (3.29) with (3.16),

$$\mathrm{diam}_\Omega(S_i) \sim \mathrm{diam}_\Omega(\widetilde{S}_i) \sim \mathrm{diam}_\Omega(\widetilde{S}_j) \sim \mathrm{diam}_\Omega(S_j). \tag{3.36}$$

Hence, the second condition of (2.39) together with (3.36) tells us that

$$\mathrm{dist}(Q, \partial\Omega_\epsilon) \gtrsim \mathrm{diam}_\Omega(\widetilde{S}_i)$$

if $Q \cap \Gamma \neq \emptyset$.

Thus, the diameters of $Q \in W$ with $Q \cap \Gamma \neq \emptyset$ are bounded from below and from above by fixed multiples of $\mathrm{diam}_\Omega(\widetilde{S}_i)$, and hence (3.35) follows as $\mathrm{le}(\Gamma_{i,j}) \lesssim \mathrm{diam}_\Omega(\widetilde{S}_i)$. □

Given $u \in L^1(\Omega_\epsilon)$, set

$$a_i = \fint_{S_i} u(x)dx = \frac{1}{|S_i|}\int_{S_i} u(x)dx,$$

where $S_i \in W$ is the square associated with $\widetilde{S}_i \in \widetilde{W}$. Recall our partition of unity consisting of the functions ϕ_i, see the discussion before (3.28). Define

$$E_\epsilon u(x) = \sum_i a_i \phi_i(x), \quad x \in \widetilde{\Omega}_\epsilon \tag{3.37}$$

for any given function $u \in W^{1,p}(\Omega_\epsilon)$, which is Lipschitz in $\overline{\Omega}_\epsilon$, and set $E_\epsilon u(x) = u(x)$ when $x \in \overline{\Omega}_\epsilon$.

Lemma 3.14. *Let E_ϵ be given by (3.37). Given $\widetilde{S} \in \widetilde{W}$, we have the estimate*

$$\|\nabla(E_\epsilon u(x))\|^p_{L^p(\widetilde{S})} \lesssim \sum_{\widetilde{S}_k \cap \widetilde{S} \neq \emptyset} \int_{G(\widetilde{S}, \widetilde{S}_k)} |\nabla u(x)|^p dx \tag{3.38}$$

with a constant that depends only on p and J.

Proof. Fix $\widetilde{S} \in \widetilde{W}$ and set $a = \fint_S u(x)dx$. Then

$$\nabla E_\epsilon u(x) = \nabla(E_\epsilon u(x) - a) = \nabla\bigg(\sum_i \phi_i(x)(a_i - a)\bigg)$$

in $\widetilde{S}$. By (3.35), $G(\widetilde{S}, \widetilde{S}_k)$ consists of no more than $C(J)$ squares and the side lengths of all of them are comparable to $\mathrm{diam}_\Omega(S)$ modulo a multiplicative constant that depends only on J. Hence, (3.28), (3.29) and the Poincaré inequality (see, e.g., [23, Lemma 3.1] for the use of the Poincaré inequality) applied to the chain $G(\widetilde{S}, \widetilde{S}_k)$ of squares give

$$\|\nabla(E_\epsilon u(x))\|^p_{L^p(\widetilde{S})} \lesssim \int_{\widetilde{S}} \sum_{\widetilde{S}_k \cap \widetilde{S} \neq \emptyset} |a_k - a|^p |\nabla\phi_k(x)|^p dx$$

$$
\begin{aligned}
&\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset} |a_k-a|^p(\mathrm{diam}_\Omega(S))^{2-p}\\
&\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset} (\mathrm{diam}_\Omega(S))^{2-p}(\mathrm{diam}_\Omega(S))^{p-2}\int_{G(\widetilde{S},\widetilde{S}_k)}|\nabla u(x)|^p dx\\
&\lesssim \sum_{\widetilde{S}_k\cap\widetilde{S}\neq\emptyset}\int_{G(\widetilde{S},\widetilde{S}_k)}|\nabla u(x)|^p dx. \qquad (3.39)
\end{aligned}
$$

This completes the proof. □

We are now ready to prove our norm estimate.

Lemma 3.15. *Let E_ϵ be given by* (3.37)*. We have*

$$
\|E_\epsilon u\|^p_{L^p(\widetilde{\Omega}_\epsilon)}+\|\nabla(E_\epsilon u)\|^p_{L^p(\widetilde{\Omega}_\epsilon)}\lesssim \|u\|^p_{L^p(\Omega_\epsilon)}+\|\nabla u\|^p_{L^p(\Omega_\epsilon)}
$$

with a constant depending only on p and J.

Proof. We begin by estimating the overlaps of $G(\widetilde{S}_k,\widetilde{S}_i)$. Towards this, for a fixed S_i, we first bound the number of distinct $\widetilde{S}$ for which S_i is associated with $\widetilde{S}$.

To begin with, (3.29) implies that for a fixed $S_i\in W$ and for every $\widetilde{S}\in\widetilde{W}$ associated with it, we have

$$
\mathrm{dist}_\Omega(\widetilde{S},S_i)\lesssim \mathrm{diam}_\Omega(S_i). \qquad (3.40)
$$

We claim that there are no more than $N(J)$ sets $\widetilde{S}\in\widetilde{W}$ associated with a fixed $S_i\in W$. Towards this, first note that for any $x\in\Omega$ and $0<r<\mathrm{diam}(\Omega)$, the hyperbolic segment Γ of Ω joining x to a point $y\in\partial B_\Omega(x,r)$ satisfies

$$
r=\mathrm{dist}_\Omega(x,y)\leqslant \mathrm{le}(B_\Omega(x,r)\cap\Gamma).
$$

Let $z\in\Gamma$ be such that $\mathrm{le}(\Gamma[x,z])=\frac{r}{2}$. Then since, by Lemma 2.28, hyperbolic segments of Ω satisfy (2.39) with a constant $0<\epsilon_0=\epsilon_0(J)<1$, we have

$$
B\left(z,\frac{1}{2}\epsilon_0 r\right)\subset B_\Omega(x,r).
$$

Thus,

$$
C(J)r^2\leqslant |B_\Omega(x,r)|\leqslant \pi r^2, \qquad (3.41)
$$

where the upper bound comes from

$$
B_\Omega(x,r)\subset B(x,r).
$$

By (3.14), each $\widetilde{S}$ associated with a fixed S_i contains a Euclidean disk of radius comparable to $\mathrm{diam}_\Omega(\widetilde{S})$, which in turn, by (3.29), is comparable to $\mathrm{dist}_\Omega(\widetilde{S},S_i)$ and to $\mathrm{diam}_\Omega(S_i)$. Hence, the number of such $\widetilde{S}$ is no more than a constant $N(J)$.

Since $\widetilde{S}_i$ has no more than 8 neighbors and the number of the sets $\widetilde{S}$ associated with any $S\in W$ is no more than $N(J)$, by (3.29), (3.41) and (3.35), we conclude that

$$
\sum_{\widetilde{S}_i\in\widetilde{W}}\sum_{\widetilde{S}_i\cap\widetilde{S}_k\neq\emptyset}\chi_{G(\widetilde{S}_k,\widetilde{S}_i)}(x)\lesssim 1 \qquad (3.42)
$$

for all x; notice that (3.35) with (3.29) and (3.41) implies that each Whitney square is contained in at most uniformly finitely many chains. Inequality (3.42) is the counterpart of [23, (3.2), p. 80].

Now Lemma 3.14 together with (3.42) gives

$$
\begin{aligned}
\|\nabla(E_\epsilon u)\|^p_{L^p(\widetilde{\Omega}_\epsilon)}&=\sum_{\widetilde{S}_i\in\widetilde{W}}\|\nabla(E_\epsilon u)\|^p_{L^p(\widetilde{S}_i)}\\
&\lesssim \sum_{\widetilde{S}_i\in\widetilde{W}}\sum_{\widetilde{S}_k\cap\widetilde{S}_i\neq\emptyset}\int_{G(\widetilde{S}_i,\widetilde{S}_k)}|\nabla u(x)|^p dx
\end{aligned}
$$

$$= \int_{\Omega_\epsilon} \sum_{\widetilde{S}_i \in \widetilde{W}} \sum_{\widetilde{S}_k \cap \widetilde{S}_i \neq \emptyset} \chi_{G(\widetilde{S}_k, \widetilde{S}_i)}(x) |\nabla u(x)|^p dx$$
$$\lesssim \|\nabla u\|^p_{L^p(\Omega_\epsilon)}$$

with the constant depending only on p and J.

We are left to estimate the integral of $|Eu|^p$ over $\widetilde{\Omega}_\epsilon$. By the definition of Eu, we have

$$\int_{\widetilde{S}_i} |Eu|^p dx \lesssim \sum_{\widetilde{S}_i \cap \widetilde{S}_k \neq \emptyset} \int_{S_k} |u|^p dx$$

and the desired bound follows via (3.42). □

Finally, we prove Theorem 3.9.

Proof of Theorem 3.9. Let E_ϵ be given by (3.37). Let us first show that the above procedure gives us an extension of our Lipschitz function u to a function $E_\epsilon u$ in $W^{1,p}(\Omega)$ with the desired norm bound. Towards this, we claim that $E_\epsilon u$ is locally Lipschitz in Ω.

According to our construction, $E_\epsilon u$ is smooth in $\widetilde{\Omega}_\epsilon$. Hence, to show the local Lipschitz continuity, we only need to consider the case where $z_1 \in \overline{\Omega}_\epsilon$ and $z_2 \in \widetilde{\Omega}_\epsilon$ with

$$B(z_2, 2|z_1 - z_2|) \subset \Omega.$$

Suppose that $z_2 \in \widetilde{S}$ for some $\widetilde{S} \in \widetilde{W}$. Then by (3.29) and the Lipschitz continuity of u, we have

$$\begin{aligned} |E_\epsilon u(z_2) - u(z_1)| &\leqslant \sum_{\widetilde{S}_k \cap \widetilde{S} \neq \emptyset} \phi_k(z_2) |a_k - u(z_1)| \\ &\lesssim \sum_{\widetilde{S}_k \cap \widetilde{S} \neq \emptyset} \phi_k(z_2)(\operatorname{dist}(z_1, S_k) + \operatorname{diam}(S_k)) \\ &\lesssim \sum_{\widetilde{S}_k \cap \widetilde{S} \neq \emptyset} \phi_k(z_2)(|z_1 - z_2| + \operatorname{diam}_\Omega(\widetilde{S}_k)) \lesssim |z_1 - z_2|, \end{aligned}$$

where in the last inequality we applied the facts that for $\widetilde{S}_k \cap \widetilde{S} \neq \emptyset$, it holds that

$$\operatorname{diam}_\Omega(\widetilde{S}_k) \sim \operatorname{dist}_\Omega(S, \Omega_\epsilon) \sim \operatorname{dist}_\Omega(z_2, \Omega_\epsilon) \leqslant |z_1 - z_2|.$$

Therefore, we obtain the local Lipschitz continuity of $E_\epsilon u$.

Recall that $\partial\Omega_\epsilon$ has Lebesgue measure zero by Lemma 2.24. Hence, Lemma 3.15 and the local Lipschitz continuity of $E_\epsilon u$ give that $Eu \in W^{1,p}(\Omega)$ with

$$\|u\|_{L^p(\Omega)} + \|\nabla(E_\epsilon u)\|_{L^p(\Omega)} \leqslant C(J,p)(\|u\|_{L^p(\Omega_\epsilon)} + \|\nabla u\|_{L^p(\Omega_\epsilon)}).$$

Consequently, E_ϵ is a bounded linear operator from $W^{1,p}(\Omega_\epsilon) \cap \operatorname{Lip}(\overline{\Omega}_\epsilon)$ to $W^{1,p}(\Omega)$. Next, $W^{1,p}(\Omega_\epsilon) \cap \operatorname{Lip}(\overline{\Omega}_\epsilon)$ is dense in $W^{1,p}(\Omega_\epsilon)$ for $1 < p < 2$: even $C^\infty(\mathbb{R}^2)$ is dense in $W^{1,p}(G)$ for $1 < p < \infty$ if G is a planar Jordan domain [31]. This allows us to extend E_ϵ (uniquely) to a bounded linear operator from $W^{1,p}(\Omega_\epsilon)$ to $W^{1,p}(\Omega)$. Thus, the claim of Theorem 3.9 follows. □

3.3 Proof of the general case

In this subsection, we prove the necessity of (1.1) in the general case where Ω is a bounded simply connected $W^{1,p}$-extension domain.

First of all, Ω is necessarily J-John, where the constant J depends only on p and the norm of the extension operator $\|E\|$ for Ω (see, e.g., [14, Theorem 3.4]). Fix $z_1, z_2 \in \mathbb{R}^2 \setminus \Omega$. Let $\Omega_n = \varphi(B(0, 1 - \frac{1}{n}))$ for $n \geqslant 2$, where $\varphi\colon \mathbb{D} \to \Omega$ is a conformal map with $\varphi(0)$ the John center of Ω. Let $\widetilde{\Omega}_n$ be the complementary domain of Ω_n. Then

$$\bigcap_{n=4}^{\infty} \widetilde{\Omega}_n = \mathbb{R}^2 \setminus \Omega.$$

By Theorem 3.9, we know that each Ω_n is a $W^{1,p}$-extension domain with the norm of the operator depending only on p, J and $\|E\|$. Hence, by Theorem 3.1, there is a curve $\gamma_n \subset \widetilde{\Omega}_n \cup \partial\Omega_n$ connecting z_1 and z_2 so that

$$\int_{\gamma_n} \operatorname{dist}(z, \partial\Omega_n)^{1-p} ds \leqslant C(J, \|E\|, p)|z_1 - z_2|^{2-p}. \tag{3.43}$$

We now proceed as in the proof of Lemma 2.1 to find a limit curve satisfying (1.1). Unlike in Lemma 2.1, here our integrand is not fixed. For this reason, we repeat the argument with a small modification.

Notice that we may assume that $\operatorname{le}(\gamma_n) \leqslant C(J, \|E\|, p)|z_1 - z_2| =: M$ by Lemma 2.2, and thus also $\gamma_n \subset \overline{B}(z_1, M)$. Therefore, by Lemma 2.1, a subsequence, not relabeled, converges uniformly to a limiting curve γ. We use the constant speed parametrization from $[0, 1]$ for each γ_n, and by taking a further subsequence if necessary, assume that $\operatorname{le}(\gamma_n) \to l$ as $n \to \infty$.

Since φ is continuous up to the boundary (see Remark 2.26), we have that Ω_n converges to Ω (in the Hausdorff distance with respect to the Euclidean metric). Thus, for any $t \in [0, 1]$, we have

$$\operatorname{dist}(\gamma(t), \partial\Omega) = \lim_{n\to\infty} \operatorname{dist}(\gamma(t), \partial\Omega_n). \tag{3.44}$$

By the (uniform) convergence of γ_n to γ, we also have

$$\lim_{n\to\infty} |\gamma(t) - \gamma_n(t)| = 0. \tag{3.45}$$

Now, (3.44) together with (3.45) gives

$$\operatorname{dist}(\gamma(t), \partial\Omega)^{1-p} = \lim_{n\to\infty} \operatorname{dist}(\gamma_n(t), \partial\Omega_n)^{1-p}.$$

Combined with Fatou's Lemma and (3.43), this yields

$$\begin{aligned}
\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-p} ds(z) &= \int_0^1 \operatorname{dist}(\gamma(t), \partial\Omega)^{1-p} |\gamma'(t)| dt \leqslant \int_0^1 \operatorname{dist}(\gamma(t), \partial\Omega)^{1-p} l dt \\
&= \int_0^1 \lim_{n\to\infty} \operatorname{dist}(\gamma_n(t), \partial\Omega_n)^{1-p} |(\gamma_n'(t))| dt \\
&\leqslant \liminf_{n\to\infty} \int_0^1 \operatorname{dist}(\gamma_n(t), \partial\Omega_n)^{1-p} |(\gamma_n'(t))| dt \\
&= \liminf_{n\to\infty} \int_{\gamma_n} \operatorname{dist}(z, \partial\Omega_n)^{1-p} ds(z) \\
&\leqslant C(J, \|E\|, p)|z_1 - z_2|^{2-p}.
\end{aligned}$$

We have completed the proof of the general case.

4 Proof of sufficiency

In this section, we prove the sufficiency of the condition (1.1) in Theorem 1.1, but begin with an auxiliary version. Namely, let $1 < p < \hat{p} < 2$ and suppose that Ω is a Jordan domain with the property that there exists a constant C such that for every pair of points $z_1, z_2 \in \mathbb{R}^2 \setminus \overline{\Omega}$, one can find a curve γ joining them in $\mathbb{R}^2 \setminus \overline{\Omega}$ with

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{1-\hat{p}} ds(z) \leqslant C|z_1 - z_2|^{2-\hat{p}}. \tag{4.1}$$

We claim that Ω is a $W^{1,p}$-extension domain. Write $\widetilde{\Omega}$ for the complementary domain of Ω.

Theorem 4.1. *Let $1 < p < \hat{p} < 2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded Jordan domain. Suppose that for all $z_1, z_2 \in \widetilde{\Omega}$, there exists a curve $\gamma \subset \widetilde{\Omega}$ joining z_1 and z_2 such that* (4.1) *holds. Then Ω is a $W^{1,p}$-extension domain and the norm of the extension operator depends only on $p, \hat{p}$ and the constant C in* (4.1).

The proof of Theorem 4.1 is given in several steps. In the first step, in the following subsection, we show that (4.1) also holds for initial arcs of hyperbolic rays $\Gamma \subset \widetilde{\Omega}$, up to a multiplicative constant. In the second subsection, we introduce shadows of sets and use them to assign a collection of Whitney squares of the domain Ω to each such Whitney square $\widetilde{Q}$ of its complementary domain $\widetilde{\Omega}$ that satisfies $\ell(\widetilde{Q}) \leqslant 3\mathrm{diam}(\Omega)$. In the third subsection, we use these Whitney squares to construct our extension operator. The fourth subsection gives a basic estimate that we use to control the extension. The fifth subsection deals with the construction of additional intermediate squares and estimates for them. The sixth subsection completes the proof.

Eventually, in the final subsection of this section, we prove Theorem 1.1 via Theorem 4.1 and an approximation and compactness argument. For this, it is crucial that the norm of the extension operator in Theorem 4.1 depends only on $p, \hat{p}$ and C in the inequality (4.1) and that a uniform version of (4.1) for some $\hat{p} > p$ and for all of our approximating Jordan domains follows from (1.1) by Lemma 2.3 (see Lemma 4.16 below).

Since we rely on compactness arguments, we do not obtain an explicit form for the extension operator for Theorem 1.1. On the other hand, once we know that Ω is indeed a $W^{1,p}$-extension domain, an explicit extension operator (a version of the Whitney extension operator) can be given [16, 17, 38]. We do not see a way to directly show that this kind of concrete procedure works under our assumptions.

4.1 Transferring the condition to hyperbolic rays

According to the Riemann mapping theorem, there is a conformal map $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \widetilde{\Omega}$. We refer to this fixed map through Subsection 4.6 by $\widetilde{\varphi}$. Since $\partial\Omega$ is a Jordan curve, the Carathéodory-Osgood theorem allows us to extend $\widetilde{\varphi}$ continuously to the boundary of $\mathbb{D}$ as a homeomorphism. We denote the extension still by $\widetilde{\varphi}$. Recall the definition of a hyperbolic ray from Subsection 2.3.

Lemma 4.2. *Assume that* (4.1) *holds for* $\widetilde{\Omega}$ *for a bounded Jordan domain* Ω. *Let* $z_1 \in \partial\Omega$ *and* $[z_2, z_3]$ *be an arc of the hyperbolic ray* $\Gamma \subset \overline{\widetilde{\Omega}}$ *corresponding to* z_1. *Then*

$$\int_{[z_2,z_3]} \mathrm{dist}(z, \partial\Omega)^{1-\hat{p}} ds(z) \leqslant C'|z_2 - z_3|^{2-\hat{p}}, \tag{4.2}$$

where C' *depends only on* $\hat{p}$ *and the constant* C *in* (4.1).

Proof. By symmetry, we may assume that z_3 is after z_2 on Γ when one moves towards infinity. Suppose first that $z_2 \neq z_1$. Let γ be a curve from (4.1) for the pair z_2, z_3. We use the notation from Subsection 2.7; especially, we let γ_k be a subcurve of γ that joins the inner and outer boundaries of $\widetilde{\varphi}(A(z_1, k))$, provided that $[z_2, z_3]$ intersects at least three such sets. If $[z_2, z_3]$ is contained in the union of two of these sets, we claim that (4.2) follows from the bi-Lipschitz estimate from Lemma 2.13. Indeed, by the definition of hyperbolic rays in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$, $\widetilde{\varphi}^{-1}([z_2, z_3])$ is contained in $B(\widetilde{\varphi}^{-1}(z_3), \frac{3}{4}(|\widetilde{\varphi}^{-1}(z_3)| - 1))$ if $[z_2, z_3]$ is contained in the union of two consecutive $\widetilde{\varphi}(A(z_1, k))$. Write $G = B(\widetilde{\varphi}^{-1}(z_3), \frac{4}{5}(|\widetilde{\varphi}^{-1}(z_3)| - 1))$. Then the version of (4.2) (with $\partial\Omega$ replaced by ∂G) holds for the radial segment $\widetilde{\varphi}^{-1}([z_1, z_3]) = [\widetilde{\varphi}^{-1}(z_2), \widetilde{\varphi}^{-1}(z_3)] \subset G$ with a constant depending only on $\hat{p}$. Hence, a change of variable together with Lemma 2.13 applied to $\widetilde{\varphi}$ in the set G ensures us that (4.2) holds for $[z_2, z_3]$ with a constant depending only on $\hat{p}$; notice that $\mathrm{dist}(z, \partial\widetilde{\varphi}(G)) \leqslant \mathrm{dist}(z, \partial\Omega)$ when $z \in \widetilde{\varphi}(G)$ since $\widetilde{\varphi}$ is a homeomorphism.

Suppose then that $[z_2, z_3]$ intersects at least three of the sets $\widetilde{\varphi}(A(z_1, k))$. For each $k \in \mathbb{Z}$ with

$$|\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)| \leqslant 2^{k-1} \leqslant 2^k \leqslant |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_3)|,$$

let

$$Z_k = \widetilde{\varphi}(S_k^1) \cap \Gamma_k,$$

where S_k^1 is the circle centered at $\widetilde{\varphi}^{-1}(z_1)$ and with radius $3 \times 2^{k-2}$.

Fix $k \leqslant 2$ as above. According to Lemma 2.31,

$$\mathrm{dist}(\Gamma_k, \partial\Omega) \sim \mathrm{dist}(Z_k, \partial\Omega) \tag{4.3}$$

and

$$\mathrm{le}(\Gamma_k) \sim \mathrm{dist}(\Gamma_k, \partial\Omega) \tag{4.4}$$

with absolute constants. Hence,

$$\int_{\Gamma_k} \mathrm{dist}(z, \partial\Omega)^{1-\hat{p}} ds(z) \lesssim \mathrm{dist}(Z_k, \partial\Omega)^{2-\hat{p}}. \tag{4.5}$$

Next, we claim that

$$\mathrm{dist}(Z_k, \partial\Omega) \gtrsim \mathrm{dist}(\gamma_k, \partial\Omega) \tag{4.6}$$

for some absolute constant. Indeed, let $B_k = \overline{B}(Z_k, \frac{1}{4}\mathrm{dist}(Z_k, \partial\Omega))$. If $\gamma_k \cap B_k \neq \emptyset$, then by the triangle inequality, we obtain the claim. For the other case, notice that B_k is a 4-Whitney-type set, and then by Lemma 2.12, $\widetilde{\varphi}^{-1}(B_k)$ is of λ-Whitney type for some absolute constant λ. Hence, (2.13) gives

$$\mathrm{dist}(\widetilde{\varphi}^{-1}(Z_k), \mathbb{S}^1) \sim \mathrm{diam}(\widetilde{\varphi}^{-1}(B_k)) \tag{4.7}$$

with an absolute constant. By the geometry of $A(z_1, k)$ in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$, we have

$$\mathrm{dist}(\widetilde{\varphi}^{-1}(Z_k), \widetilde{\varphi}^{-1}(\gamma_k)) \leqslant 4\mathrm{dist}(\widetilde{\varphi}^{-1}(Z_k), \mathbb{S}^1)$$

and

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\gamma_k)) \geqslant 2\mathrm{dist}(\widetilde{\varphi}^{-1}(Z_k), \mathbb{S}^1).$$

Hence, with the version of (2.25) for $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and (4.7), we conclude that

$$\mathrm{Cap}(\widetilde{\varphi}^{-1}(B_k), \widetilde{\varphi}^{-1}(\gamma_k), \mathbb{R}^2 \setminus \overline{\mathbb{D}}) \geqslant \delta(\lambda) > 0,$$

and the conformal invariance of capacity gives

$$\mathrm{Cap}(B_k, \gamma_k, \widetilde{\Omega}) \geqslant \delta(\lambda).$$

This estimate together with Lemma 2.17 yields

$$\mathrm{dist}(B_k, \gamma_k) \leqslant C(\lambda)\mathrm{diam}(B_k).$$

We then conclude (4.6) also in this case by the definition of B_k and the triangle inequality; indeed,

$$\begin{aligned}\mathrm{dist}(\gamma_k, \partial\Omega) &\leqslant \mathrm{dist}(B_k, \partial\Omega) + \mathrm{dist}(B_k, \gamma_k)\\ &\leqslant \mathrm{dist}(B_k, \partial\Omega) + C(\lambda)\mathrm{diam}(B_k)\\ &\leqslant C(\lambda)\mathrm{dist}(Z_k, \partial\Omega).\end{aligned}$$

By Lemma 2.31,

$$\ell(\gamma_k) \gtrsim \ell(\Gamma_k)$$

with an absolute constant. Then, by (4.6), (4.3) and (4.4), this gives that there is a subcurve $\gamma_k' \subset \gamma_k$ such that

$$\mathrm{dist}(Z_k, \partial\Omega) \gtrsim \mathrm{dist}(\gamma_k', \partial\Omega)$$

and

$$\mathrm{le}(\gamma_k') \sim \mathrm{le}(\Gamma_k)$$

with absolute constants. By combining this with (4.3) and (4.4), we conclude that

$$\int_{\gamma_k} \mathrm{dist}(z, \partial\Omega)^{1-\hat{p}} ds(z) \gtrsim \int_{\gamma_k'} \mathrm{dist}(z, \partial\Omega)^{1-\hat{p}} ds(z) \gtrsim \mathrm{dist}(Z_k, \partial\Omega)^{2-\hat{p}}. \tag{4.8}$$

Now (4.5) and (4.8) give us the inequality

$$\int_{\Gamma_k} \operatorname{dist}(z,\partial\Omega)^{1-\hat{p}} ds(z) \leqslant C(\lambda) \int_{\gamma_k} \operatorname{dist}(z,\partial\Omega)^{1-\hat{p}} ds(z). \tag{4.9}$$

Let us consider the remaining values of k. If $k \geqslant 2$, then $A(z_1,k)$ is a full annulus and of 8-Whitney type. Especially,

$$\operatorname{dist}(z,\partial\Omega) \sim \operatorname{dist}(w,\partial\Omega) \tag{4.10}$$

for all $z,w \in \widetilde{\varphi}(A(z_1,k))$ since this set is of λ-Whitney type for an absolute λ by Lemma 2.12. Moreover, $\operatorname{le}(\widetilde{\varphi}^{-1}(\gamma_k)) \geqslant \operatorname{le}(\widetilde{\varphi}^{-1}(\Gamma_k))$ since the former crosses $A(z_1,k)$ and the latter is a radial segment. Hence, the bi-Lipschitz estimate from Lemma 2.13 gives that $\operatorname{le}(\Gamma_k) \leqslant C \operatorname{le}(\gamma_k)$ with an absolute constant and (4.9) follows from (4.10). The only remaining values of k to consider are those potential k with

$$2^{k-1} \leqslant |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_3)| \leqslant 2^k$$

or

$$2^{k-1} \leqslant |\widetilde{\varphi}^{-1}(z_1) - \widetilde{\varphi}^{-1}(z_2)| \leqslant 2^k.$$

For such a k, (4.5) still holds and Lemma 2.6 shows that $\operatorname{dist}(Z_k,\partial\Omega) \sim \operatorname{dist}(Z_{k-1},\partial\Omega)$ with absolute constants. By our assumption, $[z_2,z_3]$ is not contained in the union of two of our sets $\widetilde{\varphi}(A(z_1,k))$, and hence these additional integrals over Γ_k are controlled by the earlier terms for which (4.8) holds.

We conclude from the previous paragraph and (4.9) that summing over k together with the first paragraph of the proof yields (4.2) when $z_1 \neq z_2$.

Finally if $z_1 = z_2$, we deduce (4.2) by picking $w_j \in [z_1,z_3] \cap \widetilde{\Omega}$ with $w_j \to z_1$ and by applying the conclusion from the case $z_1 \neq z_2$ (to $[w_j,z_3]$) and the monotone convergence theorem. □

4.2 Shadows of Whitney-type sets

Let Ω be a Jordan domain whose complementary domain $\widetilde{\Omega}$ satisfies (4.1). According to [30] (see also [39, Lemma 2.1]), $\widetilde{\Omega}$ is then quasiconvex with a constant that depends only on $\hat{p}$ and the constant C in (4.1). Consequently, by the second part of Lemma 2.1, also the complement of Ω is quasiconvex with the same constant. We conclude from Lemma 2.22 that Ω is J-John, where the John constant J depends only on $\hat{p}$ and the constant C in (4.1). We fix a John center x_0 for Ω and a conformal map $\varphi\colon \mathbb{D} \to \Omega$ with $\varphi(0) = x_0$. By the Carathéodory-Osgood theorem, φ extends homeomorphically up to the boundary and we refer also to the extension by φ. Our map φ will be fixed through Subsection 4.6. Recall from Subsection 4.1 that $\widetilde{\varphi}$ refers to a fixed exterior conformal map.

We assign a collection of "reflected" squares in the Whitney decomposition W of Ω to squares $\widetilde{Q}_i$ in the Whitney decomposition $\widetilde{W} = \{\widetilde{Q}_i\}$ of the complementary domain $\widetilde{\Omega}$. This will actually only be needed for those $\widetilde{Q}_i$ for which $\ell(\widetilde{Q}_i) \leqslant 3\operatorname{diam}(\Omega)$. The construction of our extension operator will then rely on these squares. We continue under the assumption that $\widetilde{\Omega}$ satisfies (4.1) and with the above φ and $\widetilde{\varphi}$. In what follows, we usually use the notation $\widetilde{A}$ to indicate that the set in question is contained in $\widetilde{\Omega}$.

Given a set $\widetilde{A} \subset \widetilde{\Omega}$, we consider all the hyperbolic rays in $\widetilde{\Omega}$ passing through $\widetilde{A}$, and define the *shadow* $S_{\widetilde{\Omega}}(\widetilde{A})$ as the set of all the points, where these rays hit the boundary $\partial\Omega$. Equivalently, by the invariance of hyperbolic rays,

$$S_{\widetilde{\Omega}}(\widetilde{A}) = \widetilde{\varphi}(\pi_r(\widetilde{\varphi}^{-1}(\widetilde{A}))),$$

where π_r is the radial projection to the unit circle.

Similarly, we define $S_\Omega(A)$ for $A \subset \Omega$, with the difference that the hyperbolic rays now begin from $\varphi(0)$. Then

$$S_\Omega(A) = \varphi(\pi_r(\varphi^{-1}(A))).$$

When there is no risk of confusion, we drop the subindices and simply write $S(\cdot)$ for the respective shadow.

We have the following properties.

Lemma 4.3. *Let $A \subset \Omega$ be a closed λ-Whitney-type set. Then $S(A)$ is connected and*

$$\operatorname{diam}_\Omega(S(A)) \sim_J \operatorname{diam}(S(A)) \sim_{J,\lambda} \operatorname{diam}(A),$$

where the constant J is the John constant. Moreover, for any fixed $M \geqslant 1$ and each closed λ-Whitney-type set $\widetilde{A}$ in the exterior domain $\widetilde{\Omega}$ of Ω with

$$\operatorname{diam}(\widetilde{A}) \leqslant M\operatorname{diam}(\Omega),$$

$S(\widetilde{A})$ is connected and

$$\operatorname{diam}(S(\widetilde{A})) \geqslant c(\lambda, M)\operatorname{diam}(\widetilde{A}).$$

Proof. Let us begin with the case $A \subset \Omega$. By the definition of Whitney-type sets, A is connected and thus also $\varphi^{-1}(A)$ is connected. Therefore, $\varphi^{-1}(S(A)) = \pi_r(\varphi^{-1}(A))$ is connected, and so is $S(A)$.

Next, by Lemma 2.12, $\varphi^{-1}(A)$ is a λ'-Whitney-type set with $\lambda' = \lambda'(\lambda)$. Moreover,

$$\operatorname{dist}(\varphi^{-1}(A), \varphi^{-1}(S(A))) = \operatorname{dist}(\varphi^{-1}(A), \pi_r(\varphi^{-1}(A))) = \operatorname{dist}(\varphi^{-1}(A), \partial\mathbb{D}),$$

and hence the conformal capacity between $\varphi^{-1}(S(A))$ and $\varphi^{-1}(A)$ in $\mathbb{D}$ is bounded from below by a positive constant depending only on λ (see (2.25)). By conformal invariance of capacity, also

$$\operatorname{Cap}(A, S(A), \Omega) \geqslant \delta(\lambda).$$

Let us prove that $C(\lambda)\operatorname{diam}(S(A)) \geqslant \operatorname{diam}(A)$. By the monotonicity of capacity, we have

$$\delta(\lambda) \leqslant \operatorname{Cap}(A, S(A), \Omega) \leqslant \operatorname{Cap}(A, S(A), \mathbb{R}^2), \tag{4.11}$$

which by Lemma 2.17 yields that

$$\operatorname{dist}(A, S(A)) \leqslant C(\lambda)\operatorname{diam}(S(A)). \tag{4.12}$$

Hence, by the definition of Whitney-type sets,

$$\operatorname{diam}(A) \lesssim \operatorname{dist}(A, \partial\Omega) \leqslant \operatorname{dist}(A, S(A)) \lesssim \operatorname{diam}(S(A)) \tag{4.13}$$

with constants depending only on λ.

Since Ω is John, hyperbolic rays are John curves by Lemma 2.19. Then for each hyperbolic ray $\Gamma \subset \Omega$ ending at $w \in \partial\Omega$ with $\Gamma \cap A \neq \emptyset$, the property (2.13) of λ-Whitney-type sets and the definition of John curves give

$$\operatorname{dist}(w, A) \leqslant C(J, \lambda)\operatorname{dist}(A, \partial\Omega) \leqslant C(J, \lambda)\operatorname{diam}(A).$$

Thus,

$$\operatorname{diam}(S(A)) \leqslant C(J, \lambda)\operatorname{diam}(A),$$

and hence, by (4.13), we can find another constant $C(J, \lambda)$ such that

$$\frac{1}{C(J, \lambda)}\operatorname{diam}(A) \leqslant \operatorname{diam}(S(A)) \leqslant C(J, \lambda)\operatorname{diam}(A). \tag{4.14}$$

Finally, it follows from Lemma 2.29 that

$$\operatorname{diam}_\Omega(S(A)) \sim \operatorname{diam}(S(A))$$

with constants depending only on J, and the asserted estimate follows by combining this with (4.14).

The connectivity of $S(\widetilde{A})$ follows analogously to the above argument. Regarding the desired estimate for $\operatorname{diam}(S(\widetilde{A}))$, we first notice that $\widetilde{A}$ contains a disk $B = B(z_0, r)$ with $r = \frac{1}{\lambda}\operatorname{diam}(\widetilde{A})$ since it is of λ-Whitney type. By the monotonicity of capacity, we know that

$$\operatorname{Cap}(\widetilde{A}, \partial\Omega, \widetilde{\Omega}) \geqslant \operatorname{Cap}(\partial B, \partial\Omega, \widetilde{\Omega} \setminus \overline{B}). \tag{4.15}$$

Next, the Möbius transformation $\phi : z \mapsto \frac{r^2}{z-z_0}$, given in complex notation, is bi-Lipschitz with a constant depending only on λ in $B(z_0, Cr) \setminus B(z_0, r)$ for $C = 2\lambda(\lambda+1)$, and $B(z_0, Cr) \setminus B(z_0, r)$ contains an arc of $\partial\Omega$ of the diameter at least $\lambda r/M$. We conclude that

$$\mathrm{dist}(\phi(\partial B), \phi(\partial\Omega)) \leqslant C(\lambda, M)\mathrm{diam}(\partial(\phi(\Omega))).$$

Hence, (2.27) (with $U_0 = \phi(B)$) gives

$$\mathrm{Cap}(\phi(\partial B), \phi(\partial\Omega), \mathbb{R}^2) = \mathrm{Cap}(\partial(\phi(B)), \partial(\phi(\Omega)), \phi(\mathbb{R}^2 \setminus (\overline{B} \cup \overline{\Omega}))) \geqslant \delta(\lambda, M). \tag{4.16}$$

Monotonicity, conformal invariance of capacity and (4.15) and (4.16) allow us to conclude that

$$\mathrm{Cap}(\widetilde{A}, \partial\Omega, \mathbb{R}^2) \geqslant \mathrm{Cap}(\widetilde{A}, \partial\Omega, \widetilde{\Omega}) \geqslant \delta(\lambda, M). \tag{4.17}$$

Now, Lemma 2.17 together with conformal invariance of capacity and (4.17) gives

$$\mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}), \partial\mathbb{D}) \leqslant C(M, \lambda).$$

Since $\widetilde{\varphi}^{-1}(\widetilde{A})$ is of $C(\lambda)$-Whitney type by Lemma 2.12, we conclude that

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{A})) \sim_\lambda \mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}), \partial\mathbb{D}) \leqslant C(M, \lambda).$$

This together with the version of (2.25) for $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and conformal invariance imply that

$$\delta(\lambda, M) \leqslant \mathrm{Cap}(\widetilde{\varphi}^{-1}(\widetilde{A}), \widetilde{\varphi}^{-1}(S(\widetilde{A})), \mathbb{R}^2 \setminus \overline{\mathbb{D}}) = \mathrm{Cap}(\widetilde{A}, S(\widetilde{A}), \widetilde{\Omega}).$$

By monotonicity of capacity, we further conclude that

$$\delta(\lambda, M) \leqslant \mathrm{Cap}(\widetilde{A}, S(\widetilde{A}), \mathbb{R}^2).$$

This estimate is the analog of (4.11) and hence we may complete the argument exactly as in the case of Ω above. □

The following lemma associates a Whitney square of Ω with a given closed boundary arc.

Lemma 4.4. *For each closed nondegenerate subarc $\gamma \subset \partial\Omega$, there exists a Whitney square $Q \in W$ satisfying*

$$\mathrm{diam}(S(Q)) \leqslant C(J)\mathrm{diam}(\gamma), \tag{4.18}$$
$$\mathrm{diam}(\gamma) \leqslant C(J)\mathrm{diam}(S(Q) \cap \gamma) \tag{4.19}$$

and

$$\mathrm{dist}(Q, \gamma) \leqslant C(J)\mathrm{diam}(\gamma). \tag{4.20}$$

Here, $C(J)$ depends only on J.

Proof. Given a closed nondegerate subarc γ, let $\alpha = \varphi^{-1}(\gamma)$. Suppose first that $\mathrm{le}(\alpha) > \frac{1}{2}$. By Lemma 2.30, φ is quasisymmetric with respect to the inner distance of Ω with η depending only on J. Pick $z_1, z_2 \in \alpha$ such that

$$\mathrm{dist}_\Omega(\varphi(z_1), \varphi(0)) = \mathrm{dist}_\Omega(\varphi(0), \gamma)$$

and

$$|z_1 - z_2| = \frac{1}{4}.$$

Recall that $\varphi(z_i)$ is rectifiably joinable, i.e., to $\varphi(0)$ by Remark 2.20 for $i = 1, 2$. Since φ is homeomorphic up to the boundary, we may pick points w_1^j and w_2^j along these rectifiable curves so that $\mathrm{dist}_\Omega(w_1^j, \varphi(0))$ tends to $\mathrm{dist}_\Omega(z_1, \varphi(0))$, $\mathrm{dist}_\Omega(w_1^j, w_2^j)$ tends to $\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_2))$, $\varphi^{-1}(w_1^j)$ tends to z_1 and $\varphi^{-1}(w_2^j)$ tends to z_2. Now

$$|0 - \varphi^{-1}(w_1^j)| \leqslant 8|\varphi^{-1}(w_1^j) - \varphi^{-1}(w_2^j)|$$

for all sufficiently large j and then the quasisymmetry of φ gives the estimate

$$\mathrm{dist}_\Omega(\varphi(0), w_1^j) \leqslant \eta(8)\mathrm{dist}_\Omega(w_1^j, w_2^j).$$

By letting j tend to infinity, we conclude that

$$\mathrm{dist}_\Omega(\varphi(0),\gamma) = \mathrm{dist}_\Omega(\varphi(z_1),\varphi(0)) \leqslant \eta(8)\mathrm{dist}_\Omega(\varphi(z_1),\varphi(z_2)) \leqslant \eta(8)\mathrm{diam}_\Omega(\gamma). \tag{4.21}$$

By the John property (see Lemma 2.19), for each hyperbolic ray $\Gamma \subset \Omega$, we have

$$\mathrm{dist}_\Omega(\varphi(0), \partial\Omega) \geqslant J\,\mathrm{le}(\Gamma).$$

Then the triangle inequality gives

$$\mathrm{dist}_\Omega(\varphi(0),\gamma) \geqslant \mathrm{dist}_\Omega(\varphi(0),\partial\Omega) \geqslant \frac{J}{2}\mathrm{diam}(\Omega). \tag{4.22}$$

Moreover, Lemma 2.29 implies that

$$\mathrm{diam}(\gamma) \sim_J \mathrm{diam}_\Omega(\gamma). \tag{4.23}$$

By combining (4.23) with (4.21) and (4.22), we conclude that

$$\mathrm{diam}(\gamma) \geqslant \frac{1}{C(J)}\mathrm{diam}(\partial\Omega).$$

Therefore, if one chooses a Whitney square Q containing $\varphi(0)$, then its shadow is $\partial\Omega$, and (4.18) follows; in this case, (4.19) holds trivially and (4.20) follows from (4.21) together with (4.23) since $\varphi(0) \in Q$.

When $\mathrm{le}(\alpha) \leqslant \frac{1}{2}$, we denote the midpoint of α by w, let

$$r = \frac{\sin(\frac{\mathrm{le}(\alpha)}{2})}{1 + 2\sin(\frac{\mathrm{le}(\alpha)}{2})}, \quad z = (1-2r)w$$

and set $B = \overline{B(z,r)}$ (see Figure 6). Observe that by the assumption $\mathrm{le}(\alpha) \leqslant \frac{1}{2}$, the set B satisfies

$$2\mathrm{dist}(B, \partial\mathbb{D}) = 2r = \mathrm{diam}(B),$$

and is of 2-Whitney type, and the radial projection of B is precisely α. Moreover, quasisymmetry of φ easily gives

$$\mathrm{dist}(\varphi(B),\gamma) \leqslant C(J)\mathrm{diam}(\varphi(B)). \tag{4.24}$$

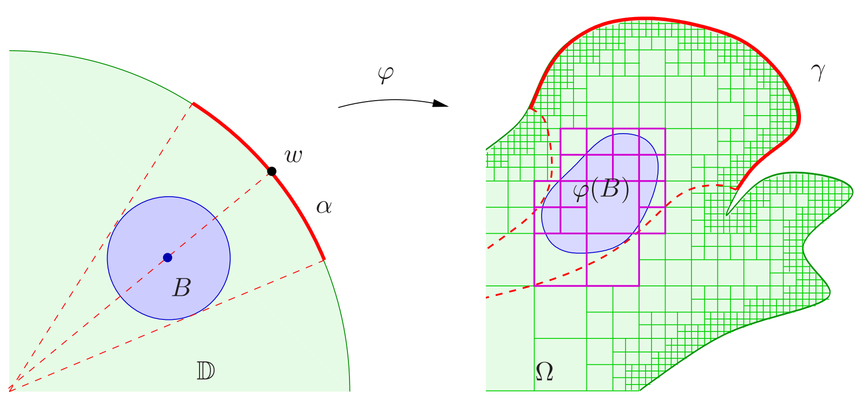

Figure 6 (Color online) The set $B \subset \mathbb{D}$ is chosen to be a Whitney-type set whose shadow is exactly α. Since $\varphi(B)$ is also of Whitney type, there are at most a fixed number of Whitney squares intersecting it. Therefore, one of these squares must have a large shadow

Consider the collection W_B of all Whitney squares in W that intersects $\varphi(B)$. Since $\varphi(B)$ is a λ-Whitney-type set by Lemma 2.12, for some absolute constant λ, this collection has no more than N squares where $N = N(\lambda)$ (see the discussion after Definition 2.10). Since φ is homeomorphic up to the boundary, the shadow of $\varphi(B)$ is precisely $\varphi(\alpha) = \gamma$. We claim that the shadow of one of the Whitney squares in W_B, call it Q, satisfies

$$\mathrm{diam}(S(Q) \cap \gamma) \geqslant \mathrm{diam}(\gamma)/N.$$

Towards this, since $\varphi(B) \subset \bigcup_{Q' \in W_B} Q'$, we have

$$\gamma = S(\varphi(B)) \subset \bigcup_{Q' \in W_B} S(Q').$$

Suppose that for every $Q' \in W_B$, we have

$$\mathrm{diam}(S(Q') \cap \gamma) < \mathrm{diam}(\gamma)/N.$$

Recall that γ is an arc and each $S(Q')$ is connected and hence also an arc. Since $\gamma \subset \bigcup_{Q' \in W_B} S(Q')$, we deduce by the triangle inequality that

$$\mathrm{diam}(\gamma) \leqslant \sum_{Q' \in W_B} \mathrm{diam}(S(Q') \cap \gamma) < \mathrm{diam}(\gamma).$$

This gives a contradiction, and hence (4.19) follows.

Towards (4.18), first notice that $\varphi(B)$ is of λ-Whitney type for an absolute λ by Lemma 2.12. Also, Q as a Whitney square is of $4\sqrt{2}$-Whitney type. Since Q intersects $\varphi(B)$, the property (2.14) of intersecting Whitney-type sets ensures that

$$\mathrm{diam}(Q) \sim_\lambda \mathrm{diam}(\varphi(B)). \tag{4.25}$$

By Lemma 4.3, we further have

$$\mathrm{diam}(S(Q)) \sim_J \mathrm{diam}(Q) \tag{4.26}$$

and

$$\mathrm{diam}(\varphi(B)) \sim_J \mathrm{diam}(\gamma) \tag{4.27}$$

since

$$S(\varphi(B)) = \gamma.$$

By combining (4.26) and (4.27) with (4.25), we conclude that

$$\mathrm{diam}(S(Q)) \leqslant C(J)\mathrm{diam}(\gamma),$$

as desired.

Finally, (4.20) follows by combining (4.24) with (4.25). □

The definition of our extension operator in Subsection 4.3 will rely on the following existence result.

Lemma 4.5. *Let Ω be a Jordan John domain with a constant J. There is a constant $C(J)$ that depends only on J so that the following holds. Given $\widetilde{Q} \in \widetilde{W}$, there exists $Q \in W$ so that*

$$\mathrm{diam}(S(Q))/C(J) \leqslant \mathrm{diam}(S(\widetilde{Q})) \leqslant C(J)\mathrm{diam}(S(Q) \cap S(\widetilde{Q})) \tag{4.28}$$

and

$$\mathrm{dist}(Q, S(\widetilde{Q})) \leqslant C(J)\mathrm{diam}(S(\widetilde{Q})). \tag{4.29}$$

Moreover, if $\ell(\widetilde{Q}) \leqslant 3\mathrm{diam}(\Omega)$, then

$$\mathrm{diam}(\widetilde{Q}) \leqslant C(J)\mathrm{diam}(Q). \tag{4.30}$$

Proof. Since $\widetilde{Q}$ is of $4\sqrt{2}$ -Whitney type, Lemma 4.3 shows that $S(\widetilde{Q})$ is a nondegenerate subarc of $\partial\Omega$. Thus, by Lemma 4.4, there exists a Whitney square $Q \in W$ that satisfies both (4.28) and (4.29) with constants depending only on J. Finally, (4.30) follows from these properties of Q together with Lemma 4.3. □

Notice that a single $Q \in W$ may well satisfy the requirement in Lemma 4.5 for many distinct $\widetilde{Q}$, of different sizes: $S(\widetilde{Q})$ can be much larger in size than $\widetilde{Q}$ (see Figure 7). We close this subsection with technical lemmas that will eventually allow us to deal with the distribution of such squares $\widetilde{Q}$.

Lemma 4.6. *Let $C \geqslant 1$. Suppose that $Q \in W$ and suppose that $\gamma_2, \ldots, \gamma_n \subset S(Q)$ are pairwise disjoint arcs so that*

$$\operatorname{diam}(S(Q)) \leqslant C\operatorname{diam}(\gamma_j)$$

for each $1 \leqslant j \leqslant n$. Then $n \leqslant N$, where N depends only on C and the John constant J of Ω.

Proof. Let $\gamma_1, \ldots, \gamma_n$ be pairwise disjoint arcs contained in $S(Q)$ so that $\operatorname{diam}(S(Q)) \leqslant C\operatorname{diam}(\gamma_j)$ for each $1 \leqslant j \leqslant n$. In order to bound n, it suffices to associate each γ_j with a disk B_j of radius $r \geqslant \operatorname{diam}(S(Q))/C'$ so that these disks are pairwise disjoint and all have the distance to $S(Q)$ no more than $C'\operatorname{diam}(S(Q))$, for a constant C' depending only on C and J.

Let w_j be the midpoint of $\varphi^{-1}(\gamma_j)$,

$$r_j = \frac{\sin(\frac{\operatorname{le}(\varphi^{-1}(\gamma_j))}{2})}{1 + 2\sin(\frac{\operatorname{le}(\varphi^{-1}(\gamma_j))}{2})}, \quad z_j = (1 - 2r_j)w_j$$

and set $B_j = \overline{B}(z_j, r_j)$. Then the radial projection of B_j is precisely $\varphi^{-1}(\gamma_j)$ and each B_j is of 4-Whitney type. Since the arcs γ_j are pairwise disjoint, so are also $\varphi^{-1}(\gamma_j)$ and consequently also the sets B_j. Then the sets $\varphi(B_j)$ are also pairwise disjoint. From Lemma 4.3, it follows that

$$\operatorname{diam}(\varphi(B_j)) \geqslant C(J)\operatorname{diam}(\gamma_j)$$

and (by (4.12))

$$\operatorname{dist}(\varphi(B_j), S(Q)) \leqslant \operatorname{dist}(\varphi(B_j), \gamma_j) \leqslant C(J)\operatorname{diam}(\gamma_j) \leqslant C(J)\operatorname{diam}(S(Q)).$$

The claim follows by recalling that a λ-Whitney-type set A contains a disk of radius $\frac{1}{\lambda}\operatorname{diam}(A)$ and that $C\operatorname{diam}(\gamma_j) \geqslant \operatorname{diam}(S(Q))$; the sets $\varphi(B_j)$ are of λ-Whitney type for an absolute λ by Lemma 2.12. □

For a Whitney-type set $\widetilde{A} \subset \widetilde{\Omega}$ and a hyperbolic ray Γ with $\Gamma \cap \widetilde{A} \neq \emptyset$, corresponding to a point $z \in \partial\Omega$, we define the *tail* of Γ with respect to $\widetilde{A}$ to be the arc of Γ between z and $\widetilde{A}$, i.e., $\Gamma_{z,w} \subset \Gamma$ with w the first point in $\widetilde{A}$ when travelled towards infinity from z. Denote this set by $T_{\widetilde{\Omega}}(\Gamma, \widetilde{A})$.

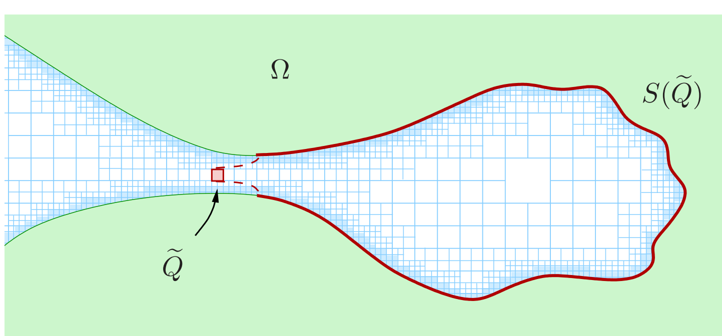

Figure 7 (Color online) The shadow $S(\widetilde{Q})$ of a Whitney square $\widetilde{Q}$ of the complementary domain $\widetilde{\Omega}$ may have a much larger diameter than the square in question

Lemma 4.7. *Let $\widetilde{A} \subset \widetilde{\Omega}$ be a closed λ-Whitney-type set so that $\widetilde{\Omega} \setminus \widetilde{A}$ is connected and let Γ be a hyperbolic ray with $\Gamma \cap \widetilde{A} \neq \emptyset$. Set $\widetilde{W}(\widetilde{A}, \Gamma) = \{\widetilde{Q}_j \in \widetilde{W} : \widetilde{Q}_j \cap T_{\widetilde{\Omega}}(\Gamma, \widetilde{A}) \neq \emptyset\}$. Then*

$$\sum_{\widetilde{Q}_j \in \widetilde{W}(\widetilde{A},\Gamma)} \ell(\widetilde{Q}_j)^{2-\hat{p}} \leqslant C \mathrm{diam}(S(\widetilde{A}))^{2-\hat{p}},$$

where C depends only on $\hat{p}$, λ and the constant in (4.1).

In order to prove this, we need an auxiliary lemma and a definition.

We define the tail of $\widetilde{A}$ by setting

$$T_{\widetilde{\Omega}}(\widetilde{A}) = \{y \in \widetilde{\Omega} \mid y \in T_{\widetilde{\Omega}}(\Gamma, \widetilde{A}) \text{ for some hyperbolic ray } \Gamma\}.$$

Equivalently,

$$T_{\widetilde{\Omega}}(\widetilde{A}) = \widetilde{\varphi}(T_{\mathbb{R}^2 \setminus \overline{\mathbb{D}}}(\widetilde{\varphi}^{-1}(\widetilde{A}))).$$

When there is no danger of confusion, we simply write T instead of $T_{(\cdot)}$.

We need an estimate for the sizes of those Whitney squares that intersect a given tail. Such estimates follow rather easily in the complement of the disk (see Figure 8), but our exterior domain case requires work.

Lemma 4.8. *Let $\widetilde{A} \subset \widetilde{\Omega}$ be a closed λ-Whitney-type set with* $\mathrm{diam}(\widetilde{A}) \leqslant 3\mathrm{diam}(\Omega)$. *Assume additionally that $\widetilde{\Omega} \setminus \widetilde{A}$ is connected. Let $\widetilde{Q} \in \widetilde{W}$ satisfy $\widetilde{Q} \cap T(\widetilde{A}) \neq \emptyset$. Then*

$$\ell(\widetilde{Q}) \leqslant C(\lambda) \mathrm{diam}(S(\widetilde{A})).$$

Proof. Fix $\widetilde{Q} \in \widetilde{W}$ with $\widetilde{Q} \cap T(\widetilde{A}) \neq \emptyset$. We may assume that $\lambda \geqslant 4\sqrt{2}$ so that also $\widetilde{Q}$ is of λ-Whitney type. Let us first prove that

$$\mathrm{diam}(\widetilde{Q}) \lesssim \mathrm{diam}(\Omega) \tag{4.31}$$

with a constant depending only on λ.

Towards this claim, recall from the definition of λ-Whitney type that there exists a disk

$$B\left(z_0, \frac{1}{\lambda}\mathrm{diam}(\widetilde{A})\right) \subset \widetilde{A}.$$

Next, by (2.26), we have

$$\mathrm{Cap}(\widetilde{A}, \partial\Omega, \widetilde{\Omega}) = \mathrm{Cap}(\partial\widetilde{A}, \partial\Omega, \widetilde{\Omega} \setminus \widetilde{A}). \tag{4.32}$$

We continue by arguing as in the proof of Lemma 4.3.

Since the Möbius transformation $\phi\colon z \mapsto \frac{\mathrm{diam}(\widetilde{A})^2}{(z-z_0)}$ is $C(\lambda)$-bi-Lipschitz in the set

$$B(z_0, (2+\lambda)\mathrm{diam}(\widetilde{A})) \setminus B(z_0, \mathrm{diam}(\widetilde{A})/\lambda)$$

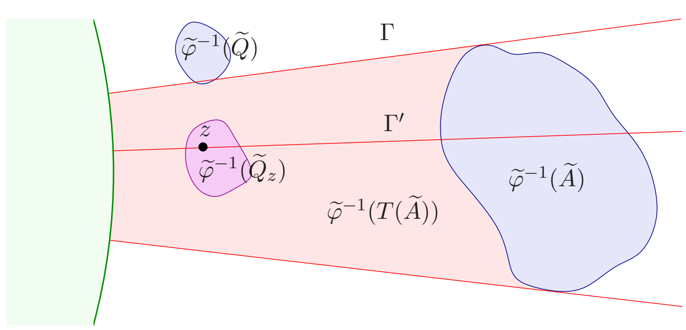

Figure 8 (Color online) In the case $\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})) < c_1\mathrm{diam}(\tilde{\varphi}^{-1}(S(\widetilde{A})))$, we argue using an extra Whitney-type set $\widetilde{\varphi}^{-1}(\widetilde{Q}_z) \subset \widetilde{\varphi}^{-1}(T(\widetilde{A}))$ of roughly the same size as $\widetilde{\varphi}^{-1}(\widetilde{A})$ that is also near $\widetilde{\varphi}^{-1}(\widetilde{A})$

and this set contains both $\partial\widetilde{A}$ and an arc of $\partial\Omega$ of a diameter at least $\mathrm{diam}(\widetilde{A})/3$, we have

$$\mathrm{dist}(\phi(\widetilde{A}),\phi(\partial\Omega))\leqslant C'(\lambda)\mathrm{diam}(\phi(\partial\Omega)).$$

Hence, (2.27) (with $U_0=\mathbb{R}^2\setminus\phi(\widetilde{A})$) gives the estimate

$$\mathrm{Cap}(\phi(\partial\widetilde{A}),\phi(\partial\Omega),\phi(\widetilde{\Omega}\setminus\widetilde{A}))\geqslant\delta(\lambda).$$

Thus,

$$\mathrm{Cap}(\widetilde{A},\partial\Omega,\widetilde{\Omega}\setminus\widetilde{A})\geqslant\delta(\lambda) \tag{4.33}$$

by the conformal invariance of capacity; notice that ϕ is conformal in the ring domain $\widetilde{\Omega}\setminus\widetilde{A}$.

Next, as $\widetilde{\varphi}^{-1}$ also preserves conformal capacity, monotonicity together with the inequalities (4.32) and (4.33) gives

$$\mathrm{Cap}(\widetilde{\varphi}^{-1}(\widetilde{A}),\partial\mathbb{D},\mathbb{R}^2)\geqslant\mathrm{Cap}(\widetilde{\varphi}^{-1}(\widetilde{A}),\partial\mathbb{D},\mathbb{R}^2\setminus\overline{\mathbb{D}})\geqslant\delta(\lambda). \tag{4.34}$$

Hence, Lemma 2.17 and the fact that $\varphi^{-1}(\widetilde{A})$ is of λ'-Whitney type by Lemma 2.12 yield

$$\mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}),\partial\mathbb{D})\leqslant C(\lambda). \tag{4.35}$$

By (2.13) and the fact that $\widetilde{\varphi}^{-1}(T_{\widetilde{\Omega}}(\widetilde{A}))=T_{\mathbb{R}^2\setminus\overline{\mathbb{D}}}\widetilde{\varphi}^{-1}(\widetilde{A})$, we deduce that

$$\mathrm{dist}(w,\mathbb{D})\leqslant C(\lambda) \tag{4.36}$$

for every $w\in\widetilde{\varphi}^{-1}(T(\widetilde{A}))$. Since $\widetilde{Q}\cap T(\widetilde{A})\neq\emptyset$ and $\widetilde{\varphi}^{-1}(\widetilde{Q})$ is also of λ'-Whitney type by Lemma 2.12, (4.36) gives us the estimate

$$\mathrm{diam}(\varphi^{-1}(\widetilde{Q}))\leqslant C(\lambda)\mathrm{dist}(\varphi^{-1}(\widetilde{Q}),\mathbb{D})\leqslant C(\lambda). \tag{4.37}$$

Now monotonicity and conformal invariance of capacity together with (2.25) and (4.37) yield

$$\mathrm{Cap}(\widetilde{Q},\partial\Omega,\mathbb{R}^2)\geqslant\mathrm{Cap}(\widetilde{Q},\partial\Omega,\widetilde{\Omega})=\mathrm{Cap}(\varphi^{-1}(\widetilde{Q}),\partial\mathbb{D},\mathbb{R}^2\setminus\overline{\mathbb{D}})\geqslant\delta(\lambda).$$

Since $\widetilde{Q}$ is a Whitney square, (4.31) follows from this by Lemma 2.17.

Recall again that the preimages of both $\widetilde{A}$ and $\widetilde{Q}$ are of λ'-Whitney type with $\lambda'=\lambda'(\lambda)$. Hence, if $\widetilde{Q}\cap\widetilde{A}\neq\emptyset$, then $\ell(\widetilde{Q})\sim\mathrm{diam}(\widetilde{A})$ by (2.14), and our asserted estimate follows from Lemma 4.3. Hence, we may assume that $\widetilde{Q}\cap\widetilde{A}=\emptyset$.

We prove the claim of the lemma first under the additional assumption that

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}))\geqslant c_1\mathrm{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{A}))), \tag{4.38}$$

where

$$c_1=\min\left\{\frac{1}{9},\frac{1}{6\lambda'},\frac{1}{8\lambda'^2}\right\}. \tag{4.39}$$

To begin, since $\widetilde{\varphi}^{-1}(\widetilde{A})$ is of λ'-Whitney type, (4.35) together with Lemma 4.3 gives

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{A}))\leqslant C(\lambda,\lambda')\mathrm{diam}(S(\widetilde{\varphi}^{-1}(\widetilde{A}))). \tag{4.40}$$

Since $\widetilde{Q}\cap T_{\widetilde{\Omega}}(\widetilde{A})\neq\emptyset$, we can pick a point

$$z\in\widetilde{\varphi}^{-1}(\widetilde{Q})\cap\widetilde{\varphi}^{-1}(T_{\widetilde{\Omega}}(\widetilde{A}))=\varphi^{-1}(\widetilde{Q})\cap T_{\mathbb{D}\setminus\overline{\mathbb{D}}}(\widetilde{\varphi}^{-1}(\widetilde{A})).$$

Then the hyperbolic ray (radial line) through z intersects $\widetilde{\varphi}^{-1}(\widetilde{A})$. Since $\widetilde{\varphi}^{-1}(\widetilde{A})$ is of λ'-Whitney type, the length of the segment of this radial line between $\partial\mathbb{D}$ and $\widetilde{\varphi}^{-1}(\widetilde{A})$ is no more than $C(\lambda')\mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}),S(\widetilde{\varphi}^{-1}(\widetilde{A})))$. Let I be the subsegment between z and $\widetilde{\varphi}^{-1}(\widetilde{A})$. Then

$$\mathrm{le}(I)\lesssim C(\lambda')\mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}),S(\widetilde{\varphi}^{-1}(\widetilde{A})))=C(\lambda')\mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}),\partial\mathbb{D}).$$

Recalling that $\widetilde{\varphi}^{-1}(\widetilde{A})$ is of λ'-Whitney type with $\lambda' = \lambda'(\lambda)$, we see that this in combination with (4.38) and (4.40) gives

$$\mathrm{le}(I) \lesssim \mathrm{diam}(\varphi^{-1}(\widetilde{A})) \lesssim \mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})) \tag{4.41}$$

with constants depending only on λ. Since $\widetilde{\varphi}^{-1}(\widetilde{Q})$ is of λ'-Whitney type, we deduce that $\mathrm{le}(I) \lesssim |z| - 1$. Because I is also a radial segment with z the closest point to $\partial\mathbb{D}$, it follows that the number of the Whitney squares of $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ that intersects I is at most $N = N(\lambda')$. Recalling that $\lambda' = \lambda'(\lambda)$, we can join $\widetilde{\varphi}^{-1}(\widetilde{A})$ and $\widetilde{\varphi}^{-1}(\widetilde{Q})$ by a chain of no more than $N(\lambda)$ Whitney squares. Then by Lemma 2.12 and the fact that a $\hat{\lambda}$-Whitney-type set intersects at most $N(\hat{\lambda})$ Whitney squares, there also exists a chain of no more than $N' = N'(\lambda)$ Whitney squares of $\widetilde{\Omega}$ joining $\widetilde{A}$ and $\widetilde{Q}$. Since both $\widetilde{A}$ and $\widetilde{Q}$ are of λ-Whitney type, their diameters are comparable by (2.14) to diameters of those Whitney squares that intersect them and the diameters of any two consecutive Whitney squares in our chain are comparable. It follows that $\mathrm{diam}(\widetilde{Q}) \leqslant C(\lambda)\mathrm{diam}(\widetilde{A})$. By Lemma 4.3 and the assumption that

$$\mathrm{diam}(\widetilde{A}) \leqslant 3\mathrm{diam}(\Omega),$$

we conclude that $\mathrm{diam}(\widetilde{Q}) \leqslant C(\lambda)\mathrm{diam}(\widetilde{A}) \leqslant C(\lambda)\mathrm{diam}(S(\widetilde{A}))$.

We are left to consider the case where

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q})) < c_1\mathrm{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{A}))). \tag{4.42}$$

If $\widetilde{Q} \subset T(\widetilde{A})$, then by Lemma 4.3 with (4.31), we have

$$\mathrm{diam}(\widetilde{Q}) \lesssim \mathrm{diam}(S(\widetilde{Q})) \lesssim \mathrm{diam}(S(\widetilde{A})).$$

If $\widetilde{Q}$ is not contained in $T(\widetilde{A})$, let $d = \mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}))$. Let w_1 and w_2 be the endpoints of $\widetilde{\varphi}^{-1}(S(\widetilde{A}))$. By (4.42) and (4.39), we have

$$d < 6\lambda' d \leqslant \mathrm{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{A}))), \tag{4.43}$$

and it follows that $\widetilde{\varphi}^{-1}(\widetilde{Q})$ intersects only one of the hyperbolic rays from w_1 and w_2 to infinity. Let Γ be this hyperbolic ray. Also, let Γ' be the hyperbolic ray in $\mathbb{R}^2 \setminus \mathbb{D}$, which intersects $\widetilde{\varphi}^{-1}(T(\widetilde{A}))$ and satisfies

$$\mathrm{dist}(\Gamma, \Gamma') = 2\lambda' d; \tag{4.44}$$

the existence of Γ' follows from (4.43). Let z be the point on Γ' with $|z| = 1 + d$ (see Figure 8). Let $\widetilde{Q}_z$ be a Whitney square so that $z \in \widetilde{\varphi}^{-1}(\widetilde{Q}_z)$. Then $\widetilde{\varphi}^{-1}(\widetilde{Q}_z)$ is also of λ'-Whitney type as $\widetilde{Q}_z$ is of $4\sqrt{2}$-Whitney type and we assume that $\lambda \geqslant 4\sqrt{2}$. Hence, by Definition 2.10 of λ'-Whitney-type sets, (4.39) and (4.42), we conclude that

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_z)) + \mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}_z), \partial\mathbb{D}) \leqslant \lambda' d + d < \frac{1}{4\lambda'}\mathrm{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{A}))), \tag{4.45}$$

where we used the fact that $c_1 \leqslant \frac{1}{8\lambda'^2} \leqslant \frac{1}{4\lambda'(\lambda'+1)}$.

Next, $\widetilde{\varphi}^{-1}(S(\widetilde{A})) = S(\varphi^{-1}(\widetilde{A})) = \pi_r(\widetilde{\varphi}^{-1}(\widetilde{A}))$, where π_r is the radial projection. Since π_r is a contraction, $\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{A})) > 0$ and $\widetilde{\varphi}^{-1}(\widetilde{A})$ is of λ'-Whitney type, we have

$$\frac{1}{4\lambda'}\mathrm{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{A}))) < \frac{1}{\lambda'}\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{A})) \leqslant \mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}), \partial\mathbb{D}).$$

By combining this estimate with (4.45), we conclude that for any point $x \in \widetilde{\varphi}^{-1}(\widetilde{Q}_z)$,

$$\mathrm{dist}(x, \partial\mathbb{D}) \leqslant \mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_z)) + \mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{Q}_z), \partial\mathbb{D}) < \mathrm{dist}(\widetilde{\varphi}^{-1}(\widetilde{A}), \partial\mathbb{D}); \tag{4.46}$$

especially,

$$\widetilde{\varphi}^{-1}(\widetilde{Q}_z) \cap \widetilde{\varphi}^{-1}(\widetilde{A}) = \emptyset.$$

Furthermore, since

$$\mathrm{diam}(\widetilde{\varphi}^{-1}(\widetilde{Q}_z)) \leqslant \lambda' d,$$

by (4.43) and (4.44), we know that $\widetilde{\varphi}^{-1}(\widetilde{Q}_z)$ does not intersect either of our two hyperbolic rays in $\mathbb{R}^2\setminus\mathbb{D}$ from the endpoints w_1 and w_2 of $\widetilde{\varphi}^{-1}(S(\widetilde{A}))$. This implies that

$$S(\widetilde{\varphi}^{-1}(\widetilde{Q}_z)) \subset \widetilde{\varphi}^{-1}(S(\widetilde{A})).$$

This, together with (4.46), yields $\widetilde{\varphi}^{-1}(\widetilde{Q}_z) \subset \widetilde{\varphi}^{-1}(T(\widetilde{A}))$, or equivalently, $\widetilde{Q}_z \subset T(\widetilde{A})$. Since $\widetilde{Q}_z \subset T(\widetilde{A})$ and $\operatorname{diam}(\widetilde{Q}_z) \lesssim \operatorname{diam}(\Omega)$ by (4.31), Lemma 4.3 gives

$$\operatorname{diam}(\widetilde{Q}_z) \lesssim \operatorname{diam}(S(\widetilde{Q}_z)) \lesssim \operatorname{diam}(S(\widetilde{A})). \tag{4.47}$$

Pick $\hat{z} \in \widetilde{\varphi}^{-1}(\widetilde{Q})\cap\Gamma$. Since $\widetilde{\varphi}^{-1}(\widetilde{Q})$ is of λ'-Whitney type, we have $|\hat{z}|-1 \sim d$ with a constant depending only on λ'. Let z_1 be the point on Γ with $|z_1| = 1+2d$ and let z_2 be a point on Γ' with $|z_2| = 1+2d$. Consider the curve γ obtained by concatenation from the part of Γ between $\hat{z}$ and z_1, the part of Γ' between z and z_2 and a shorter one of the circular arcs on $S(0, 1+2d)$ joining z_1 and z_2. Then the number of Whitney squares of $\mathbb{R}^2\setminus\overline{\mathbb{D}}$ intersecting γ is at most $N(\lambda')$. We again rely on Lemma 2.12 and the fact that a $\hat{\lambda}$-Whitney-type set intersects at most $N(\hat{\lambda})$ Whitney squares to conclude that there also exists a chain of no more than $N' = N'(\lambda)$ Whitney squares of $\widetilde{\Omega}$ joining $\widetilde{Q}_z$ to $\widetilde{Q}$. It follows that $\operatorname{diam}(\widetilde{Q}_z) \sim \operatorname{diam}(\widetilde{Q})$ and hence the desired estimate follows from (4.47). □

Proof of Lemma 4.7. Let Γ be a hyperbolic ray that intersects $\widetilde{A}$. Denote by Γ_0 the tail of Γ with respect to $\widetilde{A}$.

We claim that $\operatorname{le}(\Gamma_0) \leqslant C\operatorname{diam}(S(\widetilde{A}))$ with a constant that depends only on our data: $\hat{p}$ and the constant C in (4.1). To begin, suppose that $\widetilde{Q} \in \widetilde{W}$ intersects Γ_0. Then $\widetilde{Q}\cap T(\widetilde{A}) \neq \emptyset$, and hence Lemma 4.8 gives

$$\ell(\widetilde{Q}) \leqslant C\operatorname{diam}(S(\widetilde{A})) \tag{4.48}$$

with a constant that depends only on λ. Next, (4.48) yields that

$$\operatorname{dist}(z,\partial\Omega) \leqslant 4\sqrt{2}\,C\operatorname{diam}(S(\widetilde{A})) \tag{4.49}$$

whenever $z \in \Gamma_0$.

By (4.49) and Lemma 4.2, we have

$$\operatorname{diam}(S(\widetilde{A}))^{1-\hat{p}}\operatorname{le}(\Gamma_0) \leqslant C\int_{\Gamma_0}\operatorname{dist}(z,\partial\Omega)^{1-\hat{p}}ds(z) \leqslant C_1\ell(\Gamma_0)^{2-\hat{p}}, \tag{4.50}$$

where C depends only on λ and C_1 depends only on $\hat{p}$, λ and the constant in (4.1). This together with the assumption that $\hat{p} > 1$ results in

$$\operatorname{le}(\Gamma_0) \leqslant C_1^{1/(\hat{p}-1)}\operatorname{diam}(S(\widetilde{A})). \tag{4.51}$$

By combining (4.50) with (4.51), we conclude that

$$\int_{\Gamma_0}\operatorname{dist}(z,\partial\Omega)^{1-\hat{p}}ds(z) \leqslant C_1^{(2-\hat{p})/(\hat{p}-1)}\operatorname{diam}(\widetilde{S}(\widetilde{A}))^{2-\hat{p}}. \tag{4.52}$$

We now employ (4.52) to prove our claim.

Recall that $\widetilde{W}(A,\Gamma)$ consists of those $\widetilde{Q}_j \in \widetilde{W}$ that intersect Γ_0. Since each Whitney square has at most 20 neighboring squares, we can distribute the squares in $\widetilde{W}(A,\Gamma)$ into no more than 21 subcollections $\{\widetilde{W}_k\}_{k=1}^{21}$ such that in each of the subcollections, the squares are pairwise disjoint. Next, for any two distinct $\widetilde{Q}_i, \widetilde{Q}_j \in \widetilde{W}_k$, by Lemma 2.9, we have

$$\frac{11}{10}\widetilde{Q}_i \cap \frac{11}{10}\widetilde{Q}_j = \emptyset.$$

Clearly, for each $\widetilde{Q}_j \in \widetilde{W}(A,\Gamma)$, we have

$$\mathcal{H}^1\left(\frac{11}{10}\widetilde{Q}_j\cap\Gamma_0\right) \geqslant \frac{1}{10}\ell(\widetilde{Q}_j),$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. Recall that

$$\ell(\widetilde{Q}_j) \leqslant \mathrm{dist}(\widetilde{Q}_j, \partial\Omega) \leqslant 4\sqrt{2}\,\ell(\widetilde{Q}_j).$$

Hence, (4.52) gives

$$\begin{aligned}\sum_{\widetilde{Q}_j \in \widetilde{W}(\widetilde{A},\Gamma)} \ell(\widetilde{Q}_j)^{2-\hat{p}} &\lesssim \sum_{k=1}^{13} \sum_{\widetilde{Q}_j \in \widetilde{W}_k} \int_{\Gamma_0 \cap \frac{11}{10}\widetilde{Q}_j} \mathrm{dist}(z, \partial\Omega)^{1-\hat{p}} ds \\ &\lesssim \int_{\Gamma_0} \mathrm{dist}(z, \partial\Omega)^{1-\hat{p}} ds \lesssim \mathrm{diam}(S(\widetilde{A}))^{2-\hat{p}}.\end{aligned}$$

This completes the proof. □

4.3 Definition of the extension operator in the Jordan case

Recall from Subsection 4.2 that our conformal map $\varphi\colon \mathbb{D} \to \Omega$ satisfies $\varphi(0) = x_0$, where x_0 is a fixed John center of Ω. Let

$$B_\Omega = B(x_0, \mathrm{diam}(\Omega)).$$

Then $\overline{\Omega} \subset B_\Omega$. Recall from Lemma 2.9 that

$$\ell(\widetilde{Q}) \leqslant \mathrm{dist}(\widetilde{Q}, \partial\Omega)$$

for each $\widetilde{Q} \in \widetilde{W}$, the Whitney decomposition of $\widetilde{\Omega}$. Then, if $\widetilde{Q} \cap B_\Omega \neq \emptyset$, we obtain by definition that

$$\ell(\widetilde{Q}) \leqslant \mathrm{dist}(\widetilde{Q}, \partial\Omega) \leqslant \mathrm{diam}(\Omega).$$

Also, if $\widetilde{Q}' \in \widetilde{W}$ is a neighbor of $\widetilde{Q}$ with $\widetilde{Q} \cap B_\Omega \neq \emptyset$, then

$$\ell(\widetilde{Q}') \leqslant \mathrm{dist}(\widetilde{Q}', \partial\Omega) \leqslant (1+\sqrt{2})\mathrm{dist}(\widetilde{Q}, \partial\Omega) \leqslant 3\mathrm{diam}(\Omega).$$

Hence, the side lengths of all the Whitney squares $\widetilde{Q}$ that intersect B_Ω and of all their neighbors are at most $3\mathrm{diam}(\Omega)$.

Let $C(J)$ be the constant from Lemma 4.5. For each $\widetilde{Q}_i \in \widetilde{W}$ with $\ell(\widetilde{Q}_i) \leqslant 3\mathrm{diam}(\Omega)$, we consider the collection W_i of all squares $Q \in W$ that satisfy the conclusions of Lemma 4.5 for this value of $C(J)$. Then this collection is non-empty. We have to choose one Q from this collection. Since any choice will work, we may proceed as follows. Recall that W can be written as $\{Q_1, Q_2, \dots\}$. We pick the $Q_j \in W_i$ of smallest index j and define

$$\mathcal{R}(\widetilde{Q}_i) = Q_j.$$

It may happen that $\mathcal{R}(\widetilde{Q}_i) = \mathcal{R}(\widetilde{Q}_k)$ even when $k \neq i$ and there may well be squares $Q \in W$ for which there is no $\widetilde{Q}_i$ with $\mathcal{R}(\widetilde{Q}_i) = Q$. In fact, the number of distinct $\widetilde{Q}_i$ with $\mathcal{R}(\widetilde{Q}_i) = Q$ is always finite (see Lemma 4.12 in Subsection 4.5) but we do not have a uniform bound on the number of them. Nevertheless, Lemma 4.7 with work would allow us to control the sum of $\ell(\widetilde{Q}_i)^{2-\hat{p}}$ for the $\widetilde{Q}_i$ that satisfy $\mathcal{R}(\widetilde{Q}_i) = Q$. However, this would not suffice for our final estimate, as certain intermediate Whitney squares also come into the estimate. To overcome this, we eventually prove Lemma 4.13 that takes into consideration also these intermediate squares.

Pick a collection of functions $\phi_i \in C^\infty(\widetilde{\Omega})$ so that each ϕ_i is compactly supported in $\frac{11}{10}\widetilde{Q}_i$, $|\nabla\phi_i| \lesssim \ell(\widetilde{Q}_i)^{-1}$ and

$$\sum_i \phi_i(x) = 1$$

for all $x \in \widetilde{\Omega}$. Then the support of ϕ_i and that of ϕ_j have no intersection unless $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$ (see [23] for the existence of such a partition of unity $\{\phi_i\}$).

Given $u \in W^{1,p}(\Omega)$ and $\widetilde{Q}_i \in \widetilde{W}$ with $\ell(\widetilde{Q}_i) \leqslant 3\mathrm{diam}(\Omega)$, we set

$$a_i = \fint_{\mathcal{R}(\widetilde{Q}_i)} u(z)dz = \frac{1}{|\mathcal{R}(\widetilde{Q}_i)|}\int_{\mathcal{R}(\widetilde{Q}_i)} u(z)dz,$$

and we define $Eu(x) = u(x)$ in Ω and

$$Eu(x) = \sum_i a_i \phi_i(x) \tag{4.53}$$

for $x \in B_\Omega \setminus \overline{\Omega}$. Here, the sum runs over those i for which $\ell(\widetilde{Q}_i) \leqslant 3\mathrm{diam}(\Omega)$. We prove that $\|Eu\|_{W^{1,p}(B_\Omega\setminus\overline{\Omega})} \lesssim \|u\|_{W^{1,p}(\Omega)}$. We have not yet defined Eu on $\partial\Omega$. Since $\partial\Omega$ is of area zero by Lemma 2.24, this is not an issue, and we simply let $Eu(x) = 0$ for points in $\partial\Omega$.

Remark 4.9. A reader familiar with the extension operator employed in [23] perhaps wonders why we have chosen $\mathcal{R}(\widetilde{Q}_i)$ via the shadow of $\widetilde{Q}_i$ instead of picking a Whitney square Q of the diameter comparable to that of $\widetilde{Q}_i$ and at distance comparable to the diameter of $\widetilde{Q}_i$. Actually, such a square can be found as Ω is John, but we have not been able to establish useful estimates for the difference of averages over reflections of neighboring squares under this kind of a choice. One should view our construction of $\mathcal{R}$ as a kind of reflection via harmonic measure. In fact, the Jordan case in the setting of [23] is that of a quasidisk and for them our choice of $\mathcal{R}(\widetilde{Q}_i)$ can be checked to conform with the one used in [23]. We control the above difference of averages via suitable John subdomains of Ω. In our setting, these subdomains may well have bad overlaps contrary to what happens in [23] and in our adaptation of this technique in Section 3 (see (3.42)). The key point in what follows will be to obtain control on the overlaps in terms of the squares $\widetilde{Q}_i$.

4.4 Basic estimate

In order to estimate $|\nabla Eu|$ for the operator defined in (4.53), we need control on the differences of the averages of u over pairs of Whitney squares. Towards this, denote by $\widehat{|\nabla u|}$ the zero extension of $|\nabla u|$, and by M the Hardy-Littlewood maximal operator.

Lemma 4.10. *Given distinct Whitney squares $Q, Q' \subset \Omega$ such that*

$$\mathrm{dist}_\Omega(S(Q), S(Q')) \lesssim \ell(Q) \sim \ell(Q'), \tag{4.54}$$

we have

$$\left|\fint_Q u(z)dz - \fint_{Q'} u(z)dz\right| \leqslant C_0 \ell(Q)^{-1}\int_Q M(\widehat{|\nabla u|})(z)dz.$$

Here, C_0 depends only on J and the constants in (4.54).

Proof. Since Ω is John and $\varphi(0)$ is a John center of Ω, φ is η-quasisymmetric with respect to the inner distance by Lemma 2.30, where η depends only on the John constant J. Next, $\mathrm{dist}(A, \partial\mathbb{D}) = \mathrm{dist}(A, S(A))$ for each $A \subset \mathbb{D}$. Since $\varphi^{-1}(Q), \varphi^{-1}(Q')$ are of λ-Whitney type for some absolute constant λ by Lemma 2.12, we conclude that

$$\mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(S(Q))) \leqslant C(\lambda)\mathrm{diam}(\varphi^{-1}(Q)) \tag{4.55}$$

and

$$\mathrm{dist}(\varphi^{-1}(Q'), \varphi^{-1}(S(Q'))) \leqslant C(\lambda)\mathrm{diam}(\varphi^{-1}(Q')).$$

Let us show that quasisymmetry of φ allows us to translate (4.55) and its analog for Q' to Ω. Pick $z_1 \in \varphi^{-1}(Q)$ and $z_2 \in \varphi^{-1}(S(Q))$ such that

$$\mathrm{dist}(\varphi^{-1}(Q), \varphi^{-1}(S(Q))) = |z_1 - z_2|, \tag{4.56}$$

and let $z_3 \in \varphi^{-1}(Q)$ be a point such that

$$\mathrm{diam}(\varphi^{-1}(Q)) \leqslant 2|z_1 - z_3|. \tag{4.57}$$

Recall that $\varphi(z_2)$ is rectifiably joinable, i.e., to $\varphi(0)$ by Remark 2.20. Pick points w_j along this rectifiable curve so that w_j tend to $\varphi(z_2)$ and $\mathrm{dist}_\Omega(\varphi(z_1), w_j)$ tends to $\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_2))$. Since φ is homeomorphic up to boundary, it follows that $\varphi^{-1}(w_j)$ tend to z_2. Hence, by (4.55)–(4.57), we have

$$|z_1 - \varphi^{-1}(w_j)| \leqslant C(\lambda)|z_1 - z_3|$$

when j is sufficiently large. Then the quasisymmetry of φ gives

$$\mathrm{dist}_\Omega(\varphi(z_1), w_j) \leqslant C(J, \lambda)\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_3))$$

for all sufficiently large j. Since $\mathrm{dist}_\Omega(\varphi(z_1), w_j)$ tends to $\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_2))$, we deduce that

$$\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_2)) \leqslant C(J, \lambda)\mathrm{dist}_\Omega(\varphi(z_1), \varphi(z_3)).$$

Hence,

$$\mathrm{dist}_\Omega(Q, S(Q)) \lesssim \mathrm{diam}_\Omega(Q) \sim \ell(Q) \tag{4.58}$$

with constants depending only on λ and J. Similarly,

$$\mathrm{dist}_\Omega(Q', S(Q')) \lesssim \mathrm{diam}_\Omega(Q') \sim \ell(Q'). \tag{4.59}$$

By the triangle inequality (see Lemma 2.16), (4.58), (4.59), (4.54) and Lemma 4.3, we conclude that

$$\begin{aligned}\mathrm{dist}_\Omega(Q, Q') &\lesssim \mathrm{dist}_\Omega(Q, S(Q)) + \mathrm{diam}_\Omega(S(Q)) + \mathrm{dist}_\Omega(S(Q), S(Q'))\\ &\quad + \mathrm{diam}_\Omega(S(Q')) + \mathrm{dist}_\Omega(Q, S(Q'))\\ &\lesssim \ell(Q)\end{aligned}$$

with constants depending only on λ and J. By Lemma 2.7, we deduce from this that the length of the hyperbolic segment Γ between the centers of Q and Q' is no more than a constant (depending only on the constants in (4.54) and the John constant J) multiple of $\ell(Q)$.

Next, we construct a John subdomain $\Omega_{Q,Q'} \subset \Omega \cap CQ$ of the diameter no more than $C\ell(Q)$, containing both Q and Q', where C depends only on the John constant J. Towards this, set

$$\Omega_{Q,Q'} = Q \cup Q' \cup \bigcup_{z\in\Gamma} B(z, 3^{-1}\mathrm{dist}(z, \partial\Omega)),$$

where Γ is the above hyperbolic segment between the centers of Q and Q'. To see that $\Omega_{Q,Q'}$ is John, let z_0 be the middle point (in the sense of the length) of Γ and consider, for a given $z \in \Omega_{Q,Q'}$, the following curve γ: the first part of the curve is a line segment from z to $z_1 \in \Gamma$, where $z \in B(z_1, 3^{-1}\mathrm{dist}(z_1, \partial\Omega))$ or z_1 is the center of Q (or Q') if $z \in Q$ (or $z \in Q'$), and the second part coincides with $\Gamma[z_1, z_0]$. Since a simply connected John domain Ω is (quantitatively) inner uniform and we can use hyperbolic segments as the curves required in (2.39) (see Definition 2.27 and Lemma 2.28), it follows that the above curve is a John curve of $\Omega_{Q,Q'}$ between z and z_0, with a constant depending only on J.

By letting

$$a = \fint_{\Omega_{Q,Q'}} u dz, \quad a_Q = \fint_Q u(z)dz, \quad a_{Q'} = \fint_{Q'} u(z)dz,$$

the Poincaré inequality on $\Omega_{Q,Q'}$ from [2] (with a constant depending only on J) and (4.54) imply

$$\begin{aligned}|a_Q - a_{Q'}| &\leqslant |a_Q - a| + |a_{Q'} - a| \lesssim \fint_Q |u - a|dz + \fint_{Q'} |u - a|dz\\ &\lesssim \ell(Q)^{-1}\int_{\Omega_{Q,Q'}} |\nabla u(z)|dz \lesssim \ell(Q)\fint_{CQ} \widehat{|\nabla u|}(z)dz\\ &\lesssim \ell(Q)\fint_Q M(\widehat{|\nabla u|})(z)dz \lesssim \ell(Q)^{-1}\int_Q M(\widehat{|\nabla u|})(z)dz.\end{aligned}$$

This completes the proof. □

4.5 Intermediate Whitney squares

We employ Lemma 4.10 to estimate $|a_{\mathcal{R}(\widetilde{Q}_i)} - a_{\mathcal{R}(\widetilde{Q}_j)}|$ for pairs of neighboring squares $\widetilde{Q}_i$ and $\widetilde{Q}_j$. Unfortunately, the reflected squares need not have comparable size (see Figure 9), and hence we cannot always directly rely on Lemma 4.10. To fix this problem, we construct chains of suitable intermediate Whitney squares in order to be able to use our estimate. Each of these chains consists of a finite number of elements, but there is no uniform bound for these numbers.

Lemma 4.11. *Let $\widetilde{Q}_i$ and $\widetilde{Q}_j$ be distinct squares so that $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$ and*

$$\ell(\widetilde{Q}_i), \ell(\widetilde{Q}_j) \leqslant 3\mathrm{diam}(\Omega).$$

Suppose that $\mathrm{diam}(S(\widetilde{Q}_i)) \leqslant \mathrm{diam}(S(\widetilde{Q}_j))$. *Then there exist*

$$l = l(i,j) \in \mathbb{N} \quad \textit{and} \quad G(\widetilde{Q}_i, \widetilde{Q}_j) := \{Q^0, \dots, Q^l\}$$

consisting of squares of W so that

$$Q^0 = \mathcal{R}(\widetilde{Q}_i) \quad \textit{and} \quad Q^l = \mathcal{R}(\widetilde{Q}_j), \tag{4.60}$$

for $0 \leqslant n \leqslant l-1$, we have the estimate

$$\mathrm{dist}_\Omega(S(Q^n), S(Q^{n+1})) \lesssim \ell(Q^n) \sim \ell(Q^{n+1}), \tag{4.61}$$

and for $0 \leqslant m \leqslant l$, we have the estimate

$$\ell(Q^m) \sim 2^{-m}\mathrm{diam}(S(\widetilde{Q}_j)) \tag{4.62}$$

with constants depending only on J.

Proof. Let distinct squares $\widetilde{Q}_i$ and $\widetilde{Q}_j$ with $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$ satisfy both $\mathrm{diam}(S(\widetilde{Q}_i)) \leqslant \mathrm{diam}(S(\widetilde{Q}_j))$ and $\ell(\widetilde{Q}_i), \ell(\widetilde{Q}_j) \leqslant 3\mathrm{diam}(\Omega)$. If

$$\frac{1}{8}\mathrm{diam}(S(\widetilde{Q}_j)) \leqslant \mathrm{diam}(S(\widetilde{Q}_i)), \tag{4.63}$$

we set $l(i,j) = 1$ and define $Q^0 = \mathcal{R}(\widetilde{Q}_i)$ and $Q^1 = \mathcal{R}(\widetilde{Q}_j)$. Then by Lemmas 4.3 and 4.4 and the fact that $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$, we have that (4.61) holds with constants depending only on J. Moreover, (4.62) holds with an absolute constant.

Suppose that (4.63) fails. Set $Q^0 = \mathcal{R}(\widetilde{Q}_i)$. Pick a connected closed set $\widetilde{F}^1$ (referred to as a fake square) such that $\widetilde{\Omega} \setminus \widetilde{F}^1$ is connected,

$$\widetilde{Q}_i \subset \widetilde{F}^1 \subset \widetilde{Q}_i \cup \widetilde{Q}_j, S(\widetilde{Q}_i) \subset S(\widetilde{F}^1)$$

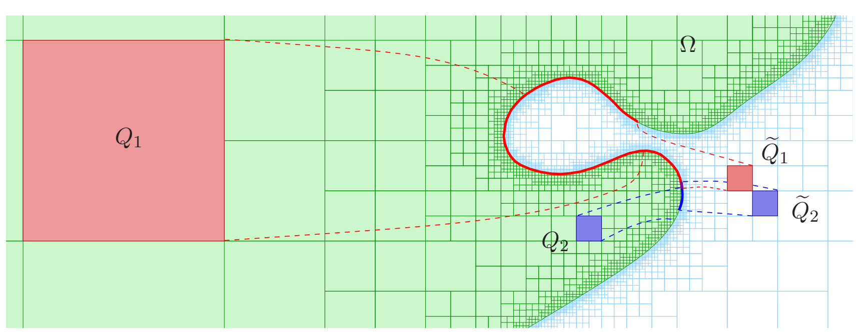

Figure 9 (Color online) The shadows of neighboring squares $\widetilde{Q}_1$ and $\widetilde{Q}_2$ can differ significantly in size from each other. Consequently, the reflected squares Q_1 and Q_2 may be of very different sizes

and

$$2\mathrm{diam}(S(\widetilde{F}^1)) = \mathrm{diam}(S(\widetilde{Q}_i \cup \widetilde{Q}_j)). \tag{4.64}$$

The existence of $\widetilde{F}^1$ is clear since $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \mathbb{D} \to \overline{\widetilde{\Omega}}$ is a homeomorphism and conformal outside $\overline{\mathbb{D}}$. For example, we can construct $\widetilde{F}^1$ in the following way. Since $\widetilde{\varphi}$ is a homeomorphism, we know that both $\widetilde{\varphi}^{-1}(\partial\widetilde{Q}_i)$ and $\widetilde{\varphi}^{-1}(\partial\widetilde{Q}_j)$ are Jordan curves, and they intersect each other. Pick $z \in \partial\widetilde{Q}_i \cap \partial\widetilde{Q}_j$. Then parameterizing $\widetilde{\varphi}^{-1}(\partial\widetilde{Q}_j)$ via $\gamma\colon [0,1] \to \widetilde{\varphi}^{-1}(\partial\widetilde{Q}_j)$ with $\gamma(0) = \gamma(1) = z$, by continuity, we see that there is $0 < t < 1$ such that by letting $\widetilde{F}^1 = \widetilde{\varphi}(\gamma[0,t] \cup \widetilde{Q}_i)$, we have that (4.64) holds; notice that the preimages under $\widetilde{\varphi}$ of hyperbolic rays are radial rays, and then $\widetilde{\varphi}^{-1}(S(\partial\widetilde{Q}_j)) = \widetilde{\varphi}^{-1}(S(\widetilde{Q}_j))$. Then by our construction, it is clear that $\widetilde{Q}_i \subset \widetilde{F}^1 \subset \widetilde{Q}_i \cup \widetilde{Q}_j$ and $\widetilde{\Omega} \setminus \widetilde{F}^1$ is connected. Hence, $\widetilde{F}^1$ is a desired set.

Notice that $\widetilde{F}^1$ is a Whitney-type set since $\ell(\widetilde{Q}_i) \sim \ell(\widetilde{Q}_j) \sim \mathrm{diam}(\widetilde{F}^1)$ and $\widetilde{Q}_i \subset \widetilde{F}^1$. By Lemma 4.4, there is a Whitney square $Q^1 \in W$ such that

$$\mathrm{diam}(S(Q^1)) \leqslant C(J)\mathrm{diam}(S(\widetilde{F}^1))$$

and

$$\mathrm{diam}(S(\widetilde{F}^1)) \leqslant C(J)\mathrm{diam}(S(Q^1) \cap S(\widetilde{F}^1)),$$

where $C(J)$ depends only on J. We did not need the assumption that $\widetilde{\Omega} \setminus \widetilde{F}^1$ is connected above; we will later use it in order to apply Lemma 4.7.

Next, we pick a connected closed set $\widetilde{F}^2$ such that $\widetilde{\Omega} \setminus \widetilde{F}^2$ is connected, $\widetilde{Q}_i \subset \widetilde{F}^2 \subset \widetilde{F}^1 \subset \widetilde{Q}_i \cup \widetilde{Q}_j$, $S(\widetilde{Q}_i) \subset S(\widetilde{F}^2)$ and

$$4\mathrm{diam}(S(\widetilde{F}^2)) = \mathrm{diam}(S(\widetilde{Q}_i \cup \widetilde{Q}_j)),$$

and select a Whitney square $Q^2 \subset \Omega$ such that

$$\mathrm{diam}(S(Q^2)) \leqslant C(J)\mathrm{diam}(S(\widetilde{F}^2))$$

and

$$\mathrm{diam}(S(\widetilde{F}^2)) \leqslant C(J)\mathrm{diam}(S(Q^2) \cap S(\widetilde{F}^2)),$$

where $C(J)$ depends only on J. We continue this process to find squares $Q^l \in W$ until we have

$$\frac{1}{2}\mathrm{diam}(S(\widetilde{F}^l)) \leqslant \mathrm{diam}(S(\widetilde{Q}_i)) \leqslant \mathrm{diam}(S(\widetilde{F}^l))$$

for some $l \in \mathbb{N}$.

By our construction,

$$2^m\mathrm{diam}(S(\widetilde{F}^m)) = \mathrm{diam}(S(\widetilde{Q}_i \cup \widetilde{Q}_j)) \tag{4.65}$$

for $m = 1, \dots, l$. Next, Q^m was obtained via Lemma 4.4, where the corresponding square satisfies by (4.26) the additional requirement that

$$\mathrm{diam}(S(Q^m)) \sim_J \ell(Q^m). \tag{4.66}$$

Taking into account the estimate

$$\mathrm{diam}(S(\widetilde{F}^m)) \lesssim \mathrm{diam}(S(Q^m)) \lesssim \mathrm{diam}(S(\widetilde{F}^m)) \tag{4.67}$$

with constants depending only on J that follows from our choice of Q^m, we conclude with (4.62).

Regarding (4.61), we recall from the construction that $S(Q^m) \cap S(\widetilde{F}^m) \neq \emptyset$ and $S(\widetilde{F}^n) \cap S(\widetilde{F}^{n+1}) \neq \emptyset$ for all relevant n and m. Since dist_Ω satisfies a triangle inequality by Lemma 2.16, we conclude that

$$\begin{aligned}&\mathrm{dist}_\Omega(S(Q^n), S(Q^{n+1}))\\ &\quad\lesssim \mathrm{diam}_\Omega(S(Q^n)) + \mathrm{diam}_\Omega(S(\widetilde{F}^n)) + \mathrm{diam}_\Omega(S(\widetilde{F}^{n+1})) + \mathrm{diam}_\Omega(S(Q^{n+1})).\end{aligned} \tag{4.68}$$

Hence, (4.65)–(4.68) together with Lemma 2.29 give (4.61). □

From now on, in this subsection, we always assume that $\ell(\widetilde{Q}_i) \leqslant 3\mathrm{diam}(\Omega)$ and $\ell(\widetilde{Q}_j) \leqslant 3\mathrm{diam}(\Omega)$. We call such Whitney squares allowable. The preceding lemma gives the chain $G(\widetilde{Q}_i, \widetilde{Q}_j)$ when

$$\mathrm{diam}(S(\widetilde{Q}_i)) \leqslant \mathrm{diam}(S(\widetilde{Q}_j)).$$

Especially, both $G(\widetilde{Q}_i, \widetilde{Q}_j)$ and $G(\widetilde{Q}_j, \widetilde{Q}_i)$ have been constructed when

$$\mathrm{diam}(S(\widetilde{Q}_i)) = \mathrm{diam}(S(\widetilde{Q}_j)).$$

Even though the claim of the lemma does not imply that these chains coincide as sets, the construction in the proof of the lemma gives this. In order not to make our notation overly complicated, we abuse notation and extend our definition also to the case where

$$\mathrm{diam}(S(\widetilde{Q}_i)) > \mathrm{diam}(S(\widetilde{Q}_j))$$

by setting $G(\widetilde{Q}_i, \widetilde{Q}_j) := G(\widetilde{Q}_j, \widetilde{Q}_i)$. Under this convention, Q^0 is one of the squares $\mathcal{R}(\widetilde{Q}_i)$ and $\mathcal{R}(\widetilde{Q}_j)$, Q^l, $l = l(i,j) = l(j,i)$ is the other one, and (4.61) holds as stated, but for (4.62), we need to replace $\mathrm{diam}(S(\widetilde{Q}_j))$ with the maximum of $\mathrm{diam}(S(\widetilde{Q}_j))$ and $\mathrm{diam}(S(\widetilde{Q}_j))$.

Given an allowable $\widetilde{Q}_i \in \widetilde{W}$, we define $G(\widetilde{Q}_i) = \bigcup_j G(\widetilde{Q}_i, \widetilde{Q}_j)$, where the union runs over all the squares $\widetilde{Q}_j \in \widetilde{W}$ that intersect $\widetilde{Q}_i$.

Our next lemma gives estimates for the overlaps of our chains.

Lemma 4.12. *There is a positive integer $N = N(J)$ so that*

$$\sum_{Q \in G(\widetilde{Q}_i, \widetilde{Q}_j)} \chi_Q(x) \leqslant N \tag{4.69}$$

for all i and j and every $x \in \Omega$. Moreover,

$$\sum_j \sum_{Q \in G(\widetilde{Q}_i, \widetilde{Q}_j)} \chi_Q(x) \leqslant 20 \sum_{Q \in G(\widetilde{Q}_i)} \chi_Q(x) \tag{4.70}$$

for each i, and

$$\sum_i \sum_{Q \in G(\widetilde{Q}_i)} \chi_Q(x) < \infty \tag{4.71}$$

for all $x \in \Omega$.

Proof. The first claim follows from (4.62). The second claim is an immediate consequence of the fact that the Whitney square $\widetilde{Q}_i$ has at most 20 neighbors. Towards the final claim, recall that $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \mathbb{D} \to \overline{\widetilde{\Omega}}$ is a homeomorphism (and conformal in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$). This implies that the diameter of the shadow of $\tilde{A}$ tends to zero uniformly when $\mathrm{diam}(\tilde{A}) \to 0$. Consequently, given $\delta > 0$, there can be only a finite number of $\widetilde{Q}_j \in \widetilde{W}$ with $\ell(\widetilde{Q}_i) \leqslant 3\mathrm{diam}(\Omega)$ for which $\mathrm{diam}(S(\widetilde{Q}_i \cup \widetilde{Q}_j)) \geqslant \delta$ for some neighbor $\widetilde{Q}_i$ of $\widetilde{Q}_j$. The final claim follows from this together with our first claim, (4.66) and (4.67). □

Notice that we are not claiming a uniform bound for the number of distinct $\widetilde{Q}_i$ for which a given Q belongs to $G(\widetilde{Q}_i)$. In fact, such a bound does not necessarily exist. The following lemma provides us with a crucial substitute for such an estimate.

Lemma 4.13. *For each $Q \in W$, we have*

$$\sum_{Q \in G(\widetilde{Q}_i)} \ell(\widetilde{Q}_i)^{2-\hat{p}} \lesssim \ell(Q)^{2-\hat{p}},$$

where the constant depends only on $\hat{p}$ and the constant C in (4.1).

Proof. Recall that Ω is J-John with a constant that depends only on $\hat{p}$ and the constant C in (4.1) by Lemma 2.23.

Fix $Q \in W$ so that $Q \in G(\widetilde{Q}_i)$ for at least one i. By Lemma 4.12, the number of such indices i is finite, and for each of the at most 20 neighbors $\widetilde{Q}_j$, Q corresponds to at most $N(J)$ different fake squares $\widetilde{F}^m_{i,j}$ used in the construction of $G(\widetilde{Q}_i, \widetilde{Q}_j)$. Consider this finite collection of the sets $\widetilde{F}^m_{i,j}$. We relabel them as $\widetilde{F}_n$ with respect to n, i.e., $1 \leqslant n \leqslant k$, so that the diameters of $\widetilde{\varphi}^{-1}(S(\widetilde{F}_n))$ decrease when n increases.

We set $\widetilde{F}^1_Q := \widetilde{F}_k$ and stop the construction if $k = 1$ or the shadow of each F_n with $1 \leqslant n \leqslant k-1$ intersects the shadow of $\widetilde{F}^1_Q$. If this is not the case and $S(\widetilde{F}_{k-1}) \cap S(\widetilde{F}^1_Q) = \emptyset$, we set $\widetilde{F}^2_Q = \widetilde{F}_{k-1}$. Otherwise, we consider $\widetilde{F}_{k-2}$ as a candidate for $\widetilde{F}^2_Q$ and continue inductively via the following procedure. We choose $\widetilde{F}^2_Q$ to be $\widetilde{F}_n$ for the largest integer n smaller than k for which $S(\widetilde{F}_n) \cap S(\widetilde{F}^1_Q) = \emptyset$. We stop the process if $n = 1$ or the shadow of each F_m with $1 \leqslant m \leqslant n-1$ intersects $S(\widetilde{F}^1_Q)$ or $S(\widetilde{F}^2_Q)$. Otherwise, we choose $\widetilde{F}^3_Q$ to be $\widetilde{F}_m$ with the largest $m \leqslant n-1$ such that its shadow does not intersect $S(\widetilde{F}^1_Q)$ nor $S(\widetilde{F}^2_Q)$, and continue this process. This gives us $\widetilde{F}^1_Q, \ldots, \widetilde{F}^{n_0}_Q$ with pairwise disjoint shadows. By the construction of these sets, Lemma 4.6 gives us a universal bound on n_0 in terms of $C(J)$ (see (4.18) and (4.19)).

Let $\widetilde{F}_n$ be a set from above, which was not chosen as one of the sets $\widetilde{F}^i_Q$. By the construction in the previous paragraph, there is an index l so that $S(\widetilde{F}_n) \cap S(\widetilde{F}^l_Q) \neq \emptyset$. Since $\widetilde{\varphi}^{-1}(S(\widetilde{F}_n)) = S(\widetilde{\varphi}^{-1}(\widetilde{F}_n))$ and $\widetilde{\varphi}^{-1}(S(\widetilde{F}^l_Q)) = S(\widetilde{\varphi}^{-1}(\widetilde{F}^l_Q))$ are closed arcs of the unit circle, at least one of the endpoints of $S(\widetilde{F}^l_Q)$ is contained in $S(\widetilde{F}_n)$; otherwise, $S(\widetilde{F}_n)$ is strictly contained in $S(\widetilde{F}^l_Q)$, which means that

$$\operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{F}^l_Q))) > \operatorname{diam}(\widetilde{\varphi}^{-1}(S(\widetilde{F}_n))),$$

contradicting our selection of the sets $\widetilde{F}^l_Q$. Therefore, by assigning two hyperbolic rays to each $\widetilde{F}^l_Q$, we obtain a collection of $2n_0$ hyperbolic rays that intersect all of our sets $\widetilde{F}^m_{i,j}$ with $i \in I(Q)$.

Let Γ be one of our $2n_0$ hyperbolic rays. Denote by Γ_0 the tail of Γ with respect to a set in

$$\{\widetilde{F}^m_{i,j} \mid Q \in G(\widetilde{Q}_i), \Gamma \cap \widetilde{F}^m_{i,j} \neq \emptyset\},$$

whose preimage under $\widetilde{\varphi}$ is furthest away from the origin, i.e., a last set that Γ hits towards infinity. Let $\widetilde{F}_0$ be such a set. Then $\ell(Q) \sim \operatorname{diam}(S(\widetilde{F}_0))$ by (4.62) as $\widetilde{F}_0$ is one of the sets $\widetilde{F}^m_{i,j}$. Moreover, $\widetilde{F}_0$ is of $8\sqrt{2}$-Whitney type and $\widetilde{\Omega} \setminus \widetilde{F}_0$ is connected since $\widetilde{F}_0$ is one of the sets $\widetilde{F}^m_{i,j}$ (see Figure 10 for an illustration). Hence, Lemma 4.7 gives the estimate

$$\sum_{\widetilde{Q}_l \in \widetilde{W}, \widetilde{Q}_l \cap \Gamma_0 \neq \emptyset} \ell(\widetilde{Q}_l)^{2-\hat{p}} \lesssim \ell(Q_m)^{2-\hat{p}} \tag{4.72}$$

with a constant that depends only on $\hat{p}$ and the constant C in (4.1).

Since each $\widetilde{F}^m_{i,j}$ is a subset of $\widetilde{Q}_i \cup \widetilde{Q}_j$ where $\widetilde{Q}_i \cap \widetilde{Q}_j \neq \emptyset$, each Whitney square has at most 20 neighbors, and the number n_0 of our hyperbolic rays is bounded in terms of J, our claim follows from (4.72). $\square$

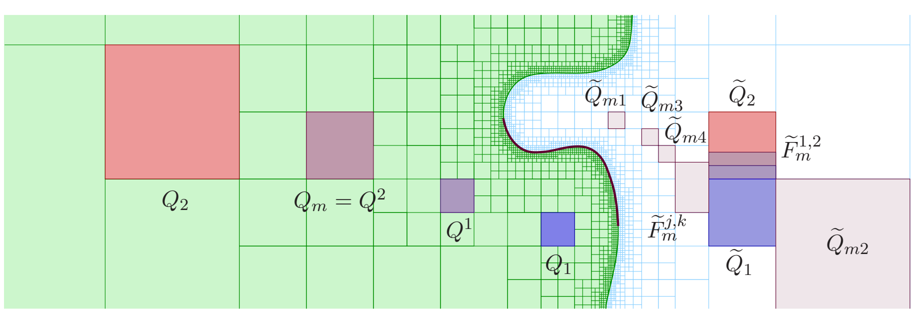

Figure 10 (Color online) A square $Q \in W$ might be associated with several squares $\widetilde{Q}_l$ as well as to fake squares $\widetilde{F}^{j,k}_Q$. In the illustration, the squares $\widetilde{Q}_1$ and $\widetilde{Q}_2$ give rise to two fake squares, one of which is associated with Q. Another fake square as well as four (real) squares that are associated with Q are exhibited. Also, the shadow of Q is shown

4.6 Sufficiency in the Jordan case

Recall the definition of Eu via (4.53) from Subsection 4.3 and of the chains $G(\widetilde{Q}_i,\widetilde{Q}_j)$ and the sets $G(\widetilde{Q}_i)$ from Subsection 4.5. We begin by estimating the norm of the gradient of our extension over each square $\widetilde{Q}\in\widetilde{W}$ with $\widetilde{Q}\cap B_\Omega\neq\emptyset$.

Lemma 4.14. *For all $\widetilde{Q}_i\in\widetilde{W}$ with $\widetilde{Q}_i\cap B_\Omega\neq\emptyset$, we have*

$$\|\nabla Eu\|^p_{L^p(\widetilde{Q}_i)}\leqslant C\sum_i\sum_{Q\in G(\widetilde{Q}_i,\widetilde{Q}_k)}\ell(\widetilde{Q}_i)^{2-\hat{p}}\ell(Q)^{\hat{p}-2}\int_Q(M(|\widehat{\nabla u}|)(z))^pdz,$$

where the sum is over all the indices k for which $\widetilde{Q}_k\cap\widetilde{Q}_j\neq\emptyset$. Here, C depends only on $p,\hat{p}$ and the constant C in (4.1).

Proof. Recall that $\widetilde{\varphi}:\mathbb{R}^2\setminus\overline{\mathbb{D}}\to\widetilde{\Omega}$ extends homeomorphically up to the boundary.

Fix $\widetilde{Q}_j$ with $\widetilde{Q}_j\cap\widetilde{Q}_i\neq\emptyset$. Let $Q^n,Q^{n+1}\in G(\widetilde{Q}_i,\widetilde{Q}_j)$ be consecutive squares. Then

$$\operatorname{dist}_\Omega(S(Q^n),S(Q^{n+1}))\lesssim_J\ell(Q^n)\sim_J\ell(Q^{n+1})\tag{4.73}$$

by (4.61).

Let $q>0$. Then by (4.62), together with Lemma 4.3, we have the estimate

$$\begin{aligned}\sum_{Q^n\in G(\widetilde{Q}_i,\widetilde{Q}_j)}\ell(Q^n)^{-q}&\leqslant C(q,J)\min\{\operatorname{diam}(S(\widetilde{Q}_i)),\operatorname{diam}(S(\widetilde{Q}_j))\}^{-q}\\&\leqslant C(q,J)\ell(\widetilde{Q}_i)^{-q}.\end{aligned}\tag{4.74}$$

Recall that $\{\phi_k\}$ is a partition of unity with $\phi_k=0$ in $\widetilde{Q}_j$ if $\widetilde{Q}_j\cap\widetilde{Q}_k=\emptyset$. Hence, for each $x\in\widetilde{Q}_i$, we have

$$\nabla Eu(x)=\nabla\bigg(\sum_{\widetilde{Q}_k\cap\widetilde{Q}_i\neq\emptyset}a_k\phi_k(x)\bigg)=\nabla\bigg(\sum_{\widetilde{Q}_k\cap\widetilde{Q}_i\neq\emptyset}(a_k-a_i)\phi_k(x)\bigg),$$

where a_l refers to the average of u over $\mathcal{R}(\widetilde{Q}_l)$. Since $|\nabla\phi_k|\lesssim\ell(\widetilde{Q}_i)^{-1}$ whenever $\widetilde{Q}_k\cap\widetilde{Q}_i\neq\emptyset$, we further have

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(\widetilde{Q}_i)}&\lesssim\int_{\widetilde{Q}_i}\sum_{\widetilde{Q}_k\cap\widetilde{Q}_i\neq\emptyset}|a_k-a_i|^p|\nabla\phi_k(x)|^pdx\\&\lesssim\sum_{\widetilde{Q}_k\cap\widetilde{Q}_i\neq\emptyset}|a_k-a_i|^p\ell(\widetilde{Q}_j)^{-p}|\widetilde{Q}_i|\\&\lesssim\sum_{\widetilde{Q}_k\cap\widetilde{Q}_i\neq\emptyset}|a_k-a_i|^p\ell(\widetilde{Q}_i)^{2-p}\end{aligned}\tag{4.75}$$

with an absolute constant.

Let $\epsilon=\frac{\hat{p}-p}{p}>0$. We apply Lemma 4.10 via (4.73), Hölder's inequality and (4.74) with $q=\frac{\epsilon p}{p-1}$ to get

$$\begin{aligned}|a_k-a_i|^p&\lesssim\bigg(\sum_{Q^n\in G(\widetilde{Q}_k,\widetilde{Q}_i)}|a_{Q^n}-a_{Q^{n+1}}|\bigg)^p\\&\lesssim\bigg(\sum_{Q^n\in G(\widetilde{Q}_k,\widetilde{Q}_i)}\ell(Q^n)\fint_{Q^n}M(|\widehat{\nabla u}|)(z)dz\bigg)^p\\&\lesssim\bigg[\sum_{Q^n\in G(\widetilde{Q}_k,\widetilde{Q}_i)}\ell(Q^n)^{1+\epsilon-\epsilon}\bigg(\fint_{Q^n}(M(|\widehat{\nabla u}|)(z))^pdz\bigg)^{\frac{1}{p}}\bigg]^p\\&\lesssim\bigg(\sum_{Q_m\in G(\widetilde{Q}_k,\widetilde{Q}_i)}\ell(Q^n)^{p+p\epsilon}\fint_{Q^n}(M(|\widehat{\nabla u}|)(z))^pdz\bigg)\bigg(\sum_{Q^n\in G(\widetilde{Q}_k,\widetilde{Q}_i)}\ell(Q^n)^{-\frac{\epsilon p}{p-1}}\bigg)^{p-1}\end{aligned}$$

$$\lesssim \ell(\widetilde{Q}_i)^{-\epsilon p} \sum_{Q^n \in G(\widetilde{Q}_k, \widetilde{Q}_i)} \ell(Q^n)^{p+p\epsilon-2} \int_{Q^n} (M(\widehat{|\nabla u|})(z))^p dz.$$

Above, the constants depend only on $p, \hat{p}$ and the constant C in (4.1).

By recalling that $\epsilon p = \hat{p} - p$ and inserting the above estimate into (4.75), we obtain

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(\widetilde{Q}_i)} &\lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q}_i \neq \emptyset} |a_k - a_i|^p \ell(\widetilde{Q}_i)^{2-p} \\ &\lesssim \sum_{\widetilde{Q}_k \cap \widetilde{Q}_i \neq \emptyset} \sum_{Q^n \in G(\widetilde{Q}_i, \widetilde{Q}_k)} \ell(\widetilde{Q}_i)^{2-\hat{p}} \ell(Q^n)^{\hat{p}-2} \int_{Q^n} (M(\widehat{|\nabla u|})(z))^p dz\end{aligned}$$

with the desired control on the constants. □

Proof of Theorem 4.1. Recall that $B_\Omega = B(x_0, \operatorname{diam}(\Omega))$, Eu is defined on B_Ω as in (4.53), and $\ell(\widetilde{Q}) \leqslant 3\operatorname{diam}(\Omega)$ whenever $\widetilde{Q} \in \widetilde{W}$ intersects B_Ω or is a neighbor of such a square. By Lemma 4.14, we have

$$\|\nabla Eu\|^p_{L^p(B_\Omega \setminus \overline{\Omega})} \lesssim \sum_{\widetilde{Q}_i \cap B_\Omega \neq \emptyset} \sum_{\widetilde{Q}_k \cap \widetilde{Q}_i \neq \emptyset} \sum_{Q \in G(\widetilde{Q}_i, \widetilde{Q}_k)} \ell(\widetilde{Q}_i)^{2-\hat{p}} \ell(Q)^{\hat{p}-2} \int_Q (M(\widehat{|\nabla u|})(z))^p dz$$

with a constant depending only on our data: $p, \hat{p}$ and the constant C in (4.1).

Towards interchanging the order of summation, we notice that a fixed $Q \in W$ appears in our triple sum only when $Q \in G(\widetilde{Q}_i)$ in which case it is counted for each of the at most 20 neighbors $\widetilde{Q}_j$ at most $N(J)$ times by Lemma 4.12. Hence, by interchanging the order of summation (Tonelli's theorem), we obtain by Lemma 4.13 the estimate

$$\begin{aligned}\|\nabla Eu\|^p_{L^p(B_\Omega \setminus \overline{\Omega})} &\lesssim \sum_{\widetilde{Q}_i \cap B_\Omega \neq \emptyset} \sum_{\widetilde{Q}_k \cap \widetilde{Q}_i \neq \emptyset} \sum_{Q \in G(\widetilde{Q}_i, \widetilde{Q}_k)} \ell(\widetilde{Q}_i)^{2-\hat{p}} \ell(Q)^{\hat{p}-2} \int_Q (M(\widehat{|\nabla u|})(z))^p dz \\ &\lesssim \sum_{Q \in W} \sum_{Q \in G(\widetilde{Q}_i)} \ell(\widetilde{Q}_i)^{2-\hat{p}} \ell(Q)^{\hat{p}-2} \int_Q (M(\widehat{|\nabla u|})(z))^p dz \\ &\lesssim \sum_{Q \in W} \int_Q (M(\widehat{|\nabla u|})(z))^p dz \\ &\lesssim \int_{\mathbb{R}^2} \widehat{|\nabla u|}^p(z) dz \leqslant \int_\Omega |\nabla u|^p dz. \end{aligned} \tag{4.76}$$

Here, the constants depend only on our data.

Next, recall that $Eu(x) = \sum_j a_j \phi_j(x)$ when $x \in B_\Omega \setminus \overline{\Omega}$, where a_j is the average over $\mathcal{R}(\widetilde{Q}_j) \in W$ with $\ell(\widetilde{Q}_j) \leqslant 3\operatorname{diam}(\Omega)$. Write $\mathcal{R}^{-1}(Q)$ for the collection of all $\widetilde{Q}_j \in \widetilde{W}$ with $Q = \mathcal{R}(\widetilde{Q}_j)$. Now

$$\sum_{\widetilde{Q}_j \in \mathcal{R}^{-1}(Q)} \ell(\widetilde{Q}_j)^2 \leqslant C(J) \ell(Q)^2$$

since for every $\widetilde{Q}_j \in \mathcal{R}^{-1}(Q)$, we have $\widetilde{Q}_j \subset C(J)Q$ by Lemma 4.3, (4.20) and the triangle inequality. Then, by the definition of Eu, Tonelli's theorem for series and Hölder's inequality, we obtain

$$\begin{aligned}\|Eu\|^p_{L^p(B_\Omega \setminus \overline{\Omega})} &\lesssim \sum_{Q \in W} \sum_{\widetilde{Q}_j \in \mathcal{R}(Q)} \ell(\widetilde{Q}_j)^2 \left(\fint_Q |u| dx \right)^p \\ &\lesssim \sum_{Q \in W} \sum_{\widetilde{Q}_j \in \mathcal{R}^{-1}(Q)} \ell(\widetilde{Q}_j)^2 \ell(Q)^{-2} \int_Q |u|^p dx \\ &\lesssim \sum_{Q \in W} \int_Q |u|^p dx \lesssim \int_\Omega |u|^p dx \end{aligned} \tag{4.77}$$

with constants depending only on our data. By combining (4.76) and (4.77), we conclude that

$$\int_{B_\Omega \setminus \partial\Omega} (|\nabla Eu|^p + |Eu|^p) dx \leqslant C\|u\|^p_{W^{1,p}(\Omega)},$$

where C depends only on $p, \hat{p}$ and the constant C in (4.1).

Suppose now that $u \in W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega})$. We extend Eu to all of B_Ω by letting

$$\hat{E}u(x) = Eu(x) \quad \text{when } x \in B_\Omega \setminus \partial\Omega, \quad \hat{E}u(x) = u(x) \quad \text{when } x \in \partial\Omega.$$

We claim that $\hat{E}u(x)$ is continuous in B_Ω.

Notice that Eu is clearly continuous (even smooth) in $B_\Omega \setminus \overline{\Omega}$ and smooth in Ω. Hence, we are reduced to show continuity at every $x \in \partial\Omega$. Recall that Ω is Jordan. This implies that $\operatorname{diam}(S(\widetilde{Q}))$ tends to zero uniformly when $\ell(\widetilde{Q})$ tends to zero. Given $x \in \partial\Omega$ and points x_k converging to x from within $\widetilde{\Omega}$, we pick Whitney squares $\widetilde{Q}_k$ containing x_k. Then by the fact that $\{\phi_j\}$ forms a partition of unity, we have

$$\begin{aligned} |\hat{E}u(x_k) - u(x)| &= \left| \sum_{\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset} a_j \phi_j(x_k) - \sum_{\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset} \phi_j(x_k) u(x) \right| \\ &\leqslant \sum_{\widetilde{Q}_j \cap \widetilde{Q}_k \neq \emptyset} \phi_j(x_k) |a_j - u(x)|. \end{aligned}$$

Since $\widetilde{Q}_k$ tend to x, the neighboring squares of $\widetilde{Q}_k$ also tend to x. We claim that their shadows also converge to x. Towards this, it suffices to check that the preimages of their shadows tend to $\widetilde{\varphi}^{-1}(x)$ under our homeomorphism $\widetilde{\varphi}\colon \mathbb{R}^2 \setminus \mathbb{D} \to \overline{\widetilde{\Omega}}$ that is conformal in $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$. Now the preimages of the shadows of these squares are the radial projections of the preimages $\widetilde{\varphi}^{-1}(\widetilde{A}_k)$ of these squares and the desired conclusion follows since $\widetilde{\varphi}^{-1}(\widetilde{A}_k)$ tend to $\widetilde{\varphi}^{-1}(x)$. Hence, it follows from Lemmas 4.3 and 4.4 that the Whitney squares of Ω associated with the neighboring squares of $\widetilde{Q}_k$ also tend to x. Thus, we have

$$\hat{E}u(x_k) \to u(x)$$

by the assumption that u is the restriction of a smooth (especially continuous) function to Ω and $Eu(x_k)$ is defined via averages over the squares associated with the neighboring squares of $\widetilde{Q}_k$.

Recall that Ω is John and the Lebesgue measure of $\partial\Omega$ is zero Lemma 2.24. With the continuity of $\hat{E}u$, [24, Theorem 4] then guarantees that the above definition gives a Sobolev function with the desired norm control. Also by Lemma 2.24, we know that $\hat{E}u = Eu$ as Sobolev functions. Thus,

$$E\colon W^{1,p}(\Omega) \cap C^\infty(\overline{\Omega}) \to W^{1,p}(B_\Omega)$$

is a bounded operator, and it is also linear by its definition.

Recall that $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ for $1 < p < \infty$ if Ω is a planar Jordan domain (see [31]). By our norm estimates above, we can (uniquely) extend E to entire $W^{1,p}(\Omega)$ as a bounded operator. This extension is given by the original definition of E. Since B_Ω is an extension domain, we conclude that the claim of the theorem follows. □

Remark 4.15. The norm of our extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(B_\Omega)$ depends only on $p, \hat{p}$ and the constant C in (4.1), both for the homogeneous and the full Sobolev norms (see (4.76) and (4.77)). Here, $B_\Omega = B(x_0, \operatorname{diam}(\Omega))$ and x_0 is a chosen John center of Ω. If we wish to extend to entire $\mathbb{R}^2$, then the norm of the extension operator will also necessarily depend on the diameter of Ω if we use the full Sobolev norm.

4.7 Proof of the general case

We establish the existence of an extension operator in the general case of a bounded simply connected domain Ω via an approximation process, relying on our earlier results.

Recall that we are claiming the existence of a bounded extension operator under the condition (1.1) for a given bounded simply connected domain Ω. We have already verified a version of this if Ω is Jordan.

In order to be able to prove the general case by using the result for the Jordan case, we need a sequence of approximating Jordan domains to have extension operators with uniform norm bounds. For this purpose, we have stated the dependence of the norm of the extension operator in Theorem 4.1 explicitly in Remark 4.15.

From now on, Ω is a bounded simply connected domain that satisfies (1.1). Towards the existence of a suitable approximating sequence, recall that (1.1) guarantees that Ω is John (see Corollary 2.23). Fix a conformal map $\varphi\colon \mathbb{D}\to\Omega$ so that $\varphi(0)$ is a John center of Ω. By Remark 2.26, we may extend φ continuously up to the boundary. We still denote the extended map by φ.

Let $B_n = B(0, 1-\frac{1}{n})$ for $n \geqslant 2$. Then $\Omega_n = \varphi(B_n)$ are Jordan John domains (with constant independent of n) contained in Ω by Lemma 2.30, and converge to Ω uniformly in Hausdorff distane because of the uniform continuity of φ up to the boundary. Actually, φ is even uniformly Hölder continuous (see [11, 35]).

Before giving the proof of Theorem 1.1, we establish a technical result according to which the complementary domain of Ω_n satisfies the condition (4.1) with $\hat{p} > p$ and C that are independent of n. This eventually allows us to apply Theorem 4.1 to Ω_n so as to complete the proof by a compactness argument.

Lemma 4.16. *Each of the complementary domains $\widetilde{\Omega}_n$ of Ω_n satisfies the condition* (4.1) *with curves $\gamma\subset\widetilde{\Omega}_n$ for a fixed $\hat{p} > p$ and a constant independent of n.*

Proof. Fix $n \geqslant 2$ and let $z_1, z_2 \in \widetilde{\Omega}_n$. Write $R = 1 - \frac{1}{n}$. We begin by noticing that if z_1 and z_2 are both outside Ω, then the condition (4.1) with a fixed $\hat{p} > p$ and C follows immediately from (1.1) and the self-improvement property for Ω from Lemma 2.3 since $\operatorname{dist}(z, \partial\Omega) \leqslant \operatorname{dist}(z, \partial\Omega_n)$ for $z \in \mathbb{R}^2 \setminus \Omega$. Hence, switching z_1 and z_2 if necessary, we may assume that $z_1 \in \Omega\setminus\Omega_n$.

We fix $\hat{p} > p$ as in the first paragraph of the proof. Suppose first that also $z_2 \in \Omega\setminus\Omega_n$. Let us consider the case where $\varphi^{-1}(z_2) \in B(\varphi^{-1}(z_1), \delta(1-|\varphi^{-1}(z_1)|))$, where δ is as in Lemma 2.4. Then Lemma 2.4 gives us a curve α joining $\varphi^{-1}(z_1)$ to $\varphi^{-1}(z_2)$ in $B(\varphi^{-1}(z_1), (1-|z_1|)/2)\setminus \overline{B}(0, R)$ so that

$$\int_\alpha \operatorname{dist}(z, \partial B(0,R)\cup\partial B(\varphi^{-1}(z_1), (1-|\varphi^{-1}(z_1)|)/2))^{1-\hat{p}}ds(z) \leqslant C(\hat{p})|\varphi^{-1}(z_1)-\varphi^{-1}(z_2)|^{2-\hat{p}} \tag{4.78}$$

(see Figure 11). Because $B := B(\varphi^{-1}(z_1), (1-|\varphi^{-1}(z_1)|)/2)$ is of 2-Whitney type, Lemma 2.13 gives us the estimate

$$C^{-1}|\varphi'(\varphi^{-1}(z_1))||w_2-w_1| \leqslant |\varphi(w_2)-\varphi(w_1)| \leqslant C|\varphi'(\varphi^{-1}(z_1))||w_2-w_1|$$

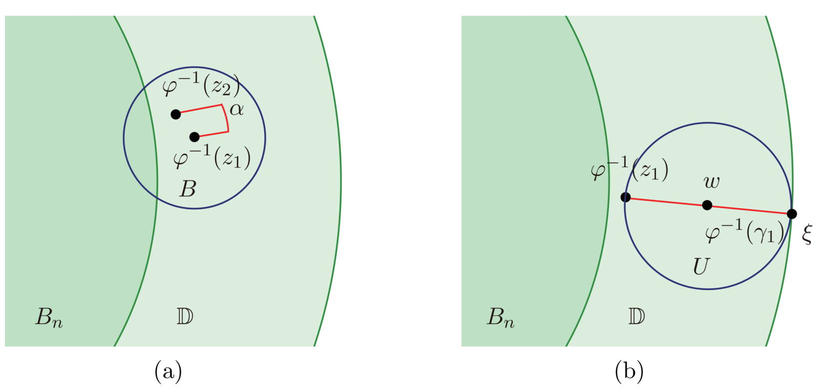

Figure 11 (Color online) The proof of the existence of the curve satisfying (4.1) for the domain $\widetilde{\Omega}_n$ is split into two cases. In (a), we have the case where the preimages of the points z_1 and z_2 are close enough so that one can use a curve α from Lemma 2.4 connecting them in the annular domain $\Omega\setminus\Omega_n$. In (b), it is the case where the preimages are far from each other and the constructed curve exits the annular domain

for all $w_1, w_2 \in B$ for an absolute constant C. Since $\alpha \subset B$, we may apply this bi-Lipschitz estimate to the above integral over α so as to conclude for $\gamma = \varphi \circ \alpha$ that

$$\int_\gamma \operatorname{dist}(z, \varphi(S(0,R) \cap B) \cup \varphi(\partial B))^{1-\hat{p}} ds(z) \leqslant C'(\hat{p})|z_2 - z_1|^{2-\hat{p}},$$

where $C'(\hat{p})$ depends only on $\hat{p}$. The desired inequality follows since φ is a homeomorphism and hence $\operatorname{dist}(z, \partial\widetilde{\Omega}_n) \geqslant \operatorname{dist}(z, \varphi((S(0,R) \cap B) \cup \partial B))$ when $z \in \varphi(\alpha) = \gamma$. The desired conclusion also follows if the roles of z_1 and z_2 above are reversed.

Next, (2.16) (applied to φ^{-1}) gives us an absolute constant C such that if

$$C|z_1 - z_2| \leqslant \max\{\operatorname{dist}(z_1, \partial\Omega), \operatorname{dist}(z_2, \partial\Omega)\},$$

then we are in a situation covered by the previous paragraph. Thus, we may assume that

$$C|z_1 - z_2| \geqslant \max\{\operatorname{dist}(z_1, \partial\Omega), \operatorname{dist}(z_2, \partial\Omega)\}. \tag{4.79}$$

Recall from Lemma 2.25 that φ is η-quasisymmetric with respect to the inner distance with η that depends only on the John constant of Ω. Define

$$U = B\left(\frac{1 + |\varphi^{-1}(z_1)|}{2} \frac{\varphi^{-1}(z_1)}{|\varphi^{-1}(z_1)|}, \frac{1 - |\varphi^{-1}(z_1)|}{2}\right).$$

Then the disk U is contained in $\mathbb{D} \setminus \overline{B_n}$, $z_1 \in \varphi(\overline{U})$, $\varphi(\overline{U}) \cap \partial\Omega \neq \emptyset$, and Lemma 2.30 gives that $\varphi(U)$ is J'-John with the center $\varphi(w)$, where w is the center of U, and J' depends only on the John constant J of Ω.

We claim that

$$\operatorname{diam}(\varphi(U)) \leqslant C(J)\operatorname{dist}(z_1, \partial\Omega). \tag{4.80}$$

Towards this, let $\xi = \varphi^{-1}(z_1)/|\varphi^{-1}(z_1)|$, the tangent point of U with the unit circle, and pick a point $z_3 \in \partial\Omega$ satisfying

$$\operatorname{dist}(z_1, \partial\Omega) = |z_1 - z_3|.$$

Pick a sequence of points x_j along the Euclidean segment between z_1 and z_3 so that $x_j \to z_3$. Then

$$\operatorname{dist}_\Omega(z_1, x_j) = |z_1 - x_j| \leqslant \operatorname{dist}_\Omega(z_1, x_j). \tag{4.81}$$

Since φ is a homeomorphism of the unit disk onto Ω, there is a subsequence of the sequence (x_j) so that the preimages converge to some $w_3 \in \partial\mathbb{D}$. For simplicity, we refer to the elements of this subsequence still by x_j. Write r for the radius of U. By the definitions of U and ξ, we have

$$2r = |\varphi^{-1}(z_1) - \xi| = \operatorname{dist}(\varphi^{-1}(z_1), \partial\mathbb{D}),$$

and since $w_3 \in \partial\mathbb{D}$, we conclude that

$$2r \leqslant |\varphi^{-1}(z_1) - w_3|.$$

In particular, for any $w_2 \in U$, we have

$$|\varphi^{-1}(z_1) - w_2| \leqslant 2r \leqslant |\varphi^{-1}(z_1) - w_3|,$$

and consequently,

$$|\varphi^{-1}(z_1) - w_2| \leqslant 2|\varphi^{-1}(z_1) - \varphi^{-1}(x_j)| \tag{4.82}$$

for all sufficiently large j. Quasisymmetry of φ together with (4.82) and (4.81) now gives for all sufficiently large j the estimate

$$|z_1 - \varphi(w_2)| \leqslant \operatorname{dist}_\Omega(z_1, \varphi(w_2)) \leqslant \eta(2)\operatorname{dist}_\Omega(z_1, x_j) \leqslant \eta(2)\operatorname{dist}(z_1, \partial\Omega).$$

Hence, (4.80) follows.

By connecting z_1 to the John center $\varphi(w)$ of $\varphi(U)$ and then the John center to $\varphi(\xi) \in \partial\Omega$ via hyperbolic segments in $\varphi(U)$, we obtain by Remark 2.20, (4.80) and (4.79) a curve $\gamma_1 \subset \varphi(U)$ consisting of two John curves and joining z_1 to $\partial\Omega$ so that

$$\begin{aligned}\int_{\gamma_1} \operatorname{dist}(z, \partial\Omega_n)^{1-\hat{p}} ds(z) &\leqslant \int_{\gamma_1} \operatorname{dist}(z, \partial(\varphi(U)))^{1-\hat{p}} ds(z)\\ &\lesssim \operatorname{dist}(\varphi(w), \partial(\varphi(U))^{2-s} \lesssim \operatorname{diam}(\varphi(U))^{2-\hat{p}}\\ &\lesssim \operatorname{dist}(z_1, \partial\Omega)^{2-\hat{p}} \lesssim |z_2 - z_1|^{2-\hat{p}}\end{aligned}$$

(see Figure 11). Here, the constants depend only on J. Analogously, we find a corresponding curve γ_2 for z_2. It remains to join the two endpoints $\widetilde{z}_1$ and $\widetilde{z}_2$ of γ_1 and γ_2 in $\partial\Omega$ by a curve γ_3 outside Ω as in the first paragraph of the proof; notice here that (4.79) guarantees that

$$|\widetilde{z}_1 - \widetilde{z}_2| \leqslant C|z_1 - z_2|.$$

By the triangle inequality, the curve obtained by concatenating γ_1, γ_3 and γ_2 satisfies our requirements.

We are left to consider the case where $z_1 \in \Omega \setminus \overline{\Omega}_n$ and $z_2 \notin \Omega$. In this case, we simply use the curve γ_1 constructed above for z_1 together with a curve γ_3 outside Ω joining z_2 and the endpoint of γ_1 in $\partial\Omega$ as above. □

Proof of Theorem 1.1. By Section 3, we only need to prove the sufficiency of (1.1). Recall the conformal map φ and the domains

$$\Omega_n = \varphi(B_n)$$

from the beginning of this subsection. By Lemma 2.30, the domains Ω_n are John domains with a John center $x_0 = \varphi(0)$ with a John constant depending only on J.

By Lemma 4.16 and Theorem 4.1, (1.1) yields that there exist extension operators

$$E_n \colon W^{1,p}(\Omega_n) \to W^{1,p}(B(x_0, \operatorname{diam}(\Omega_n))),$$

where the norms of the extension operators E_n are independent of n (see Remark 4.15). Since $\Omega_n = \varphi(B_n)$ and φ is continuous up to the boundary, $\operatorname{diam}(\Omega_n) \to \operatorname{diam}(\Omega)$ when n tends to infinity. Hence, $B(x_0, r) \subset B(x_0, \operatorname{diam}(\Omega_n))$ for all sufficiently large n when $r = \operatorname{diam}(\Omega) - \operatorname{dist}(x_0, \partial\Omega)$. Define $B = B(x_0, r)$. We conclude that

$$E_n \colon W^{1,p}(\Omega_n) \to W^{1,p}(B)$$

is a bounded extension operator with a norm bound independent of n for all sufficiently large n.

Fix $u \in W^{1,p}(\Omega)$, and let $u_n = u|_{\Omega_n}$ for $n \geqslant 2$. Now $\|\nabla E_n u_n\|_{L^p(B)} + \|E_n u_n\|_{L^p(B)}$ is bounded independently of n for large n. Hence, by the assumption $p > 1$, there exists a subsequence that converges weakly in $L^p(B)$ to some $v \in W^{1,p}(B)$ with

$$\|\nabla v\|_{L^p(B)} + \|v\|_{L^p(B)} \leqslant \liminf_{n\to\infty}(\|\nabla E_n u_n\|_{L^p(B)} + \|E_n u_n\|_{L^p(B)}).$$

Define $Eu := v$ and notice that $\Omega \subset B$ and the sequence $\{E_n u_n\}$ converges to u pointwise a.e. on Ω. Hence, we know that Eu is an extension of u, and the desired norm bound over B follows from the uniform bound on the extension operators E_n. Since B is a $W^{1,p}$-extension domain, this completes the proof of Theorem 1.1. □

5 Proof of Corollary 1.3

Before giving the proof of Corollary 1.3, we present a lemma stating that we can always swap an unbounded domain with a compact boundary to a bounded domain (and vice versa) with the same extendability and curve properties. This is the main observation needed to conclude Corollary 1.3 from Theorems 1.1 and 1.2.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Fix $x \in \Omega$ and define an unbounded domain $\hat{\Omega} = i_x(\Omega \setminus \{x\})$ using the inversion*

$$i_x \colon \mathbb{R}^2 \setminus \{x\} \to \mathbb{R}^2 \setminus \{x\} \colon y \mapsto x + \frac{y-x}{|y-x|^2}.$$

Then

(1) *for any $1 \leqslant p \leqslant \infty$, the domain Ω is a $W^{1,p}$-extension domain if and only if $\hat{\Omega}$ is a $W^{1,p}$-extension domain*;

(2) *for any $q > 2$, there exists a constant $C > 0$ such that for all $z_1, z_2 \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining z_1 and z_2 so that*

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-q}} ds(z) \leqslant C|z_1 - z_2|^{\frac{q-2}{q-1}} \tag{5.1}$$

if and only if there exists a constant $\hat{C} > 0$ such that for every $\hat{z}_1, \hat{z}_2 \in \hat{\Omega}$ (see (1.2)), there exists a rectifiable curve $\hat{\gamma} \subset \hat{\Omega}$ joining $\hat{z}_1$ and $\hat{z}_2$ so that

$$\int_\gamma \operatorname{dist}(z, \partial\hat{\Omega})^{\frac{1}{1-q}} ds(z) \leqslant \hat{C}|\hat{z}_1 - \hat{z}_2|^{\frac{q-2}{q-1}}. \tag{5.2}$$

Proof. Let $R = 2\operatorname{diam}(\Omega)$ and $2r = \operatorname{dist}(x, \partial\Omega)$. Then $\partial\Omega \subset A(x, r, R) := B(x, R) \setminus \overline{B(x, r)}$. Notice that i_x is a bi-Lipschitz map when restricted to $A(x, r, R)$ with the bi-Lipschitz constant depending only on r and R, and $i_x(A(x, r, R)) = A(x, 1/R, 1/r)$.

(1) Notice that for $0 < r_1 < r_2 < \infty$, the annulus $A(x, r_1, r_2)$ is also a $W^{1,p}$-extension domain with an operator E_{r_1,r_2}. Now, assume that Ω is a $W^{1,p}$-extension domain with an extension operator E. Let us show that $\hat{\Omega}$ is also a $W^{1,p}$-extension domain. Towards this, take $u \in W^{1,p}(\hat{\Omega})$. By the fact that i_x is bi-Lipschitz on $A(x, r, R)$, we have $u \circ i_x|_{A(x,r,R)\cap\Omega} \in W^{1,p}(\Omega \setminus B(x, r))$. Since $A(x, r, 2r) \subset A(x, r, R) \cap \Omega$, we have $E_{r,2r}(u \circ i_x|_{A(x,r,2r)}) \in W^{1,p}(\mathbb{R}^2)$. Now, define $v \in W^{1,p}(\Omega)$ by

$$v(y) = \begin{cases} E_{r,2r}(u \circ i_x)(y), & \text{if } y \in B(x, 2r), \\ u \circ i_x(y), & \text{if } y \in \Omega \setminus B(x, 2r). \end{cases}$$

This can then be extended to $Ev \in W^{1,p}(\mathbb{R}^2)$. Again, by the bi-Lipschitz property of i_x on $A(x, r, R)$, we have

$$(Ev) \circ i_x^{-1} \in W^{1,p}(A(x, 1/R, 1/r)),$$

which finally gives the required extension $\hat{E}u \in W^{1,p}(\mathbb{R}^2)$ as

$$\hat{E}u(y) = \begin{cases} E_{1/R,1/r}((Ev) \circ i_x^{-1})(y), & \text{if } y \in B(x, 1/r), \\ u(y), & \text{if } y \notin B(x, 1/r). \end{cases}$$

This shows that $\hat{\Omega}$ is a $W^{1,p}$-extension domain.

Let us then show the converse and assume that $\hat{\Omega}$ is a $W^{1,p}$-extension domain with an extension operator $\hat{E}$. The construction of the extension is done analogously to the previous case. Let $u \in W^{1,p}(\Omega)$. Then $v \in W^{1,p}(\hat{\Omega})$, when we define

$$v(y) = \begin{cases} E_{1/(2r),1/r}(u \circ i_x^{-1})(y), & \text{if } y \notin B(x, 1/r), \\ u \circ i_x^{-1}(y), & \text{if } y \in \Omega \cap B(x, 1/r), \end{cases}$$

and the required extension $Eu \in W^{1,p}(\mathbb{R}^2)$ is then given by

$$Eu(y) = \begin{cases} E_{r,R}((\hat{E}v) \circ i_x)(y), & \text{if } y \notin B(x, r), \\ u(y), & \text{if } y \in B(x, r). \end{cases}$$

Hence, Ω is a $W^{1,p}$-extension domain.

(2) Suppose the existence of curves $\gamma \subset \Omega$ satisfying (5.1). Let us show the condition (5.2) for $\hat{\Omega}$. Towards this, take $\hat{z}_1, \hat{z}_2 \in \hat{\Omega}$. Suppose first that $\hat{z}_1, \hat{z}_2 \in \overline{B}(x, 1/r)$. Define $z_1 = i_x^{-1}(\hat{z}_1)$ and $z_2 = i_x^{-1}(\hat{z}_2)$. Let $\gamma \subset \Omega$ be a curve joining z_1 and z_2 so that (5.1) holds. By [39, Lemma 2.1], we may assume that $\mathrm{le}(\gamma) \leqslant C|z_1 - z_2|$. If $\gamma \subset A(x, r, R)$, then by the bi-Lipschitz property of i_x, the curve $\hat{\gamma} = i_x(\gamma)$ satisfies (5.2). If $\gamma \not\subset A(x, r, R)$, let $\tilde{z}_1, \tilde{z}_2 \subset \gamma \cap \partial B(x, r)$ be such that $\gamma[z_1, \tilde{z}_1]$ connects z_1 to $\tilde{z}_1$ in $A(x, r, R)$, and $\gamma[z_2, \tilde{z}_2]$ connects z_2 to $\tilde{z}_2$ in $A(x, r, R)$. Then

$$|\tilde{z}_1 - \tilde{z}_2| \leqslant \mathrm{le}(\gamma) \leqslant C|z_1 - z_2|. \tag{5.3}$$

Let α be a shorter arc of $\partial B(x, r)$ joining $\tilde{z}_1$ and $\tilde{z}_2$. Since $\mathrm{dist}(\partial B(x, r), \partial\Omega) \geqslant r$, by (5.3), we have

$$\int_\alpha \mathrm{dist}(z, \partial\Omega)^{\frac{1}{1-q}}\, ds(z) \leqslant C|z_1 - z_2|^{\frac{q-2}{q-1}}.$$

Hence, again by the bi-Lipschitz property of i_x on $A(x, r, R)$, the curve $\hat{\gamma} = i_x(\gamma[z_1, \tilde{z}_1] * \alpha * \gamma[\tilde{z}_2, z_2])$ satisfies (5.2).

Suppose then that $\hat{z}_1, \hat{z}_2 \in \overline{B}(x, 1/r)$ fails. Then if $[\hat{z}_1, \hat{z}_2] \cap \partial B(x, 1/r)$ contains two distinct points, we can use the previous case to connect these by a curve $\gamma \subset \hat{\Omega}$. For the remaining part, we can simply use the remaining parts of $[\hat{z}_1, \hat{z}_2] \setminus B(x, 1/r)$. Finally, if $[\hat{z}_1, \hat{z}_2] \cap \partial B(x, 1/r)$ is a singleton, we simply use $[\hat{z}_1, \hat{z}_2]$.

The proof of the converse implication is analogous. Towards it, let us assume that there exist curves $\hat{\gamma} \subset \hat{\Omega}$ satisfying (5.2). Let $z_1, z_2 \in \Omega$ and define $\hat{z}_1 = i_x(z_1)$ and $\hat{z}_2 = i_x(z_2)$. Let $\hat{\gamma} \subset \hat{\Omega}$ be a curve connecting $\hat{z}_1$ and $\hat{z}_2$ that satisfies (5.2). By Lemma 2.2, we may assume that $\mathrm{le}(\hat{\gamma}) \leqslant C|\hat{z}_1 - \hat{z}_2|$. If $\hat{\gamma} \subset \overline{B}(x, 1/r)$, again the bi-Lipschitz property of i_x inside $A(x, r, R)$ gives that $\gamma = i_x^{-1}(\hat{\gamma})$ satisfies (5.1). Let us then suppose that $\hat{\gamma} \not\subset \overline{B}(x, 1/r)$. If $\hat{z}_1 \in \overline{B}(x, 1/r)$, we take $\tilde{z}_1 \in \hat{\gamma} \cap \partial B(x, 1/r)$ so that $\hat{\gamma}[\hat{z}_1, \tilde{z}_1] \subset \overline{B}(x, 1/r)$. If $\hat{z}_1 \notin \overline{B}(x, 1/r)$, we define $\tilde{z}_1 = \hat{z}_1$. Similarly, if $\hat{z}_2 \in \overline{B}(x, 1/r)$, we take $\tilde{z}_2 \in \hat{\gamma} \cap \partial B(x, 1/r)$ so that $\hat{\gamma}[\hat{z}_2, \tilde{z}_2] \subset \overline{B}(x, 1/r)$, and if $\hat{z}_2 \notin \overline{B}(x, 1/r)$, we set $\tilde{z}_2 = \hat{z}_2$. Then the curve

$$\gamma = i_x^{-1}(\hat{\gamma}[\hat{z}_1, \tilde{z}_1]) * [i_x^{-1}(\tilde{z}_1), i_x^{-1}(\tilde{z}_2)] * i_x^{-1}(\hat{\gamma}[\tilde{z}_2, \hat{z}_2])$$

connects z_1 to z_2 in Ω and satisfies (5.1) because of

$$|\tilde{z}_1 - \tilde{z}_2| \leqslant \mathrm{le}(\hat{\gamma}) \leqslant C|\hat{z}_1 - \hat{z}_2|$$

and $\mathrm{dist}(x, \gamma) \geqslant r$. □

Proof of Corollary 1.3. By Lemma 5.1, it suffices to show that the complementary domain $\widetilde{\Omega}$ of a given Jordan $W^{1,p}$-extension domain Ω, where $1 < p < \infty$, is a $W^{1,q}$-extension domain for $q = p/(p-1)$.

Suppose first that our Jordan domain Ω is a $W^{1,p}$-extension domain for a given $1 < p < 2$. Then Theorem 3.1 and Remark 3.7 give the existence of curves as in (1.1) in the complementary domain $\widetilde{\Omega}$. Notice that (1.1) is precisely (1.2) with $q = p/(p-1) > 2$. Thus, by applying Lemma 5.1 (twice) and Theorem 1.2, we conclude that $\widetilde{\Omega}$ is a $W^{1,q}$-extension domain.

If Ω is a $W^{1,p}$-extension domain for some $p > 2$, then (1.2) holds by Theorem 1.2 (for points in Ω). Let $x \in \Omega$, and take i_x and $\hat{\Omega}$ as in Lemma 5.1. Now, by applying Lemma 5.1 again, we see that (1.1) holds for points in $\hat{\Omega}$. For any pair $z_1, z_2 \in \overline{\hat{\Omega}}$, there exist sequences $(x_j)_j$ and $(y_j)_j$ in $\hat{\Omega}$ that converge to z_1 and z_2, respectively. For every pair (x_j, y_j), we take a curve $\gamma_j \subset \hat{\Omega}$ satisfying (1.1) (with the obvious notational changes). By Lemma 2.1, there exists a limiting curve $\gamma \subset \overline{\hat{\Omega}}$ connecting z_1 and z_2 also satisfying (1.1). Hence, by Theorem 1.1, $\mathbb{R}^2 \setminus \overline{\hat{\Omega}}$ is a $W^{1,q}$-extension domain, and so, via Lemma 5.1, also $\tilde{\Omega}$.

We are left with the case $p = 2$. In this case, Ω is necessarily a uniform domain and hence so is $\widetilde{\Omega}$. Thus, $\widetilde{\Omega}$ is also a $W^{1,2}$-extension domain (see [13–15, 23]). □

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