A NOTE ON FINITE GROUPS SATISFYING PERMUTIZER CONDITION

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Let G be a finite group, H be a subgroup of G. The permutizer of H in G is the subgroup

$$P_G(H) = \langle x \in G \mid \langle x \rangle H = H \langle x \rangle \rangle.$$

G is said to satisfy the permutizer condition if

$$H \lneqq P_G(H), \text{ for all } H \lneqq G.$$

If G satisfies the permutizer condition, we call G a *pc*-group; clearly, all the supersolvable groups are *pc*-groups. It was shown in [1] that *pc*-groups of odd order were supersolvable. It is easy to verify that the symmetric group of degree four is a solvable *pc*-group, but it is not supersolvable. In general, solvable *pc*-groups of even order are not bound to be supersolvable. In this paper, we study the solvable *pc*-groups in detail and give a necessary and sufficient condition for the supersolvability of solvable *pc*-groups. In fact, we will prove the following theorem.

Theorem 1. *Let G be a solvable pc-group, then*

(i) *G is supersolvable iff G is S_4-free.*

(ii) *G is p-supersolvable, for any odd prime p.*

The symbols used in this paper are mainly taken from [1].

The following Lemma is essential in the process of proving our main theorem.

Lemma 1. *Let P be a p-group, p a prime, H a subgroup of index p^n and X a normal complement of H which is elementary Abelian. If P contains an element Y such that $P = \langle Y \rangle H$, then if p is odd, $n = 1$; if $p = 2$, $n \leqslant 2$.*

Proof. By [1] we know that if p is odd, the conclusion holds. Therefore we may assume that $p = 2$. Assume that the result is false and let P be a minimal counter example, so $n \geqslant 3$. Since $X \Delta P$, we can choose $1 \neq Z \in X \cap Z(P)$, clearly, for $H^* = H\langle z \rangle$, $P = XH^* = \langle y \rangle H^*$, $|P{:}\ H^*| = 2^{n-1}$. Let $P = P/\langle z \rangle$, $\bar{H}^* = H\langle z \rangle/\langle z \rangle$, $\bar{X} = X/\langle z \rangle$, $\langle \bar{y} \rangle = \langle y \rangle \langle z \rangle / \langle z \rangle$, clearly $\bar{P}, \bar{H}^*, \bar{X}, \bar{Y}$ satisfy the conditions in the lemma. By the minimality of $|P|$ we conclude $n - 1 \leqslant 2$. Hence $n = 3$, $|X| = 8$. H acts on X by conjugacy. Let K be the kernal $\langle h \in H \mid h^{-1}xh = x$, any $x \in X \rangle$. Clearly $K \Delta H$. Hence $K \Delta P (= KX)$, If $K \neq 1$, consider

$$\bar{P} = P/K,\ \bar{H} = H/K,\ \bar{X} = XK/K,\ \langle \bar{y} \rangle = \langle y \rangle K/K,\ |\bar{P} \cdot \bar{H}| = 2^n,$$

by the minimality of $|P|$, $n \leqslant 2$, contrary to the assumption, therefore $K = 1$.

Now we may take H as a subgroup of $PSL(3,2)$, the sylow 2-subgroup of PSL (3.2) is not cyclic and of order 8. So $\exp H|4$.

Since $P = HX$, $y = \sigma x$, $\sigma \in H$, $x \in X$ obviously $\sigma \neq 1$, $x \neq 1$, $O(\sigma)|4$. Let $N = [X, P]$, then $N \Delta P$ and $N \leqslant X$. So $|N||4$, σ introduces an automorphism of N by conjugacy. Since $|\mathrm{Aut}(N)||6, [\sigma^2, N] = 1$, $y^2 = \sigma x \sigma x = \sigma^2 \sigma^{-1} x^{-1} \sigma x = \sigma^2[\sigma, x]$, $y^4 = \sigma^2[\sigma, x]\sigma^2[\sigma, x] = \sigma^4[\sigma, x]^2 = 1$ (noting that exp $X = 2$ $[\sigma, x] \in N$). Hence $|P: H| = |\langle y\rangle|/l\langle y\rangle \cap H|$ is a factor of 4. $n \leqslant 2$, contrary to $n \geqslant 3$. The contradiction proves the lemma.

For conclusion, we narrate the definition of p-supersolvability, we say a solvable group G to be p-supersolvable, if every p-principal factor of G is of order p. p-supersolvability is quite analogous to supersolvability (cf. [2]).

Proof of Theorem 1. (i) We only prove the "if" part. Suppose that the result is false and let G be a minimal counter example. Clearly, the properties of G are inherited by its quotient subgroups. If $\Phi(G) \neq 1$, then $G/\Phi(G)$ is supersolvable, and so is G. But it is impossible, so $\Phi(G) = 1$. If G has two different minimal normal subgroups N_1, N_2, by the supersolvability of G/N_i, we infer that $G/(N_1 \cap N_2) \approx G$ is supersolvable. But that is contrary to the assumption. So G has only one minimal normal subgroup N. It is not hard to prove that $N = F(G)$, the fitting subgroup of G. Hence $C_G(N) = N$. Since $\Phi(G) = 1$, there exists a maximal subgroup M such that $G = NM$. Obviously, $M \cap N \Delta M$, $M \cap N \Delta N$, $M \cap N \Delta MN = G$, $M \cap N = 1$. Since G is a solvable pc-group, there exists an element Y of G such that $G = \langle y\rangle M$. Let $|N| = p^n$, y may be chosen to be of order p^m. There exists $H \in Syl_p(M)$ such that $P = \langle y\rangle H = NH \in S_y l_p(G)$. By Lemma 2, $n \leqslant 2$. If $n = 1$, G is supersolvable because of the supersolvability of G/N. Hence $n = 2, p = 2$. Since $C_G(N) = N$, M acts on N by conjugacy. But $|\mathrm{Aut}(N)| = 6$ and G is a pc-group. It is easy to show that $|M| = 6$ and $G \approx S_4$, which is contrary to the assumption. The contradiction proves the (i) part. (ii) In a similar way, we can prove (ii).

Corollary 1. *Let G be a solvable pc-group. then the sylow 2-subgroup Q of G' is normal in G and G/Q is supersolvable.*

Proof. By Theorem 1, G is p-supersolvable for odd prime P. Then G' is p-nilpotent (see [2], Th. 1, p. 716). Therefore $Q \Delta G$. Since G/Q is also a solvable pc-group and has an Abelian sylow 2-subgroup, G/Q is S_4-free. By Theoreml, G/Q is supersolvable.

Corollary 2. *Let G be a solvable pc-group, p the largest prime factor of $|G|$. If $p > 3$, then the sylow p-subgroup P of G is normal in G. Hence the $\{2,3\}'$-Hall subgroup H of G is a normal subgroup of G. Moreover, H is supersolvably embedded in G, i. e. the principal factors of G which lie in H are of prime orders.*

Proof. We employ the induction on $|G|$. If $\Phi(G) \neq 1$, then $P\Phi(G)/\Phi(G)$ is normal in $G/\Phi(G)$, $P\Phi(G) \Delta G$. Since $P\Phi(G)$ is nilpotent, P is the characteristic subgroup of $P\Phi(G)$, so $P \Delta G$, if $\Phi(G) = 1$. Let N be a minimal normal subgroup of G, M a subgroup of G such that $G = MN, M \cap N = 1$. Similar to the way used in the proof of Theorem 1, we have $|N| = q$ or 4, q is a prime. Since PN/N is normal in G/N, $PN \Delta G$, $|\mathrm{Aut}(N)| = q - 1$ or 6. Hence $P \leqslant C_G(N)$. If $q = p$, then $N \leqslant P$. If $q < p$, P char PN. In both cases, we have $P \Delta G$. Noticing the p-supersolvability of G, we can easily arrive at the remainder conclusion.

Corollary 4 shows that if we want to study the supersolvability of the solvable pc-group, it is enough to study the supersolvability of the pc-group of order $2^n 3^m$.

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References

[1] Weistein, M., *Between Nilpotent and Solvable*, Polygonal Publishing House, 1982.
[2] Huppert, B., *Endiche Gruppen I*, Springer-Verlag, 1967.