轴对称弹塑性问题的自然单元法

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摘要 基于无网格自然单元法, 提出了求解轴对称弹塑性问题的一条新途径. 自然单元法是一种基于自然邻近插值的无网格数值计算方法, 其形函数的计算不涉及复杂的矩阵求逆, 也不需要任何人为的参数. 本研究基于增量虚位移原理, 建立了增量格式的轴对称弹塑性问题的自然单元法求解方案. 由于自然邻近插值函数具有插值性, 因此可以直接施加本质边界条件. 为了避免每次迭代都形成和分解刚度矩阵, 在每个载荷增量步采用修正的Newton-Raphson法进行迭代计算. 两个经典的轴对称弹塑性数值算例结果表明, 采用增量形式的自然单元法求解轴对称弹塑性问题是行之有效的, 并且具有较高的计算精度.

关键词 无网格法, 自然单元法, 弹塑性问题, 轴对称

作为近几十年来迅速兴起的一种数值方法, 无网格法^[1~3]克服了对网格的依赖性, 且在节点不规则分布时依然保持较高的计算精度, 因此受到了国际计算力学界的高度重视. 目前发展的无网格法主要有无单元伽辽金法^[4,5]、边界无单元法^[6,7]、重构核粒子法^[8,9]和自然单元法^[10]等. 与大多数无网格法不同, 自然单元法的形函数具有插值性且在边界节点间线性变化, 可以直接准确地施加本质边界条件. 此外, 自然单元法的形函数构造简单, 不需要人为参数, 也不涉及复杂的矩阵求逆运算. 因此, 自然单元法具有良好的发展前景, 并且已经在很多领域得到了应用^[11~14].

目前, 有些学者在自然单元法解决弹塑性力学问题方面做了一些有益的工作. 朱合华等人^[15]将自然单元法应用于弹塑性力学平面问题, 王建华等人^[16]研究了在自然单元法中采用不同屈服准则解决三维弹塑性问题, 江涛等人^[17]将自然单元法应用于线性强化材料的弹塑性分析. 但目前尚未见到采用自然单元法分析

轴对称弹塑性问题的研究.

本研究依据弹塑性力学中的增量理论, 详细推导了轴对称弹塑性分析的自然单元法计算公式, 并给出了详细的数值实现过程. 在此基础上, 编制了相应的计算程序. 最后, 通过两个算例的计算和对比分析验证了本文方法的可行性和有效性.

1 自然邻近插值

自然单元法采用自然邻近插值构造近似函数. 根据插值基函数的不同, 自然邻近插值可分为Sibson插值^[18]和Laplace^[19]插值, 这两种插值都是基于计算几何中Delaunay三角剖分图和Voronoi结构. 本研究选用Sibson插值法.

若在求解域 Ω 上任意布置一组离散的节点 $N=\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$, 则任意点 $\mathbf{x}_I$ 的一阶Voronoi结构可定义为

$$T_I = \{\mathbf{x} \in R^2 : d(\mathbf{x}, \mathbf{x}_I) < d(\mathbf{x}, \mathbf{x}_J), \forall J \neq I\}, \quad (1)$$

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式中, $d(\mathbf{x}, \mathbf{x}_I)$ 表示点 $\mathbf{x}$ 到 $\mathbf{x}_I$ 的距离. 如果对某点进行插值, 还需要对插值点构造二阶Voronoi结构. 二阶Voronoi结构的数学表达式为

$$T_{IJ} = \{\mathbf{x} \in R^2 : d(\mathbf{x}, \mathbf{x}_I) < d(\mathbf{x}, \mathbf{x}_J) < d(\mathbf{x}, \mathbf{x}_K), \forall K \neq I \neq J\}. \quad (2)$$

从几何上讲, T_{IJ} 实际上表示的是以节点 $\mathbf{x}_I$ 为最近点, 以节点 $\mathbf{x}_J$ 为次近点的空间点位置的集合. 图1所示为平面中7个节点的Voronoi划分以及待插值点 $\mathbf{x}$ 的二阶Voronoi结构. 将具有公共Voronoi结构边界的节点连接形成的三角形称为Delaunay三角形. 通过Delaunay三角形顶点的外接圆都不会包含其他任何节点, 这个特性常被用于确定计算点的自然邻节点.

根据Sibson插值法, 插值点 $\mathbf{x}$ 对节点 $\mathbf{x}_I$ 的形函数 $\phi_I(\mathbf{x})$ 可写为^[20,21]

$$\phi_I(\mathbf{x}) = \frac{A_I(\mathbf{x})}{A(\mathbf{x})}, \quad (3)$$

式中 $A_I(\mathbf{x})$ 和 $A(\mathbf{x})$ 分别表示二阶Voronoi结构 T_{xI} 和一阶Voronoi结构 T_x 的面积. 形函数 $\phi_I(\mathbf{x})$ 满足正定性、插值性、单位分解性和线性一致性. 定义了各节点的插值函数后, 点 $\mathbf{x}$ 的位移函数可写为

$$u^h(\mathbf{x}) = \sum_{I=1}^n \phi_I(\mathbf{x}) u_I, \quad (4)$$

式中, $u_I (I=1, 2, \dots, n)$ 表示节点 $\mathbf{x}$ 周围自然邻节点 I 的位移, $\phi_I(\mathbf{x})$ 是对应节点的形函数.

2 轴对称塑性增量本构关系

材料进入塑性状态后继续加载时, 总应变增量可以分解为弹性和塑性两部分, 即

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}_e + d\boldsymbol{\varepsilon}_p, \quad (5)$$

此时的应力增量和应变增量之间呈现非线性关系:

$$d\boldsymbol{\sigma} = \mathbf{D}_{ep} d\boldsymbol{\varepsilon}, \quad (6)$$

式中, $\mathbf{D}_{ep}$ 表示弹塑性矩阵, 且有^[22~24]

$$\mathbf{D}_{ep} = \mathbf{D}_e - \mathbf{D}_p, \quad (7)$$

式中, $\mathbf{D}_e$ 表示弹性矩阵, $\mathbf{D}_p$ 为塑性矩阵, 且有

图1 点 $\mathbf{x}$ 的一阶和二阶Voronoi结构

Figure 1 First-order and second-order Voronoi cells about $\mathbf{x}$

$$\mathbf{D}_p = \frac{9G^2}{\sigma_s^2(3G + E_p)} \begin{bmatrix} s_r^2 & s_r s_z & s_r s_\theta & s_r \tau_{rz} \\ s_r s_z & s_z^2 & s_z s_\theta & s_z \tau_{rz} \\ s_r s_\theta & s_z s_\theta & s_\theta^2 & s_\theta \tau_{rz} \\ s_r \tau_{rz} & s_z \tau_{rz} & s_\theta \tau_{rz} & \tau_{rz}^2 \end{bmatrix}, \quad (8)$$

式中 s_r , s_z 和 s_θ 表示偏应力, E_p 和 G 分别表示材料的硬化系数和剪切模量, σ_s 为屈服应力.

对于轴对称结构, 应力增量和应变增量为

$$d\boldsymbol{\sigma} = [d\sigma_r \quad d\sigma_z \quad d\sigma_\theta \quad d\tau_{rz}]^T, \quad (9)$$

$$d\boldsymbol{\varepsilon} = [d\varepsilon_r \quad d\varepsilon_z \quad d\varepsilon_\theta \quad d\gamma_{rz}]^T. \quad (10)$$

为了使非线性问题转化为线性问题, 可以采取逐步增加荷载的方式. 在一定的应力和应变水平上施加一个足够小的增量荷载, 将产生应力增量 $\Delta\boldsymbol{\sigma}$ 和应变增量 $\Delta\boldsymbol{\varepsilon}$, 此时本构关系式(6)近似表示为

$$\Delta\boldsymbol{\sigma} = \mathbf{D}_{ep} \Delta\boldsymbol{\varepsilon}. \quad (11)$$

3 控制方程

轴对称问题通常采用圆柱坐标 (r, θ, z) 进行分析, 以 z 轴作为对称轴, 所有应力、应变和位移都只是 r 和 z 的函数, 任一点的位移只有沿 r 方向的径向位移 u 和沿 z 方向的轴向位移 w , 因此可将三维空间问题转换为二维问题. 假设在 t 时刻的条件已知, 即体载荷 $\mathbf{b}$ 、边界载荷 $\mathbf{f}$ 和位移条件 $\mathbf{u}$ 已知, 当过渡到 $t + \Delta t$ 时, 载荷和位移条件有一增量, 即

$${}^{t+\Delta t}\mathbf{f} = {}^t\mathbf{f} + \Delta\mathbf{f}, \quad (12a)$$

$${}^{t+\Delta t}\mathbf{b} = {}^t\mathbf{b} + \Delta\mathbf{b}, \quad (12b)$$

$${}^{t+\Delta t}\bar{\mathbf{u}} = {}^t\bar{\mathbf{u}} + \Delta\bar{\mathbf{u}}, \quad (12c)$$

则位移增量 $\Delta\mathbf{u}$ 和应变增量 $\Delta\boldsymbol{\varepsilon}$ 应满足如下方程:

$$\begin{aligned} & 2\pi \int_{\Omega} \delta(\Delta\boldsymbol{\varepsilon})^T ({}^t\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}) r d\Omega \\ & - 2\pi \int_{\Omega} \delta(\Delta\mathbf{u})^T ({}^t\mathbf{b} + \Delta\mathbf{b}) r d\Omega \\ & - 2\pi \int_{\Gamma_t} \delta(\Delta\mathbf{u})^T ({}^t\bar{\mathbf{t}} + \Delta\bar{\mathbf{t}}) r d\Gamma = 0, \end{aligned} \quad (13)$$

式中, $\delta(\Delta\mathbf{u})$ 和 $\delta(\Delta\boldsymbol{\varepsilon})$ 分别是增量的虚位移和虚应变. 将式(11)代入式(13)可得

$$\begin{aligned} & 2\pi \int_{\Omega} \delta(\Delta\boldsymbol{\varepsilon})^T \mathbf{D}_{ep} \Delta\boldsymbol{\varepsilon} r d\Omega - 2\pi \int_{\Omega} \delta(\Delta\mathbf{u})^T \Delta\mathbf{b} r d\Omega \\ & - 2\pi \int_{\Gamma_t} \delta(\Delta\mathbf{u})^T \Delta\bar{\mathbf{t}} r d\Gamma = -2\pi \int_{\Omega} \delta(\Delta\boldsymbol{\varepsilon})^T {}^t\boldsymbol{\sigma} r d\Omega \\ & + 2\pi \int_{\Omega} \delta(\Delta\mathbf{u})^T {}^t\mathbf{b} r d\Omega + 2\pi \int_{\Gamma_t} \delta(\Delta\mathbf{u})^T {}^t\bar{\mathbf{t}} r d\Gamma. \end{aligned} \quad (14)$$

域内任一点的增量位移场函数可由式(4)表示为

$$\Delta\mathbf{u}^h = \sum_{I=1}^n \phi_I \Delta\mathbf{u}_I. \quad (15)$$

再利用几何关系, 则得到

$$\Delta\boldsymbol{\varepsilon} = \sum_{I=1}^n \mathbf{B}_I \Delta\mathbf{u}_I, \quad (16)$$

式中,

$$\mathbf{B}_I = \begin{bmatrix} \frac{\partial \phi_I}{\partial r} & 0 & \frac{\phi_I}{r} & \frac{\partial \phi_I}{\partial z} \\ 0 & \frac{\partial \phi_I}{\partial z} & 0 & \frac{\partial \phi_I}{\partial r} \end{bmatrix}^T. \quad (17)$$

将式(15)和(16)代入式(14), 并由虚位移的任意性, 可得到下面的代数方程组:

$${}^t\mathbf{K}_{IJ}^{ep} \Delta\mathbf{u}_J = \Delta\mathbf{f}_I, \quad (18)$$

式中,

$${}^t\mathbf{K}_{IJ}^{ep} = 2\pi \int_{\Omega} \mathbf{B}_I^T \mathbf{D}_{ep} \mathbf{B}_J r d\Omega, \quad (19)$$

$$\begin{aligned} \Delta\mathbf{f}_I &= 2\pi \int_{\Gamma_t} \phi_I {}^{t+\Delta t}\bar{\mathbf{t}} r d\Gamma + 2\pi \int_{\Omega} \phi_I {}^{t+\Delta t}\mathbf{b} r d\Omega \\ &\quad - 2\pi \int_{\Omega} \mathbf{B}_I^T {}^t\boldsymbol{\sigma} r d\Omega. \end{aligned} \quad (20)$$

4 求解方案

4.1 迭代公式

为避免每次迭代都要形成和分解总体刚度矩阵, 本研究采用修正的Newton-Raphson方法求解增量自然单元法方程. 将式(18)改写为

$$\mathbf{K}_{IJ}^e \Delta\mathbf{u}_J^{(n)} = \Delta\mathbf{f}_I^{(n)} \quad (n = 0, 1, 2, \dots), \quad (21)$$

式中 n 表示迭代次数, 且有

$$\mathbf{K}_{IJ}^e = 2\pi \int_{\Omega} \mathbf{B}_I^T \mathbf{D}_e \mathbf{B}_J r d\Omega, \quad (22)$$

$$\begin{aligned} \Delta\mathbf{f}_I^{(n)} &= 2\pi \int_{\Gamma_t} \phi_I {}^{t+\Delta t}\bar{\mathbf{t}}^{(n)} r d\Gamma + 2\pi \int_{\Omega} \phi_I {}^{t+\Delta t}\mathbf{b}^{(n)} r d\Omega \\ &\quad - 2\pi \int_{\Omega} \mathbf{B}_I^T {}^{t+\Delta t}\boldsymbol{\sigma}^{(n)} r d\Omega. \end{aligned} \quad (23)$$

4.2 本构关系的积分

迭代过程中, 根据任一积分点的位移增量 $\Delta\mathbf{u}^{(n)}$ 和应变增量 $\Delta\boldsymbol{\varepsilon}^{(n)}$, 可通过如下公式求得应力增量:

$$\Delta\boldsymbol{\sigma}^{(n)} = m\mathbf{D}_e \Delta\boldsymbol{\varepsilon}^{(n)} + \int_0^{(1-m)\Delta\boldsymbol{\varepsilon}^{(n)}} \mathbf{D}_{ep} d\boldsymbol{\varepsilon}, \quad (24)$$

式中 m 称为弹性因子. 为了提高计算精度, 本研究采用切向预测径向返回子增量法^[25]完成本构关系的积分.

5 数值算例

5.1 厚壁圆筒

如图2所示, 考虑一个受均布内压 $P=10.951$ MPa的厚壁圆筒, 材料是理想弹塑性体. 内径 $a=10$ cm, 外径 $b=20$ cm. 弹性模量 $E=10$ GPa, 泊松比 $\mu=0.3$, 屈服极限 $\sigma_s=20$ MPa, 服从von-Mises屈服准则. 为了该问题的分析, 截取长 $h=10$ cm的一段, 采用的两种节点布置方案如图3所示. 图4所示为自然单元法计算的应力数值解与解析解^[26]的对比情况. 显然, 本研究采用规则节点和不规则节点的计算结果和解析解都吻合很好, 从而验证

图2 厚壁圆筒
Figure 2 The thick-walled cylinder

图 3 厚壁圆筒的节点布置. (a) 规则节点; (b) 不规则节点
Figure 3 Nodal arrangement of the thick-walled cylinder. (a) Regularly distribution; (b) irregularly distribution

图 4 (网络版彩色)厚壁圆筒的应力数值解与解析解比较
Figure 4 (Color online) Numerical stress results compared with analytical solutions for the thick-walled cylinder

证了本研究方法的有效性,同时也表明了本研究方法对节点的不规则分布不敏感.

5.2 厚壁球壳

考虑一受均布内压 $P=17$ MPa的厚壁球壳,材料取为理想弹塑性体.球壳内径 $a=10$ cm、外径 $b=16$ cm、弹性模量 $E=10$ GPa、屈服极限 $\sigma_s=20$ MPa、泊松比 $\mu=0.3$,服从von-Mises屈服准则.此问题为球对称问题,现将它作为轴对称问题进行求解,其计算模型如图5所示.采用图6所示的节点布置方案计算得到的应力数值解见图7.为了进行对比,图7还给出了相对应的解析解^[26].从图7可以看出,两者吻合得很好,这进一步验证了本研究方法的有效性.

6 结论

作为一种具有代表性的无网格方法,自然单元法的节点影响域是由节点的Voronoi结构所规定的自然相

图 5 厚壁球壳
Figure 5 The thick-walled spherical shell

图 6 厚壁球壳的节点布置
Figure 6 Nodal distribution for a thick-walled spherical shell

图 7 (网络版彩色)厚壁球壳的应力数值解与解析解比较
Figure 7 (Color online) Numerical stress results compared with analytical solutions for the thick-walled spherical shell

邻关系, 不受人为参数的影响, 具有其他无网格法不可比拟的优越性, 因而被计算力学界广泛关注. 本研究采用自然邻近插值对增量位移场函数近似, 基于增量虚位移原理建立了增量格式的轴对称弹塑性分析的自然

单元法求解方案, 并编制了相应的计算程序. 本研究分析和算例求解结果表明, 采用自然单元法进行轴对称结构的弹塑性分析是可行的, 具有精度高和稳定性好的优点.

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Summary for “轴对称弹塑性问题的自然单元法”

Natural element method for axisymmetric elastoplastic problems

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Based on the meshless natural element method and incremental theory of plasticity, a new algorithm for elastoplastic analysis of a symmetric structure is put forward. The meshless method is a recently developed numerical method, which only requires the nodal information and can effectively overcome the difficulties brought about by the meshes. As widely known, conventional meshfree methods suffer from the numerical difficulties in dealing with the essential boundary conditions. Differing from most of meshfree methods, the natural element method (NEM) is a Galerkin-based meshless method that is built upon the notion of the natural neighbor interpolation. It is noteworthy that, there are two natural neighbor interpolants: Natural neighbor-based Sibson interpolation and Laplace interpolation (non-Sibsonian interpolation). The interpolation relies on the concepts of Delaunay triangulations and Dirichlet tessellations of a set of nodes to build the shape function. Compared with the moving least squares approximation widely used in many meshless methods, the natural neighbor interpolation does not involve complex matrix inversion, needs no artificial parameter but can improve the computational efficiency. Because the constructed shape functions from the natural neighbor interpolation possess the Kronecker delta function property, the essential boundary conditions can be imposed accurately without any special techniques. Meanwhile, the NEM does not require extra efforts to generate the background because it utilizes a set of Delaunay triangles, which are automatically identified in process of the basis function definition, as its background cell. Therefore, the NEM is a most promising numerical method which can be widely applied to nonlinear analysis of materials, three-dimensional calculation and large deformation problems.

Generally, plasticity would make many projects more reasonable, and would fully utilize intensity potentially of structures. The NEM has been successfully used to solve elastoplastic problems of plane structures. In the present work, the NEM is applied to elastoplastic analysis of axisymmetric structures. Firstly, a brief description of the rudiments of Voronoi diagrams and Delaunay triangles in the context of natural neighbor interpolation is delineated. The basic principle of Sibson natural neighbor interpolation approach is also presented. A three-dimensional axisymmetric structure is created by the rotation of the cross section around the axis of symmetry and a set of properly scattered nodes within the cross section are employed. In the analysis of elastoplastic problems, the increments of stress and strain for axisymmetric problems are utilized to characterize the elastoplastic constitutive relationship. Owing to the nonlinear nature of plastic deformation, the computation at each load step employs an iterative solution method, either the Newton-Raphson method or the modified Newton-Raphson method. In order to reduce the expensive operations of stiffness matrix evaluation and factorization, the modified Newton-Raphson method with initial elastic stiffness matrix is adopted in the paper. The tangent predictor-radial return algorithm is applied to integrate the constitutive equations. The elastoplastic analyses of two classical axisymmetric structures have been successfully carried out and the fact that the present results show a good agreement with the analytical solutions demonstrate the effectiveness and accuracy of the proposed method for axisymmetric elastoplastic analyses.

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