GENERALIZED RAYLEIGH PRINCIPLE AND ITS APPLICATIONS

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ABSTRACT

In this paper, we mainly deal with eigenvalue problems of non-self-adjoint operator. To begin with, the generalized Rayleigh variational principle, the idea of which was due to Morse and Feshbach, is examined in detail and proved more strictly in mathematics. Then, other three equivalent formulations of it are presented. While applying them to approximate calculation we find the condition under which the above variational method can be identified as the same with Galerkin's one. After that we illustrate the generalized variational principle by considering the hydrodynamic stability of plane Poiseuille flow and Bénard convection. Finally, the Rayleigh quotient method is extended to the cases of non-self-adjoint matrix in order to determine its strong eigenvalue in linear algebra.

I. INTRODUCTION

A comprehensive theory for the eigenvalue problems of self-adjoint operator has long been established[1]. The well known Rayleigh variational principle, which has found its wide applications in various fields such as vibration and waves in the finite dimensional system or continuum system without dissipation, may be employed in approximate calculation[2]. Nevertheless, we often come across a coupled or dissipative system in reality. It seems to be necessary to explore generalized eigenvalue problems of non-self-adjoint operator and corresponding variational principles.

So far as non-self-adjoint operator is concerned, it is Morse and Feshbach[3] who set up the fundamental idea in 1953. In [4]. Chandrasekhar discussed the instability of the Couette flow between two rotating coaxial cylinders, and then followed the study of the adjoint variational method for ordinary differential equations of high order by Prasad[5]. In 1977 Shen[6] examined the finite element method for non-self-adjoint operators. More recently, Xu[7] has solved the classical Columbus problem and given the stability criterion for liquid-filled cavities. Based on the above achievements, the present author gave several generalized variational principles of different forms applied to a large class of integro-differential equation system with a higher accurate demonstration, thus making the Rayleigh variational principles useful and significant in the finite element method computation. They have been applied to the hydrodynamic stability problems and several variational relations are derived. In conclusion, we have developed a generalized Rayleigh quotient method in the calculation of strong eigenvalue for a non-self-adjoint matrix.

II. ADJOINT OPERATOR IN THE VECTOR FUNCTION SPACE

In order to investigate the eigenvalue problem and corresponding variational principle for a general integro-differential equation system, we must define the so-called adjoint operator and specify its representation in the vector function space. Generally speaking, an integro-differential equation system may be expressed in the operator form as follows:

$$\boldsymbol{L}\boldsymbol{u} = \boldsymbol{f}, \tag{2.1}$$

where

$$L = \begin{pmatrix} l_{11}, & l_{12}, & \cdots & l_{1n} \\ \vdots & \vdots & & \vdots \\ l_{n1}, & l_{n2}, & \cdots & l_{nn} \end{pmatrix}, \tag{2.2}$$

and each element l_{ij} of the above matrix is ordinary integral , differential operator or their combination. Therefore, L is a linear operator mapping one vector function space into another and $\boldsymbol{u}, \boldsymbol{f}$ are elements in the domains of definition and value, respectively. Then, the formal adjoint operator of L is

$$L^* = \begin{pmatrix} l_{11}^*, & l_{21}^* & \cdots & l_{n1}^* \\ \vdots & \vdots & & \vdots \\ l_{1n}^*, & l_{2n}^* & \cdots & l_{nn}^* \end{pmatrix}, \tag{2.3}$$

where l_{ij}^* denotes adjoint operators for ordinary integral and differential ones. It is well known that

$$\begin{gathered} \left[\int d\xi K(x, \xi)\right]^* = \int d\xi\, \overline{K(\xi, x)} \\ \left[\sum_{i=0}^{N} \alpha_i(x) \frac{d^i}{dx^i}\right]^* = \sum_{i=0}^{N} (-1)^i \overline{\frac{d^i}{dx^i} \alpha_i(x)} \\ \left[\sum_{i,j,k\cdots}^{N} A_{i,j,k\cdots} \frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k}\right]^* = \sum_{i,j,k,\cdots}^{N} (-1)^{i+j+k,\cdots,} \overline{\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k \cdots} A_{i,j,k}}, \end{gathered} \tag{2.4}$$

where the symbol bar means "conjugation", because we always need the inner product defined in the complex domain, i.e.

$$\langle f, g\rangle = \int \bar{f} g d\Omega. \tag{2.5}$$

Now it is easy to determine the adjoint operator of any linear one in the vector function space according to (2.3), (2.4). As for differential operator, by integral by part or Green formula, we have

$$\langle u_i, l_{ij} v_j\rangle - \langle l_{ij}^* u_i, v_j\rangle = R(u_i, v_j). \tag{2.6}$$

$R(u_i, v_j)$ usually is not equal to zero, whereas a certain condition for u_i has to be satisfied for boundary value problems. And then we could manage to find the corresponding condition for v_j so as to ensure the right-hand side of (2.6) to be zero. In this case, we have

$$\begin{aligned}\langle \boldsymbol{u}, L\boldsymbol{v}\rangle &= \sum\langle u_i, l_{ij}v_j\rangle \\ &= \sum\langle l_{ij}^* u_i, v_j\rangle = \langle L^*\boldsymbol{u}, \boldsymbol{v}\rangle,\end{aligned} \tag{2.7}$$

and the operators L and L^* are really adjoint to each other. If the boundary condition specified for the original problem is $G\boldsymbol{u}|_\Gamma = 0$, and the corresponding one obtained for $\boldsymbol{v}$ is $G_{\boldsymbol{v}}^*|_\Gamma = 0$, then the problem

$$\begin{aligned} L\boldsymbol{u} &= 0, \\ G\boldsymbol{u}|_\Gamma &= 0 \end{aligned} \tag{2.8}$$

is adjoint to the problem

$$\begin{aligned} L^*\boldsymbol{v} &= 0, \\ G^*\boldsymbol{v}|_\Gamma &= 0. \end{aligned} \tag{2.9}$$

It is evident that the term "adjoint" is reciprocal, i.e.

$$(L^*)^* = L. \tag{2.10}$$

Iff $L = L^*$ and $G = G^*$, we take the boundary value problem to be self-adjoint.

III. Generalized Eigenvalue Problem for Non-self-adjoint Operator

A rather general class of physical problems might be reduced to the following generalized eigenvalue problem for non-self-adjoint operator:

$$\begin{aligned} (A - \lambda B)\boldsymbol{u} &= 0, \\ G\boldsymbol{u}|_\Gamma &= 0, \end{aligned} \tag{3.1}$$

where

$$A = \begin{pmatrix} a_{11}, & a_{12}, & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{1n}, & a_{2n}, & \cdots & a_n{}^u \end{pmatrix},$$

$$B = \begin{pmatrix} b_{11}, & b_{12}, & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{1n}, & b_{2n}, & \cdots & b_{nn} \end{pmatrix}. \tag{3.2}$$

Both of them are linear operators in the vector functions space, $\boldsymbol{u}$ is the element in the same space and λ is the eigenvalue. Thus, the adjoint eigenvalue problem of (3.1) is

$$\begin{aligned} (A^* - \mu B^*)\boldsymbol{v} &= 0, \\ G^*\boldsymbol{v}|_\Gamma &= 0. \end{aligned} \tag{3.3}$$

The relations between the eigenvalues and eigenfunctions for the problems (3.1) and (3.3) are as follows:

Lemma 1. *If A, B are completely continuous operators in a vector function space, and the problem* (3.1) *has an eigenvalue* λ_0, *then* $\bar{\lambda}_0$ *must be one of the eigenvalues of the problem* (3.3).

Proof. We prove the lemma by reduction to absurdity. Suppose that $\bar{\lambda}_0$ is not the eigenvalue of the problem (3.3). According to the complete continuity of the operators A and B and the Fredholm alternative principle, the inhomogeneous problem

$$(A^* - \bar{\lambda}_0 \ B^*)\boldsymbol{v} = \boldsymbol{f}, \qquad G^*\boldsymbol{v}|_\Gamma = 0 \tag{3.4}$$

must have unique nonzero solution for any nonzero element $\boldsymbol{f}$. For the eigenfunction $\boldsymbol{u}_0$ corresponding to the eigenvalue λ_0 of (3.1), we have

$$\begin{aligned}\langle \boldsymbol{u}_0, f\rangle &= \langle \boldsymbol{u}_0, (A^* - \bar{\lambda}_0 \ B^*)\boldsymbol{v}\rangle \\ &= \langle (A - \lambda_0 \ B)\boldsymbol{u}_0, \boldsymbol{v}\rangle = 0.\end{aligned} \tag{3.5}$$

It follows that $\boldsymbol{u}_0 \equiv 0$, which contradicts the assumption that λ_0 is the eigenvalue of (3.1). Consequently, $\bar{\lambda}_0$ must be one of the eigenvalues of (3.3). The proof is completed.

Lemma 2. *The eigenfunctions* $\boldsymbol{u}$, $\boldsymbol{v}$ *corresponding to non-conjugate eigenvalues for the problems* (3.1) *and* (3.3) *respectively are orthogonal in the following sense*:

$$\langle \boldsymbol{v}, B\boldsymbol{u}\rangle = 0, \tag{3.6.a}$$

or

$$\langle B^*\boldsymbol{v}, \boldsymbol{u}\rangle = 0. \tag{3.6.b}$$

Proof. Substracting the inner products by left multiplying (3.1) with $\boldsymbol{v}$ and right multiplying (3.3) with $\boldsymbol{u}$, we immediately obtain

$$\langle \boldsymbol{v}, \lambda B\boldsymbol{u}\rangle - \langle \mu B^*\boldsymbol{v}, \boldsymbol{u}\rangle = 0, \tag{3.7}$$

i.e.

$$(\lambda - \bar{\mu})\langle \boldsymbol{v}, B\boldsymbol{u}\rangle = 0. \tag{3.8}$$

Then we come to the conclusion that $\langle \boldsymbol{v}, B\boldsymbol{u}\rangle = \langle B^*\boldsymbol{v}, \boldsymbol{u}\rangle = 0$ due to $\lambda \neq \bar{\mu}$. The proof is thus finished.

Lemma 3. *The eigenfunctions* $\boldsymbol{u}$, $\boldsymbol{v}$ *corresponding to non-conjugate and nonzero eigenvalues of the problems* (3.1) *and* (3.3) *respectively are orthogonal in the following sense*:

$$\langle \boldsymbol{v}, A\boldsymbol{u}\rangle = 0, \tag{3.9.a}$$

or

$$\langle A^*\boldsymbol{v}, \boldsymbol{u}\rangle = 0. \tag{3.9.b}$$

Because of $\lambda \neq 0$, the problems (3.1) and (3.3) are equivalent to the following adjoint pairs of eigenvalue problems:

$$\left(B-\frac{1}{\lambda}A\right)\boldsymbol{u}=0,$$
$$G\boldsymbol{u}|_{\Gamma}=0, \tag{3.10}$$
$$\left(B^{*}-\frac{1}{\mu}A^{*}\right)\boldsymbol{v}=0,$$
$$G^{*}\boldsymbol{v}|_{\Gamma}=0. \tag{3.11}$$

According to Lemma 2, we readily get Lemma 3.

Lemma 4. *If the eigenfunction system for a generalized eigenvalue problem of non-self-adjoint operator is complete*[1], *we have the following generalized Fourier expansion*:

$$\boldsymbol{f}=\sum_{1}^{\infty}\alpha_k\,\boldsymbol{u}_k=\sum_{1}^{\infty}\beta_k\boldsymbol{v}_k, \tag{3.12}$$

where the coefficients α_k, β_k are

$$\alpha_k=\langle\boldsymbol{v}_k, B\,\boldsymbol{f}\rangle,\quad \beta_k=\langle\boldsymbol{f}, B\,\boldsymbol{u}_k\rangle. \tag{3.13}$$

Here the eigenfunction systems $\{\boldsymbol{u}_k\}$, $\{\boldsymbol{v}_k\}$ have been normalized, i.e.

$$\langle\boldsymbol{v}_i, B\boldsymbol{u}_j\rangle=\delta_{ij}. \tag{3.14}$$

IV. Generalized Rayleigh Variational Principle

By virtue of the properties of the adjoint eigenvalue problems, we first prove the variational principle 1 more strictly and then present other three equivalent formulations:

Variational Principle 1. *For generalized eigenvalue problems* (3.1), (3.3), *the functions* $\boldsymbol{u}$, $\boldsymbol{v}$, *which make the functional*

$$J(\boldsymbol{u},\ \boldsymbol{v})=\frac{\langle\boldsymbol{v}, A\boldsymbol{u}\rangle}{\langle\boldsymbol{v}, B\boldsymbol{u}\rangle} \tag{4.1}$$

to be stationary, among the function class satisfying the conditions $G\boldsymbol{u}|_{\Gamma}=0$, $G^{*}\boldsymbol{v}|_{\Gamma}=0$, *must be the eigenfunctions of the problems* (3.1), (3.3) *respectively. And the value of functional J is the very eigenvalue of* (3.1) *corresponding to the above eigenfunction* $\boldsymbol{u}$.

Proof. $\boldsymbol{u}_0$, $\boldsymbol{v}_0$ are assumed to be the functions making the functional J stationary. Let

$$\lambda=J(\boldsymbol{u}_0,\ \boldsymbol{v}_0)=\frac{\langle\boldsymbol{v}_0, A\boldsymbol{u}_0\rangle}{\langle\boldsymbol{v}_0,\ B\boldsymbol{u}_0\rangle}, \tag{4.2}$$

then we produce

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1) See [8] for the conditions of completeness.

$$\delta J = \frac{\langle \delta \boldsymbol{v}_0, Au_0\rangle + \langle \boldsymbol{v}_0, A\delta \boldsymbol{u}_0\rangle}{\langle \boldsymbol{v}_0, B\boldsymbol{u}_0\rangle} - \frac{\langle \boldsymbol{v}_0, A\boldsymbol{u}_0\rangle(\langle \delta \boldsymbol{v}_0, B\boldsymbol{u}_0\rangle + \langle \boldsymbol{v}_0, B\delta \boldsymbol{u}_0\rangle)}{\langle \boldsymbol{v}_0, B\boldsymbol{u}_0\rangle^2}$$

$$= \frac{\langle \delta \boldsymbol{v}_0, (A-\lambda B)\boldsymbol{u}_0\rangle + \langle (A^* - \bar{\lambda} B^*)\boldsymbol{v}_0, \delta \boldsymbol{u}_0\rangle}{\langle \boldsymbol{v}_0, B\boldsymbol{u}_0\rangle} = 0. \tag{4.3}$$

According to the fundamental theorem in variational calculus, we arrive at

$$(A - \lambda B)\boldsymbol{u}_0 = 0,$$
$$(A^* - \bar{\lambda} B^*)\boldsymbol{v}_0 = 0. \tag{4.4}$$

Considering $G\boldsymbol{u}|_\Gamma = 0$, $G^*\boldsymbol{v}|_\Gamma = 0$ and Lemma 1, we come to the conclusion that $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are the eigenfunction of (3.1) and (3.3) respectively and λ is the eigenvalue of (3.1) corresponding to $\boldsymbol{u}_0$.

Variational Principle 2. *For generalized eigenvalue problems* (3.1) (3.3), *the functions* $\boldsymbol{u}$, $\boldsymbol{v}$, *which make the functional*

$$I(\boldsymbol{u}, \boldsymbol{v}) = \frac{\langle \boldsymbol{v}, B\boldsymbol{u}\rangle}{\langle \boldsymbol{v}, A\boldsymbol{u}\rangle} \tag{4.5}$$

to be stationary, among the function class satisfying the conditions $G\boldsymbol{u}|_\Gamma = 0$, $G^*\boldsymbol{v}|_\Gamma = 0$, *must be the eigenfunctions of the problems* (3.1), (3.3) *corresponding to the nonzero eigenvalue respectively. And the value of the functional I is the reciprocal of the very eigenvalue of* (3.1) *corresponding to the above eigenfunction* $\boldsymbol{u}$.

Proof. Because of $\lambda \neq 0$ and $\bar{\lambda} \neq 0$, the problems (3.1), (3.3) can be reduced to

$$\left(B - \frac{1}{\lambda} A\right)\boldsymbol{u} = 0, \tag{4.6}$$

$$G\boldsymbol{u}|_\Gamma = 0,$$

$$\left(B^* - \frac{1}{\mu} A^*\right)\boldsymbol{v} = 0,$$

$$G^*\boldsymbol{v}|_\Gamma = 0. \tag{4.7}$$

Following the variational principle 1, the variational principle 2 is proved.

Variational Principle 3. *For the generalized eigenvalue problems* (3.1), (3.3), *the functions* $\boldsymbol{u}$, $\boldsymbol{v}$, *which make the functional*

$$P(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{v}, A\boldsymbol{u}\rangle. \tag{4.8}$$

to be stationary under the constraint

$$\langle \boldsymbol{v}, B\boldsymbol{u}\rangle = 1 \tag{4.9}$$

among the function class satisfying the conditions $G\boldsymbol{u}|_\Gamma=0$, $G^*\boldsymbol{v}|_\Gamma=0$, *must be the eigenfunctions of the problems* (3.1), (3.3) *respectively.*

Proof. By means of Lagrange multiplier method, we assume

$$P'(\boldsymbol{u}, \boldsymbol{v}) = P(\boldsymbol{u}, \boldsymbol{v}) + \lambda(1 - \langle \boldsymbol{v}, B\boldsymbol{u}\rangle). \tag{4.10}$$

Provided that $\boldsymbol{u}, \boldsymbol{v}$ are the functions making the functional P' stationary,

$$\begin{aligned}\delta P' &= \langle \delta\boldsymbol{v}_0, A\boldsymbol{u}_0\rangle + \langle \boldsymbol{v}_0, A\delta\boldsymbol{u}_0\rangle - \lambda(\langle \delta\boldsymbol{v}_0, B\boldsymbol{u}_0\rangle + \langle \boldsymbol{v}_0, B\delta\boldsymbol{u}_0\rangle) \\ &= \langle \delta\boldsymbol{v}_0, (A - \lambda B)\boldsymbol{u}_0\rangle + \langle (A^* - \bar{\lambda}B^*)\boldsymbol{v}_0, \delta\boldsymbol{u}_0\rangle = 0.\end{aligned} \tag{4.11}$$

It follows that $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are the eigenfunctions of (3.1), (3.3), and the Lagrange multiplier λ happens to be the very eigenvalue of (3.1) corresponding to $\boldsymbol{u}_0$.

Variational Principle 4. *For the generalized eigenvalue problems* (3.1), (3.3), *the functions* $\boldsymbol{u}, \boldsymbol{v}$, *which make the functional.*

$$Q(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{v}, B\boldsymbol{u}\rangle \tag{4.12}$$

to be stationary under the constraint

$$\langle \boldsymbol{v}, A\boldsymbol{u}\rangle = 1 \tag{4.13}$$

among the function class satisfying the conditions $G\boldsymbol{u}|_\Gamma = 0$, $G^*\boldsymbol{v}|_\Gamma = 0$, *must be the eigenfunctions of* (3.1) *and* (3.3) *corresponding to the nonzero eigenvalue respectively.*

Proof. We might do it following the variational principle 2. The process of it is omitted.

From the above-mentioned variational principles, it is easy to see that if A is self-adjoint operator and B is identity one, then we are able to derive the original Rayleigh variational principle without difficulty. Since the eigenvalue of non-self-adjoint boundary value problems need not be real, we have not shown any extremum property above in contrast to self-adjoint operators.

Now we intend to do some approximate calculation applying the variational principle 3. Assume

$$\boldsymbol{u} = \sum_{i=1}^{n} c_i\boldsymbol{\phi}_i,$$

$$\boldsymbol{v} = \sum_{j=1}^{n} d_j\boldsymbol{\phi}_j, \tag{4.14}$$

where $\{\boldsymbol{\phi}_i\}$, $\{\boldsymbol{\phi}_j\}$ are the base function systems satisfying the boundary conditions of (3.1) and (3.3) respectively. Substituting (4.14) into (4.10) and letting $\partial P'/\partial d_j = 0$ we get the linear algebraic equation system for c as,

$$(\mathscr{A} - \lambda\ \beta)\ \boldsymbol{c} = 0, \tag{4.15}$$

where the elements of the matrices $\mathscr{A}$, $\mathscr{B}$ of order $n \times n$ are

$$a_{ij} = \langle \boldsymbol{\phi}_i, A\boldsymbol{\phi}_j\rangle,$$

$$b_{ij} = \langle \boldsymbol{\phi}_i, B\boldsymbol{\phi}_j\rangle, \tag{4.16}$$

and $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^T$, in which the symbol T means "transpose". Then the equation for determining the eigenvalues is

$$|\mathscr{A} - \lambda \mathscr{B}| = 0. \tag{4.17}$$

Likewise, substituting (4.14) into (4.10) and letting $\partial P' / \partial C_i = 0$, we have the linear algebraic equation system for $\boldsymbol{d}$ as,

$$(\bar{\mathscr{A}}^T - \mu \bar{\mathscr{B}}^T)\boldsymbol{d} = 0. \tag{4.18}$$

It is evident that $\lambda = \bar{\mu}$.

Remark that even if the operators A, B are non-self-adjoint, it might be assumed $\boldsymbol{\phi}_i = \boldsymbol{\phi}_i$ with $G \equiv G^*$, and the above technique is equivalent to Galërkin method. For this reason, we say that the above variational principles are the theoretical basis, which underlies the Palërkin method.

V. Applications in Hydrodynamic Stability

(1) The linear stability of the plane Poiseuille flow could be reduced to the eigenvalue problem of Oir-Sommerfeld equation as follows:

$$\{(D^2 - \alpha^2)^2 - i\alpha R[(U - c)(D^2 - \alpha^2) - D^2 U]\}\phi = 0, \tag{5.1}$$

$$\phi(\pm 1) = \phi'(\pm 1) = 0, \tag{5.2}$$

where $D = d/dy$ is differential operator, α is wavenumber, R is Reynolds number, and c is eigenvalue with its real part indicating phase velocity and its imaginary part multiplied by α indicating the growth rate of disturbances. $U = (1 - y^2)$ represents the velocity profile of the basic flow, ϕ is the complex amplitude of the disturbance stream function.

In [8], several numerical methods have been mentioned. According to the generalized variational principle, we might solve the above problem numerically as well.

We first find the formulation of the corresponding adjoint problem:

$$\{(D^2 - \alpha^2)^2 + i\alpha R[(D^2 - \alpha^2)(U - c) - D^2 U]\phi^* = 0, \tag{5.3}$$

$$\phi^*(\pm 1) = \phi^{*\prime}(\pm 1) = 0. \tag{5.4}$$

On account of the variational principle 1, it is concluded that the eigenvalue c is the stationary value of the functional,

$$J(\phi^*, \phi) = \frac{i \int_{-1}^{1} \bar{\phi}^*\{(D^2 - \alpha^2)^2 - i\alpha R[U(D^2 - \alpha^2) - D^2 U]\phi\} \, dy}{\alpha R \int_{-1}^{1} \bar{\phi}^*(D^2 - \alpha^2)\phi dy.}. \tag{5.5}$$

The eigenfunctions of the problems (5.1), (5.2) and (5.3), (5.4) constitute a biothogonal function system, i.e.

$$\int_{-1}^{1} \bar{\phi}_i^*(D^2 - \alpha^2)\phi_j dy = \delta_{ij}. \tag{5.6}$$

An alternative formulation could be derived according to the variational principle 3 in a similar way.

(2) Seeing that the principle of stability exchange holds true for Bénard convection, we are able to derive the following eigenvalue problem:

$$
\begin{aligned}
(D^2 - a^2)^2 w - \theta &= 0, \\
(D^2 - a^2)\theta &= -Ra^2 w,
\end{aligned}
\tag{5.7}
$$

with the boundary conditions

$$
\begin{aligned}
&w = \theta = 0, && (z = 0,\ 1) \\
&Dw = 0 \quad \text{or} \quad D^2 w = 0, && (z = 0, 1)
\end{aligned}
\tag{5.8}
$$

where w is the disturbance velocity in z direction, θ is disturbance temperature, a is wavenumber, $R = -g\alpha\beta d^4/k\nu$ denotes Rayleigh number, in which g represents the acceleration of gravity, k is thermal conductivity, ν is kinetic viscosity, α is the thermal expansivity, β is the temperature gradient. The Rayleigh number implies the comparison of the unstable effects of buoyance due to heat expansion with the stable ones due to viscosity.

Likewise, we might derive the corresponding adjoint eigenvalue problem:

$$
\begin{aligned}
(D^2 - a^2)^2 w^* \qquad\qquad &= -Ra^2\theta^*, \\
- w^* + (D^2 - a^2)\theta^* &= 0,
\end{aligned}
\tag{5.9}
$$

with the same boundary conditions as (5.8),

$$
\begin{aligned}
&w^* = \theta^* = 0, && (z = 0, 1) \\
&Dw^* = 0 \text{ or } D^2 w^* = 0. && (z = 0, 1)
\end{aligned}
\tag{5.10}
$$

Then the operators A, B corresponding to (3.1) are

$$
A = \begin{pmatrix} (D^2 - a^2)^2 & -1 \\ 0 & (D^2 - a^2) \end{pmatrix},
\tag{5.11}
$$

$$
B = \begin{pmatrix} 0 & 0 \\ -a^2 & 0 \end{pmatrix}.
\tag{5.12}
$$

In the light of the variational principle 1, the eigenvalues of the Bénard problem are the stationary values of the functional,

$$
J(\theta, \theta^*, w, w^*) = \frac{-\int_0^1 [D^2 w D^2 w^* + 2a^2 Dw^* Dw + a^4 ww^* - w^*\theta - D\theta D\theta^* - a^2\theta\theta^*]\, dy}{a^2 \int_0^1 \theta^* w dy}.
\tag{5.13}
$$

We might obtain other variational formulations in a similar way.

VI. Generalized Rayleigh Quotient Method

For self-adjoint matrices, the convergence could be accelerated by using the

Rayleigh quotient method, which will be extended to the cases of Non-self-adjoint matrices.

If a non-self-adjoint matrix A has a complete eigenvector system $\{\boldsymbol{u}_k\}$, $k=1, 2, \ldots, N$, then its adjoint counterpart must have another complete one $\{\boldsymbol{v}_k\}$[10]. As a result, an arbitrary vector $\boldsymbol{z}_0$ in the same space may be expressed as follows:

$$\boldsymbol{z}_0 = \alpha_1\boldsymbol{u}_1 + \alpha_2\boldsymbol{u}_2 + \cdots + \alpha_N\boldsymbol{u}_N \tag{6.1}$$

$$\boldsymbol{z}_0 = \beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 + \cdots + \beta_N\boldsymbol{v}_N, \tag{6.2}$$

which lead to the following vector sequences:

$$\boldsymbol{P}_k = A^k\boldsymbol{z}_0 = \alpha_1\lambda_1^k\boldsymbol{u}_1 + \alpha_2\lambda_2^k\boldsymbol{u}_2 + \cdots + \alpha_N\lambda_N^k\boldsymbol{u}_N, \tag{6.3}$$

$$\boldsymbol{Q}_k = A^{*k}\boldsymbol{z}_0 = \beta_1\bar{\lambda}_1^k\boldsymbol{v}_1 + \beta_2\bar{\lambda}_2^k\boldsymbol{v}_2 + \cdots + \beta_N\bar{\lambda}_N^k\boldsymbol{v}_N. \tag{6.4}$$

When $\{\boldsymbol{u}_k\}$ and $\{\boldsymbol{v}_k\}$ are orthogonal and normalized, we have

$$m_k = \frac{\langle \boldsymbol{Q}_k, A\boldsymbol{P}_k\rangle}{\langle \boldsymbol{Q}_k, \boldsymbol{P}_k\rangle} = \frac{\alpha_1\bar{\beta}_1\lambda_1^{2k+1} + \alpha_2\bar{\beta}_2\lambda_2^{2k+1} + \cdots + \alpha_N\bar{\beta}_N\lambda_N^{2k+1}}{\alpha_1\bar{\beta}_1\lambda_1^{2k} + \alpha_2\bar{\beta}_2\lambda_2^{2k} + \cdots + \alpha_N\bar{\beta}_N\lambda_N^{2k}}. \tag{6.5}$$

Suppose that λ_1 is the strong eigenvalue of the matrix A, i.e.

$$|\lambda_1| \gg \mathrm{Max}\ (|\lambda_2|, |\lambda_3|, \cdots, |\lambda_N|). \tag{6.6}$$

It is clear that

$$\lambda_1 = \lim_{k\to\infty} m_k = \lim_{k\to\infty} \frac{\langle \boldsymbol{Q}_k, A\boldsymbol{P}_k\rangle}{\langle \boldsymbol{Q}_k, \boldsymbol{P}_k\rangle}. \tag{6.7}$$

If the absolute value of λ_1 is not much greater than those of other eigenvalues, the convergence can be improved by eigenvalue shift technique.

In order to determine the next strong eigenvalue λ_2, $\boldsymbol{z}_0$ should be selected to be orthogonal to $\boldsymbol{u}_1$ or $\boldsymbol{v}_1$ so that α_1 or β_1 is equal to zero. It follows from (6.5) that

$$\lambda_2 = \lim_{k\to\infty} m_k = \lim_{k\to\infty} \frac{\langle \boldsymbol{Q}_k, A\boldsymbol{P}_k\rangle}{\langle \boldsymbol{Q}_k, \boldsymbol{P}_k\rangle}. \tag{6.8}$$

When only one or two strong eigenvalues are required, the above method is superior to the usual LR, QR[11] method with much less numerical work.

Example. We have computed the eigenvalues of the following matrix:

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & i & 0 \\ 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{6.9}$$

When choosing $\boldsymbol{z}_0 = (1, 1, 1)$, we obtain

k	Approximate Eigenvalue
7	$4.99999957 + 8.56071269 \times 10^{-8} i$
8	$4.99999996 + 8.52944476 \times 10^{-9} i$
9	$4.99999999 + 8.53278719 \times 10^{-10} i$
10	$4.99999999 + 8.53540864 \times 10^{-11} i$
Exact	5

When choosing $z_0 = (1, 0, 0)$, we obtain the next strong eigenvalue with $\alpha_1 = 0$, $\beta_1 = 0$. With the help of eigenvalue shift, we may arrive at

k	Approximate Eigenvalue
6	$1.41420319 + 0.70710441\ i$
7	$1.41421271 + 0.70710567\ i$
8	$1.41421357 + 0.70710659\ i$
9	$1.41421356 + 0.70710676\ i$
10	$1.41421356 + 0.70710678\ i$
Exact	$1.41421356 + 0.70710678\ i$

Thus, it is concluded that the result of the tenth iteration is sufficiently accurate and the convergence is rapid enough. If the eigenvalue shift method is not applied, more than twenty iterations would be needed so that the same accuracy might be achieved owing to the slight difference between the absolute values of the eigenvalues.

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