可压缩 MHD 方程的正则性与能量守恒

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摘要 本文研究可压缩理想 MHD (magnetohydrodynamics) 系统解的正则性与能量守恒问题, 给出使得该系统保持能量守恒时其解关于正则性的两个充分条件, 其中 $(t, x) \in (0, T) \times \mathbb{T}^d$ ($d \geq 1$). 结论的证明主要利用类似于 Constantin 等 (1994) 研究匀质不可压缩 Euler 方程时所采用的交换子估计.

关键词 可压缩 MHD 方程 正则性 能量守恒

MSC (2020) 主题分类 35D30, 35B65, 35Q35, 76W05

1 引言

在磁流体动力学中, MHD 方程刻画了液体金属、等离子体、盐水和电解质中导电流体的动力学行为 (参见文献 [1–3]). 本文研究如下可压缩理想 MHD 系统的正则性与能量守恒的关系:

$$\begin{cases} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \operatorname{curl} B \times B, \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t B = \operatorname{curl}(u \times B), \\ \operatorname{div} B = 0, \end{cases} \quad (1.1)$$

其中, u 和 B 分别表示速度和磁场, ρ 是密度, p 是压力, 压力 p 是密度 ρ 满足一定条件下的给定函数. 这里所考虑的空间维数是 d ($d \geq 1$) 维.

流体动力学方程弱解的能量守恒问题是近年来一个热点问题. 对于不可压缩 Euler 方程, 自从 1949 年 Onsager [4] 提出其著名的 Onsager 猜想后, 已经有了许多重要的研究成果. 受湍流定律的启发, Onsager 首先猜想: 存在一个阈值, 当弱解的正则性超过这个阈值时能量守恒. 具体地, 对于 3 维 Euler

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方程在 Hölder 空间 C^α 中的任意弱解, 当指标 $\alpha > \frac{1}{3}$ 时能量守恒, 而当 $\alpha < \frac{1}{3}$ 时存在能量耗散的弱解. Eyink [5] 在较强的假设条件下证明了解的的能量守恒. 紧接着, Constantin 等 [6] 证明了在周期区域上当速度场 $u \in B_3^{\alpha, \infty}$ ($\alpha > \frac{1}{3}$) 时能量守恒. Duchon 和 Robert [7] 得到了在较弱条件下的能量守恒. 2008 年, Cheskidov 等 [8] 利用 Littlewood-Paley 分解进一步提高了之前的结果. 需要指出的是, 上述结果均是在全空间和周期区域上 Euler 方程的能量守恒. 对于 3 维不可压缩 Euler 方程在有界区域上的能量守恒, 可参见文献 [9–11]. 另外对于不可压缩或可压缩 Euler 方程及 Navier-Stokes 方程的能量守恒问题可参见文献 [12–14] 等.

对于 MHD 方程的能量守恒问题最近也有了一些结果 (参见文献 [15–18]). Gao 等 [17] 考虑了不可压缩 MHD 方程弱解的局部能量方程. Guo 和 Tan [18] 在能量耗散方法的基础上, 从纵向和横向分别研究了不可压缩 MHD 系统的能量方程. Wu 和 Tan [15] 证明了在 Besov 空间的正则性条件下, 非匀质不可压缩 MHD 系统弱解的全局能量守恒. 而对于在有界区域上的 3 维不可压缩理想 MHD 方程, Wang 和 Zuo [16] 得到了具有内部 Besov 正则性弱解的能量守恒.

上述关于 MHD 方程的能量守恒及耗散问题, 考虑的均是不可压缩情形, 而关于可压缩情形的能量守恒的结果还很少. 最近, 文献 [19] 利用文献 [6, 13] 中的方法研究了可压缩 MHD 方程的正则性及能量守恒问题. 本文同样利用文献 [6] 中研究不可压缩 Euler 方程时所采用的交换子估计, 以及文献 [13, 20] 中的正则化方法, 得到了可压缩理想 MHD 方程能量守恒的两个充分条件. 具体地, 为了得到第一个充分条件 (定理 1.1), 不同于匀质 Euler 方程中的项 $\partial_t u$ (参见文献 [6]), 系统 (1.1) 中的项 $\partial_t(\rho u)$ 是非线性的, 这里采用在时间和空间上同时进行正则化的方法来处理该项. 由此我们在对余项进行交换子估计时, 除了空间上的 Besov 正则性假设外, 还需要加上关于时间的 Besov 正则性假设. 在第二个充分条件 (定理 1.2), 对于 ρ, u 和 B 均不需要关于时间的 Besov 正则性要求, 但需要假设条件 $\rho, u, B \in BV \cap C$, 其中 BV 和 C 分别表示有界变差函数和连续函数空间. 另外, 相比于可压缩 Euler 方程, 可压缩 MHD 方程还需要考虑磁场 B 的正则性对系统能量守恒的影响, 即需要处理非线性项 $\text{curl} B \times B$ 以及 $\text{curl}(u \times B)$. 在定理 1.1 中, 我们利用条件 $\text{div} B = 0$, 将 Lorentz 力项 $\text{curl} B \times B$ 改写成导数形式, 并将该导数通过分部积分转移到实验函数上, 从而去掉了文献 [19] 中关于 $\text{curl} B$ 的正则性假设条件.

在给出本文的主要结论之前, 首先给出压力势的定义如下:

$$P(\rho) \stackrel{\text{def}}{=} \rho \int_1^\rho \frac{p(r)}{r^2} dr.$$

另外, 用记号 $BV(\Omega)$ 表示有界变差函数空间. 由有界变差函数空间的拓扑性质可知, 若一个光滑函数序列 $\{\omega_n\}_{n=1}^\infty$ 在 $BV(\Omega)$ 上收敛到 ω , 则意味着 ω_n 在 $L^1(\Omega)$ 上强收敛到 ω , 且其梯度 $\nabla \omega_n$ 在有界测度空间 $\mathcal{M}(\Omega)$ 上弱 * 收敛到 $\nabla \omega$.

本文的结论是如下两个定理:

定理 1.1 令 (ρ, u, B) 是系统 (1.1) 在分布意义下的弱解, 假设

$$u \in (B_3^{\alpha, \infty} \cap L^\infty)((0, T) \times \mathbb{T}^d), \quad B \in B_3^{\beta, \infty}((0, T) \times \mathbb{T}^d), \quad \rho \in (B_3^{\gamma, \infty} \cap L^\infty)((0, T) \times \mathbb{T}^d), \quad (1.2)$$

且 $0 \leq \underline{\rho} \leq \rho \leq \bar{\rho}$ 对 $(t, x) \in (0, T) \times \mathbb{T}^d$ 几乎处处成立, 这里 $\underline{\rho}, \bar{\rho}, 0 < \gamma \leq \alpha < 1$ 和 $0 < \beta < 1$ 是常数并满足

$$\min\{2\alpha + \gamma, \alpha + 2\beta, \alpha + 2\gamma\} > 1.$$

另外假设 $p \in C^2[\underline{\rho}, \bar{\rho}]$, 则系统 (1.1) 在分布意义下满足局部能量守恒, 即

$$\partial_t \left(\frac{1}{2} \rho |u|^2 + \frac{1}{2} |B|^2 + P(\rho) \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |u|^2 + p(\rho) + P(\rho) \right) u \right] - \operatorname{div}((u \times B) \times B) = 0,$$

其中 $(t, x) \in (0, T) \times \mathbb{T}^d$.

注 1.1 注意到条件 $\alpha \geq \gamma$, 则 $3\alpha \geq \alpha + 2\gamma > 1$, 因此 α 需满足大于 $\frac{1}{3}$. 从而与文献 [6, 13, 21] 等研究不可压缩或可压缩 Euler 方程能量守恒时对速度场的正则性要求一致. 从该意义上来说, 定理 1.1 中为保证可压缩 MHD 方程能量守恒其解所需满足的正则性条件是最优的.

注 1.2 文献 [19] 中关于磁场 B 的条件需满足 $\operatorname{curl} B \in B_3^{\beta, \infty}((0, T) \times \mathbb{T}^d)$, 这里去掉了这个条件.

下面给出使得系统 (1.1) 保持能量守恒另一个充分条件, 这里不需要关于时间的 Besov 正则性假设条件.

定理 1.2 假设压力 p 满足

$$p \in C^2(0, \infty) \cap C[0, \infty), \quad p(0) = 0. \quad (1.3)$$

$(\rho, u, B) \in L^\infty((0, T) \times \mathbb{T}^d)$ 是系统 (1.1) 在分布意义下的一个解, 且对几乎处处 $t \in [0, T]$ 满足

$$\rho(t), u(t), B(t) \in BV \cap C(\mathbb{T}^d)$$

及

$$\rho, u, B \in L^\infty(0, T; C(\mathbb{T}^d)), \quad \nabla \rho, \nabla u, \operatorname{curl} B \in L_{\text{weak}^*}^\infty(0, T; \mathcal{M}(\mathbb{T}^d)).$$

则系统 (1.1) 在分布意义下满足如下局部能量方程:

$$\partial_t \left(\frac{1}{2} \rho |u|^2 + \frac{1}{2} |B|^2 + P(\rho) \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |u|^2 + p(\rho) + P(\rho) \right) u \right] - \operatorname{div}[(u \times B) \times B] = 0,$$

其中 $(t, x) \in (0, T) \times \mathbb{T}^d$.

注 1.3 定理 1.1 和 1.2 的证明方法同样适用于可压缩理想 Hall-MHD 方程, 并可得到该方程相应的局部能量守恒充分条件.

2 预备知识

2.1 Besov 空间

假设 $\Omega = (0, T) \times \mathbb{T}^d$ 或者 $\Omega = \mathbb{T}^d$. Besov 空间 $B_p^{\alpha, \infty}(\Omega)$ 由具有如下范数的函数 f 所组成:

$$\|f\|_{B_p^{\alpha, \infty}(\Omega)} \stackrel{\text{def}}{=} \|f\|_{L^p(\Omega)} + \sup_{y \in \Omega} \frac{\|f(\cdot + y) - f\|_{L^p(\Omega \cap (\Omega - y))}}{|y|^\alpha} < \infty, \quad (2.1)$$

这里 $\Omega - y = \{x - y : x \in \Omega\}$.

2.2 磨光化

设 $\eta \in C_c^\infty(\mathbb{R}^N)$ 是一个标准的磨光核, 这里 $N = 1 + d$ 或者 $N = d$. 令

$$\eta^\epsilon(x) = \frac{1}{\epsilon^N} \eta\left(\frac{x}{\epsilon}\right). \quad (2.2)$$

对任意函数 f , 规定记号 $f^\epsilon = \eta^\epsilon * f$, 这里 f^ϵ 在 $\Omega^\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$ 是有意义的.

由 Besov 空间的定义, 可得到如下结论:

$$\|f^\epsilon - f\|_{L^p(\Omega)} \leq C\epsilon^\alpha \|f\|_{B_p^{\alpha,\infty}(\Omega)} \quad (2.3)$$

和

$$\|\nabla f^\epsilon\|_{L^p(\Omega)} \leq C\epsilon^{\alpha-1} \|f\|_{B_p^{\alpha,\infty}(\Omega)}. \quad (2.4)$$

其证明可参见文献 [15].

3 主要结论的证明

3.1 定理 1.1 的证明

类似于文献 [13] 的方法, 磨光动量方程得

$$\partial_t(\rho u)^\epsilon + \text{div}(\rho u \otimes u)^\epsilon + \nabla p(\rho)^\epsilon = (\text{curl} B \times B)^\epsilon.$$

设 $\varphi \in C_0^\infty((0, T) \times \mathbb{T}^d)$ 是实验函数, 用 φu^ϵ 乘以上述磨光后的方程, 然后关于时间和空间积分得

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho u)^\epsilon \cdot \varphi u^\epsilon dx dt + \int_0^T \int_{\mathbb{T}^d} \text{div}(\rho u \otimes u)^\epsilon \cdot \varphi u^\epsilon dx dt \\ & - \int_0^T \int_{\mathbb{T}^d} (\text{curl} B \times B)^\epsilon \cdot \varphi u^\epsilon dx dt + \int_0^T \int_{\mathbb{T}^d} \varphi u^\epsilon \cdot \nabla p(\rho)^\epsilon dx dt \\ & = 0. \end{aligned} \quad (3.1)$$

取 ϵ 足够小, 使得 $\text{supp } \varphi \subset (\epsilon, T - \epsilon) \times \mathbb{T}^d$. 通过适当的交换子可将 (3.1) 改写为

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon) \cdot \varphi u^\epsilon dx dt + \int_0^T \int_{\mathbb{T}^d} \text{div}((\rho u)^\epsilon \otimes u^\epsilon) \cdot \varphi u^\epsilon dx dt \\ & - \int_0^T \int_{\mathbb{T}^d} (\text{curl} B^\epsilon \times B^\epsilon) \cdot \varphi u^\epsilon dx dt + \int_0^T \int_{\mathbb{T}^d} \varphi u^\epsilon \cdot \nabla p(\rho^\epsilon) dx dt \\ & \stackrel{\text{def}}{=} R_1^\epsilon + R_2^\epsilon + R_3^\epsilon + R_4^\epsilon, \end{aligned} \quad (3.2)$$

其中,

$$\begin{aligned} R_1^\epsilon & \stackrel{\text{def}}{=} \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) \cdot \varphi u^\epsilon dx dt, \\ R_2^\epsilon & \stackrel{\text{def}}{=} \int_0^T \int_{\mathbb{T}^d} \text{div}[(\rho u)^\epsilon \otimes u^\epsilon - (\rho u \otimes u)^\epsilon] \cdot \varphi u^\epsilon dx dt, \\ R_3^\epsilon & \stackrel{\text{def}}{=} \int_0^T \int_{\mathbb{T}^d} [(\text{curl} B \times B)^\epsilon - (\text{curl} B^\epsilon \times B^\epsilon)] \cdot \varphi u^\epsilon dx dt, \\ R_4^\epsilon & \stackrel{\text{def}}{=} \int_0^T \int_{\mathbb{T}^d} \nabla(p(\rho^\epsilon) - p(\rho)^\epsilon) \cdot \varphi u^\epsilon dx dt. \end{aligned}$$

下面依次计算 (3.2) 中的各项. 对于左边的第 1 项, 有

$$\int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon) \cdot \varphi u^\epsilon dx dt = \int_0^T \int_{\mathbb{T}^d} \left(\varphi \partial_t \rho^\epsilon |u^\epsilon|^2 + \frac{1}{2} \varphi \rho^\epsilon \partial_t |u^\epsilon|^2 \right) dx dt. \quad (3.3)$$

对于 (3.2) 左边的第 2 项, 首先磨光连续性方程 (1.1)₂, 有

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho u)^\epsilon = 0. \quad (3.4)$$

直接计算, 可得

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon) \cdot \varphi u^\epsilon dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} ((\rho u)^\epsilon \otimes u^\epsilon) : \nabla(\varphi u^\epsilon) dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} \varphi((\rho u)^\epsilon \otimes u^\epsilon) : \nabla u^\epsilon dx dt - \int_0^T \int_{\mathbb{T}^d} ((\rho u)^\epsilon \otimes u^\epsilon) : (\nabla \varphi \otimes u^\epsilon) dx dt \\ &= - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \varphi(\rho u)^\epsilon \cdot \nabla |u^\epsilon|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} ((\rho u)^\epsilon \cdot \nabla \varphi) |u^\epsilon|^2 dx dt \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \operatorname{div}(\varphi(\rho u)^\epsilon) |u^\epsilon|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} ((\rho u)^\epsilon \cdot \nabla \varphi) |u^\epsilon|^2 dx dt \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} (\nabla \varphi \cdot (\rho u)^\epsilon + \varphi \operatorname{div}(\rho u)^\epsilon) |u^\epsilon|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} ((\rho u)^\epsilon \cdot \nabla \varphi) |u^\epsilon|^2 dx dt \\ &= - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \varphi \partial_t \rho^\epsilon |u^\epsilon|^2 dx dt - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} ((\rho u)^\epsilon \cdot \nabla \varphi) |u^\epsilon|^2 dx dt. \end{aligned} \quad (3.5)$$

为了计算 (3.2) 左边的第 3 项, 我们磨光 (1.1)₃ 并乘以 B^ϵ , 得

$$\frac{1}{2} \partial_t |B^\epsilon|^2 = \operatorname{curl}(u \times B)^\epsilon \cdot B^\epsilon. \quad (3.6)$$

上式结合公式 $\operatorname{div}(a \times b) = \operatorname{curl} a \cdot b - a \cdot \operatorname{curl} b$, 有

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} (\operatorname{curl} B^\epsilon \times B^\epsilon) \cdot \varphi u^\epsilon dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} \varphi(u^\epsilon \times B^\epsilon) \cdot \operatorname{curl} B^\epsilon dx dt \\ &= \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{div}((u^\epsilon \times B^\epsilon) \times B^\epsilon) dx dt - \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{curl}(u^\epsilon \times B^\epsilon) \cdot B^\epsilon dx dt \\ &= \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{div}((u^\epsilon \times B^\epsilon) \times B^\epsilon) dx dt - \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{curl}(u \times B)^\epsilon \cdot B^\epsilon dx dt \\ &\quad - \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{curl}[(u^\epsilon \times B^\epsilon) - (u \times B)^\epsilon] \cdot B^\epsilon dx dt \\ &\stackrel{\text{def}}{=} - \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \varphi \partial_t |B^\epsilon|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} (u^\epsilon \times B^\epsilon) \times B^\epsilon \cdot \nabla \varphi dx dt + R_5^\epsilon \\ &= \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \partial_t \varphi |B^\epsilon|^2 dx dt - \int_0^T \int_{\mathbb{T}^d} (u^\epsilon \times B^\epsilon) \times B^\epsilon \cdot \nabla \varphi dx dt + R_5^\epsilon. \end{aligned} \quad (3.7)$$

对于 (3.2) 左边第 4 项, 利用磨光的质量方程 (3.4) 及压力势 P 的定义, 有

$$P'(\rho^\epsilon)(\partial_t \rho^\epsilon + u^\epsilon \cdot \nabla \rho^\epsilon + \rho^\epsilon \operatorname{div} u^\epsilon) = P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon).$$

再由链式法则, 上式变为

$$\partial_t P(\rho^\epsilon) + u^\epsilon \cdot \nabla P(\rho^\epsilon) + P'(\rho^\epsilon) \rho^\epsilon \operatorname{div} u^\epsilon = P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon). \quad (3.8)$$

另外, 由 P 的定义, 有

$$\rho^\epsilon P'(\rho^\epsilon) = P(\rho^\epsilon) + p(\rho^\epsilon).$$

由此可计算 (3.2) 左边第 4 项如下:

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \varphi u^\epsilon \cdot \nabla p(\rho^\epsilon) dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot u^\epsilon p(\rho^\epsilon) dx dt - \int_0^T \int_{\mathbb{T}^d} \varphi p(\rho^\epsilon) \operatorname{div} u^\epsilon dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot u^\epsilon p(\rho^\epsilon) dx dt - \int_0^T \int_{\mathbb{T}^d} \varphi [\rho^\epsilon P'(\rho^\epsilon) - P(\rho^\epsilon)] \operatorname{div} u^\epsilon dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot u^\epsilon p(\rho^\epsilon) dx dt + \int_0^T \int_{\mathbb{T}^d} \varphi [\partial_t P(\rho^\epsilon) + u^\epsilon \cdot \nabla P(\rho^\epsilon) + P(\rho^\epsilon) \operatorname{div} u^\epsilon] dx dt \\ &\quad - \int_0^T \int_{\mathbb{T}^d} \varphi P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dx dt \\ &= - \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot u^\epsilon p(\rho^\epsilon) dx dt - \int_0^T \int_{\mathbb{T}^d} \partial_t \varphi P(\rho^\epsilon) dx dt - \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot P(\rho^\epsilon) u^\epsilon dx dt \\ &\quad - \int_0^T \int_{\mathbb{T}^d} \varphi P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dx dt. \end{aligned} \quad (3.9)$$

联立 (3.2)、(3.3)、(3.5)、(3.7) 和 (3.9), 得

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t \varphi \left(\frac{1}{2} \rho^\epsilon |u^\epsilon|^2 + \frac{1}{2} |B^\epsilon|^2 + P(\rho^\epsilon) \right) dx dt \\ &+ \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot \left(\frac{1}{2} (\rho u)^\epsilon |u^\epsilon|^2 + p(\rho^\epsilon) u^\epsilon + P(\rho^\epsilon) u^\epsilon - (u^\epsilon \times B^\epsilon) \times B^\epsilon \right) dx dt \\ &=: -R_1^\epsilon - R_2^\epsilon - R_3^\epsilon - R_4^\epsilon - \int_0^T \int_{\mathbb{T}^d} \varphi P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dx dt. \end{aligned} \quad (3.10)$$

为了证明定理 1.1, 我们需证明当 $\epsilon \rightarrow 0$ 时, R_i^ϵ ($i = 1, 2, 3, 4$) 和 $\int_0^T \int_{\mathbb{T}^d} \varphi P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dx dt$ 趋于 0. 事实上, 当 $\epsilon \rightarrow 0$ 时, 利用磨光函数的性质, 得

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t \varphi \left(\frac{1}{2} \rho^\epsilon |u^\epsilon|^2 + \frac{1}{2} |B^\epsilon|^2 + P(\rho^\epsilon) \right) dx dt \\ &+ \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot \left(\frac{1}{2} (\rho u)^\epsilon |u^\epsilon|^2 + p(\rho^\epsilon) u^\epsilon + P(\rho^\epsilon) u^\epsilon - (u^\epsilon \times B^\epsilon) \times B^\epsilon \right) dx dt \end{aligned}$$

趋近于

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \varphi \left(\frac{1}{2} \rho |u|^2 + \frac{1}{2} |B|^2 + P(\rho) \right) dx dt$$

$$+ \int_0^T \int_{\mathbb{T}^d} \nabla \varphi \cdot \left(\frac{1}{2} \rho |u|^2 u + p(\rho) u + P(\rho) u - (u \times B) \times B \right) dx dt.$$

对于 R_1^ϵ , 注意到

$$\begin{aligned} \rho^\epsilon u^\epsilon - (\rho u)^\epsilon &= (\rho^\epsilon - \rho)(u^\epsilon - u) - \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (\rho(t - \tau, x - \xi) - \rho(t, x)) \\ &\quad \times (u(t - \tau, x - \xi) - u(t, x)) d\xi d\tau. \end{aligned} \quad (3.11)$$

由此, R_1^ϵ 的第 1 部分可由分部积分公式、(2.3)、(2.4) 及定理 1.1 的假设, 当 $\epsilon \rightarrow 0$ 时, 有

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^d} \varphi \partial_t [(\rho^\epsilon - \rho)(u^\epsilon - u)] \cdot u^\epsilon dx dt \right| \\ &\leq \int_0^T \int_{\mathbb{T}^d} |\partial_t \varphi (\rho^\epsilon - \rho)(u^\epsilon - u) \cdot u^\epsilon| dx dt + \int_0^T \int_{\mathbb{T}^d} |\varphi (\rho^\epsilon - \rho)(u^\epsilon - u) \cdot \partial_t u^\epsilon| dx dt \\ &\leq C \|\varphi\|_{C^1} \epsilon^\gamma \epsilon^\alpha \|\rho\|_{B_3^{\gamma, \infty}} \|u\|_{B_3^{\alpha, \infty}}^2 + C \|\varphi\|_{C^0} \epsilon^\gamma \epsilon^\alpha \epsilon^{\alpha-1} \|\rho\|_{B_3^{\gamma, \infty}} \|u\|_{B_3^{\alpha, \infty}}^2 \rightarrow 0. \end{aligned}$$

对于 R_1^ϵ 的第 2 部分, 根据 Fubini 定理、(2.3)、(2.4) 和分部积分公式, 可得当 $\epsilon \rightarrow 0$ 时,

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^d} \varphi \partial_t \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (\rho(t - \tau, x - \xi) - \rho(t, x)) (u(t - \tau, x - \xi) - u(t, x)) d\xi d\tau \cdot u^\epsilon dx dt \right| \\ &= \left| \int_0^T \int_{\mathbb{T}^d} \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (\rho(t - \tau, x - \xi) \right. \\ &\quad \left. - \rho(t, x)) (u(t - \tau, x - \xi) - u(t, x)) \cdot \partial_t (\varphi(t, x) u^\epsilon(t, x)) d\xi d\tau dx dt \right| \\ &\leq \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \int_0^T \int_{\mathbb{T}^d} |\eta^\epsilon(\tau, \xi) (\rho(t - \tau, x - \xi) \\ &\quad - \rho(t, x)) (u(t - \tau, x - \xi) - u(t, x)) \cdot \partial_t \varphi(t, x) u^\epsilon(t, x)| dx dt d\xi d\tau \\ &\quad + \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \int_0^T \int_{\mathbb{T}^d} |\eta^\epsilon(\tau, \xi) (\rho(t - \tau, x - \xi) \\ &\quad - \rho(t, x)) (u(t - \tau, x - \xi) - u(t, x)) \cdot \varphi(t, x) \partial_t u^\epsilon(t, x)| dx dt d\xi d\tau. \end{aligned}$$

进而有

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^d} \varphi \partial_t \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (\rho(t - \tau, x - \xi) - \rho(t, x)) (u(t - \tau, x - \xi) - u(t, x)) d\xi d\tau \cdot u^\epsilon dx dt \right| \\ &\leq \|\varphi\|_{C^1} \frac{1}{\epsilon^N} \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta \left(\frac{\tau}{\epsilon}, \frac{\xi}{\epsilon} \right) \|\rho(t - \tau, x - \xi) - \rho(t, x)\|_{L^3(Q)} \|u(t - \tau, x - \xi) - u(t, x)\|_{L^3(Q)} \\ &\quad \times \|u^\epsilon(t, x)\|_{L^3(Q)} d\xi d\tau \\ &\quad + \|\varphi\|_{C^0} \frac{1}{\epsilon^N} \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta \left(\frac{\tau}{\epsilon}, \frac{\xi}{\epsilon} \right) \|\rho(t - \tau, x - \xi) - \rho(t, x)\|_{L^3(Q)} \|u(t - \tau, x - \xi) - u(t, x)\|_{L^3(Q)} \\ &\quad \times \|\partial_t u^\epsilon(t, x)\|_{L^3(Q)} d\xi d\tau \\ &\leq \|\varphi\|_{C^1} \frac{1}{\epsilon^N} \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta \left(\frac{\tau}{\epsilon}, \frac{\xi}{\epsilon} \right) \epsilon^\gamma \|\rho\|_{B_3^{\gamma, \infty}(\Omega)} \epsilon^\alpha \|u\|_{B_3^{\alpha, \infty}(\Omega)}^2 d\xi d\tau \\ &\quad + \|\varphi\|_{C^0} \frac{1}{\epsilon^N} \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta \left(\frac{\tau}{\epsilon}, \frac{\xi}{\epsilon} \right) \epsilon^\gamma \|\rho\|_{B_3^{\gamma, \infty}(\Omega)} \epsilon^\alpha \|u\|_{B_3^{\alpha, \infty}(\Omega)} \epsilon^{\alpha-1} \|u\|_{B_3^{\alpha, \infty}(\Omega)} d\xi d\tau \end{aligned}$$

$$\leq C\|\varphi\|_{C^1}\epsilon^\gamma\epsilon^\alpha\|\rho\|_{B_3^{\gamma,\infty}}\|u\|_{B_3^{\alpha,\infty}}^2 + C\|\varphi\|_{C^0}\epsilon^\gamma\epsilon^\alpha\epsilon^{\alpha-1}\|\rho\|_{B_3^{\gamma,\infty}}\|u\|_{B_3^{\alpha,\infty}}^2 \rightarrow 0,$$

这里 $Q = \Omega \cap (\Omega + (\tau, \xi))$, $|(\tau, \xi)| < \epsilon$.

类似地处理 R_2^ϵ . 首先有

$$\begin{aligned} (\rho u)^\epsilon \otimes u^\epsilon - (\rho u \otimes u)^\epsilon &= ((\rho u)^\epsilon - \rho u) \otimes (u^\epsilon - u) - \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (\rho u(t - \tau, x - \xi) - \rho u(t, x)) \\ &\quad \otimes (u(t - \tau, x - \xi) - u(t, x)) d\xi d\tau. \end{aligned}$$

要利用上述分解来估计 R_2^ϵ , 我们要求 $\rho u \in B_3^{\alpha,\infty}((0, T) \times \mathbb{T}^d)$. 另外, 当 $\alpha \geq \gamma$ 时, 有 $B_3^{\alpha,\infty} \hookrightarrow B_3^{\gamma,\infty}$, 并且

$$\begin{aligned} \frac{\|(\rho u)(\cdot + \xi) - \rho u\|_{L^3}}{|\xi|^\gamma} &\leq \frac{\|\rho(u(\cdot + \xi) - u)\|_{L^3}}{|\xi|^\gamma} + \frac{\|(\rho(\cdot + \xi) - \rho)u(\cdot + \xi)\|_{L^3}}{|\xi|^\gamma} \\ &\leq \|\rho\|_{L^\infty}\|u\|_{B_3^{\gamma,\infty}} + \|u\|_{L^\infty}\|\rho\|_{B_3^{\gamma,\infty}} \\ &\leq \|\rho\|_{L^\infty}\|u\|_{B_3^{\alpha,\infty}} + \|u\|_{L^\infty}\|\rho\|_{B_3^{\gamma,\infty}}. \end{aligned}$$

对于 R_2^ϵ 的第 1 部分, 当 $\epsilon \rightarrow 0$ 时, 有

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^d} \varphi \operatorname{div}[(\rho u)^\epsilon - \rho u] \otimes (u^\epsilon - u) \cdot u^\epsilon dx dt \right| \\ &\leq \|\varphi\|_{C^0} \int_0^T \int_{\mathbb{T}^d} |((\rho u)^\epsilon - \rho u) \otimes (u^\epsilon - u) : \nabla u^\epsilon| dx dt \\ &\quad + \|\varphi\|_{C^1} \int_0^T \int_{\mathbb{T}^d} |(((\rho u)^\epsilon - \rho u) \otimes (u^\epsilon - u)) \cdot u^\epsilon| dx dt \\ &\leq C\|\varphi\|_{C^0}\epsilon^\gamma\epsilon^\alpha\epsilon^{\alpha-1}\|\rho u\|_{B_3^{\gamma,\infty}}\|u\|_{B_3^{\alpha,\infty}}^2 + C\|\varphi\|_{C^1}\epsilon^\gamma\epsilon^\alpha\|\rho u\|_{B_3^{\gamma,\infty}}\|u\|_{B_3^{\alpha,\infty}}^2 \\ &\leq C\|\varphi\|_{C^0}\epsilon^{\gamma+2\alpha-1}(\|\rho\|_{L^\infty}\|u\|_{B_3^{\alpha,\infty}} + \|u\|_{L^\infty}\|\rho\|_{B_3^{\gamma,\infty}})\|u\|_{B_3^{\alpha,\infty}}^2 \\ &\quad + C\|\varphi\|_{C^1}\epsilon^{\gamma+\alpha}(\|\rho\|_{L^\infty}\|u\|_{B_3^{\alpha,\infty}} + \|u\|_{L^\infty}\|\rho\|_{B_3^{\gamma,\infty}})\|u\|_{B_3^{\alpha,\infty}}^2 \rightarrow 0. \end{aligned}$$

同样地, 对于 R_2^ϵ 的第 2 部分, 当 $\epsilon \rightarrow 0$ 时, 有

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{T}^d} \operatorname{div} \left[\int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (\rho u(t - \tau, x - \xi) - \rho u(t, x)) \right. \right. \\ &\quad \left. \left. \otimes (u(t - \tau, x - \xi) - u(t, x)) d\xi d\tau \right] \cdot \varphi u^\epsilon dx dt \right| \\ &\leq C\|\varphi\|_{C^0}\epsilon^\gamma\epsilon^\alpha\epsilon^{\alpha-1}(\|\rho\|_{L^\infty}\|u\|_{B_3^{\alpha,\infty}} + \|u\|_{L^\infty}\|\rho\|_{B_3^{\gamma,\infty}})\|u\|_{B_3^{\alpha,\infty}}^2 \\ &\quad + C\|\varphi\|_{C^1}\epsilon^\gamma\epsilon^\alpha(\|\rho\|_{L^\infty}\|u\|_{B_3^{\alpha,\infty}} + \|u\|_{L^\infty}\|\rho\|_{B_3^{\gamma,\infty}})\|u\|_{B_3^{\alpha,\infty}}^2 \rightarrow 0. \end{aligned}$$

考虑 R_3^ϵ , 令

$$\begin{aligned} R_3^\epsilon &= \int_0^T \int_{\mathbb{T}^d} [(\operatorname{curl} B \times B)^\epsilon - \operatorname{curl} B^\epsilon \times B^\epsilon] \cdot \varphi u^\epsilon dx dt \\ &= \int_0^T \int_{\mathbb{T}^d} \left[(B \cdot \nabla B)^\epsilon - \frac{1}{2} \nabla(|B|^2)^\epsilon \right] \cdot \varphi u^\epsilon dx dt - \int_0^T \int_{\mathbb{T}^d} \left[(B^\epsilon \cdot \nabla B^\epsilon) - \frac{1}{2} \nabla(|B^\epsilon|^2) \right] \cdot \varphi u^\epsilon dx dt \\ &= \int_0^T \int_{\mathbb{T}^d} [(B \cdot \nabla B)^\epsilon - (B^\epsilon \cdot \nabla B^\epsilon)] \cdot \varphi u^\epsilon dx dt - \int_0^T \int_{\mathbb{T}^d} \left[\frac{1}{2} \nabla(|B|^2)^\epsilon - \frac{1}{2} \nabla(|B^\epsilon|^2) \right] \cdot \varphi u^\epsilon dx dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^T \int_{\mathbb{T}^d} [\partial_i(B_i B_j)^\epsilon - \partial_i(B_i^\epsilon B_j^\epsilon)] \cdot \varphi u^\epsilon dx dt - \int_0^T \int_{\mathbb{T}^d} \left[\frac{1}{2} \partial_i(|B_i|^2)^\epsilon - \frac{1}{2} \partial_i(|B_i|^\epsilon)^2 \right] \cdot \varphi u^\epsilon dx dt \\
&\stackrel{\text{def}}{=} R_{31}^\epsilon + R_{32}^\epsilon.
\end{aligned} \tag{3.12}$$

因为 R_{32}^ϵ 与 R_{31}^ϵ 的计算方法相同, 所以我们只对 R_{31}^ϵ 进行计算. 与前面相同, 先作如下分解:

$$\begin{aligned}
B_i^\epsilon B_j^\epsilon - (B_i B_j)^\epsilon &= (B_i^\epsilon - B_i)(B_j^\epsilon - B_j) - \int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (B_i(t - \tau, x - \xi) - B_i(t, x)) \\
&\quad \times (B_j(t - \tau, x - \xi) - B_j(t, x)) d\xi d\tau.
\end{aligned}$$

对于 R_{31}^ϵ 的第 1 部分, 由 Hölder 不等式, 得

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^d} \partial_i[(B_i^\epsilon - B_i)(B_j^\epsilon - B_j)] \cdot \varphi u^\epsilon dx dt \right| \\
&\leq C \|\varphi\|_{C^0} \|B_i^\epsilon - B_i\|_{L^3} \|B_j^\epsilon - B_j\|_{L^3} \|\operatorname{div} u^\epsilon\|_{L^3} + C \|\varphi\|_{C^1} \|B_i^\epsilon - B_i\|_{L^3} \|B_j^\epsilon - B_j\|_{L^3} \|u^\epsilon\|_{L^3} \\
&\leq C \|\varphi\|_{C^0} \epsilon^\beta \epsilon^{\alpha-1} \|B\|_{B_3^{\beta, \infty}}^2 \|u\|_{B_3^{\alpha, \infty}} + C \|\varphi\|_{C^1} \epsilon^\beta \epsilon^\beta \epsilon^\alpha \|B\|_{B_3^{\beta, \infty}}^2 \|u\|_{B_3^{\alpha, \infty}}.
\end{aligned}$$

至于 R_{31}^ϵ 的第 2 部分, 利用 Fubini 定理和 Hölder 不等式, 得

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{T}^d} \partial_i \left(\int_{-\epsilon}^\epsilon \int_{\mathbb{T}^d} \eta^\epsilon(\tau, \xi) (B_i(t - \tau, x - \xi) - B_i(t, x)) \right. \right. \\
&\quad \left. \left. \times (B_j(t - \tau, x - \xi) - B_j(t, x)) d\xi d\tau \right) \cdot \varphi u^\epsilon dx dt \right| \\
&\leq C \|\varphi\|_{C^0} \epsilon^\beta \epsilon^\beta \epsilon^{\alpha-1} \|B\|_{B_3^{\beta, \infty}}^2 \|u\|_{B_3^{\alpha, \infty}} + C \|\varphi\|_{C^1} \epsilon^\beta \epsilon^\beta \epsilon^\alpha \|B\|_{B_3^{\beta, \infty}}^2 \|u\|_{B_3^{\alpha, \infty}}.
\end{aligned}$$

R_5^ϵ 的估计类似于 R_{31}^ϵ 的估计.

下面考虑 R_4^ϵ . 首先注意到, 若 $p \in C^2([a, b])$, 则对任意 $h, h_0 \in [a, b]$, 由中值定理得

$$|p(h) - p(h_0) - p'(h_0)(h - h_0)| \leq C(h - h_0)^2,$$

这里 C 是不依赖于 h 和 h_0 的常数. 因此,

$$|p(\rho^\epsilon(t, x)) - p(\rho(t, x)) - p'(\rho(t, x))(\rho^\epsilon(t, x) - \rho(t, x))| \leq C(\rho^\epsilon(t, x) - \rho(t, x))^2. \tag{3.13}$$

类似地,

$$|p(\rho(t, y)) - p(\rho(t, x)) - p'(\rho(t, x))(\rho(t, y) - \rho(t, x))| \leq C(\rho(t, y) - \rho(t, x))^2. \tag{3.14}$$

将 (3.14) 两边关于 y 作卷积并由 Jensen 不等式, 得

$$|p(\rho)^\epsilon(t, x) - p(\rho(t, x)) - p'(\rho(t, x))(\rho^\epsilon(t, x) - \rho(t, x))| \leq C(\rho(t, \cdot) - \rho(t, x))^2 *_y \eta^\epsilon. \tag{3.15}$$

联立 (3.13) 和 (3.15) 得

$$|p(\rho^\epsilon(t, x)) - p(\rho)^\epsilon(t, x)| \leq C(\rho^\epsilon(t, x) - \rho(t, x))^2 + C(\rho(t, \cdot) - \rho(t, x))^2 *_y \eta^\epsilon.$$

至此, 与前面类似, 当 $\epsilon \rightarrow 0$ 时, 我们可估计 R_4^ϵ 如下:

$$|R_4^\epsilon| = \left| \int_0^T \int_{\mathbb{T}^d} \nabla(p(\rho^\epsilon) - p(\rho)^\epsilon) \cdot \varphi u^\epsilon dx dt \right|$$

$$\begin{aligned}
 &\leq \int_0^T \int_{\mathbb{T}^d} |\varphi(p(\rho^\epsilon) - p(\rho)^\epsilon) \operatorname{div} u^\epsilon| dx dt + \int_0^T \int_{\mathbb{T}^d} |(p(\rho^\epsilon) - p(\rho)^\epsilon) u^\epsilon \cdot \nabla \varphi| dx dt \\
 &\leq C \|\varphi\|_{C^0} \epsilon^{2\gamma} \epsilon^{\alpha-1} \|\rho\|_{B_3^{\gamma,\infty}}^2 \|u\|_{B_3^{\alpha,\infty}} + C \|\varphi\|_{C^1} \epsilon^{2\gamma} \|\rho\|_{B_3^{\gamma,\infty}}^2 \|u\|_{B_3^{\alpha,\infty}} \rightarrow 0.
 \end{aligned}$$

最后估计 $\int_0^T \int_{\mathbb{T}^d} \varphi P'(\rho^\epsilon) \operatorname{div}(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon) dx dt$. 利用 (3.11) 将其分成两部分进行估计. 对于第 1 部分, 当 $\epsilon \rightarrow 0$ 时,

$$\begin{aligned}
 &\left| \int_0^T \int_{\mathbb{T}^d} \varphi P'(\rho^\epsilon) \operatorname{div}[(\rho^\epsilon - \rho)(u^\epsilon - u)] dx dt \right| \\
 &\leq \int_0^T \int_{\mathbb{T}^d} |\nabla \varphi(\rho^\epsilon - \rho)(u^\epsilon - u) P'(\rho^\epsilon)| dx dt + \int_0^T \int_{\mathbb{T}^d} |\varphi(\rho^\epsilon - \rho)(u^\epsilon - u) \cdot P''(\rho^\epsilon) \nabla \rho^\epsilon| dx dt \\
 &\leq C \|\varphi\|_{C^1} \epsilon^\gamma \epsilon^\alpha \|\rho\|_{B_3^{\gamma,\infty}} \|u\|_{B_3^{\alpha,\infty}} + C \|\varphi\|_{C^0} \epsilon^\gamma \epsilon^\alpha \epsilon^{\gamma-1} \|\rho\|_{B_3^{\gamma,\infty}}^2 \|u\|_{B_3^{\alpha,\infty}} \rightarrow 0.
 \end{aligned}$$

第 2 部分的证明与之类似. 至此完成了定理 1.1 的证明.

3.2 定理 1.2 的证明

首先将系统 (1.1) 关于时间和空间正则化. 不同于定理 1.1 证明过程中所采用的正则化, 这里对时间和空间分别取不同的参数. 具体地, 对于空间, 我们利用如下关于空间的卷积:

$$w^\epsilon(t, x) = \int_{\mathbb{T}^d} \eta^\epsilon(y) w(t, x - y) dy,$$

这里 η^ϵ 由 (2.2) 给出. 对于时间的正则化, 利用文献 [20] 中的方法关于时间作卷积, 得

$$w^h(t, x) = \int_0^T \chi_h(t - s) w(s, x) ds,$$

这里 $\chi_h = \frac{1}{h} 1_{[-h, 0]}$. 若 $w \in L^1_{\text{loc}}(0, T; X)$ 且 t 和 $t + h$ 是函数 w 的 Lebesgue 点, 则有

$$\partial_t w^h = \frac{w(t + h) - w(t)}{h} \in X. \quad (3.16)$$

特别地, 函数 w^h 在 $[h, T - h]$ 上是绝对连续的, 且对于几乎处处的 $t \in (h, T - h)$ 其导数满足 (3.16).

记

$$w^{\epsilon, h} = (w^\epsilon)^h.$$

对 (1.1)₂ 进行正则化, 得

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho u)^\epsilon = 0, \quad \partial_t \rho^{\epsilon, h} + \operatorname{div}(\rho u)^{\epsilon, h} = 0, \quad (3.17)$$

其中第 2 个等式在 $t \in (h, T - h)$ 内成立.

同样, 我们有

$$\begin{aligned}
 &\partial_t (\rho u)^\epsilon + \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon) + \nabla p(\rho)^\epsilon \\
 &= \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon) - \operatorname{div}(\rho u \otimes u)^\epsilon + (\operatorname{curl} B \times B)^\epsilon
 \end{aligned} \quad (3.18)$$

以及

$$\partial_t (\rho^\epsilon u^\epsilon)^h + \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon)^h + \nabla (p(\rho))^\epsilon{}^h$$

$$= (\operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon) - \operatorname{div}(\rho u \otimes u)^\epsilon)^h + \partial_t(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon)^h + (\operatorname{curl} B \times B)^\epsilon{}^h. \quad (3.19)$$

将 (3.19) 两边乘以 $\varphi u^{\epsilon,h}$, 这里 $\varphi \in C_c^\infty((h, T-h) \times \mathbb{T}^d)$, 并在两边加上关于磁场 B 的项 $((u \times B)^\epsilon \cdot \operatorname{curl} B^\epsilon)^h \varphi$, 再将所得方程在 $(0, T) \times \mathbb{T}^d$ 上积分, 得

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt + \int_0^T \int_{\mathbb{T}^d} \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} \nabla(p(\rho))^\epsilon{}^h \cdot u^{\epsilon,h} \varphi dx dt + \int_0^T \int_{\mathbb{T}^d} ((u \times B)^\epsilon \cdot \operatorname{curl} B^\epsilon)^h \varphi dx dt \\ & = \int_0^T \int_{\mathbb{T}^d} \operatorname{div}[(\rho u)^\epsilon \otimes u^\epsilon - (\rho u \otimes u)^\epsilon]^h \cdot u^{\epsilon,h} \varphi dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon - (\rho u)^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} [(\operatorname{curl} B \times B)^\epsilon - (\operatorname{curl} B^\epsilon \times B^\epsilon)]^h \cdot u^{\epsilon,h} \varphi dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} [(u \times B)^\epsilon \cdot \operatorname{curl} B^\epsilon - (u^\epsilon \times B^\epsilon) \cdot \operatorname{curl} B^\epsilon]^h \varphi dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} [(\operatorname{curl} B^\epsilon \times B^\epsilon)^h u^{\epsilon,h} + ((u^\epsilon \times B^\epsilon) \cdot \operatorname{curl} B^\epsilon)^h] \varphi dx dt. \end{aligned} \quad (3.20)$$

首先处理 (3.20) 左端的第 1 项. 基于 (3.16), 有

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt \\ & = \int_{\mathbb{R}} \int_{\mathbb{T}^d} \frac{\rho^\epsilon(t+h)u^\epsilon(t+h) - \rho^\epsilon(t)u^\epsilon(t)}{h} \cdot u^{\epsilon,h} \varphi dx dt \\ & \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \rho^\epsilon \frac{u^\epsilon(t+h) - u^\epsilon(t)}{h} \cdot u^{\epsilon,h} \varphi dx dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \rho^\epsilon \partial_t(|u^{\epsilon,h}|^2) \varphi dx dt \\ & = \int_{\mathbb{R}} \int_{\mathbb{T}^d} \frac{\rho^\epsilon(t+h) - \rho^\epsilon(t)}{h} u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi dx dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \rho^\epsilon \partial_t(|u^{\epsilon,h}|^2) \varphi dx dt \\ & = \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_t \rho^\epsilon u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi dx dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \rho^\epsilon \partial_t(|u^{\epsilon,h}|^2) \varphi dx dt. \end{aligned}$$

再由正则化的连续性方程 (3.17), 得

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^d} \partial_t(\rho^\epsilon u^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt \\ & = - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \operatorname{div}(\rho u)^\epsilon{}^h u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi dx dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \rho^\epsilon \partial_t(|u^{\epsilon,h}|^2) \varphi dx dt \\ & = \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^\epsilon{}^h \cdot \nabla(u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi) dx dt + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \rho^\epsilon \partial_t(|u^{\epsilon,h}|^2) \varphi dx dt \\ & = \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^\epsilon{}^h \cdot \nabla(u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi) dx dt - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\rho u)^\epsilon \cdot \nabla(\varphi |u^{\epsilon,h}|^2) + \rho^\epsilon |u^{\epsilon,h}|^2 \partial_t \varphi] dx dt. \end{aligned} \quad (3.21)$$

对于 (3.20) 左边第 2 项, 有

$$\int_0^T \int_{\mathbb{T}^d} \operatorname{div}((\rho u)^\epsilon \otimes u^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt = - \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((\rho u)^\epsilon \otimes u^\epsilon)^h : \nabla(u^{\epsilon,h} \varphi) dx dt. \quad (3.22)$$

对于 (3.20) 左边第 4 项, 借助于公式 $\operatorname{div}(a \times b) = \operatorname{curl} a \cdot b - a \cdot \operatorname{curl} b$ 及 (3.6), 得

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{T}^d} [(u \times B)^\epsilon \cdot \operatorname{curl} B^\epsilon]^h \varphi dx dt \\
 &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} [\operatorname{curl}(u \times B)^\epsilon \cdot B^\epsilon]^h \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \operatorname{div}((u \times B)^\epsilon \times B^\epsilon)^h \varphi dx dt \\
 &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} \left(\frac{1}{2} \partial_t |B^\epsilon|^2 \right)^h \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \operatorname{div}((u \times B)^\epsilon \times B^\epsilon)^h \varphi dx dt \\
 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_t \varphi \left(\frac{1}{2} |B^\epsilon|^2 \right)^h dx dt + \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((u \times B)^\epsilon \times B^\epsilon)^h \cdot \nabla \varphi dx dt.
 \end{aligned} \tag{3.23}$$

下面考虑 (3.20) 左边第 3 项. 根据定理 1.2 中的假设 (1.3), 可将压力 $p(\rho)$ 表示如下: 对任意 $\delta > 0$, 有

$$\begin{aligned}
 p(\rho) &= p_\delta(\rho) + (p(\rho) - p_\delta(\rho)), \quad p_\delta \in C^2[0, \infty), \\
 p_\delta(0) &= p'_\delta(0) = 0, \quad |p - p_\delta| < \delta.
 \end{aligned} \tag{3.24}$$

相应地, 有

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{T}^d} (\nabla p(\rho))^{\epsilon, h} \cdot u^{\epsilon, h} \varphi dx dt \\
 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p(\rho))^{\epsilon, h} \operatorname{div} u^{\epsilon, h} \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p(\rho))^{\epsilon, h} u^{\epsilon, h} \cdot \nabla \varphi dx dt \\
 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho))^{\epsilon, h} \operatorname{div} u^{\epsilon, h} \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho))^{\epsilon, h} u^{\epsilon, h} \cdot \nabla \varphi dx dt + \mathcal{O}(\delta) \\
 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} p_\delta(\rho^{\epsilon, h}) \operatorname{div} u^{\epsilon, h} \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho))^{\epsilon, h} u^{\epsilon, h} \cdot \nabla \varphi dx dt \\
 &\quad + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho^{\epsilon, h}) - (p_\delta(\rho))^{\epsilon, h}) \operatorname{div} u^{\epsilon, h} \varphi dx dt + \mathcal{O}(\delta).
 \end{aligned} \tag{3.25}$$

为了计算上式右端第 1 项, 将连续性方程 (1.1)₂ 关于时间和空间分别磨光后成为

$$\partial_t \rho^{\epsilon, h} + \operatorname{div}(\rho^{\epsilon, h} u^{\epsilon, h}) = -\operatorname{div}(\rho u)^{\epsilon, h} + \operatorname{div}(\rho^{\epsilon, h} u^{\epsilon, h}),$$

因此, 类似于 (3.8), 可得

$$\begin{aligned}
 & \partial_t P_\delta(\rho^{\epsilon, h}) + \operatorname{div}(P_\delta(\rho^{\epsilon, h}) u^{\epsilon, h}) + p_\delta(\rho^{\epsilon, h}) \operatorname{div} u^{\epsilon, h} \\
 &= -P'_\delta(\rho^{\epsilon, h}) [\operatorname{div}(\rho u)^{\epsilon, h} - \operatorname{div}(\rho^{\epsilon, h} u^{\epsilon, h})],
 \end{aligned} \tag{3.26}$$

这里

$$P_\delta(\rho) = \rho \int_1^\rho \frac{p_\delta(z)}{z^2} dz.$$

由 (3.24), 有 $P_\delta \in C^2[0, \infty)$. 回到 (3.25), 可推导出

$$\begin{aligned}
 & \int_0^t \int_{\mathbb{T}^d} (\nabla p(\rho))^{\epsilon, h} \cdot u^{\epsilon, h} \varphi dx dt \\
 &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} [\partial_t P_\delta(\rho^{\epsilon, h}) + \operatorname{div}(P_\delta(\rho^{\epsilon, h}) u^{\epsilon, h})] \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho))^{\epsilon, h} u^{\epsilon, h} \cdot \nabla \varphi dx dt \\
 &\quad + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho^{\epsilon, h}) - (p_\delta(\rho))^{\epsilon, h}) \operatorname{div} u^{\epsilon, h} \varphi dx dt
 \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}} \int_{\mathbb{T}^d} P'_\delta(\rho^{\epsilon,h}) [\operatorname{div}(\rho u)^{\epsilon,h} - \operatorname{div}(\rho^{\epsilon,h} u^{\epsilon,h})] \varphi dx dt + \mathcal{O}(\delta) \\
& = - \int_{\mathbb{R}} \int_{\mathbb{T}^d} P(\rho^{\epsilon,h}) \partial_t \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} P(\rho^{\epsilon,h}) u^{\epsilon,h} \cdot \nabla \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p(\rho))^{\epsilon,h} u^{\epsilon,h} \cdot \nabla \varphi dx dt + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho^{\epsilon,h}) - (p_\delta(\rho))^{\epsilon,h}) \operatorname{div} u^{\epsilon,h} \varphi dx dt \\
& \quad + \int_{\mathbb{R}} \int_{\mathbb{T}^d} P'_\delta(\rho^{\epsilon,h}) [\operatorname{div}(\rho u)^{\epsilon,h} - \operatorname{div}(\rho^{\epsilon,h} u^{\epsilon,h})] \varphi dx dt + \mathcal{O}(\delta). \tag{3.27}
\end{aligned}$$

将 (3.21)–(3.23) 和 (3.27) 代入 (3.20), 得

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^{\epsilon,h} \cdot \nabla (u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi) dx dt - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\rho u)^\epsilon \cdot \nabla (\varphi |u^{\epsilon,h}|^2) + \rho^\epsilon |u^{\epsilon,h}|^2 \partial_t \varphi] dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((\rho u)^\epsilon \otimes u^\epsilon)^h : \nabla (u^{\epsilon,h} \varphi) dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} P(\rho^{\epsilon,h}) \partial_t \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} P(\rho^{\epsilon,h}) u^{\epsilon,h} \cdot \nabla \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p(\rho))^{\epsilon,h} u^{\epsilon,h} \cdot \nabla \varphi dx dt + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_\delta(\rho^{\epsilon,h}) - (p_\delta(\rho))^{\epsilon,h}) \operatorname{div} u^{\epsilon,h} \varphi dx dt \\
& \quad + \int_{\mathbb{R}} \int_{\mathbb{T}^d} P'_\delta(\rho^{\epsilon,h}) [\operatorname{div}(\rho u)^{\epsilon,h} - \operatorname{div}(\rho^{\epsilon,h} u^{\epsilon,h})] \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \frac{1}{2} (|B^\epsilon|^2)^h \partial_t \varphi dx dt + \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((u \times B)^\epsilon \times B^\epsilon)^h \cdot \nabla \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \operatorname{div} [((\rho u)^\epsilon \otimes u^\epsilon) - (\rho u \otimes u)^\epsilon]^h \cdot u^{\epsilon,h} \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_t (\rho^\epsilon u^\epsilon - (\rho u)^\epsilon)^h \cdot u^{\epsilon,h} \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\operatorname{curl} B \times B)^\epsilon - (\operatorname{curl} B^\epsilon \times B^\epsilon)]^h \cdot u^{\epsilon,h} \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(u \times B)^\epsilon \cdot \operatorname{curl} B^\epsilon - (u^\epsilon \times B^\epsilon) \cdot \operatorname{curl} B]^\epsilon \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\operatorname{curl} B^\epsilon \times B^\epsilon)^h u^{\epsilon,h} + ((u^\epsilon \times B^\epsilon) \cdot \operatorname{curl} B^\epsilon)^h] \varphi dx dt \\
& = \mathcal{O}(\delta).
\end{aligned}$$

将上式进一步变形, 有

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{T}^d} \left(\frac{1}{2} \rho^\epsilon |u^{\epsilon,h}|^2 + P(\rho^{\epsilon,h}) + \frac{1}{2} (|B^\epsilon|^2)^h \right) \partial_t \varphi dx dt \\
& \quad + \int_{\mathbb{R}} \int_{\mathbb{T}^d} \left(((\rho u)^\epsilon \otimes u^\epsilon)^h \cdot u^{\epsilon,h} + \frac{1}{2} (\rho u)^\epsilon |u^{\epsilon,h}|^2 - (\rho u)^{\epsilon,h} (u^\epsilon(t+h) \cdot u^{\epsilon,h}) \right) \cdot \nabla \varphi dx dt \\
& \quad + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (P(\rho^{\epsilon,h}) + (p(\rho))^{\epsilon,h}) u^{\epsilon,h} \cdot \nabla \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((u \times B)^\epsilon \times B^\epsilon)^h \cdot \nabla \varphi dx dt \\
& =: \sum_{i=1}^8 H_{\epsilon,h}^i + \mathcal{O}(\delta), \tag{3.28}
\end{aligned}$$

其中,

$$\begin{aligned}
H_{\epsilon,h}^1 & = \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^{\epsilon,h} \cdot \nabla (u^\epsilon(t+h) \cdot u^{\epsilon,h} \varphi) dx dt - \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^\epsilon \cdot \nabla (|u^{\epsilon,h}|^2) \varphi dx dt \\
& \quad - \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((\rho u)^\epsilon \otimes u^\epsilon)^h : \nabla u^{\epsilon,h} \varphi dx dt,
\end{aligned}$$

$$\begin{aligned}
 H_{\epsilon,h}^2 &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} (p_{\delta}(\rho^{\epsilon,h}) - (p_{\delta}(\rho))^{\epsilon,h}) \operatorname{div} u^{\epsilon,h} \varphi dx dt, \\
 H_{\epsilon,h}^3 &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} P'_{\delta}(\rho^{\epsilon,h}) [\operatorname{div}(\rho u)^{\epsilon,h} - \operatorname{div}(\rho^{\epsilon,h} u^{\epsilon,h})] \varphi dx dt, \\
 H_{\epsilon,h}^4 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \operatorname{div}[(\rho u)^{\epsilon} \otimes u^{\epsilon} - (\rho u \otimes u)^{\epsilon}]^h \cdot u^{\epsilon,h} \varphi dx dt, \\
 H_{\epsilon,h}^5 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_t(\rho^{\epsilon} u^{\epsilon} - (\rho u)^{\epsilon})^h \cdot u^{\epsilon,h} \varphi dx dt, \\
 H_{\epsilon,h}^6 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\operatorname{curl} B \times B)^{\epsilon} - (\operatorname{curl} B^{\epsilon} \times B^{\epsilon})]^h \cdot u^{\epsilon,h} \varphi dx dt, \\
 H_{\epsilon,h}^7 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(u \times B)^{\epsilon} \cdot \operatorname{curl} B^{\epsilon} - (u^{\epsilon} \times B^{\epsilon}) \cdot \operatorname{curl} B^{\epsilon}]^h \varphi dx dt, \\
 H_{\epsilon,h}^8 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\operatorname{curl} B^{\epsilon} \times B^{\epsilon})^h u^{\epsilon,h} + ((u^{\epsilon} \times B^{\epsilon}) \cdot \operatorname{curl} B^{\epsilon})^h] \varphi dx dt.
 \end{aligned}$$

当 $\epsilon, h \rightarrow 0$ 时, 根据磨光性质, 可得 (3.28) 左端的极限为

$$\begin{aligned}
 &\int_{\mathbb{R}} \int_{\mathbb{T}^d} \partial_t \varphi \left(\frac{1}{2} \rho |u|^2 + \frac{1}{2} |B|^2 + P(\rho) \right) dx dt \\
 &+ \int_{\mathbb{R}} \int_{\mathbb{T}^d} \nabla \varphi \cdot \left(\frac{1}{2} \rho |u|^2 u + p(\rho) u + P(\rho) u - (u \times B) \times B \right) dx dt.
 \end{aligned}$$

为证明定理 1.2, 需证明当 $\epsilon, h \rightarrow 0$ 时, $H_{\epsilon,h}^i$ ($i = 1, 2, \dots, 8$) 均趋于 0. 这里首先考虑 $\epsilon \rightarrow 0$ 时 $H_{\epsilon,h}^i$ ($i = 1, 2, \dots, 8$) 的极限, 再考虑 $h \rightarrow 0$ 时的极限. 对于 $H_{\epsilon,h}^1$, 可将其改写为

$$\begin{aligned}
 H_{\epsilon,h}^1 &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^{\epsilon,h}(t-h) \cdot \nabla u^{\epsilon} \cdot u^{\epsilon,h}(t-h) \varphi(t-h) dx dt + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^{\epsilon,h} \cdot u^{\epsilon}(t+h) \cdot \nabla u^{\epsilon,h} \varphi dx dt \\
 &- \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^{\epsilon} \cdot \nabla u^{\epsilon,h} \cdot u^{\epsilon,h} \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((\rho u)^{\epsilon} \otimes u^{\epsilon})^h : \nabla u^{\epsilon,h} \varphi dx dt.
 \end{aligned}$$

由定理 1.2 的假设条件

$$\operatorname{ess\,sup}_{t \in [0, T]} (\|\rho(t, \cdot)\|_{BV \cap C(\mathbb{T}^d)} + \|u(t, \cdot)\|_{BV \cap C(\mathbb{T}^d; \mathbb{R}^d)} + \|B(t, \cdot)\|_{BV \cap C(\mathbb{T}^d; \mathbb{R}^d)}) < \infty$$

可知, 当 h 固定而 $\epsilon \rightarrow 0$ 时, 对任意 $1 \leq p < \infty$, 有

$$\begin{aligned}
 &(\rho u)^{\epsilon,h} \rightarrow (\rho u)^h, \quad \text{在 } C([h, T-h] \times \mathbb{T}^d; \mathbb{R}^d) \text{ 中,} \\
 &u^{\epsilon,h} \rightarrow u^h, \quad \text{在 } C([h, T-h] \times \mathbb{T}^d; \mathbb{R}^d) \text{ 中,} \\
 &\nabla u^{\epsilon} \rightarrow \nabla u, \quad \text{在 } L^p_{\text{weak}^*}(0, T; \mathcal{M}(\mathbb{T}^d; \mathbb{R}^{d \times d})) \text{ 中,} \\
 &\nabla u^{\epsilon,h} \rightarrow \nabla u^h, \quad \text{在 } C_{\text{weak}^*}([h, T-h]; \mathcal{M}(\mathbb{T}^d; \mathbb{R}^{d \times d})) \text{ 中,} \\
 &(\rho u)^{\epsilon} \rightarrow \rho u, \quad \text{在 } L^p(0, T; C(\mathbb{T}^d; \mathbb{R}^d)) \text{ 中,} \\
 &u^{\epsilon} \rightarrow u, \quad \text{在 } L^p(0, T; C(\mathbb{T}^d; \mathbb{R}^d)) \text{ 中.}
 \end{aligned}$$

因此, 当 $\epsilon \rightarrow 0$ 时可推导出

$$H_{\epsilon,h}^1 \rightarrow H_h^1 = \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^h(t-h) \cdot \nabla u \cdot u^h(t-h) \varphi(t-h) dx dt + \int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u)^h \cdot u(t+h) \cdot \nabla u^h \varphi dx dt$$

$$-\int_{\mathbb{R}} \int_{\mathbb{T}^d} (\rho u) \cdot \nabla u^h \cdot u^h \varphi dx dt - \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((\rho u) \otimes u)^h : \nabla u^h \varphi dx dt.$$

注意到, 若 X 是一个 Banach 空间, 则对任意 $v \in L^\infty(0, T; X)$ 和任意 $1 \leq p < \infty$, 当 $h \rightarrow 0$ 时, 有

$$v^h \rightarrow v \quad \text{和} \quad v(\cdot \pm h) \rightarrow v, \quad \text{在 } L^p(0, T; X) \text{ 中}.$$

由此得到 $H_h^1 \rightarrow 0$.

类似地, 可以处理 $H_{\epsilon, h}^2$ 、 $H_{\epsilon, h}^4$ 、 $H_{\epsilon, h}^6$ 和 $H_{\epsilon, h}^7$. 对于 $H_{\epsilon, h}^3$, 由分部积分公式, 得

$$H_{\epsilon, h}^3 = - \int_{\mathbb{R}} \int_{\mathbb{T}^d} \nabla(P'_\delta(\rho^{\epsilon, h})\varphi)[(\rho u)^{\epsilon, h} - \rho^{\epsilon, h} u^{\epsilon, h}] dx dt,$$

这里

$$\nabla P'_\delta(\rho^{\epsilon, h}) = P''_\delta(\rho^{\epsilon, h}) \nabla \rho^{\epsilon, h}.$$

再经过上述相同的论证可得 $H_{\epsilon, h}^3$ 趋于 0.

对于 $H_{\epsilon, h}^5$, 同样由分部积分公式可得

$$H_{\epsilon, h}^5 = \int_{\mathbb{R}} \int_{\mathbb{T}^d} ((\rho^\epsilon u^\epsilon - (\rho u)^\epsilon)^h) \cdot \partial_t(u^{\epsilon, h} \varphi) dx dt.$$

对固定的 h , 由于 $u \in L^\infty$, 因此 $\partial_t(u^{\epsilon, h} \varphi)$ 一致有界. 另外, 同样由于 $\rho, u \in L^\infty$, 因此当 $\epsilon \rightarrow 0$ 时, 有

$$\rho^\epsilon u^\epsilon - (\rho u)^\epsilon \rightarrow 0, \quad \text{在 } L^p((0, T) \times \mathbb{T}^d) \text{ (} 1 \leq p < \infty \text{) 中},$$

从而 $H_{\epsilon, h}^5$ 趋于 0.

最后处理 $H_{\epsilon, h}^8$. 由等式 $(u^\epsilon \times B^\epsilon) \cdot \text{curl} B^\epsilon = -u^\epsilon \cdot (\text{curl} B^\epsilon \times B^\epsilon)$ 可将 $H_{\epsilon, h}^8$ 变为

$$\begin{aligned} H_{\epsilon, h}^8 &= - \int_{\mathbb{R}} \int_{\mathbb{T}^d} [(\text{curl} B^\epsilon \times B^\epsilon)^h u^{\epsilon, h} + ((u^\epsilon \times B^\epsilon) \cdot \text{curl} B^\epsilon)^h] \varphi dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{T}^d} [((\text{curl} B^\epsilon \times B^\epsilon) u^\epsilon)^h - (\text{curl} B^\epsilon \times B^\epsilon)^h u^{\epsilon, h}] \varphi dx dt. \end{aligned}$$

类似地, $H_{\epsilon, h}^8$ 趋于 0. 最后, 令 (3.28) 中的 $\delta \rightarrow 0$. 至此定理 1.2 证毕.

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Regularity and energy conservation for the compressible MHD equations

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Abstract In this paper, we are concerned with the regularity and energy conservation of the weak solutions for inhomogeneous compressible ideal MHD (magnetohydrodynamics) systems. We give two sufficient conditions on the regularity of solutions in order to ensure the energy to be conserved, where $(t, x) \in (0, T) \times \mathbb{T}^d$ ($d \geq 1$). The main tool we have employed relies on the commutator estimates similar to those used by Constantin et al. (1994) for the homogeneous incompressible Euler equations.

Keywords compressible MHD equations, regularity, energy conservation

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