Evolution of solitons of nonlinear Schrödinger equation with variable parameters*

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Abstract The nonlinear Schrödinger equation with variable parameters is solved by means of variational technique. A set of evolution equations for the solitary-wave solution is derived. The propagation properties of the solitons in an adiabatic amplification system and in a dispersion-decreasing fiber are analyzed. An explicit analytical approximate soliton solution in the exponentially dispersion-decreasing fiber is obtained using the derived dynamical equations.

Keywords: nonlinear Schrödinger equation, optical soliton, optical fiber.

The soliton concept is a sophisticated mathematical structure based on the integrability of a class of nonlinear differential equations[1]. Integrable nonlinear differential equations have one feature in common: they are all conservative and are thus derivable from a Hamiltonian. The nonlinear Schrödinger equation is one member of the class of integrable equations. It reads as

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}+|u|^2u=0, \tag{1}$$

where u is the envelope of the electric field as a function of propagation distance z and retarded time t. It is well known that in a perfect, lossless optical fiber, the fundamental soliton, corresponding to solitary-wave solution of eq. (1), has the desirable property that its shape and amplitude are invariant during propagation along the fiber. This characteristic makes it possible to use soliton pulses as information carriers in the high-bit-rate optical communication systems. In a practical fiber, however, the power of the pulses decreases exponentially with distance, and the nonlinear effect is weakened accordingly due to fiber loss. Ultimately the pulse loses its identity as a soliton and becomes subject to ordinary dispersion deterioration. For soliton communication systems, the loss compensation can be achieved by slowly varying the fiber parameters (dispersion or nonlinearity) axially[2—7] or by using adiabatic amplification[8,9]. Thus in a practical soliton-based communication system, the propagation of the pulse is described by the following nonlinear Schrödinger equation with variable parameters:

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\frac{\partial^2 u}{\partial t^2}+\gamma(z)|u|^2u=\mathrm{i}\alpha(z)u, \tag{2}$$

where $\beta(z)$ and $\gamma(z)$ are the group-velocity dispersion and nonlinear coefficients respectively; $\alpha(z)$ is the fiber loss or the amplification rate. If $\beta(z)=\gamma(z)=1$, eq. (2) describes dynamics

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of solitons in the transmission system with periodical or adiabatic amplification[9]; in this case, the soliton solution of the equation has been obtained by means of the perturbation theory[8], and the evolution equations of the parameters of the soliton have been derived using the variational approach[9]. In ref. [2], Tajima showed numerically that optical solitons in the so-called dispersion-decreasing fibers (corresponding to the case of $\alpha(z)=0$, $\gamma(z)=1$) can propagate with their width and shape unchanged. In ref. [3] the propagation of solitons in dispersion-decreasing fibers was investigated numerically and analytically, and a transformed soliton solution, corresponding to the case of $\beta(z)=\gamma(z)=1$, $\alpha(z)=\Lambda$ (constant equivalent gain), was obtained. Kuehl[4] has studied the propagation properties of solitons on an axially nonuniform optical fiber and obtained an analytical solution in the case of $\alpha(z)=0$, $\beta(z)=1$. In the present paper, we consider a more general case which takes into consideration all the three variable parameters of eq. (2). We will derive the evolution equations for the solitary-wave solution of eq. (2) by means of variational technique[9,10] and apply the obtained evolution equations to the adiabatic amplification system and the dispersion-decreasing fiber. We have obtained an explicit analytical approximate soliton solution in the exponentially dispersion-decreasing fiber for the first time to our knowledge.

1 Evolution equations of the parameters of the soliton

We omit the gain or loss term on the right side of eq. (2) for simplicity. When this term has to be taken into consideration, it can be included in $\beta(z)$ or $\gamma(z)$ term by a proper transformation. It is easy to show that eq. (2) can be restated as a variational problem in terms of the following Lagrangian:

$$\mathscr{L}=\frac{\mathrm{i}}{2}\left[u\frac{\partial u^*}{\partial z}-u^*\frac{\partial u}{\partial z}\right]+\frac{\beta(z)}{2}\left|\frac{\partial u}{\partial t}\right|^2-\frac{\gamma(z)}{2}|u|^4, \tag{3}$$

where the asterisk denotes the complex conjugation. This means that eq. (2) results from the variational equations corresponding to the variational principle

$$\delta\left[\iint\mathscr{L}\left(u,u^*,\frac{\partial u}{\partial z},\frac{\partial u^*}{\partial z},\frac{\partial u}{\partial t},\frac{\partial u^*}{\partial t}\right)\mathrm{d}z\mathrm{d}t\right]=0, \tag{4}$$

i.e. the equation

$$\frac{\delta\mathscr{L}}{\delta u^*}=\frac{\partial}{\partial z}\frac{\partial\mathscr{L}}{\partial\left(\dfrac{\partial u^*}{\partial z}\right)}+\frac{\partial}{\partial t}\frac{\partial\mathscr{L}}{\partial\left(\dfrac{\partial u^*}{\partial t}\right)}-\frac{\partial\mathscr{L}}{\partial u^*}=0 \tag{5}$$

is equivalent to equation (2).

In order to describe the evolution of a pulse, we assume the following trial function:

$$u(z,t)=A\operatorname{sech}\left[\frac{t}{a}\right]\exp[\mathrm{i}\varphi+\mathrm{i}bt^2], \tag{6}$$

where A, a, φ, b are the amplitude, the pulse width, the phase and the chirp parameter, respectively. They are all the real functions of the propagation distance z and the behavior of which is determined by the variational equation (eq. (4)). We also assume that the initial value of the soliton parameters are A_0, a_0, φ_0, b_0, respectively.

Inserting eq. (6) into eq. (3) and integrating out the explicit t dependence, we arrive at an effective Lagrangian:

$$L=\int_{-\infty}^{\infty}\mathscr{L}\mathrm{d}t=2A^2a\frac{\mathrm{d}\varphi}{\mathrm{d}z}+\frac{\pi^2A^2a^3}{6}\frac{\mathrm{d}b}{\mathrm{d}z}+\frac{\beta(z)A^2}{3a}+\frac{\pi^2\beta(z)A^2a^3b^2}{3}-\frac{2\gamma(z)A^4a}{3}. \tag{7}$$

The Lagrange-Euler variational equations deduced from eqs. (7) and (4) produce the system of evolution equations of the parameters of the soliton:

$$\frac{\partial L}{\partial A}-\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial A_z}\right)=0\Rightarrow 2\frac{\mathrm{d}\varphi}{\mathrm{d}z}+\frac{\pi^2a^2}{6}\frac{\mathrm{d}b}{\mathrm{d}z}+\frac{\pi^2\beta(z)a^2b^2}{3}+\frac{\beta(z)}{3a^2}-\frac{4\gamma(z)A^2}{3}=0, \tag{8a}$$

$$\frac{\partial L}{\partial a}-\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial a_z}\right)=0\Rightarrow 2\frac{\mathrm{d}\varphi}{\mathrm{d}z}+\frac{\pi^2a^2}{2}\frac{\mathrm{d}b}{\mathrm{d}z}+\pi^2\beta(z)a^2b^2-\frac{\beta(z)}{3a^2}-\frac{2\gamma(z)A^2}{3}=0, \tag{8b}$$

$$\frac{\partial L}{\partial \varphi}-\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \varphi_z}\right)=0\Rightarrow \frac{\mathrm{d}}{\mathrm{d}z}(A^2a)=0, \tag{8c}$$

$$\frac{\partial L}{\partial b}-\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial b_z}\right)=0\Rightarrow 4\beta(z)bA^2a^3-\frac{\mathrm{d}}{\mathrm{d}z}(A^2a^3)=0, \tag{8d}$$

where the subscript z denotes the partial derivative. From (8a)—(8d), we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\beta(z)}\frac{\mathrm{d}a}{\mathrm{d}z}\right]=\frac{4}{\pi^2a}\left[\frac{\beta(z)}{a^2}-\frac{\gamma(z)W}{a}\right], \tag{9a}$$

$$b=\frac{1}{2\beta(z)a}\frac{\mathrm{d}a}{\mathrm{d}z}, \tag{9b}$$

$$A^2a=A_0^2a_0=W(\text{const}), \tag{9c}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z}=-\frac{\beta(z)}{3a^2}+\frac{5\gamma(z)W}{6a}. \tag{9d}$$

It is obvious that if the profiles of $\beta(z)$ and $\gamma(z)$ are given, one can obtain a set of explicit evolution equations or a soliton solution. For $\beta(z)=\gamma(z)=1$, the set of eqs. (9a)—(9d) becomes the evolution equations of the parameters of the soliton of the standard nonlinear Schrödinger equation[10], i.e. equation (1).

Now let us take into consideration the $\alpha(z)$ term. If $\alpha(z)>0$, this term represents the gain; while if $\alpha(z)<0$, it represents the loss. Take the following transformation:

$$u=q\exp\left[\int_0^z\alpha(z')\mathrm{d}z'\right], \tag{10}$$

eq. (2) is changed into

$$\mathrm{i}\frac{\partial q}{\partial z}+\frac{1}{2}\beta(z)\frac{\partial^2q}{\partial t^2}+\gamma(z)\exp\left[2\int_0^z\alpha(z')\mathrm{d}z'\right]|q|^2q=0. \tag{11}$$

Thus, if we replace $\gamma(z)$ in eqs. (9a)—(9d) with $\gamma(z)\exp\left[2\int_0^z\alpha(z')\mathrm{d}z'\right]$, we obtain the evolution equations of the parameters of the soliton of the nonlinear Schrödinger equation (eq. (11)):

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\beta(z)}\frac{\mathrm{d}a}{\mathrm{d}z}\right]=\frac{4}{\pi^2a}\left\{\frac{\beta(z)}{a^2}-\frac{\gamma(z)\exp\left[2\int_0^z\alpha(z')\mathrm{d}z'\right]W}{a}\right\}, \tag{12a}$$

$$b=\frac{1}{2\beta(z)a}\frac{\mathrm{d}a}{\mathrm{d}z}, \tag{12b}$$

$$A^2a=A^2(0)a(0)=W(\text{const.}), \tag{12c}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z}=-\frac{\beta(z)}{3a^2}+\frac{5\gamma(z)\exp\left[2\int_0^z\alpha(z')\mathrm{d}z'\right]W}{6a}. \tag{12d}$$

Finally, the analytical approximate soliton solution of eq. (2) can be arrived at by using equation

(10).

2 Results and discussion

Obviously, the evolution equations of the parameters of the soliton of the nonlinear Schrödinger equation (eq. (2)) can be reduced to a single second-order equation for the pulse width (eq. (12a)); since once $a(z)$ is known, the other parameters are determined by eqs. (12b)—(12d). So we will mainly analyze the dynamical behavior of the soliton pulse width. We rewrite eq. (12a) as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\beta(z)}\frac{\mathrm{d}a}{\mathrm{d}z}\right] = -\frac{\partial V}{\partial a}, \tag{13}$$

where

$$V = \frac{2}{\pi^2}\left\{\frac{\beta(z)}{a^2} - \frac{2\gamma(z)\exp\left[2\int_0^z \alpha(z')\mathrm{d}z'\right]W}{a}\right\}.$$

Thus eq. (13) may be regarded as an equation of motion for a particle with the variable mass $1/\beta(z)$ in the variable potential V. In an ideal lossless fiber, the equilibrium position, $a' = \beta/(\gamma W)$, at the bottom of the potential well V corresponds to the exact soliton[11]. From eq. (12b), we can see that the chirped solitons correspond to oscillations above the equilibrium position, $a' = \beta/(\gamma W)$. In a fiber with variable parameters, the dynamical behavior of the soliton can be obtained numerically from a Runge-Kutta integration of eq. (13). In some special cases, we can even obtain the approximate analytical soliton solution of eq. (2). Here we analyze two typical kinds of transmission systems with variable parameters.

2.1 Transmission system with adiabatic amplification[9]

Setting $\beta(z) = \gamma(z) = 1$, and introducing the normalized pulse width $y = a(z)/a_0$, we obtain from eq. (13),

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = \frac{4}{\pi^2 a_0^4}\left\{\frac{1}{y^3} - \frac{N^2\exp\left[2\int_0^z \alpha(z')\mathrm{d}z'\right]}{y^2}\right\}, \tag{14}$$

where $N = A_0 a_0$, in agreement with reference [9].

2.2 Dispersion-decreasing fiber

We assume an exponentially dispersion-decreasing fiber with the group-velocity dispersion $\beta(z) = \beta(0)\mathrm{e}^{-\theta z}$, where θ is the altering rate of the dispersion, and set $\gamma(z) = \gamma$ (constant), $\alpha(z) = 0$. Take the following transformations:

$$\tau = t/a_0, \quad \zeta = z/L_\mathrm{D}, \quad q = \sqrt{\gamma L_\mathrm{D}}\, u, \tag{15}$$

where $L_\mathrm{D} = a_0^2/\beta(0)$ is the dispersion distance of the fiber. Eq. (2) is transformed into the normalized equation:

$$\mathrm{i}\frac{\partial q}{\partial \zeta} + \frac{1}{2}\mathrm{e}^{-\theta L_\mathrm{D}\zeta}\frac{\partial^2 q}{\partial \tau^2} + |q|^2 q = 0. \tag{16}$$

Comparing eq. (16) with eq. (11), we obtain the following equation for normalized pulse width $y = a(z)/a_0$ from eq. (12a):

$$\frac{d}{d\zeta}\left(e^{\theta L_D \zeta} \frac{dy}{d\zeta}\right) = \frac{4}{\pi^2 a_0^4}\left(\frac{e^{-\theta L_D \zeta}}{y^3} - \frac{1}{y^2}\right). \tag{17}$$

The solution of eq. (17) is $y = e^{-\theta L_D \zeta}$, the other parameters are obtained from eqs. (12b)—(12d), $b = -\frac{\theta L_D}{2}(e^{\theta L_D \zeta} - 1)$, $A = e^{\frac{\theta L_D}{2}\zeta}$, $\varphi = \frac{1}{2\theta L_D}(e^{\theta L_D \zeta} - 1)$. Thus for an initial fundamental soliton, we obtain an explicit approximate analytical soliton solution of the nonlinear Schrödinger equation in the exponentially dispersion-decreasing fiber:

$$q(\zeta, \tau) = e^{\frac{\theta L_D}{2}\zeta} \operatorname{sech}(e^{\theta L_D \zeta}\tau) \exp\left[\frac{i}{2\theta L_D}(e^{\theta L_D \zeta} - 1) - i\frac{\theta L_D}{2}(e^{\theta L_D \zeta} - 1)\tau^2\right]. \tag{18}$$

The soliton solution $u(z, t)$ can be obtained using eq. (15). It is clearly shown from eq. (18) that the amplitude of the soliton in the dispersion-decreasing fiber increases exponentially with distance, and the pulse width decreases exponentially with distance. Thus the effect of the fiber loss on the propagation of the soliton can be compensated for by properly altering the dispersion of the fiber. When the fiber loss is excessively compensated for, the solitons in the dispersion-decreasing fiber can be effectively compressed. These results are in agreement with the numerical simulations in references [12, 13].

So far, several kinds of fibers with various variable dispersion profiles, such as dispersion-managed fiber[5], dispersion-adapted fiber[6] and dispersion-compensating fiber[7], have been proposed. Recently, a transform-limited pulse train has been successfully generated with a dispersion-decreasing, erbium-doped fiber amplifier[14]. The obtained result in this paper is expected to be useful in these transmission systems.

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