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On spectral theory of the Riemann zeta function

Dedicated to Professor Lo Yang on the Occasion of His 80th Birthday

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Abstract Every nontrivial zero of the Riemann zeta function is associated as eigenvalue with an eigenfunction of the fundamental differential operator on a Hilbert-Pólya space. It has geometric multiplicity one. A relation between nontrivial zeros of the zeta function and eigenvalues of the convolution operator is given. It is an analogue of the Selberg transform in Selberg's trace formula. Elements of the Hilbert-Pólya space are characterized by the Poisson summation formula.

Keywords Hilbert-Pólya space, spectrum of operators, zeros of zeta function

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1 Introduction

We denote by $C_c^\infty(0,\infty)$ the space of compactly supported smooth functions on $(0,\infty)$. Let g be a real-valued function in $C_c^\infty(0,\infty)$. If

$$h(x)=\int_0^\infty g(xy)g(y)dy,$$

then $h\in C_c^\infty(0,\infty)$. Its Mellin transform is

$$\widehat{h}(s)=\int_0^\infty h(x)x^{s-1}dx=\widehat{g}(s)\widehat{g}(1-s).$$

The convolution operator $\upsilon(h)$ is defined by

$$\upsilon(h)f(x)=\int_0^\infty h(\lambda)f(\lambda^{-1}x)d^\times\lambda,\quad d^\times\lambda=\lambda^{-1}d\lambda.$$

The Schwartz space $S(\mathbb{R})$ is the set of all rapidly decreasing, infinitely differentiable functions f on $\mathbb{R}$ with $\sup_{x\in\mathbb{R}}|x^af^{(b)}(x)|<\infty$ for $a,b=0,1,2,\dots$ We use natural logarithm to identify $\mathbb{R}_+^\times=(0,\infty)$ with $\mathbb{R}$. This induces an isomorphism between spaces $S(\mathbb{R}_+^\times)$ and $S(\mathbb{R})$. The fundamental differential operator on $S(\mathbb{R}_+^\times)$ is

$$Df(x)=-xf'(x).$$

We denote by $\mathcal{H}_\cap$ the subspace of all even functions $f \in S(\mathbb{R})$ such that $f(0) = 0$ and $\mathfrak{F}f(0) = 0$, where the Fourier transform of f is

$$\mathfrak{F}f(y) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{2\pi \mathrm{i}xy}dx.$$

The strong Schwartz space is

$$\mathcal{H}_- := \{f \mid \mathbb{R}_+^\times \to \mathbb{C} \text{ such that } x^\alpha f(x) \in S(\mathbb{R}_+^\times) \text{ for every real number } \alpha\}.$$

The zeta operator Z is defined by $Zf(x) = \sum_{n=1}^{\infty} f(nx)$. We call the quotient space

$$\mathcal{H} := \mathcal{H}_-/Z\mathcal{H}_\cap$$

a Hilbert-Pólya space. Let D_- be the operator induced by D on $\mathcal{H}$.

The Hilbert-Pólya conjecture says that the Riemann hypothesis is true because non-trivial zeros of the zeta function correspond (in a certain canonical way) to the eigenvalues of some positive operator.

Note that the convolution operator $\upsilon(h)$ is a positive operator on $L^2(0,\infty)$. In direction of the Hilbert-Pólya conjecture, a spectral interpretation for critical zeros of the zeta-function is given by Connes [2]. He constructed a closed unbounded differential operator D_χ and a Hilbert-Pólya space $\mathcal{H}_\chi$. The discrete spectrum of D_χ acting on $\mathcal{H}_\chi$ is the set of imaginary parts of critical zeros of the L-function with Grössencharakter χ [2, Theorem 1, p. 40].

Motivated by Connes' construction, Meyer [3, Corollary 4.2, p. 8] proved that the eigenvalues of the transpose D_-^t of D_- acting on the space of continuous linear functionals $\mathcal{H} \to \mathbb{C}$ are exactly the nontrivial zeros of $\zeta(s)$ and that the algebraic multiplicity of eigenvalues of D_-^t is the order of corresponding zero of $\zeta(s)$.

In infinite dimensional spaces an operator and its transpose may not have the same spectrum. If D_- was a compact operator on $\mathcal{H}$, then D_- and its transpose would have the same spectrum. However, D_- is unbounded and hence is not compact on $\mathcal{H}$. Thus we do not know whether or not every nontrivial zero of the Riemann zeta function is an eigenvalue of D_-. This question leads us to confirm by explicit construction in Theorem 1.1 that every nontrivial zero of the zeta function is indeed an eigenvalue of D_-, which suffices for the purpose of the Riemann hypothesis [1, 9].

Theorem 1.1. *If ρ is a nontrivial zero of $\zeta(s)$, then*

$$F_\rho(x) = \int_1^{\infty} Z\eta(tx)t^{\rho-1}dt$$

is an eigenfunction of D_- on the Hilbert-Pólya space $\mathcal{H}$ associated with the eigenvalue ρ, where $\eta(x) = 8\pi x^2(\pi x^2 - \frac{3}{2})\mathrm{e}^{-\pi x^2}$, i.e.,

$$DF_\rho(x) = \rho F_\rho(x) + Z\eta(x), \quad \eta \in \mathcal{H}_\cap.$$

A fundamental result in [4, (1.8), p. 55] is that if λ is an eigenvalue of the fundamental differential operator, then $h(\lambda)$ is an eigenvalue of invariant integral operators. Since Connes' discussion is modelled on Selberg's trace formula, analogous statement for Connes' approach to the Riemann hypothesis would be that if ρ is nontrivial zero of the zeta function, then $\widehat{h}(\rho)$ is an eigenvalue of $\upsilon(h)$. We confirm this analogy by explicit construction in Theorem 1.2.

Theorem 1.2. *If ρ is a nontrivial zero of $\zeta(s)$, then $\widehat{h}(\rho)$ is an eigenvalue of $\upsilon(h)$ on the Hilbert-Pólya space $\mathcal{H}$ associated with the eigenfunction F_ρ.*

In the following theorem we give a characterization of eigenfunctions associated with nontrivial zeros of the Riemann zeta function via the Poisson summation formula.

Theorem 1.3. *Let $\delta_{l,F}(x) = \int_0^{\infty} Z^{-1}F(xt)\frac{\sin(2l+1)\pi t}{\pi t}dt$. Then*

$$\lim_{l\to\infty}\left(\frac{1}{x}\sum_{m=1}^{l}\mathfrak{F}Z^{-1}F\left(\frac{m}{x}\right) - \delta_{l,F}(x)\right) = F(x)$$

for every $F \in \mathcal{H}_-$. An element $F \in \mathcal{H}_-$ belongs to the subspace $Z\mathcal{H}_\cap$ if and only if $JZ\mathfrak{F}Z^{-1}F(x) = F(x)$ for all x, where $Jf(x) = x^{-1}f(x^{-1})$.

By using Theorem 1.3, we prove the following theorem about geometric multiplicity of nontrivial zeros of the Riemann zeta function.

Theorem 1.4. *Let ρ be a nontrivial zero of $\zeta(s)$. Then the eigenvalue ρ of D_- has geometric multiplicity one.*

If the geometric multiplicity of each eigenvalue of D_- on the Hilbert-Pólya space $\mathcal{H}$ is the same as its algebraic multiplicity, then Theorem 1.4 implies the simplicity of zeros of the Riemann zeta function.

2 Proofs of Theorems 1.1 and 1.2

We denote by $\mathbb{N}$ the set of all positive integers. For any interval I on $\mathbb{R}$, we define

$$S(\mathbb{R}_+^\times)_I = \{f \mid \mathbb{R}_+^\times \to \mathbb{C} \text{ such that } x^\alpha f(x) \in S(\mathbb{R}_+^\times) \text{ for every } \alpha \in I\}.$$

Lemma 2.1. *If f is an even function in $S(\mathbb{R})$, then both f and $\mathfrak{F}f$ belong to $S(\mathbb{R}_+^\times)_{(0,\infty)}$.*

Proof. By definition of $S(\mathbb{R})$ and $S(\mathbb{R}_+^\times)_{(0,\infty)}$ we have $f \in S(\mathbb{R}_+^\times)_{(0,\infty)}$. Since the Fourier transform $\mathfrak{F}$ maps $S(\mathbb{R})$ into itself, we also have $\mathfrak{F}f \in S(\mathbb{R}_+^\times)_{(0,\infty)}$.

This completes the proof of the lemma. □

Proof of Theorem 1.1. Since

$$\eta(x) = 8\pi x^2\left(\pi x^2 - \frac{3}{2}\right)\mathrm{e}^{-\pi x^2}, \tag{2.1}$$

we have $\eta \in \mathcal{H}_\cap$, $\widehat{\eta}(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)$, and $\mathfrak{F}\eta(x) = \eta(x)$ by calculation. Thus by Poisson's summation formula,

$$JZ\eta(x) = Z\eta(x). \tag{2.2}$$

Put

$$F(x) = \int_1^\infty Z\eta(tx)t^{\rho-1}dt. \tag{2.3}$$

Since $\eta \in \mathcal{H}_\cap$, by Lemma 2.1 and analytic continuation,

$$\int_0^\infty Z\eta(tx)t^{s-1}dt = x^{-s}\zeta(s)s(s-1)\pi^{-s/2}\Gamma(s/2) = x^{-s}\xi(s)$$

for $\Re s > 0$. In particular, $\int_0^\infty Z\eta(tx)t^{\rho-1}dt = 0$ where the integral is convergent for fixed x. Hence by (2.2),

$$F(x) = -\int_0^1 Z\eta(tx)t^{\rho-1}dt = -\int_1^\infty \frac{1}{x}\, Z\eta\left(\frac{t}{x}\right)t^{-\rho}dt. \tag{2.4}$$

We write $F(x) = x^{-\rho}\int_x^\infty Z\eta(t)t^{\rho-1}dt$. By the product formula, $-xF'(x) = \rho F(x) + Z\eta(x)$, i.e.,

$$DF(x) = \rho F(x) + Z\eta(x), \quad \eta \in \mathcal{H}_\cap \tag{2.5}$$

so that ρ is an eigenvalue of D_- on the Hilbert-Pólya space if we can show $F \in \mathcal{H}_-$.

Since $\mathbb{R}$ is open and since $x^\sigma D^m F(x)$ is a continuous function of $x \in (0,\infty)$, for every $k, m+1 \in \mathbb{N}$, $\sigma \in \mathbb{R}$ it suffices to show that

$$x^\sigma D^m F(x) \ll 1 \quad \text{as} \quad x \to 0 \quad \text{or} \quad x \to \infty.$$

If $0 < x \leqslant 1$, then $X := 1/x \geqslant 1$. By (2.4),

$$|x^\sigma D^m F(x)| = \left| - X^{-\sigma}\,\frac{d^m}{d\,(\log X)^m}\left(X\int_1^\infty Z\eta(tX)t^{-\rho}dt\right)\right|$$

$$\leqslant 8\pi \sum_{n=1}^{\infty} \int_1^{\infty} X^{-\sigma} \left| \frac{d^m}{d(\log X)^m} \left(X(ntX)^2 \left(\pi(ntX)^2 - \frac{3}{2} \right) \mathrm{e}^{-\pi(ntX)^2} \right) \right| t^{-\Re\rho} dt.$$

As $\pi > 3$ and $nt \geqslant 1$, there exists a constant $C_{\sigma,m} > 0$ such that

$$X^{-\sigma} \left| \frac{d^m}{d(\log X)^m} \left(X(ntX)^2 \left(\pi(ntX)^2 - \frac{3}{2} \right) \mathrm{e}^{-\pi(ntX)^2} \right) \right| \mathrm{e}^{2(ntX)^2} \leqslant C_{\sigma,m}$$

for all $t \geqslant 1$ and all $n \in \mathbb{N}$ as $X \to \infty$ (i.e., $x \to 0+$). It follows that

$$|x^\sigma D^m F(x)| \leqslant 8\pi C_{\sigma,m} \left(\sum_{n=1}^{\infty} \mathrm{e}^{-n^2} \right) \int_1^{\infty} \mathrm{e}^{-t^2} t^{-\Re\rho} dt \ll 1$$

as $x \to 0$.

If $x \geqslant 1$, by (2.3) we derive similarly to the case $x \to 0$ that

$$\begin{aligned} |x^\sigma D^m F(x)| &= \left| x^\sigma \frac{d^m}{d(\log x)^m} \left(\int_1^{\infty} Z\eta(tx) t^{\rho-1} dt \right) \right| \\ &\leqslant 8\pi \sum_{n=1}^{\infty} \int_1^{\infty} \left| x^\sigma \frac{d^m}{d(\log x)^m} \left((ntx)^2 \left(\pi(ntx)^2 - \frac{3}{2} \right) \mathrm{e}^{-\pi(ntx)^2} \right) \right| t^{\Re\rho-1} dt \ll 1 \end{aligned}$$

as $x \to \infty$. Therefore $F \in \mathcal{H}_-$.

As $\eta \in \mathcal{H}_\cap$, for any fixed $x > 0$ and for all $t \geqslant 1$ we have $|\eta(tx)| \leqslant c_x (tx)^{-2}$ for a constant c_x depending on x. Thus

$$\int_1^{\infty} \left| \sum_{n \geqslant N+1} \eta(ntx) \right| t^{\Re\rho-1} dt \leqslant \frac{c_x}{x^2 N(2-\Re\rho)} \to 0$$

as $N \to \infty$, so that we can change the order of summation and integration to obtain

$$F(x) = Z \int_1^{\infty} \eta(tx) t^{\rho-1} dt. \tag{2.6}$$

Hence

$$Z^{-1} F(x) = \int_1^{\infty} \eta(tx) t^{\rho-1} dt = x^{-\rho} \int_x^{\infty} \eta(t) t^{\rho-1} dt \sim \frac{\widehat{\eta}(\rho)}{x^\rho}$$

as $x \to 0+$. Since $\widehat{\eta}(\rho) \neq 0$, this implies that F is not an element in $Z\mathcal{H}_\cap$. Hence F is a nontrivial element in $\mathcal{H}$.

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let F be given as in (2.3). By (2.6),

$$F(x) = Z \int_1^{\infty} \eta(tx) t^{\rho-1} dt.$$

Since h is a compact support on $(0, \infty)$,

$$\begin{aligned} &\left| \int_0^{\infty} h(\lambda) d^\times \lambda \sum_{n=N+1}^{\infty} \int_1^{\infty} \eta(t\lambda^{-1} nx) t^{\rho-1} dt \right| \\ &\quad \leqslant \frac{c_x}{x^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \int_0^{\infty} |h(\lambda)\lambda^2| d^\times \lambda \int_1^{\infty} t^{\Re\rho-3} dt \to 0 \end{aligned}$$

as $N \to \infty$. Thus we can change the order of integration and summation and derive

$$\begin{aligned} \upsilon(h) F(x) &= \int_0^{\infty} h(\lambda) d^\times \lambda Z \int_1^{\infty} \eta(t\lambda^{-1} x) t^{\rho-1} dt \\ &= Z \int_0^{\infty} h(\lambda) d^\times \lambda \int_1^{\infty} \eta(t\lambda^{-1} x) t^{\rho-1} dt \end{aligned}$$

$$= \widehat{h}(\rho)F(x) + Zg_1(x), \tag{2.7}$$

where

$$\begin{aligned}
g_1(x) &= \int_0^\infty h(\lambda)d^\times\lambda \int_1^\infty \eta(t\lambda^{-1}x)t^{\rho-1}dt - \widehat{h}(\rho)\int_1^\infty \eta(tx)t^{\rho-1}dt \\
&= \int_1^\infty (\upsilon(h)\eta(tx) - \widehat{h}(\rho)\eta(tx))t^{\rho-1}dt \\
&= x^{-\rho}\{\widehat{h}(\rho)\widehat{\eta}(\rho) - \widehat{h}(\rho)\widehat{\eta}(\rho)\} - \int_0^1 (\upsilon(h)\eta(tx) - \widehat{h}(\rho)\eta(tx))t^{\rho-1}dt \\
&= -\int_0^1 (\upsilon(h)\eta(tx) - \widehat{h}(\rho)\eta(tx))t^{\rho-1}dt
\end{aligned} \tag{2.8}$$

is an element in $\mathcal{H}_\cap$ (shown next). Therefore, $\widehat{h}(\rho)$ is an eigenvalue of $\upsilon(h)$ associated with the eigenfunction $F(x)$ on the Hilbert-Pólya space.

We now show that $g_1 \in \mathcal{H}_\cap$. First notice that $\upsilon(h)\eta$ and η are elements in $\mathcal{H}_\cap$. Also $g_1(0) = 0$ and

$$\begin{aligned}
\mathfrak{F}g_1(0) &= \int_{-\infty}^\infty g_1(x)dx = \int_1^\infty t^{\rho-1}dt \int_{-\infty}^\infty (\upsilon(h)\eta(tx) - \widehat{h}(\rho)\eta(tx))dx \\
&= 2\int_1^\infty t^{\rho-2}dt \int_0^\infty (\upsilon(h)\eta(x) - \widehat{h}(\rho)\eta(x))dx = \frac{2}{1-\rho}(\widehat{h}(1) - \widehat{h}(\rho))\widehat{\eta}(1) = 0,
\end{aligned}$$

where the change of order of integration is permissible as the double integral is absolute integrable.

Let $a, b = 0, 1, 2, 3, \ldots$ Since $\upsilon(h)\eta - \widehat{h}(\rho)\eta \in \mathcal{H}_\cap$, there exists a positive constant M such that

$$\sup_{x\in\mathbb{R}} |(\upsilon(h)\eta - \widehat{h}(\rho)\eta)^{(b)}(x)| \leqslant M$$

and

$$\sup_{x\in\mathbb{R}} |x^{\max(a,b)+1}(\upsilon(h)\eta - \widehat{h}(\rho)\eta)^{(b)}(x)| \leqslant M.$$

For $|x| \geqslant 1$, we have

$$\begin{aligned}
|x^a g_1^{(b)}(x)| &= \left| \int_1^\infty x^a t^b ([\upsilon(h)\eta]^{(b)}(tx) - \widehat{h}(\rho)\eta^{(b)}(tx))t^{\rho-1}dt \right| \\
&\leqslant \int_1^\infty |tx|^{\max(a,b)+1} |([\upsilon(h)\eta]^{(b)}(tx) - \widehat{h}(\rho)\eta^{(b)}(tx))| t^{\Re\rho-2}dt \\
&\leqslant \frac{M}{1-\Re\rho}.
\end{aligned}$$

For $|x| \leqslant 1$, we have

$$\begin{aligned}
|x^a g_1^{(b)}(x)| &= \left| -\int_0^1 x^a t^b ([\upsilon(h)\eta]^{(b)}(tx) - \widehat{h}(\rho)\eta^{(b)}(tx))t^{\rho-1}dt \right| \\
&\leqslant \int_1^\infty |([\upsilon(h)\eta]^{(b)}(tx) - \widehat{h}(\rho)\eta^{(b)}(tx))| t^{\Re\rho-1}dt \leqslant \frac{M}{\Re\rho}.
\end{aligned}$$

Thus $g_1 \in S(\mathbb{R})$, and therefore $g_1 \in \mathcal{H}_\cap$ as η is even.

This completes the proof of Theorem 1.2. □

3 Properties of $\delta_{l,F}$

Lemma 3.1. *Let F be any element in $\mathcal{H}_-$. Then $Z^{-1}F \in S(\mathbb{R}_+^\times)_{[1,\infty)}$. For every $k, m+1 \in \mathbb{N}$, $\sigma > 0$ we have*

$$x^\sigma D^m Z^{-1}F(x) \ll |\log x|^{-k} \quad as \quad x \to \infty.$$

Proof. Let F be any element in $\mathcal{H}_-$. Then $\widehat{F}(s)$ is an entire function. For $s=\sigma+\mathrm{i}t$, $t\neq 0$ by partial integration

$$\widehat{F}(s)=\int_0^\infty F(x)x^{s-1}dx=\frac{(-1)^\nu}{(\mathrm{i}t)^\nu}\int_0^\infty \frac{\partial^\nu[x^\sigma F(x)]}{\partial(\log x)^\nu}\,x^{\mathrm{i}t-1}dx \tag{3.1}$$

for any real σ and $\nu\in\mathbb{N}$. Thus $\widehat{F}(s)$ is of rapid decay for s in any vertical strip. By Mellin's inversion formula

$$x^\sigma Z^{-1}F(x)=\frac{1}{2\pi\mathrm{i}}\int_{\Re s=\sigma+1}\frac{\widehat{F}(s)}{\zeta(s)}\,x^{\sigma-s}ds\ll x^{-1}\ll|\log x|^{-k}$$

for $x>1$, $\sigma>0$ and $k\in\mathbb{N}$.

By [8, (3.11.8), p. 60], a constant A exists such that

$$1/\zeta(s)=O(\log|t|) \tag{3.2}$$

for $\Re(s)\geqslant 1-\frac{A}{\log|t|}$ with $|t|\geqslant t_0$. Let $\delta=A/\log t_0$, $C_1=\{s=1-\delta+\mathrm{i}t\mid |t|<t_0\}$, and

$$C_2=\left\{s=\sigma+\mathrm{i}t\,\middle|\,\sigma=1-\frac{A}{\log|t|},\ |t|\geqslant t_0\right\}.$$

If $x<1$, we can move the line of integration to obtain

$$xZ^{-1}F(x)=\frac{1}{2\pi\mathrm{i}}\int_{C_1\cup C_2}\frac{\widehat{F}(s)}{\zeta(s)}x^{1-s}\,ds=O(x^\delta)+\frac{1}{2\pi\mathrm{i}}\int_{C_2}\frac{\widehat{F}(s)}{\zeta(s)}\,x^{1-s}ds. \tag{3.3}$$

Since $\widehat{F}(s)$ is of rapid decay for s any vertical strip, by (3.2) a constant M exists such that $|\widehat{F}(s)/\zeta(s)|\leqslant M|t|^{-2}$ for all $s\in C_2$. It follows that

$$\begin{aligned}\int_{C_2}\frac{\widehat{F}(s)}{\zeta(s)}\,x^{1-s}ds&\ll x^{\frac{1}{\sqrt{|\log x|}}}+\int_{|t|>\mathrm{e}^{A\sqrt{|\log x|}}}\frac{1}{|t|^2}\,dt\\&\ll\frac{1}{\mathrm{e}^{\sqrt{|\log x|}}}+\frac{1}{\mathrm{e}^{A\sqrt{|\log x|}}}\ll|\log x|^{-k},\end{aligned}$$

when $x\to 0$ for $k\in\mathbb{N}$. Hence $xZ^{-1}F(x)\ll|\log x|^{-k}$ when $x<1$ for $k\in\mathbb{N}$. Therefore for any $\sigma\geqslant 1$,

$$x^\sigma Z^{-1}F(x)\ll|\log x|^{-k}$$

as $x\to 0$ or ∞. We also have $x^\sigma Z^{-1}F(x)\ll|\log x|^{-k}$ as $x\to\infty$ for every $k\in\mathbb{N}$, $\sigma>0$.

Since $\widehat{F}(s)$ is of rapid decay for s any vertical strip and

$$D^mZ^{-1}F(x)=\frac{1}{2\pi\mathrm{i}}\int_{C_1\cup C_2}\frac{\widehat{F}(s)s^m}{\zeta(s)}x^{-s}ds,$$

we derive similarly to the above that

$$x^\sigma D^mZ^{-1}F(x)\ll|\log x|^{-k} \tag{3.4}$$

as $x\to 0$ or ∞ for every $k,m+1\in\mathbb{N}$, $\sigma\in[1,\infty)$, i.e., $Z^{-1}F\in S(\mathbb{R}_+^\times)_{[1,\infty)}$. We also have

$$x^\sigma D^mZ^{-1}F(x)\ll|\log x|^{-k}\quad\text{as}\quad x\to\infty$$

for every $k,m+1\in\mathbb{N}$, $\sigma>0$.

This completes the proof of the lemma. □

Lemma 3.2. *Let*

$$\delta_{l,F}(x)=\int_0^\infty Z^{-1}F(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}dt.$$

Then

$$\delta_{l,F}(x)=\int_0^{1+\frac{\frac{1}{2}}{x}}\mathfrak{F}Z^{-1}F(t)dt. \tag{3.5}$$

Proof. Since

$$\mathfrak{F}Z^{-1}F(u) = 2\int_0^\infty Z^{-1}F(t)\cos 2\pi utdt,$$

we can write

$$\begin{aligned}\int_0^{\frac{l+\frac{1}{2}}{x}} \mathfrak{F}Z^{-1}F(u)du &= 2\int_0^\infty Z^{-1}F(t)dt\int_0^{\frac{l+\frac{1}{2}}{x}} \cos 2\pi utdu\\ &= \int_0^\infty Z^{-1}F(t)\,\frac{\sin\frac{(2l+1)\pi t}{x}}{\pi t}\,dt\\ &= \int_0^\infty Z^{-1}F(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}dt,\end{aligned}$$

where the change of order of integration is permissible because the double integral is absolute integrable as $Z^{-1}F \in L^1(0,\infty)$ plus one finite interval of integration for a bounded function.

This completes the proof of the lemma. □

Lemma 3.3. *We can write*

$$\delta_{l,F}(x) = \frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{(\frac{l+\frac{1}{2}}{x})^s\widehat{F}(s)}{s\zeta(1-s)}\,ds.$$

In particular, if $F = Zg$ *for an element* $g \in \mathcal{H}_\cap$ *then*

$$\delta_{l,Zg}(x) = \frac{1}{2\pi \mathrm{i}}\int_{\Re s=K} \frac{(\frac{l+\frac{1}{2}}{x})^{1-s}\widehat{\mathfrak{F}g}(s)}{s-1}ds$$

for any $K > 0$, *where* $\widehat{\mathfrak{F}g}(s)$ *is of rapid decay for* s *any vertical strip of the half-plane* $\Re s > 0$.

Proof. By (3.3),

$$Z^{-1}F(x) = \frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{\widehat{F}(s)}{\zeta(s)}x^{-s}\,ds.$$

Hence

$$\int_0^\infty Z^{-1}F(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}\,dt = \int_0^\infty \frac{\sin(2l+1)\pi t}{\pi t}\,dt\frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{\widehat{F}(s)}{\zeta(s)}\,(xt)^{-s}ds,$$

where the double integral is absolute integrable because of the contour $C_1 \cup C_2$ (using $|\sin(2l+1)\pi t| \leqslant 1$ when $t \geqslant 1$ and $|\sin(2l+1)\pi t| \leqslant (2l+1)\pi t$ when $t < 1$). Thus we can change the order of integration and derive

$$\begin{aligned}\int_0^\infty Z^{-1}F(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}\,dt &= \frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{\widehat{F}(s)}{\zeta(s)}\,x^{-s}ds\int_0^\infty \frac{\sin(2l+1)\pi t}{\pi t^{1+s}}dt\\ &= \frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{\pi^{s-1}(2l+1)^s\widehat{F}(s)}{\zeta(s)}x^{-s}ds\int_0^\infty \frac{\sin t}{t^{1+s}}\,dt.\end{aligned}$$

By [6, Example 10, p. 162],

$$\int_0^\infty \frac{\sin y}{y^{1+s}}\,dy = -\Gamma(-s)\sin\,\frac{\pi s}{2} \tag{3.6}$$

for $-1 < \Re s < 1$. It follows that

$$\int_0^\infty Z^{-1}F(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}dt = -\frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{\pi^{s-1}(2l+1)^s\widehat{F}(s)}{\zeta(s)}x^{-s}\Gamma(-s)\sin\,\frac{\pi s}{2}\,ds.$$

Inserting $\frac{\Gamma(-s)\sin\,\frac{\pi s}{2}}{\zeta(s)} = -\frac{\pi^{1-s}2^{-s}}{s\zeta(1-s)}$ (see [8, (2.1.8), p. 16]) into the right-hand side of the above identity, we find that

$$\int_0^\infty Z^{-1}F(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}\,dt = \frac{1}{2\pi \mathrm{i}}\int_{C_1\cup C_2} \frac{(\frac{l+\frac{1}{2}}{x})^s\widehat{F}(s)}{s\zeta(1-s)}\,ds.$$

By an argument similar to that made in the paragraph containing (3.3),

$$\delta_{l,F}\left(\left(l+\frac{1}{2}\right)x\right) \ll \frac{1}{x|\log x|^k} \tag{3.7}$$

for any positive integer k as $x \to 0$. It follows that $\delta_{l,F} \in L^1(0,\infty)$.

Next, if $F = Zg$ for an element $g \in \mathcal{H}_\cap$ then

$$\delta_{l,Zg}(x) = -\frac{1}{2\pi \mathrm{i}} \int_{(1)-C_1\cup C_2} \frac{(\frac{l+\frac{1}{2}}{x})^{1-s}\widehat{Zg}(1-s)}{(1-s)\zeta(s)}\, ds.$$

Taking Mellin transform for both sides of the Poisson summation formula for the function $\mathfrak{F}g \in \mathcal{H}_\cap$ we obtain that $\widehat{Zg}(1-s) = \zeta(s)\widehat{\mathfrak{F}g}(s)$. Consequently,

$$\delta_{l,Zg}(x) = \frac{1}{2\pi \mathrm{i}} \int_{(1)-C_1\cup C_2} \frac{(\frac{l+\frac{1}{2}}{x})^{1-s}\widehat{\mathfrak{F}g}(s)}{s-1}\, ds.$$

As $g \in \mathcal{H}_\cap$, by Lemma 2.1 we have $\mathfrak{F}g \in S(\mathbb{R}_+^\times)_{(0,\infty)}$. Similarly to (3.1) we obtain that $\widehat{\mathfrak{F}g}(s)$ is of rapid decay for s in any vertical strip of the half-plane $\Re s > 0$. As $\widehat{\mathfrak{F}g}(1) = 0$ because $\mathfrak{F}g \in \mathcal{H}_\cap$ we can move the line of integration to $\Re s = K$ and derive that

$$\delta_{l,Zg}(x) = \frac{1}{2\pi \mathrm{i}} \int_{\Re s = K} \frac{(\frac{l+\frac{1}{2}}{x})^{1-s}\widehat{\mathfrak{F}g}(s)}{s-1}\, ds$$

for any $K > 0$.

This completes the proof of the lemma. □

4 Proofs of Theorems 1.3 and 1.4

Lemma 4.1. *Let F be any element in $\mathcal{H}_-$. Then for $m+1 \in \mathbb{N}$,*

$$\int_0^\infty D^m Z^{-1}F(u)du = 0.$$

Proof. By Mellin's inversion formula,

$$Z^{-1}F(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\widehat{F}(s)}{\zeta(s)} x^{-s} ds \tag{4.1}$$

for $c > 1$. For any $t > 0$, by (3.1) we can change the order of integration to obtain

$$\int_t^\infty Z^{-1}F(x)dx = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\widehat{F}(s)}{(s-1)\zeta(s)}\, t^{1-s} ds.$$

By (3.1) and (3.2), we can move the line of integration and get

$$\int_t^\infty Z^{-1}F(x)dx = \frac{1}{2\pi \mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \frac{\widehat{F}(s)}{(s-1)\zeta(s)}\, t^{1-s} ds.$$

As the integral is absolutely integrable, by the Riemann-Lebesgue lemma [7, Theorem 1, p. 11],

$$\lim_{t\to 0+} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \frac{\widehat{F}(s)}{(s-1)\zeta(s)}\, t^{1-s} ds = 0.$$

Hence

$$\int_0^\infty Z^{-1}F(x)dx = 0.$$

Also

$$D^m Z^{-1}F(x)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\widehat{F}(s)s^m}{\zeta(s)}x^{-s}\,ds$$

and

$$\int_t^\infty D^m Z^{-1}F(x)dx=\frac{1}{2\pi\mathrm{i}}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty}\frac{\widehat{F}(s)s^m}{(s-1)\zeta(s)}\,t^{1-s}ds.$$

By the Riemann-Lebesgue lemma,

$$\lim_{t\to 0+}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty}\frac{\widehat{F}(s)s^m}{(s-1)\zeta(s)}\,t^{1-s}ds=0.$$

Hence

$$\int_0^\infty D^m Z^{-1}F(x)dx=0.$$

This completes the proof of the lemma. □

Lemma 4.2. *Let $f\in L^1(\mathbb{R})$ be an even function. If $\mathfrak{F}f(0)=0$, then*

$$\frac{1}{\alpha}\sum_{m=1}^{l}\mathfrak{F}f\left(\frac{m}{\alpha}\right)=\sum_{m=1}^{\infty}\frac{f(m\alpha+0)+f(m\alpha-0)}{2}+\int_0^{\frac12}f(\alpha t)\,\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt+R(f,\alpha),$$

where

$$\begin{aligned}R(f,\alpha)=&\sum_{m=1}^{\infty}\int_m^{m+\frac12}\{f(t\alpha)-f(m\alpha+0)\}\,\frac{\sin(2l+1)\pi}{\sin\pi t}\,dt\\&+\sum_{m=1}^{\infty}\int_{m-\frac12}^{m}\{f(t\alpha)-f(m\alpha-0)\}\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt\end{aligned}$$

for $l\in\mathbb{N}$, where $\mathfrak{F}f(x)=\int_{-\infty}^{\infty}f(y)\mathrm{e}^{2\pi\mathrm{i}yx}dy$.

Proof. The following argument is a minor modification of [7, lines 15–21, p. 61].

As $\mathfrak{F}f(0)=0$ and $f\in L^1(0,\infty)$, from the trigonometric identity

$$2\sum_{m=1}^{l}\cos 2\pi m\alpha t=\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}-1,$$

we derive

$$\begin{aligned}\sum_{m=1}^{l}\mathfrak{F}f(m\alpha)&=\int_0^\infty f(t)2\left(\sum_{m=1}^{l}\cos 2\pi m\alpha t\right)dt=\int_0^\infty f(t)\left\{\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}-1\right\}dt\\&=\lim_{M\to\infty}\int_0^{\frac{M+\frac12}{\alpha}}f(t)\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}\,dt.\end{aligned}$$

We can write

$$\begin{aligned}&\int_0^{\frac{M+\frac12}{\alpha}}f(t)\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}dt\\&=\frac{1}{\alpha}\sum_{m=1}^{M}\frac{f(\frac{m}{\alpha}+0)+f(\frac{m}{\alpha}-0)}{2}\\&\quad+\int_0^{\frac{1}{2\alpha}}f(t)\,\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}\,dt+\sum_{m=1}^{M}\int_{\frac{m}{\alpha}}^{\frac{m+\frac12}{\alpha}}\left\{f(t)-f\left(\frac{m}{\alpha}+0\right)\right\}\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}\,dt\end{aligned}$$

$$+\sum_{m=1}^{M}\int_{\frac{m-\frac{1}{2}}{\alpha}}^{\frac{m}{\alpha}}\left\{f(t)-f\left(\frac{m}{\alpha}-0\right)\right\}\frac{\sin(2l+1)\pi\alpha t}{\sin\pi\alpha t}\,dt.$$

Therefore

$$\frac{1}{\alpha}\sum_{m=1}^{l}\mathfrak{F}f\left(\frac{m}{\alpha}\right)=\sum_{m=1}^{\infty}\frac{f(m\alpha+0)+f(m\alpha-0)}{2}+\int_{0}^{\frac{1}{2}}f(\alpha t)\,\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt+R(f,\alpha),$$

where

$$\begin{aligned}R(f,\alpha)&=\sum_{m=1}^{\infty}\int_{m}^{m+\frac{1}{2}}\{f(t\alpha)-f(m\alpha+0)\}\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt\\&\quad+\sum_{m=1}^{\infty}\int_{m-\frac{1}{2}}^{m}\{f(t\alpha)-f(m\alpha-0)\}\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.3. *Let F be any element in $\mathcal{H}_-$ and $f(t)=Z^{-1}F(|t|)$. Then*

$$\lim_{l\to\infty}\sum_{m=1}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f(t\alpha+m\alpha)-f(m\alpha))\,\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt=0$$

for any fixed real α.

Proof. By (4.1),

$$f(t)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\widehat{F}(s)}{\zeta(s)}t^{-s}\,ds,\quad t>0,$$

for a constant $c\geqslant 1$. Since

$$(t\alpha+m\alpha)^{-s}-(m\alpha)^{-s}=-s\int_{m\alpha}^{t\alpha+m\alpha}u^{-s-1}du,$$

we have

$$\begin{aligned}&\left|(f(t\alpha+m\alpha)-f(m\alpha))\frac{\sin(2l+1)\pi t}{\sin\pi t}v\right|\\&\leqslant\left|\left(\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{-s\widehat{F}(s)}{\zeta(s)}\left[\int_{m\alpha}^{t\alpha+m\alpha}u^{-s-1}du\right]ds\right)\frac{\sin(2l+1)\pi t}{\sin\pi t}\right|\\&\leqslant\frac{(2l+1)\pi t}{2\pi\cdot 2t}\;\frac{t}{m^{1+c}\alpha^{c}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left|\frac{-s\widehat{F}(s)}{\zeta(s)}\right||ds|\ll\frac{1}{m^{1+c}}\end{aligned}$$

for $|t|\leqslant 1/2$, where the implied constant depends only on α and l, not on t. The uniform convergence of the series

$$\sum_{m=1}^{\infty}(f(t\alpha+m\alpha)-f(m\alpha))\frac{\sin(2l+1)\pi t}{\sin\pi t}$$

for $|t|\leqslant 1/2$ justifies the following change of integration and summation:

$$\begin{aligned}&\sum_{m=1}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f(t\alpha+m\alpha)-f(m\alpha))\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt\\&=\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{-s\widehat{F}(s)}{\zeta(s)}\left[\sum_{m=1}^{\infty}\int_{m\alpha}^{t\alpha+m\alpha}u^{-s-1}du\right]ds\right)\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt.\end{aligned}$$

Since there exists a constant c_3 depending on α such that

$$\left|\left(\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{-s\widehat{F}(s)}{\zeta(s)}\left[\sum_{m=1}^{\infty}\int_{m\alpha}^{t\alpha+m\alpha}u^{-s-1}du\right]ds\right)\frac{1}{\sin\pi t}\right|\leqslant c_3$$

for all $|t| \leqslant 1/2$, applying the Riemann-Lebesgue lemma to the right-hand side of the above identity we derive

$$\lim_{l\to\infty}\sum_{m=1}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f(t\alpha+m\alpha)-f(m\alpha))\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt=0$$

for any fixed real α.

This completes the proof of the lemma. □

Lemma 4.4. *Let $\delta_{l,F}(x)$ be given as in Lemma 3.2. Then*

$$\lim_{l\to\infty}\left(\frac{1}{x}\sum_{m=1}^{l}\mathfrak{F}Z^{-1}F\left(\frac{m}{x}\right)-\delta_{l,F}(x)\right)=F(x)$$

for every element F in $\mathcal{H}_-$.

Proof. Let F be any element in $\mathcal{H}_-$. Put $f(t)=Z^{-1}F(|t|)$. Then f is an even function on $\mathbb{R}$. By (3.4) with $\sigma=1, m=0, k=2$, f belongs to $L^1(0,\infty)$. By Lemma 4.1 with $m=0$, $\mathfrak{F}f(0)=0$. Thus by Lemma 4.2,

$$\frac{1}{x}\sum_{m=1}^{l}\mathfrak{F}f\left(\frac{m}{x}\right)=\sum_{m=1}^{\infty}\frac{f(mx+0)+f(mx-0)}{2}+\int_0^{\frac{1}{2}}f(xt)\,\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt+R(f,x),$$

where

$$R(f,x)=\sum_{m=1}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}(f(tx+mx)-f(mx))\,\frac{\sin(2l+1)\pi t}{\sin\pi t}dt.$$

By Lemma 4.3, for $x\in(0,\infty)$ we have

$$\lim_{l\to\infty}\left(\frac{1}{x}\sum_{m=1}^{l}\mathfrak{F}Z^{-1}F\left(\frac{m}{x}\right)-\int_0^{\frac{1}{2}}f(xt)\,\frac{\sin(2l+1)\pi t}{\sin\pi t}dt\right)=F(x).$$

By (3.4) and the Riemann-Lebesgue lemma,

$$\lim_{l\to\infty}\int_0^{1/2}f(xt)\left(\frac{1}{\sin\pi t}-\frac{1}{\pi t}\right)\sin(2l+1)\pi t dt=0$$

and

$$\lim_{l\to\infty}\int_{1/2}^{\infty}f(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}\,dt=0.$$

Therefore,

$$\lim_{l\to\infty}\int_0^{\frac{1}{2}}f(xt)\,\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt=\lim_{l\to\infty}\int_0^{\infty}f(xt)\,\frac{\sin(2l+1)\pi t}{\pi t}\,dt.$$

The stated identity then follows.

This completes the proof of the lemma. □

Lemma 4.5 (See [5, Proposition 4.1, p. 87])**.** *Suppose $f\in L^1(\mathbb{R})$. Then $\mathfrak{F}f$ is continuous and bounded on $\mathbb{R}$.*

Lemma 4.6 (See [5, Theorem 4.2, p. 87])**.** *Suppose $f\in L^1(\mathbb{R})$ and assume also that $\mathfrak{F}f\in L^1(\mathbb{R})$. Then $f(x)=\mathfrak{F}^t\mathfrak{F}f(x)$ for almost every x.*

Lemma 4.7 (See [3, (10), p. 4])**.** *The set of all even functions in $S(\mathbb{R})$ equals*

$$\mathcal{H}_+:=\{f\in L^2(\mathbb{R}_+^{\times},dx)\mid \textit{both } f \textit{ and } \mathfrak{F}f \textit{ belong to } S(\mathbb{R}_+^{\times})_{(0,\infty)}\}.$$

Proof of Theorem 1.3. The first part of Theorem 1.3 is Lemma 4.4.

We now show that, for any element in $F \in \mathcal{H}_-$, F belongs to $Z\mathcal{H}_\cap$ if and only if $JZ\mathfrak{F}Z^{-1}F(x) = F(x)$ holds for every $x \in (0,\infty)$.

If $F \in Z\mathcal{H}_\cap$, by the Poisson summation formula the stated identity is true.

Conversely, if $JZ\mathfrak{F}Z^{-1}F(x) = F(x)$ for every $x \in (0,\infty)$ then

$$\mathfrak{F}Z^{-1}F(x) = Z^{-1}JF(x) \tag{4.2}$$

for all $x \in (0,\infty)$. As $JF \in \mathcal{H}_-$, we have $Z^{-1}JF \in S(\mathbb{R}_+^\times)_{[1,\infty)}$ by Lemma 3.1. Thus (4.2) says that $\mathfrak{F}Z^{-1}F \in S(\mathbb{R}_+^\times)_{[1,\infty)}$. Also $Z^{-1}F \in S(\mathbb{R}_+^\times)_{[1,\infty)}$ by Lemma 3.1.

Next, we show that both $\mathfrak{F}Z^{-1}F$ and $Z^{-1}F$ are in $S(\mathbb{R}_+^\times)_{(0,1)}$. By Lemmas 3.1 and 4.5, $\mathfrak{F}Z^{-1}F$ is continuous on $\mathbb{R}$. Moreover $\mathfrak{F}Z^{-1}F(0) = 0$ by Lemma 4.1.

By (4.2) and Lemma 4.6,

$$Z^{-1}F(x) = \mathfrak{F}^t Z^{-1}JF(x) \tag{4.3}$$

for almost every x. As the right-hand side of (4.3) is continuous on $\mathbb{R}$ by Lemma 4.5, $Z^{-1}F(x)$ can be extended uniquely to a continuous function on $\mathbb{R}$. We use the same notation for the extended function. Since $\mathfrak{F}^t Z^{-1}JF(0) = 0$ by Lemma 4.1, by (4.3) we also have $Z^{-1}F(0) = 0$.

For $m \in \mathbb{N}$, by (4.3),

$$D^m Z^{-1}F(x) = D^m \mathfrak{F}^t Z^{-1}JF(x) = (2\pi \mathrm{i} x)^m \mathfrak{F}^t (y^m Z^{-1}JF(y))(x). \tag{4.4}$$

By choosing $\sigma = m+1$, $k=2$ in (3.4) and $m=0$ in (3.4), we find that $y^m Z^{-1}JF(y) \in L^1(0,\infty)$. Then Lemma 4.5 together with (4.4) implies that $D^m Z^{-1}F(x)$ is a continuous on $\mathbb{R}$ and that $D^m Z^{-1}F(0) = 0$.

For $m \in \mathbb{N}$,

$$D^m \mathfrak{F}Z^{-1}F(x) = (-2\pi \mathrm{i} x)^m \mathfrak{F}(y^m Z^{-1}F(y))(x).$$

This identity implies that $D^m \mathfrak{F}Z^{-1}F(x)$ is continuous on $\mathbb{R}$ and $D^m \mathfrak{F}Z^{-1}F(0) = 0$.

Summarizing the above four paragraphs, $D^m Z^{-1}F(x)$ and $D^m \mathfrak{F}Z^{-1}F(x)$ are continuous on $\mathbb{R}$ and vanish at $x=0$ for $m = 0,1,2,\ldots$ Hence they are bounded on $(0,1)$, so that for every $k, m+1 \in \mathbb{N}$, $\sigma \in (0,1)$ we have

$$x^\sigma D^m Z^{-1}F(x) \ll |\log x|^{-k} \quad \text{and} \quad x^\sigma D^m \mathfrak{F}Z^{-1}F(x) \ll |\log x|^{-k} \quad \text{as} \quad x \to 0.$$

This together with the 2nd part of Lemma 3.1 implies that $\mathfrak{F}Z^{-1}F \in S(\mathbb{R}_+^\times)_{(0,1)}$ and $Z^{-1}F \in S(\mathbb{R}_+^\times)_{(0,1)}$. Thus by the end of 2nd paragraph of this proof we have shown that $\mathfrak{F}Z^{-1}F \in S(\mathbb{R}_+^\times)_{(0,\infty)}$ and $Z^{-1}F \in S(\mathbb{R}_+^\times)_{(0,\infty)}$. Thus it follows from Lemma 4.7 that $Z^{-1}F$ belongs to $\mathcal{H}_+$. We have already shown that $\mathfrak{F}^t Z^{-1}JF(0) = 0$ and $Z^{-1}F(0) = 0$. Therefore, $Z^{-1}F$ is an element in $\mathcal{H}_\cap$ so that $F \in Z\mathcal{H}_\cap$.

This completes the proof of Theorem 1.3. □

Lemma 4.8 (See [8, Theorem 9.7, p. 218])**.** *There is a constant A such that each interval $(n, n+1)$ contains a value t_n for which*

$$|\zeta(1-s)| > t_n^{-A}$$

for $-1 \leqslant \Re s \leqslant 2$ and $\Im s = t_n$.

Lemma 4.9. *Assume that ρ is a nontrivial zero of $\zeta(s)$. If $F \in \mathcal{H}_-$ is an eigenfunction of D_- in $\mathcal{H}$ with the eigenvalue ρ, then ρ is the only pole of $\frac{\widehat{F}(s)}{s\zeta(1-s)}$ and is a simple pole inside the strip $0 < \Re s < 1$.*

Proof. By Theorem 1.1, there exists an element $F \in \mathcal{H}_-$ such that $-xF'(x) = \rho F(x) + Zg(x)$ for some $g \in \mathcal{H}_\cap$. Taking their Mellin transforms we get

$$s\widehat{F}(s) = \rho \widehat{F}(s) + \zeta(s)\widehat{g}(s)$$

for $\Re s > 1$. Since $F \in \mathcal{H}_-$, $\widehat{F}(s)$ is entire. As $g \in \mathcal{H}_\cap$, $\widehat{g}(1) = 0$ and by Lemma 2.1, $\widehat{g}(s)$ is analytic for $\Re s > 0$. By analytic continuation,

$$s\widehat{F}(s) = \rho \widehat{F}(s) + \zeta(s)\widehat{g}(s)$$

for $\Re s > 0$, i.e.,

$$\frac{\widehat{F}(s)}{\zeta(s)} = \frac{\widehat{g}(s)}{s-\rho}$$

for $\Re s > 0$. By the functional equation of $\zeta(s)$, this identity becomes

$$\frac{\widehat{F}(s)}{s\zeta(1-s)} = -2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(-s)\,\frac{\widehat{g}(s)}{s-\rho} \tag{4.5}$$

for $\Re s > 0$.

By (3.1), $\widehat{F}(s)$ is an entire function of rapid decay for s in any vertical strip. By Lemma 3.3 and moving the contour of integration to $\Re s = 1$ we obtain

$$\delta_{l,F}\left(\left(l+\frac{1}{2}\right)x\right) = \frac{1}{2\pi \mathrm{i}}\int_{\Re s=1}\frac{x^{-s}\widehat{F}(s)}{s\zeta(1-s)}\,ds. \tag{4.6}$$

If $\widehat{g}(\rho) = 0$, the above identity implies that $\frac{\widehat{F}(s)}{s\zeta(1-s)}$ is analytic inside $0 \leqslant \Re s \leqslant 1$. Note that $\widehat{F}(s)$ is of rapid decay in s in this region by (3.1). Consider the integral

$$\int \frac{(\frac{l+\frac{1}{2}}{x})^s\widehat{F}(s)}{s\zeta(1-s)}\,ds$$

taken round the rectangle $(\pm \mathrm{i}t_n, 1\pm \mathrm{i}t_n)$ with t_n's being given in Lemma 4.8, and let $n\to\infty$. From (4.5) and (4.6) we derive that

$$\delta_{l,F}(x) = \frac{1}{2\pi \mathrm{i}}\int_{\Re s=0}\frac{(\frac{l+\frac{1}{2}}{x})^s\widehat{F}(s)}{s\zeta(1-s)}\,ds.$$

Because the integrand is absolute integrable on $\Re s = 0$, by the Riemann-Lebesgue lemma,

$$\lim_{l\to\infty}\delta_{l,F}(x) = 0.$$

Thus by Lemma 4.4, $JZ\mathfrak{F}Z^{-1}F(x) = F(x)$ for all x. It follows from Theorem 1.3 that $F\in\mathcal{H}_\cap$. This contradicts to that F represents in the nontrivial element in the Hilbert-Pólya space. Therefore we must have $\widehat{g}(\rho)\neq 0$, i.e., ρ is the only pole of $\frac{\widehat{F}(s)}{s\zeta(1-s)}$ inside $0<\Re s<1$ and is a simple pole.

This completes the proof of the lemma. □

Proof of Theorem 1.4. Let $F\in\mathcal{H}_-$ be any eigenfunction of D_- on $\mathcal{H}$ with the eigenvalue ρ. By (4.5), ρ is the only pole of

$$\frac{\widehat{F}(s)}{s\zeta(1-s)} = -2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(-s)\frac{\widehat{g}(s)}{s-\rho}$$

in the strip $0<\Re s<1$, and is a simple pole.

Let

$$F_1(x) = \int_1^\infty Z\eta(tx)t^{\rho-1}dt.$$

By the proof of Theorem 1.1, $DF_1 = \rho F_1 + Z\eta$ and $\eta\in\mathcal{H}_\cap$. It follows from the proof of Lemma 4.9 and Theorem 1.3 that F_1 is a nontrivial element in $\mathcal{H}$. By (2.3) and (2.4),

$$F_1(x) = \int_1^\infty Z\eta(tx)t^{\rho-1}dt = -\frac{1}{x}\int_1^\infty Z\eta\left(\frac{t}{x}\right)t^{-\rho}dt.$$

Hence, the following double integral is absolutely integrable for $\Re s > 1$ so that we can change the order of integration to write

$$\begin{aligned}\widehat{F}_1(s) &= \int_0^\infty x^{s-1}dx\int_1^\infty Z\eta(tx)t^{\rho-1}dt\\ &= \int_1^\infty t^{\rho-s-1}\int_0^\infty Z\eta(x)x^{s-1}dx\end{aligned}$$

$$= \frac{\xi(s)}{s-\rho}.$$

Since $\widehat{F}_1(s)$ is an entire function, by analytic continuation, $\widehat{F}_1(s) = \frac{\xi(s)}{s-\rho}$ holds for all s. Thus

$$\frac{\widehat{F}_1(s)}{s\zeta(1-s)} = \frac{(s-1)\pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})}{s-\rho}.$$

Now we choose a complex number α so that

$$\frac{\widehat{F}(s) - \alpha\widehat{F}_1(s)}{s\zeta(1-s)}$$

has no pole at $s = \rho$ and hence is analytic in $0 < \Re s < 1$. The argument made in the proof of Lemma 4.9 shows that $F - \alpha F_1 \in \mathcal{H}_\cap$. Consequently, F and F_1 represent the same element in $\mathcal{H}$. Therefore the geometric multiplicity of the eigenvalue ρ of D_- on the Hilbert-Pólya space is one.

This completes the proof of Theorem 1.4. □

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