SCIENCE CHINA
Mathematics

• ARTICLES •

July 2014 Vol. 57 No. 7: 1375–1400
doi: 10.1007/s11425-014-4809-z

Congruences involving generalized central trinomial coefficients

SUN Zhi-Wei

Department of Mathematics, Nanjing University, Nanjing 210093, *China*
Email: zwsun@nju.edu.cn

Received February 6, 2013; accepted May 7, 2013; published online April 2, 2014

Abstract For integers b and c the generalized central trinomial coefficient $T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$. Those $T_n = T_n(1,1)$ $(n = 0,1,2,\ldots)$ are the usual central trinomial coefficients, and $T_n(3,2)$ coincides with the Delannoy number $D_n = \sum_{k=0}^n \binom{n}{k}\binom{n+k}{k}$ in combinatorics. We investigate congruences involving generalized central trinomial coefficients systematically. Here are some typical results: For each $n = 1,2,3,\ldots$, we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(b^2-4c)^{n-1-k} \equiv 0 \pmod{n^2}$$

and in particular $n^2 \mid \sum_{k=0}^{n-1}(2k+1)D_k^2$; if p is an odd prime then

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p} \quad \text{and} \quad \sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p},$$

where $(-)$ denotes the Legendre symbol. We also raise several conjectures some of which involve parameters in the representations of primes by certain binary quadratic forms.

Keywords congruences, central trinomial coefficients, Motzkin numbers, central Delannoy numbers

MSC(2010) 11A07, 11B75, 05A10, 05A15, 11B65, 11E25

Citation: Sun Z-W. Congruences involving generalized central trinomial coefficients. Sci China Math, 2014, 57: 1375–1400, doi: 10.1007/s11425-014-4809-z

1 Introduction

For $n \in \mathbb{N} = \{0,1,2,\ldots\}$, the n-th central trinomial coefficient $T_n = [x^n](1+x+x^2)^n$ is the coefficient of x^n in the expansion of $(1+x+x^2)^n$. Since T_n is the constant term of $(1+x+x^{-1})^n$, by the multi-nomial theorem we see that

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!k!(n-2k)!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k}.$$

Central trinomial coefficients arise naturally in enumerative combinatorics (cf. Sloane [15]), e.g., T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$. As Andrews [1] pointed out, central trinomial coefficients were first studied by Euler. In 1987, Andrews and Baxter [3] found that the q-analogues of central trinomial coefficients have applications in the hard hexagon model.

For $n \in \mathbb{N}$, the n-th Motzkin number is defined by

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k,$$

where C_k denotes the k-th Catalan number,

$$\frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1}.$$

It is known that M_n equals the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y = 0$ and are made up of the only allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$ (cf. [15]).

Surprisingly we find that central trinomial coefficients and Motzkin numbers have nice congruence properties despite their combinatorial backgrounds. For example, we have the following conjecture. (As usual, for an integer a and an odd prime p, the notation $(\frac{a}{p})$ stands for the Legendre symbol.)

Conjecture 1.1. (i) *For any $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we have*

$$\sum_{k=0}^{n-1} (8k+5)T_k^2 \equiv 0 \pmod{n}.$$

If p is a prime, then

$$\sum_{k=0}^{p-1} (8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}.$$

(ii) *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_kT_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1 - 9\left(\frac{p}{3}\right)\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{M_kT_k}{(-3)^k} \equiv \frac{p}{2}\left(\left(\frac{p}{3}\right) - 1\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{T_kH_k}{3^k} \equiv \frac{3 + (\frac{p}{3})}{2} - p\left(1 + \left(\frac{p}{3}\right)\right) \pmod{p^2},$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k} 1/j$.

Given $b, c \in \mathbb{Z}$, we define the *generalized central trinomial coefficients*

$$\begin{aligned} T_n(b,c) &:= [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k \end{aligned}$$

and introduce the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1), \quad M_n = M_n(1,1),$$

$$T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n} \quad \text{and} \quad M_n(2,1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Thus, $T_n(b,c)$ can be viewed a natural common extension of central binomial coefficients and central trinomial coefficients, while $M_n(b,c)$ can be viewed as a natural common extension of Catalan numbers and Motzkin numbers. Let $d=b^2-4c$. Wilf [22, p. 159] observed that if $\varepsilon>0$ is sufficiently small then

$$\sum_{n=0}^{\infty}T_n(b,c)x^n=\frac{1}{\sqrt{1-2bx+dx^2}},$$

for $|x|<\varepsilon$. This implies the recurrence

$$(n+1)T_{n+1}(b,c)=(2n+1)bT_n(b,c)-dnT_{n-1}(b,c),\quad n\in\mathbb{Z}^+$$

(see also Noe [13]). Also, the Zeilberger algorithm (cf. [14, pp. 101–119]) yields the recursion

$$(n+3)M_{n+1}(b,c)=b(2n+3)M_n(b,c)-dnM_{n-1}(b,c),\quad n=1,2,3,\ldots,$$

which implies that

$$2cx^2\sum_{n=0}^{\infty}M_n(b,c)x^n=1-bx-\sqrt{1-2bx+dx^2}.$$

The central Delannoy numbers (see [6]) are defined by

$$D_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k},\quad n\in\mathbb{N}.$$

Such numbers also arise in many enumeration problems in combinatorics (cf. [15]); for example, D_n is the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$. For $n\in\mathbb{N}$, we define the polynomial

$$D_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Note that $D_n((x-1)/2)$ coincides with the well-known Legendre polynomial $P_n(x)$ of degree n. It is known that

$$\sum_{n=0}^{\infty}P_n(t)x^n=\frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if $b,c\in\mathbb{Z}$ and $d=b^2-4c\neq 0$ then

$$\sum_{n=0}^{\infty}T_n(b,c)\left(\frac{x}{\sqrt{d}}\right)^n=\frac{1}{\sqrt{1-2bx/\sqrt{d}+d(x/\sqrt{d})^2}}=\sum_{n=0}^{\infty}P_n\left(\frac{b}{\sqrt{d}}\right)x^n$$

and hence,

$$T_n(b,c)=(\sqrt{d})^nP_n\left(\frac{b}{\sqrt{d}}\right).$$

It follows that

$$T_n(2x+1,x^2+x)=P_n(2x+1)=D_n(x)$$

for all $x\in\mathbb{Z}$; in particular, $D_n=T_n(3,2)$.

Motivated by Conjecture 1.1, we investigate congruences involving generalized central trinomial coefficients as well as generalized Motzkin numbers.

Now, we state the main results of this paper.

Theorem 1.2. *Let p be an odd prime and let $b,c\in\mathbb{Z}$.*

(i) *For any integer $m\not\equiv 0 \pmod p$, we have*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{m^k}\equiv\left(\frac{(m-b)^2-4c}{p}\right)\pmod p,\tag{1.1}$$

$$2c\sum_{k=0}^{p-1}\frac{M_k(b,c)}{m^k}\equiv(m-b)^2-((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right)\pmod p. \tag{1.2}$$

(ii) *If p does not divide $d=b^2-4c$, then we have*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k}\equiv\left(\frac{cd}{p}\right)\pmod p. \tag{1.3}$$

If $b\not\equiv 2c \pmod p$, then

$$\sum_{k=0}^{p-1}\frac{T_k(b,c^2)^2}{(b-2c)^{2k}}\equiv\left(\frac{-c^2}{p}\right)\pmod p. \tag{1.4}$$

(iii) *Assume that $p\nmid c$. If $d=b^2-4c\not\equiv 0 \pmod p$, then*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k}\equiv 0\pmod p. \tag{1.5}$$

If $D=b^2-4c^2\not\equiv 0 \pmod p$, then

$$\sum_{k=0}^{p-1}\frac{T_k(b,c^2)M_k(b,c^2)}{(b-2c)^{2k}}\equiv\frac{4b}{b+2c}\left(\frac{D}{p}\right)\pmod p. \tag{1.6}$$

Example 1.3. Let $p>3$ be a prime. Applying Theorem 1.2(ii)–(iii) with $b=c=1$, we get

$$\sum_{k=0}^{p-1}\frac{T_k^2}{(-3)^k}\equiv\left(\frac{p}{3}\right)\pmod p,\quad \sum_{k=0}^{p-1}\frac{T_kM_k}{(-3)^k}\equiv 0\pmod p, \tag{1.7}$$

$$\sum_{k=0}^{p-1}T_k^2\equiv\left(\frac{-1}{p}\right)\pmod p,\quad \sum_{k=0}^{p-1}T_kM_k\equiv\frac{4}{3}\left(\frac{p}{3}\right)\pmod p. \tag{1.8}$$

Corollary 1.4. *Let p be an odd prime. For any integer x, we have*

$$\sum_{k=0}^{p-1}D_k(x)^2\equiv\left(\frac{x(x+1)}{p}\right)\pmod p. \tag{1.9}$$

In particular,

$$\sum_{k=0}^{p-1}D_k^2\equiv\left(\frac{2}{p}\right)\pmod p. \tag{1.10}$$

Proof. It suffices to recall that

$$D_k(x)=T_k(2x+1,x^2+x)$$

and apply Theorem 1.2(ii). □

Theorem 1.5. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$.*

(i) *For any $n\in\mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}\equiv 0\pmod n \tag{1.11}$$

and

$$6\sum_{k=0}^{n-1}kT_k(b,c^2)(b-2c)^{n-1-k}\equiv 0\pmod n. \tag{1.12}$$

If p is an odd prime not dividing $b-2c$, then

$$\frac{2c}{p}\sum_{k=0}^{p-1}\frac{T_k(b,c^2)}{(b-2c)^k}\equiv -b+(b+2c)\left(\frac{b^2-4c^2}{p}\right)\ (\mathrm{mod}\ p), \tag{1.13}$$

$$\frac{12c^2}{p}\sum_{k=0}^{p-1}\frac{kT_k(b,c^2)}{(b-2c)^k}\equiv (b+2c)^2\left(1-\left(\frac{b^2-4c^2}{p}\right)\right)-4c^2\ (\mathrm{mod}\ p). \tag{1.14}$$

(ii) *Suppose that $d=1$, i.e., there is an $m\in\mathbb{Z}$ such that*

$$b=2m+1,\quad c=m^2+m,$$

and hence $T_k(b,c)=D_k(m)$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)=\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k\in\mathbb{Z} \tag{1.15}$$

for all $n\in\mathbb{Z}^+$. If p is a prime not dividing $b-1=2m$, then

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)\equiv p+\frac{b+1}{b-1}p\left(\left(\frac{b+1}{2}\right)^{p-1}-1\right)\ (\mathrm{mod}\ p^3) \tag{1.16}$$

and

$$\sum_{k=0}^{p-1}(2k+1)^2T_k(b,c)\equiv\frac{2}{b-1}\left(\frac{(1-b)/2}{p}\right)=\frac{1}{m}\left(\frac{-m}{p}\right)\ (\mathrm{mod}\ p). \tag{1.17}$$

Example 1.6. Putting $b=1$ and $c=\pm1$ in (1.11), we get

$$\sum_{k=0}^{n-1}(-1)^kT_k\equiv 0\ (\mathrm{mod}\ n)\quad\text{and}\quad\sum_{k=0}^{n-1}3^{n-1-k}T_k\equiv 0\ (\mathrm{mod}\ n),$$

where n is any positive integer. Also, for a prime $p>3$, (1.13) with $b=1$ and $c=\pm1$ yields

$$\sum_{k=0}^{p-1}(-1)^kT_k\quad\text{and}\quad\sum_{k=0}^{p-1}T_k/3^k$$

modulo p^2 given by Cao and Pan [5].

Remark 1.7. For any $n\in\mathbb{Z}^+$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k3^{n-1-k}=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-1-k}(k+1)\binom{2k}{k},$$

for, if a_n denotes the left-hand side or the right-hand side of the last equality, then by the Zeilberger algorithm [14, pp. 101–119], we have the recurrence

$$(n+1)(2n+1)a_{n+2}=(4n^2+10n+3)a_{n+1}+3n(2n+3)a_n,\quad n=0,1,2,\dots$$

If $b,c\in\mathbb{Z}$ with $b^2-4c=1$, then for any prime $p\nmid c$, by (1.16) we have

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)\equiv p\quad(\mathrm{mod}\ p^2).$$

Theorem 1.8. *Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.*

(i) *For any $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} \equiv 0 \pmod{n}, \tag{1.18}$$

and furthermore,

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} = nT_n(b,c)T_{n-1}(b,c). \tag{1.19}$$

(ii) *For any $n \in \mathbb{Z}^+$, we have*

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2 d^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_k c^k d^{n-1-k} \in \mathbb{Z}. \tag{1.20}$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv 1 + \frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2} \pmod{p}. \tag{1.21}$$

Now, we give one more theorem.

Theorem 1.9. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k} \equiv \left(\frac{-1}{p}\right) + \frac{p^2}{3}E_{p-3} \pmod{p^3}, \tag{1.22}$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{8^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \tag{1.23}$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-3)^2}{16^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \tag{1.24}$$

$$\sum_{k=1}^{p-1}\frac{D_k^2}{k^2} \equiv -2q_p(2)^2 \pmod{p}, \tag{1.25}$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers, and $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Remark 1.10. (1.25) was conjectured by Sun [20].

We will show Theorems 1.2 and 1.5 in Sections 2 and 3, respectively. Section 4 is devoted to our proofs of Theorems 1.8 and 1.9. In Section 5, we are going to pose more conjectures for further research.

2 Proof of Theorem 1.2

The following lemma essentially follows from [21, (1.5)], but we will give a direct proof.

Lemma 2.1. *Let p be an odd prime and let $m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p}\right) \pmod{p}, \tag{2.1}$$

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{m^k} \equiv \frac{m}{2} - \frac{m-4}{2}\left(\frac{m(m-4)}{p}\right) \pmod{p}. \tag{2.2}$$

Proof. Clearly,

$$\binom{2k}{k} = \binom{-1/2}{k}(-4)^k \equiv \binom{(p-1)/2}{k}(-4)^k \pmod p$$

for all $k = 0, \dots, p-1$. Thus,

$$\begin{aligned}\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} &\equiv \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \frac{(-4)^k}{m^k} = \left(1 - \frac{4}{m}\right)^{(p-1)/2} \\ &= \frac{(m(m-4))^{(p-1)/2}}{m^{p-1}} \equiv \left(\frac{m(m-4)}{p}\right) \pmod p.\end{aligned}$$

This proves (2.1).

Observe that

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k+1}{k}}{m^k} = \frac{\binom{p}{(p-1)/2}}{m^{(p-1)/2}} + \frac{1}{2}\sum_{k=0}^{(p-3)/2} \frac{\binom{2k+2}{k+1}}{m^k} \equiv \frac{m}{2}\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} - \frac{m}{2} \pmod p.$$

Hence,

$$\begin{aligned}\sum_{k=0}^{(p-1)/2} \frac{C_k}{m^k} &= \sum_{k=0}^{(p-1)/2} \frac{2\binom{2k}{k} - \binom{2k+1}{k}}{m^k} \equiv \left(2 - \frac{m}{2}\right)\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} + \frac{m}{2} \\ &\equiv \frac{m}{2} - \frac{m-4}{2}\left(\frac{m(m-4)}{p}\right) \pmod p.\end{aligned}$$

So (2.2) also holds. We are done. □

Proof of Theorem 1.2(i). In the case $c \equiv 0 \pmod p$, as $T_k(b,c) \equiv b^k \pmod c$ for all $k \in \mathbb{N}$, we have

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \sum_{k=0}^{p-1} \frac{b^k}{m^k} \equiv \left(\frac{(m-b)^2}{p}\right) \pmod p.$$

So (1.1) holds if $p \mid c$. Note that (1.2) is trivial when $p \mid c$.

Suppose that $c \not\equiv 0 \pmod p$. For any $n \in \mathbb{N}$, clearly,

$$T_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k \equiv \begin{cases} \binom{n}{n/2}c^{n/2} \pmod b, & \text{if } 2 \mid n, \\ 0 \pmod b, & \text{if } 2 \nmid n, \end{cases}$$

and similarly,

$$M_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k \equiv \begin{cases} C_{n/2}c^{n/2} \pmod b, & \text{if } 2 \mid n, \\ 0 \pmod b, & \text{if } 2 \nmid n. \end{cases}$$

In the case $b \equiv 0 \pmod p$, by applying Lemma 2.1 we obtain

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} &\equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}c^k}{m^{2k}} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{(m^2c^{p-2})^k} \equiv \left(\frac{m^2-4c}{p}\right) \pmod p, \\ \sum_{k=0}^{p-1} \frac{M_k(b,c)}{m^k} &\equiv \sum_{k=0}^{(p-1)/2} \frac{C_k c^k}{m^{2k}} \equiv \sum_{k=0}^{(p-1)/2} \frac{C_k}{(m^2c^{p-2})^k} \equiv \frac{m^2}{2c} - \frac{m^2-4c}{2c}\left(\frac{m^2-4c}{p}\right) \pmod p.\end{aligned}$$

So (1.1) and (1.2) hold when $p \mid b$.

Below we assume that $p \nmid bc$. Observe that

$$\sum_{n=0}^{p-1} \frac{T_n(b,c)}{m^n} = \sum_{n=0}^{p-1} \frac{1}{m^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{c^k}{b^{2k}} \sum_{n=0}^{p-1} \frac{b^n}{m^n}\binom{n}{2k},$$

similarly,

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n}=\sum_{k=0}^{(p-1)/2}C_k\frac{c^k}{b^{2k}}\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}.$$

Now we consider the case $m\equiv b\pmod p$. For $k\in\{0,1,\ldots,(p-1)/2\}$, we have

$$\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}\equiv\sum_{n=2k}^{p-1}\binom{n}{2k}=\binom{p}{2k+1}\pmod p$$

with the help of a well-known identity of Chu (see, (1.52) of Gould [8, p, 7] or (5.26) of [9, p. 169]). Thus, by the above,

$$\sum_{n=0}^{p-1}\frac{T_n(b,c)}{m^n}\equiv\binom{p-1}{(p-1)/2}\frac{c^{(p-1)/2}}{b^{p-1}}\equiv\left(\frac{-c}{p}\right)=\left(\frac{(m-b)^2-4c}{p}\right)\pmod p,$$
$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n}\equiv C_{(p-1)/2}\frac{c^{(p-1)/2}}{b^{p-1}}\equiv 2\left(\frac{-c}{p}\right)=2\left(\frac{(m-b)^2-4c}{p}\right)\pmod p.$$

So (1.1) and (1.2) are true.

Below we consider the remaining case $m\not\equiv b\pmod p$. For $k\in\{0,1,\ldots,(p-1)/2\}$, we have

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}&=[x^{2k}]\sum_{n=0}^{p-1}\frac{b^n}{m^n}(1+x)^n\\&\equiv[x^{2k}]\sum_{n=0}^{p-1}(b+bx)^n m^{p-1-n}=[x^{2k}]\frac{(b+bx)^p-m^p}{b+bx-m}\\&=[x^{2k}]\frac{(b+bx)^p-m^p}{-(m-b)^p}\cdot\frac{(bx)^p-(m-b)^p}{bx-(m-b)}\\&\equiv[x^{2k}]\frac{b^p+b^px^p-m^p}{-(m-b)^p}\sum_{j=0}^{p-1}(bx)^j(m-b)^{p-1-j}\equiv\frac{b^{2k}}{(m-b)^{2k}}\pmod p.\end{aligned}$$

Therefore, with the help of Lemma 2.1,

$$\sum_{k=0}^{p-1}\frac{T_n(b,c)}{m^n}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{c^k}{b^{2k}}\cdot\frac{b^{2k}}{(m-b)^{2k}}\equiv\left(\frac{(m-b)^2-4c}{p}\right)\pmod p.$$

This proves (1.1).

In a similar way,

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n}\equiv\sum_{k=0}^{(p-1)/2}C_k\frac{c^k}{(m-b)^{2k}}\equiv\sum_{k=0}^{(p-1)/2}\frac{C_k}{M^k}\pmod p,$$

where $M:=(m-b)^2c^{p-2}$. Applying Lemma 2.1, we get the desired (1.2). □

Lemma 2.2. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. Let p be any odd prime and let $n\in\{0,\ldots,p-1\}$. If $p\nmid d$ or $p/2<n<p$, then*

$$T_n(b,c)\equiv\left(\frac{d}{p}\right)d^nT_{p-1-n}(b,c)\pmod p. \tag{2.3}$$

Proof. If $p\mid d$, then

$$T_n(b,c)\equiv[x^n]\left(x^2+bx+\frac{b^2}{4}\right)^n=[x^n]\left(x+\frac{b}{2}\right)^{2n}=\binom{2n}{n}\frac{b^n}{2^n}\pmod p.$$

Note that for $n = (p+1)/2, \dots, p-1$, we have

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \equiv 0 \pmod p.$$

Now assume that $p \nmid d$. Then

$$\begin{aligned} d^n T_{p-1-n}(b,c) &= d^n (\sqrt{d})^{p-1-n} P_{p-1-n}\left(\frac{b}{\sqrt{d}}\right) \\ &= d^{(p-1)/2} \sum_{k=0}^{p-1-n} \binom{p-1-n+k}{2k}\binom{2k}{k}\left(\frac{b/\sqrt{d}-1}{2}\right)^k (\sqrt{d})^n \\ &= d^{(p-1)/2} \sum_{k=0}^{p-1} \binom{n+k-p}{2k}\binom{2k}{k}\left(\frac{b-\sqrt{d}}{2\sqrt{d}}\right)^k (\sqrt{d})^n \\ &\equiv d^{(p-1)/2} \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\left(\frac{b-\sqrt{d}}{2\sqrt{d}}\right)^k (\sqrt{d})^n \\ &\equiv \left(\frac{d}{p}\right)(\sqrt{d})^n P_n\left(\frac{b}{\sqrt{d}}\right) = \left(\frac{d}{p}\right) T_n(b,c) \pmod p. \end{aligned}$$

This concludes the proof. □

Remark 2.3. Lemma 2.2 in the case $p \nmid d$ is essentially known (see, e.g., [13, (14)]), but our proof is simple and direct. By Lemma 2.2, for any prime $p > 3$, we have

$$\sum_{k=0}^{p-1} \frac{T_k^2}{9^k} = \sum_{k=0}^{p-1}\left(\frac{T_k}{(-3)^k}\right)^2 \equiv \sum_{k=0}^{p-1}\left(\left(\frac{-3}{p}\right)T_{p-1-k}\right)^2 = \sum_{j=0}^{p-1} T_j^2 \pmod p$$

and hence

$$\sum_{k=0}^{p-1} T_k^2/9^k \equiv \left(\frac{-1}{p}\right) \pmod p$$

in light of Example 1.3.

Let A and B be integers. The Lucas sequence $u_n = u_n(A,B)$ ($n \in \mathbb{N}$) is defined by

$$u_0 = 0, \quad u_1 = 1, \quad \text{and} \quad u_{n+1} = Au_n - Bu_{n-1}, \quad n = 1,2,3,\dots$$

Let α and β be the two roots of the equation $x^2 - Ax + B = 0$. It is well known that if $\Delta = A^2 - 4B \neq 0$ then

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{for all} \ \ n = 0,1,2,\dots$$

Lemma 2.4. *Let A and B be integers. For any odd prime p, we have*

$$u_p(A,B) \equiv \left(\frac{A^2-4B}{p}\right) \pmod p.$$

Proof. Though this is a known result, here we provide a simple proof.

If $\Delta = A^2 - 4B \equiv 0 \pmod p$, then

$$u_n(A,B) \equiv u_n\left(A, \frac{A^2}{4}\right) = n\left(\frac{A}{2}\right)^{n-1} \pmod p, \quad \text{for } n = 1,2,3,\dots$$

and in particular $u_p(A,B) \equiv 0 \pmod p$.

When $\Delta \not\equiv 0 \pmod p$, we have

$$\Delta u_p(A,B) = (\alpha-\beta)(\alpha^p - \beta^p) \equiv (\alpha-\beta)(\alpha-\beta)^p = \Delta^{(p+1)/2} \pmod p$$

with α and β the two roots of the equation $x^2 - Ax + B = 0$, hence

$$u_p(A,B) \equiv \left(\frac{\Delta}{p}\right) \pmod p$$

as desired. □

Proof of Theorem 1.2(ii). Suppose that $d = b^2 - 4c \not\equiv 0 \pmod p$. By Lemma 2.2,

$$\left(\frac{d}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k} \equiv \sum_{k=0}^{p-1} T_k(b,c)T_{p-1-k}(b,c) = [x^{p-1}]\bigg(\sum_{n=0}^{\infty} T_n(b,c)x^n\bigg)^2$$
$$= [x^{p-1}]\frac{1}{1-2bx+dx^2} = [x^p]\frac{x}{1-2bx+dx^2} \pmod p.$$

Write

$$\frac{x}{1-2bx+dx^2} = \sum_{n=0}^{\infty} u_n x^n.$$

Then $u_0 = 0$ and $u_1 = 1$. Since

$$(1-2bx+dx^2)\sum_{n=0}^{\infty} u_n x^n = x,$$

we have

$$u_n - 2bu_{n-1} + du_{n-2} = 0$$

for $n = 2, 3, \ldots$, hence $u_n = u_n(2b, d)$ for all $n \in \mathbb{N}$. Thus, with the help of Lemma 2.4, from the above we obtain

$$\left(\frac{d}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k} \equiv u_p(2b,d) \equiv \left(\frac{4b^2-4d}{p}\right) = \left(\frac{c}{p}\right) \pmod p.$$

This proves (1.3).

Now, suppose that $b \not\equiv 2c \pmod p$ and set

$$D = b^2 - 4c^2 = (b-2c)(b+2c).$$

If $p \mid D$, then $b \equiv -2c \not\equiv 0 \pmod p$ and

$$T_k(b,c^2) \equiv [x^k](x^2+bx+b^2/4)^k = [x^k](x+b/2)^{2k} \pmod p,$$

hence,

$$\sum_{k=0}^{p-1}\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv \sum_{k=0}^{p-1}\frac{(\binom{2k}{k}(b/2)^k)^2}{(2b)^{2k}} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod p.$$

The last step can be easily explained as follows:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \sum_{k=0}^{(p-1)/2}\binom{-1/2}{k}^2 \equiv \sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\binom{(p-1)/2}{(p-1)/2-k}$$
$$= [x^{(p-1)/2}](1+x)^{(p-1)/2+(p-1)/2} = \binom{p-1}{(p-1)/2} \equiv \left(\frac{-1}{p}\right) \pmod p.$$

Below we assume that $p \nmid D$. By Lemma 2.2 and Fermat's little theorem,

$$\left(\frac{D}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv C \pmod p,$$

where

$$C = \sum_{k=0}^{p-1} D^k T_k(b,c^2)(b-2c)^{2(p-1-k)}T_{p-1-k}(b,c^2)$$

$$= [x^{p-1}]\bigg(\sum_{k=0}^{\infty} T_k(b,c^2)(Dx)^k\bigg)\sum_{l=0}^{\infty} T_l(b,c^2)(b-2c)^{2l}x^l$$
$$= [x^{p-1}]\frac{1}{\sqrt{1-2b(Dx)+D(Dx)^2}}\cdot\frac{1}{\sqrt{1-2b(b-2c)^2x+D(b-2c)^4x^2}}$$
$$= [y^{p-1}]\frac{(b-2c)^{p-1}}{\sqrt{(1-2b(b+2c)y+(b+2c)^2Dy^2)(1-2b(b-2c)y+D(b-2c)^2y^2)}}$$

(Note that y corresponds to $(b-2c)x$). Therefore,

$$C \equiv [y^{p-1}]\frac{1}{1-Dy}\cdot\frac{1}{\sqrt{(1-(b+2c)^2y)(1-(b-2c)^2y)}}$$
$$= [y^{p-1}]\sum_{n=0}^{\infty}(Dy)^n\frac{1}{\sqrt{1-2(b^2+4c^2)y+D^2y^2}} \pmod p.$$

Observe that

$$(b^2+4c^2)^2-4(4b^2c^2)=(b^2-4c^2)^2=D^2$$

and hence

$$\frac{1}{\sqrt{1-2(b^2+4c^2)y+D^2y^2}}=\sum_{k=0}^{\infty}T_k(b^2+4c^2,4b^2c^2)y^k.$$

So we have

$$C \equiv \sum_{k=0}^{p-1}T_k(b^2+4c^2,4b^2c^2)D^{p-1-k} \equiv \sum_{k=0}^{p-1}\frac{T_k(b^2+4c^2,4b^2c^2)}{D^k}$$
$$\equiv \left(\frac{(D-(b^2+4c^2))^2-4(4b^2c^2)}{p}\right)=\left(\frac{-16c^2D}{p}\right) \pmod p$$

with the help of the first part of Theorem 1.2.

Combining the above, we finally obtain (1.4). We are done. □

Lemma 2.5. *Let b and c be integers. For any odd prime p, we have*

$$T_p(b,c)\equiv b \pmod p,\quad T_{p+1}(b,c)\equiv b^2 \pmod p, \tag{2.4}$$

and

$$T_{p-1}(b,c)\equiv\left(\frac{b^2-4c}{p}\right) \pmod p. \tag{2.5}$$

Proof. Since $\binom{p}{k}\equiv 0 \pmod p$ for all $k=1,\dots,p-1$, we have

$$T_p(b,c)=\sum_{k=0}^{(p-1)/2}\binom{p}{2k}\binom{2k}{k}b^{p-2k}c^k\equiv\binom{p}{0}b^p\equiv b \pmod p$$

with the help of Fermat's little theorem. If $1<k<p$, then

$$\binom{p+1}{k}=\frac{p(p+1)}{k(k-1)}\binom{p-1}{k-2}\equiv 0 \pmod p.$$

Thus,

$$T_{p+1}(b,c)=\sum_{k=0}^{(p+1)/2}\binom{p+1}{k}\binom{p+1-k}{k}b^{p+1-2k}c^k\equiv b^{p+1}+\binom{p+1}{1}\binom{p}{1}b^{p-1}c\equiv b^2 \pmod p.$$

If $p \mid b$, then (2.5) is valid since

$$T_{p-1}(b,c)=\sum_{k=0}^{(p-1)/2}\binom{p-1}{2k}\binom{2k}{k}b^{p-1-2k}c^k$$

$$\equiv \binom{p-1}{(p-1)/2} c^{(p-1)/2} \equiv \left(\frac{-c}{p}\right) = \left(\frac{b^2-4c}{p}\right) \pmod p.$$

When $p \nmid b$, we have

$$\begin{aligned} T_{p-1}(b,c) &\equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \frac{c^k}{b^{2k}} = \sum_{k=0}^{(p-1)/2} \binom{-1/2}{k} (-4)^k \frac{c^k}{b^{2k}} \\ &\equiv \sum_{k=0}^{(p-1)/2} \binom{(p-1)/2}{k} \left(-\frac{4c}{b^2}\right)^k = \left(1-\frac{4c}{b^2}\right)^{(p-1)/2} \equiv \left(\frac{b^2-4c}{p}\right) \pmod p. \end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.2(iii). Suppose that $d = b^2 - 4c \not\equiv 0 \pmod p$. By Lemma 2.2,

$$\sum_{k=0}^{p-1} \frac{T_k(b,c) M_k(b,c)}{d^k} \equiv \left(\frac{d}{p}\right) S_1 \pmod p,$$

where

$$\begin{aligned} S_1 &= \sum_{k=0}^{p-1} T_{p-1-k}(b,c) M_k(b,c) = [x^{p-1}] \sum_{j=0}^{\infty} T_j(b,c) x^j \sum_{k=0}^{\infty} M_k(b,c) x^k \\ &= [x^{p-1}] \frac{1}{\sqrt{1-2bx+dx^2}} \times \frac{1-bx-\sqrt{1-2bx+dx^2}}{2cx^2} \\ &= \frac{1}{2c}[x^{p+1}]\left(\frac{1-bx}{\sqrt{1-2bx+dx^2}} - 1\right) = \frac{T_{p+1}(b,c) - bT_p(b,c)}{2c}. \end{aligned}$$

In light of Lemma 2.5, $S_1 \equiv 0 \pmod p$ and hence (1.5) follows.

Now, suppose that $D = b^2 - 4c^2 \not\equiv 0 \pmod p$. In view of Lemma 2.2 and Fermat's little theorem,

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2) M_k(b,c^2)}{(b-2c)^{2k}} \equiv \left(\frac{D}{p}\right) \sum_{k=0}^{p-1} \frac{D^k T_{p-1-k}(b,c^2)}{(b-2c)^{2k}} M_k(b,c^2) \equiv \left(\frac{D}{p}\right) S_2 \pmod p,$$

where

$$\begin{aligned} S_2 &= \sum_{k=0}^{p-1} (b-2c)^{p-1-k} T_{p-1-k}(b,c^2) M_k(b,c^2) (b+2c)^k \\ &= [x^{p-1}] \sum_{j=0}^{\infty} T_j(b,c^2)((b-2c)x)^j \sum_{k=0}^{\infty} M_k(b,c^2)((b+2c)x)^k \\ &= [x^{p-1}] \frac{1-b(b+2c)x - \sqrt{1-2b(b+2c)x + D(b+2c)^2x^2}}{2c^2((b+2c)x)^2 \sqrt{1-2b(b-2c)x+D(b-2c)^2x^2}} \\ &= \frac{1}{2c^2(b+2c)^2}[x^{p+1}] \frac{1-b(b+2c)x}{\sqrt{1-2b(b-2c)x+D(b-2c)^2x^2}} \\ &\quad - \frac{1}{2c^2(b+2c)^2}[x^{p+1}] \frac{\sqrt{(1-Dx)(1-(b+2c)^2x)}}{\sqrt{(1-Dx)(1-(b-2c)^2x)}}. \end{aligned}$$

Recall the identity

$$(b^2+4c^2)^2 - 4(4b^2c^2) = D^2$$

and observe that

$$2c^2(b+2c)^2 S_2 = [y^{p+1}] \frac{(b-2c)^{p+1}}{\sqrt{1-2by+Dy^2}} - b(b+2c)[y^p] \frac{(b-2c)^p}{\sqrt{1-2by+Dy^2}}$$

$$
\begin{aligned}
&- [x^{p+1}]\frac{1-(b+2c)^2x}{\sqrt{1-2(b^2+4c^2)x+D^2x^2}}\\
&\equiv (b-2c)^2T_{p+1}(b,c^2) - b(b+2c)(b-2c)T_p(b,c^2)\\
&\quad - T_{p+1}(b^2+4c^2,4b^2c^2) + (b+2c)^2T_p(b^2+4c^2,4b^2c^2) \pmod p.
\end{aligned}
$$

Applying Lemma 2.5, we get

$$2c^2(b+2c)^2S_2 \equiv (b-2c)^2b^2 - b^2D - (b^2+4c^2)^2 + (b+2c)^2(b^2+4c^2) = 8bc^2(b+2c) \pmod p.$$

Thus, $S_2 \equiv 4b/(b+2c) \pmod p$ and this concludes the proof of (1.6). □

3 Proof of Theorem 1.5

Lemma 3.1. *Let b and c be integers. For all $n = 1, 2, 3, \dots$, we have*

$$2c\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} = -nT_n(b,c^2) + (b+2c)nT_{n-1}(b,c^2). \tag{3.1}$$

Proof. In the case $n = 1$ both sides of (3.1) coincide with $2c$. Denote by $f(n)$ the right-hand side of (3.1). Clearly, it suffices to show that for any positive integer n we have

$$
\begin{aligned}
&f(n+1) - (b-2c)f(n)\\
&\quad = 2c\sum_{k=0}^{n} T_k(b,c^2)(b-2c)^{n-k} - 2c\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-k} = 2cT_n(b,c^2).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&f(n+1) - (b-2c)f(n)\\
&\quad = -(n+1)T_{n+1}(b,c^2) + (b+2c)(n+1)T_n(b,c^2) - (b-2c)(-nT_n(b,c^2) + (b+2c)nT_{n-1}(b,c^2))\\
&\quad = -(n+1)T_{n+1}(b,c^2) + (4c^2-b^2)nT_{n-1}(b,c^2) + (n(b-2c) + (n+1)(b+2c))T_n(b,c^2)\\
&\quad = -(2n+1)bT_n(b,c^2) + (n(b-2c) + (n+1)(b+2c))T_n(b,c^2) = 2cT_n(b,c^2)
\end{aligned}
$$

with the help of the recursion for $T_n(b,c^2)$.

The above proof of (3.1) is simple. However, the reader might wonder how (3.1) was found. Set $D = b^2 - 4c^2$. Then

$$
\begin{aligned}
\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} &= [x^{n-1}]\frac{1}{\sqrt{1-2bx+Dx^2}}\cdot\frac{1}{1-(b-2c)x}\\
&= [x^{n-1}](1-(b-2c)x)^{-3/2}(1-(b+2c)x)^{-1/2}
\end{aligned}
$$

and hence

$$-2c\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} = [x^{n-1}]\frac{d}{dx}\sqrt{\frac{1-(b+2c)x}{1-(b-2c)x}}.$$

Observe that

$$
\begin{aligned}
\sqrt{\frac{1-(b+2c)x}{1-(b-2c)x}} &= \frac{1-(b+2c)x}{\sqrt{1-2bx+Dx^2}} = (1-(b+2c)x)\sum_{k=0}^{\infty} T_k(b,c^2)x^k\\
&= 1 + \sum_{k=1}^{\infty}(T_k(b,c^2) - (b+2c)T_{k-1}(b,c^2))x^k
\end{aligned}
$$

and thus,

$$[x^{n-1}]\frac{d}{dx}\sqrt{\frac{1-(b+2c)x}{1-(b-2c)x}} = n(T_n(b,c^2)-(b+2c)T_{n-1}(b,c^2)).$$

Therefore, (3.1) follows. □

Lemma 3.2. *Let $b\in\mathbb{Z}$, $c\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{Z}^+$. Then*

$$\frac{3}{n}\sum_{k=0}^{n-1}kT_k(b,c^2)(b-2c)^{n-1-k}-\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k} = \frac{(b+4c)T_n(b,c^2)-(b+2c)^2T_{n-1}(b,c^2)}{4c^2}. \tag{3.2}$$

Proof. Note that for any $k\in\mathbb{N}$ we have

$$T_k(2c,c^2)=[x^k](x^2+2cx+c^2)^k=[x^k](x+c)^{2k}=\binom{2k}{k}c^k.$$

In the case $b=2c$, we can easily verify that both sides of (3.2) coincide with

$$(2-3/n)\binom{2n-2}{n-1}c^{n-1}.$$

Below we assume $b\neq 2c$ and define

$$\sigma_n:=\sum_{k=0}^{n-1}(n-k)T_k(b,c^2)(b-2c)^{n-1-k}.$$

Clearly,

$$\sigma_n=[x^{n-1}]\bigg(\sum_{k=0}^{\infty}T_k(b,c^2)x^k\bigg)\sum_{l=0}^{\infty}(l+1)(b-2c)^lx^l.$$

For $|z|<1$, we have

$$\frac{1}{(1-z)^2}=\sum_{l=0}^{\infty}\binom{-2}{l}(-z)^l=\sum_{l=0}^{\infty}\binom{l+1}{l}z^l.$$

Thus,

$$\begin{aligned}\sigma_n&=[x^{n-1}]\frac{1}{\sqrt{1-2bx+(b^2-4c^2)x^2}}\times\frac{1}{(1-(b-2c)x)^2}\\&=[x^{n-1}](1-(b+2c)x)^{-1/2}(1-(b-2c)x)^{-5/2}=[x^{n-1}]\frac{d}{dx}f(x),\end{aligned}$$

where

$$\begin{aligned}f(x)&=\bigg(-\frac{b(b+2c)}{12c^2(b-2c)}+\frac{(b+2c)^2}{12c^2}x+\frac{2}{3(b-2c)(1-(b-2c)x)}\bigg)\frac{1}{\sqrt{1-2bx+(b^2-4c^2)x^2}}\\&=\bigg(-\frac{b(b+2c)}{12c^2(b-2c)}+\frac{(b+2c)^2}{12c^2}x+\frac{2}{3(b-2c)}\sum_{j=0}^{\infty}(b-2c)^jx^j\bigg)\sum_{k=0}^{\infty}T_k(b,c^2)x^k.\end{aligned}$$

Therefore,

$$\frac{\sigma_n}{n}=[x^n]f(x)=-\frac{b(b+2c)}{12c^2(b-2c)}T_n(b,c^2)+\frac{(b+2c)^2}{12c^2}T_{n-1}(b,c^2)+\frac{2}{3(b-2c)}\sum_{k=0}^{n}T_k(b,c^2)(b-2c)^{n-k},$$

i.e.,

$$\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}-\frac{1}{n}\sum_{k=0}^{n-1}kT_k(b,c^2)(b-2c)^{n-1-k}$$

$$= \frac{2}{3}\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} + \frac{2}{3}\cdot\frac{T_n(b,c^2)}{b-2c} + \frac{b+2c}{12c^2(b-2c)}((b^2-4c^2)T_{n-1}(b,c^2) - bT_n(b,c^2)).$$

This yields the desired (3.2). □

Proof of Theorem 1.5(i). Let n be any positive integer. Since

$$T_k(b,0) = [x^k]x^k(x+b)^k = b^k$$

for all $k \in \mathbb{N}$, (1.11) and (1.12) hold trivially when $c = 0$.

Now assume that $c \neq 0$. By Lemma 3.1, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} = \frac{bT_{n-1}(b,c^2) - T_n(b,c^2)}{2c} + T_{n-1}(b,c^2).$$

Observe that

$$\begin{aligned}
&T_n(b,c^2) - bT_{n-1}(b,c^2)\\
&\quad= \sum_{k\in\mathbb{N}}\binom{n}{2k}\binom{2k}{k}b^{n-2k}(c^2)^k - \sum_{k\in\mathbb{N}}\binom{n-1}{2k}\binom{2k}{k}b^{n-2k}(c^2)^k\\
&\quad= \sum_{k=1}^{n}\binom{n-1}{2k-1}\binom{2k}{k}b^{n-2k}c^{2k} = 2c\sum_{k=1}^{n}\binom{n-1}{2k-1}\binom{2k-1}{k-1}b^{n-2k}c^{2k-1}\\
&\quad= 2c\sum_{0<k\leqslant\lfloor n/2\rfloor}\binom{n-1}{k-1}\binom{n-k}{k}b^{n-2k}c^{2k-1} \equiv 0 \pmod{2c}.
\end{aligned}$$

Therefore, (1.11) holds. In light of Lemma 3.2, (1.12) is reduced to the congruence

$$(b+4c)T_n(b,c^2) \equiv (b+2c)^2T_{n-1}(b,c^2) \pmod{2c^2}.$$

In fact, as

$$\binom{2k}{k} = 2\binom{2k-1}{k-1}$$

for all $k \in \mathbb{Z}^+$, we have

$$\begin{aligned}
&(b+4c)T_n(b,c^2) - (b+2c)^2T_{n-1}(b,c^2)\\
&\quad= (b+4c)\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^{2k} - (b+2c)^2\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2k}\binom{2k}{k}b^{n-1-2k}c^{2k}\\
&\quad\equiv (b+4c)b^n - (b+2c)^2b^{n-1} \equiv 0 \pmod{2c^2}.
\end{aligned}$$

So (1.12) is valid.

Now, write $D = b^2-4c^2$ and suppose that p is an odd prime not dividing $b-2c$. In view of Lemmas 2.5 and 3.1 and Fermat's little theorem, we have

$$\frac{2c}{p}\sum_{k=0}^{p-1}\frac{T_k(b,c^2)}{(b-2c)^k} = \frac{(b+2c)T_{p-1}(b,c^2) - T_p(b,c^2)}{(b-2c)^{p-1}} \equiv (b+2c)\left(\frac{D}{p}\right) - b \pmod{p}.$$

This proves (1.13). If $p \mid c$, then

$$\left(\frac{D}{p}\right) = \left(\frac{b^2}{p}\right) = 1$$

and hence (1.14) becomes obvious. When $p \nmid c$, by (3.2), (1.11) and Lemma 2.5, we get

$$\frac{3}{p}\sum_{k=0}^{p-1}\frac{kT_k(b,c^2)}{(b-2c)^k} \equiv \frac{(b+4c)T_p(b,c^2)-(b+2c)^2T_{p-1}(b,c^2)}{4c^2} \equiv \frac{(b+4c)b-(b+2c)^2(\frac{D}{p})}{4c^2} \pmod p$$

and hence (1.14) follows. □

Lemma 3.3. *For $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$, we have*

$$\sum_{m=0}^{n-1}(2m+1)^2\binom{m+k}{2k} = (4n^2-1)\frac{n-k}{2k+3}\binom{n+k}{2k}. \tag{3.3}$$

Proof. Observe that

$$\begin{aligned}&(4n^2-1)\frac{n-k}{2k+3}\binom{n+k}{2k}+(2n+1)^2\binom{n+k}{2k}\\ &\quad=(4n^2+8n+3)\frac{n+1+k}{2k+3}\binom{n+k}{2k}=(4(n+1)^2-1)\frac{n+1-k}{2k+3}\binom{n+1+k}{2k}.\end{aligned}$$

So we can easily prove (3.3) by induction on n. □

Proof of Theorem 1.5(ii). We prove (1.15) by induction. (1.15) is obvious when $n=1$.

Now suppose the validity of (1.15) for a fixed $n \in \mathbb{Z}^+$. Observe that

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1+k}{k}\left(\frac{b-1}{2}\right)^k - n\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k\\ &\quad=\sum_{k=0}^{n}\left((n+1+k)\binom{n+1}{k+1}-n\binom{n}{k+1}\right)\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k\\ &\quad=(2n+1)\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k=(2n+1)D_n(m)=(2n+1)T_n(b,c).\end{aligned}$$

Therefore, by the induction hypothesis, we have

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1+k}{k}\left(\frac{b-1}{2}\right)^k\\ &\quad=\sum_{k=0}^{n-1}(2k+1)T_k(b,c)+(2n+1)T_n(b,c)=\sum_{k=0}^{n}(2k+1)T_k(b,c).\end{aligned}$$

This proves (1.15) with n replaced by $n+1$.

Let p be a prime not dividing $b-1=2m$. It is easy to see that

$$\binom{2p-1}{p-1}=\prod_{j=1}^{p-1}\left(1+\frac{p}{j}\right)\equiv 1+p\sum_{j=1}^{p-1}\frac{1}{j}=1+p\sum_{j=1}^{(p-1)/2}\left(\frac{1}{j}+\frac{1}{p-j}\right)\equiv 1 \pmod{p^2}.$$

In light of (1.15),

$$\begin{aligned}\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)T_k(b,c)&=\sum_{k=0}^{p-1}\binom{p}{k+1}\binom{p+k}{k}m^k\\ &=\binom{2p-1}{p-1}m^{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}\binom{p+k}{k}m^k\\ &\equiv m^{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}m^k=m^{p-1}+\frac{(m+1)^p-m^p-1}{m}\end{aligned}$$

$$\equiv 1+\frac{(m+1)^p-(m+1)}{m}=1+\frac{b+1}{b-1}\left(\left(\frac{b+1}{2}\right)^{p-1}-1\right)\pmod{p^2}$$

and hence (1.16) follows.

Now we show (1.17). In view of Lemma 3.3,

$$\begin{aligned}\sum_{n=0}^{p-1}(2n+1)^2T_n(b,c)&=\sum_{n=0}^{p-1}(2n+1)^2\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}m^k\\&=\sum_{k=0}^{p-1}\binom{2k}{k}m^k\sum_{n=0}^{p-1}(2n+1)^2\binom{n+k}{2k}\\&=(4p^2-1)\sum_{k=0}^{p-1}\frac{p-k}{2k+3}\binom{p+k}{2k}\binom{2k}{k}m^k\\&=(4p^2-1)\sum_{k=0}^{p-1}\frac{pm^k}{2k+3}\prod_{0<j\leqslant k}\left(\frac{p^2}{j^2}-1\right)\\&\equiv-\sum_{k=0}^{p-1}\frac{p(-m)^k}{2k+3}\pmod{p^2}\\&\equiv-(-m)^{(p-3)/2}\equiv\frac{1}{m}\left(\frac{-m}{p}\right)\pmod{p}.\end{aligned}$$

This proves (1.17). □

4 Proofs of Theorems 1.8 and 1.9

Proof of Theorem 1.8(i). We first prove (1.19) by induction.

When $n=1$, both sides of (1.19) are equal to b.

Now assume that (1.19) holds for a fixed integer $n\geqslant 1$. Then

$$\begin{aligned}&b\sum_{k=0}^{(n+1)-1}(2k+1)T_k(b,c)^2(-d)^{(n+1)-1-k}\\&\quad=b(2n+1)T_n(b,c)^2-bd\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k}\\&\quad=b(2n+1)T_n(b,c)^2-dnT_n(b,c)T_{n-1}(b,c)\\&\quad=(n+1)T_n(b,c)T_{n+1}(b,c).\end{aligned}$$

This concludes the induction step.

Now, we fix a positive integer n and want to show (1.18). As in the proof of Theorem 1.2(i),

$$T_n(b,c)\equiv\begin{cases}\binom{n}{n/2}c^{n/2}\pmod{b}, & \text{if } 2\mid n,\\ 0\pmod{b}, & \text{if } 2\nmid n.\end{cases}$$

When $b\neq 0$, b divides $T_n(b,c)$ or $T_{n-1}(b,c)$ since n or $n-1$ is odd, therefore (1.18) follows from (1.19).

Now it remains to consider the case $b=0$. Note that $T_k(0,c)=0$ for $k=1,3,5,\ldots$, and

$$T_k(0,c)=\binom{k}{k/2}c^{k/2}$$

for $k=0,2,4,\ldots$ Thus,

$$\sum_{k=0}^{n-1}(2k+1)T_k(0,c)^2(4c-0^2)^{n-1-k}=\sum_{k=0}^{\lfloor(n-1)/2\rfloor}(4k+1)\left(\binom{2k}{k}c^k\right)^2(4c)^{n-1-2k}$$

$$= (4c)^{n-1}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(4k+1)\frac{\binom{2k}{k}^2}{16^k}.$$

By induction, for any $m\in\mathbb{N}$, we have the identity

$$\sum_{k=0}^{m}(4k+1)\frac{\binom{2k}{k}^2}{16^k}=\frac{(m+1)^2}{16^m}\binom{2m+1}{m}^2=\frac{(2m+1)^2}{16^m}\binom{2m}{m}^2,$$

which was pointed out to the author by Tauraso. It follows that

$$4^{n-1}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}(4k+1)\frac{\binom{2k}{k}^2}{16^k}=n^2\binom{n-1}{\lfloor n/2\rfloor}^2.$$

Therefore,

$$\sum_{k=0}^{n-1}(2k+1)T_k(0,c)^2(4c-0^2)^{n-1-k}\equiv 0 \pmod{n^2}$$

and hence (1.18) holds when $b=0$. We are done. □

Lemma 4.1. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. For any $n\in\mathbb{N}$, we have*

$$T_n(b,c)^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}. \tag{4.1}$$

Proof. If $d=0$ (i.e., $b^2=4c$), then

$$T_n(b,c)=[x^n]\left(x^2+bx+\frac{b^2}{4}\right)^n=[x^n]\left(x+\frac{b}{2}\right)^{2n}=\binom{2n}{n}\frac{b^n}{2^n}$$

and hence (4.1) holds.

Now assume that $d\neq 0$. It is known that

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2x^k(x+1)^k=\left(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k\right)^2$$

(cf. [17, Lemma 3.2]), which is actually a special case of the famous Clausen identity (cf. [2, p. 116]) for hypergeometric series. Therefore,

$$\begin{aligned}T_n(b,c)^2&=\left((\sqrt{d})^nP_n\left(\frac{b}{\sqrt{d}}\right)\right)^2=d^nD_n\left(\frac{b/\sqrt{d}-1}{2}\right)^2\\&=d^n\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\left(\frac{b/\sqrt{d}-1}{2}\right)^k\left(\frac{b/\sqrt{d}+1}{2}\right)^k\\&=d^n\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\left(\frac{b^2/d-1}{4}\right)^k=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}.\end{aligned}$$

This completes the proof. □

Lemma 4.2. *For any $k\in\mathbb{N}$ and $n\in\mathbb{Z}^+$, we have*

$$\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}=\frac{n(n-k)}{k+1}\binom{n+k}{2k}. \tag{4.2}$$

Proof. (4.2) can be easily proved by induction on n. □

Proof of Theorem 1.8(ii). Let $n \in \mathbb{Z}^+$. In view of Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)T_m(b,c)^2d^{n-1-m} &= \sum_{m=0}^{n-1}(2m+1)d^{n-1-m}\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}^2c^kd^{m-k}\\ &= \sum_{k=0}^{n-1}\binom{2k}{k}^2c^kd^{n-1-k}\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}\\ &= \sum_{k=0}^{n-1}\binom{2k}{k}^2c^kd^{n-1-k}\frac{n(n-k)}{k+1}\binom{n+k}{2k}\\ &= n\sum_{k=0}^{n-1}(n-k)\binom{n}{k}\binom{n+k}{k}C_kc^kd^{n-1-k}\\ &= n^2\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_kc^kd^{n-1-k}.\end{aligned}$$

This proves (1.20).

Now assume $c \neq 0$ and let p be an odd prime not dividing d. By (1.20),

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} = \sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k}{k}C_k\frac{c^k}{d^k}.$$

For $k = 0, 1, \ldots, p-1$, clearly,

$$\begin{aligned}\binom{p-1}{k}\binom{p+k}{k} &= \prod_{0<j\leqslant k}\left(\frac{p-j}{j}\cdot\frac{p+j}{j}\right) = (-1)^k\prod_{0<j\leqslant k}\left(1-\frac{p^2}{j^2}\right)\\ &\equiv (-1)^k(1-p^2H_k^{(2)}) \pmod{p^4},\end{aligned}$$

where $H_k^{(2)} = \sum_{0<j\leqslant k} 1/j^2$. Thus,

$$\begin{aligned}\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} &\equiv \sum_{k=0}^{p-1}C_k\left(-\frac{c}{d}\right)^k(1-p^2H_k^{(2)}) \pmod{p^4}\\ &\equiv \sum_{k=0}^{p-1}C_k\left(-\frac{c}{d}\right)^k \pmod{p^2}.\end{aligned}$$

If $p \mid c$, then

$$\left(\frac{d}{p}\right) = \left(\frac{b^2}{p}\right) = 1$$

and hence (1.21) follows. In the case $p \nmid c$, we take an integer $m \equiv -d/c \pmod{p^2}$ and then get

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv \sum_{k=0}^{p-1}\frac{C_k}{m^k} \pmod{p^2}.$$

By [20, Lemma 2.1],

$$\sum_{k=1}^{p-1}\frac{C_k}{m^k} \equiv \frac{m-4}{2}\left(1-\left(\frac{m(m-4)}{p}\right)\right) \equiv -\frac{d+4c}{2c}\left(1-\left(\frac{d(d+4c)}{p}\right)\right) = \frac{b^2}{2c}\left(\left(\frac{d}{p}\right)-1\right) \pmod{p}.$$

(Moreover, Sun [18] determined $\sum_{k=1}^{p-1} C_k/m^k \bmod p^2$ in terms of Lucas sequences.) So (1.21) is valid. We are done. □

Remark 4.3. Let $p>3$ be a prime. As $D_k=T_k(3,2)$, by refining the proof of Theorem 1.8(ii) and using two auxiliary congruences

$$\sum_{k=1}^{p-1}(-2)^kC_k\equiv -4p\,q_p(2)\ (\mathrm{mod}\ p^3),\quad \sum_{k=1}^{p-1}(-2)^kC_kH_k^{(2)}\equiv 2q_p(2)^2\ (\mathrm{mod}\ p)$$

(the author has a proof of them), we get

$$\sum_{k=0}^{p-1}(2k+1)D_k^2\equiv p^2-4p^3q_p(2)-2p^4q_p(2)^2\ (\mathrm{mod}\ p^5).$$

Lemma 4.4. *Let $b,c\in\mathbb{Z}$. Suppose that $p>3$ is a prime not dividing $d=b^2-4c$. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k}\equiv\left(\frac{16c}{d}\right)^{(p-1)/2}+p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{\binom{2k}{k}}{2k+1}\left(-\frac{c}{d}\right)^k\ (\mathrm{mod}\ p^3). \tag{4.3}$$

Proof. With the help of (4.1), we have

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n(b,c)^2}{d^n}&=\sum_{n=0}^{p-1}\frac{1}{d^n}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{c^k}{d^k}\sum_{n=k}^{p-1}\binom{n+k}{2k}=\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{p+k}{2k+1}\left(\frac{c}{d}\right)^k\\&=\sum_{k=0}^{p-1}\frac{p}{2k+1}\binom{2k}{k}\bigg(\prod_{0<j\leqslant k}\frac{p^2-j^2}{j^2}\bigg)\left(\frac{c}{d}\right)^k\end{aligned}$$

and hence,

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n(b,c)^2}{d^n}&\equiv\sum_{k=0}^{p-1}\frac{p(-1)^k}{2k+1}\binom{2k}{k}(1-p^2H_k^{(2)})\left(\frac{c}{d}\right)^k\ (\mathrm{mod}\ p^4)\\&\equiv(-1)^{\frac{p-1}{2}}\binom{p-1}{(p-1)/2}(1-p^2H_{(p-1)/2}^{(2)})\left(\frac{c}{d}\right)^{(p-1)/2}\\&\quad+p\sum_{\substack{k=0\\ k\neq\frac{p-1}{2}}}^{p-1}\frac{\binom{2k}{k}}{2k+1}\left(-\frac{c}{d}\right)^k\ (\mathrm{mod}\ p^3).\end{aligned}$$

As Wolstenholme observed, $H_{p-1}^{(2)}\equiv 0\ (\mathrm{mod}\ p)$ since

$$\sum_{j=1}^{p-1}1/(2j)^2\equiv\sum_{k=1}^{p-1}1/k^2\ \ (\mathrm{mod}\ p).$$

Therefore,

$$H_{(p-1)/2}^{(2)}\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\frac{H_{p-1}^{(2)}}{2}\equiv 0\ (\mathrm{mod}\ p).$$

Recall Morley's congruence (cf. [12])

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\ (\mathrm{mod}\ p^3).$$

So we have

$$(-1)^{(p-1)/2}\binom{p-1}{(p-1)/2}\left(1-p^2H_{(p-1)/2}^{(2)}\right)\equiv 4^{p-1}\ (\mathrm{mod}\ p^3)$$

and hence (4.3) follows. □

Proof of Theorem 1.9. (i) Applying Lemma 4.4 with $b = 6$ and $c = -3$ we get

$$\sum_{k=0}^{p-1} \frac{T_k(6,-3)^2}{48^k} \equiv \left(\frac{-1}{p}\right) + p \sum_{\substack{k=0\\ k\neq (p-1)/2}}^{p-1} \frac{\binom{2k}{k}}{(2k+1)16^k} \pmod{p^3}.$$

By [19, (1.4) and (1.5)],

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv 0 \pmod{p^2}, \qquad \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{p}{3} E_{p-3} \pmod{p^2}.$$

So (1.22) follows.

(ii) Now we prove (1.23) and (1.24). Since $p \mid \binom{2k}{k}$ for every $k = (p+1)/2, \ldots, p-1$, by Lemma 4.4 with $b = 2$ and $c \in \{-1, -3\}$, we obtain

$$\sum_{k=0}^{p-1} \frac{T_k(2,-1)^2}{8^k} \equiv (-2)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \pmod{p^2}, \tag{4.4}$$

$$\sum_{k=0}^{p-1} \frac{T_k(2,-3)^2}{16^k} \equiv (-3)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)} \left(\frac{3}{16}\right)^k \pmod{p^2}. \tag{4.5}$$

For $n \in \mathbb{N}$, define

$$u_n = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{(-2)^k}{2k+1} \quad \text{and} \quad v_n = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{(-3)^k}{2k+1}.$$

Via the Zeilberger algorithm (cf. [14]) we find the recurrence relations

$$u_n + u_{n+2} = 0 \quad \text{and} \quad v_n + v_{n+1} + v_{n+2} = 0.$$

So, by induction we have

$$u_n = (-1)^{n(n-1)/2} = \left(\frac{-2}{2n+1}\right) \quad \text{and} \quad v_n = \left(\frac{2n+1}{3}\right),$$

for every $n = 0, 1, 2, \ldots$ Taking $n = (p-1)/2$ and noting that

$$\binom{n+k}{2k} \equiv \binom{2k}{k} / (-16)^k \pmod{p^2}$$

for $k = 0, \ldots, n$ (cf. [16, Lemma 2.2]), we then obtain

$$(-2)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \equiv u_{(p-1)/2} = \left(\frac{-2}{p}\right) \pmod{p^3},$$

$$(-3)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)} \left(\frac{3}{16}\right)^k \equiv v_{(p-1)/2} = \left(\frac{p}{3}\right) \pmod{p^3}.$$

Combining these with (4.4) and (4.5) we immediately get (1.23) and (1.24).

(iii) Finally we show (1.25). Applying (4.1) with $b = 3$ and $c = 2$ we obtain

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 2^k = D_n^2.$$

Therefore,

$$\sum_{n=1}^{p-1} \frac{D_n^2 - 1}{n^2} = \sum_{n=1}^{p-1} \frac{1}{n^2} \sum_{k=1}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 2^k$$

$$= \sum_{k=1}^{p-1} 2^k \binom{2k}{k}^2 \sum_{n=k}^{p-1} \frac{\binom{n+k}{2k}}{n^2} = \sum_{k=1}^{p-1} 2^k \binom{2k}{k}^2 \sum_{r=0}^{p-1-k} \frac{\binom{2k+r}{r}}{(k+r)^2}.$$

If $k \in \{(p+1)/2, \ldots, p-1\}$ then $p \mid \binom{2k}{k}$. For each $k = 1, \ldots, (p-1)/2$, clearly,

$$\sum_{r=0}^{p-1-k} \frac{\binom{2k+r}{r}}{(k+r)^2} = 4 \sum_{r=0}^{p-1-k} \frac{(-1)^r \binom{-2k-1}{r}}{(-2k-2r)^2} \equiv 4 \sum_{r=0}^{p-1-2k} \frac{(-1)^r \binom{p-1-2k}{r}}{(p-2k-2r)^2} \pmod p.$$

By [19, (2.5)], we have the identity

$$\sum_{r=0}^{2n} \frac{(-1)^r \binom{2n}{r}}{(2n+1-2r)^2} = \frac{(-16)^n}{(2n+1)^2 \binom{2n}{n}}.$$

Also,

$$H_{p-1}^{(2)} = \sum_{k=1}^{p-1} 1/k^2 \equiv 0 \pmod p.$$

So, by the above, we have

$$\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv \sum_{k=1}^{(p-1)/2} 2^k \binom{2k}{k}^2 \frac{4(-16)^{(p-1)/2-k}}{(p-2k)^2 \binom{p-1-2k}{(p-1)/2-k}} \equiv \sum_{k=1}^{(p-1)/2} \frac{2^k \binom{2k}{k}^2 4^{(p-1)/2-k}}{k^2 \binom{p-1-2k}{(p-1)/2-k} / (-4)^{(p-1)/2-k}} \pmod p.$$

For each $k \in \{1, \ldots, (p-1)/2\}$, obviously

$$\frac{\binom{2k}{k}}{(-4)^k} = \binom{-1/2}{k} \equiv \binom{(p-1)/2}{k} = \binom{(p-1)/2}{(p-1)/2-k} \equiv \binom{-1/2}{(p-1)/2-k} = \frac{\binom{p-1-2k}{(p-1)/2-k}}{(-4)^{(p-1)/2-k}} \pmod p.$$

Therefore,

$$\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv \sum_{k=1}^{(p-1)/2} \frac{2^k \binom{2k}{k}^2 2^{p-1}/4^k}{k^2 \binom{2k}{k}/(-4)^k} \equiv \sum_{k=1}^{(p-1)/2} \frac{(-2)^k \binom{2k}{k}}{k^2} \pmod p$$

and hence,

$$\sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv \sum_{k=1}^{p-1} \frac{(-2)^k}{k^2} \binom{2k}{k} \pmod p. \tag{4.6}$$

Let $v_n = 2^n + 2^{-n}$ and $w_n = (-1)^n + 2^{-n}$, for all $n \in \mathbb{N}$. It is easy to see that

$$v_{n+1} = \frac{5}{2} v_n - v_{n-1} \quad \text{and} \quad w_{n+1} = -\frac{1}{2} w_n + \frac{1}{2} w_{n-1} \quad \text{for all } n \in \mathbb{Z}^+.$$

Thus, applying [11, (42)] with $t = -1/2$, we obtain

$$\begin{aligned} -\frac{1}{4} \sum_{k=1}^{p-1} \frac{(-2)^k}{k^2} \binom{2k}{k} &\equiv \frac{v_p + 2w_p + 2^{-p} - 2}{p^2} + \sum_{k=1}^{p-1} \frac{v_k}{k^2} \\ &= 2^{-p} \left(\frac{2^p - 2}{p} \right)^2 + \sum_{k=1}^{p-1} \frac{2^k}{k^2} + \sum_{k=1}^{p-1} \frac{2^{-(p-k)}}{(p-k)^2} \\ &\equiv 2 q_p(2)^2 + \frac{3}{2} \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod p. \end{aligned}$$

Recall that

$$\sum_{k=1}^{p-1} 2^k / k^2 \equiv -q_p(2)^2 \pmod p$$

(which was conjectured by Skula and proved by Granville [10]). So we have

$$\sum_{k=1}^{p-1} \frac{(-2)^k}{k^2} \binom{2k}{k} \equiv -2 q_p(2)^2 \pmod p. \tag{4.7}$$

Combining (4.6) and (4.7), we finally get (1.25). This ends the proof. □

5 More conjectures for further research

Motivated by Part (ii) of Theorem 1.5, we raise the following conjecture.

Conjecture 5.1. *Let x be any integer. Then*

$$\sum_{k=0}^{n-1}(2k+1)D_k(x)^m \equiv 0 \pmod n$$

for all $m,n \in \mathbb{Z}^+$. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3 \equiv p\left(\frac{-4x-3}{p}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1}(2k+1)D_k(x)^4 \equiv p \pmod{p^2}.$$

Now, we propose the following conjecture related to Theorem 1.2(ii).

Conjecture 5.2. *Let $b,c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(8ck+4c+b)T_k(b,c^2)^2(b-2c)^{2(n-1-k)} \equiv 0 \pmod n.$$

If p is an odd prime not dividing $b(b-2c)$, then

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

Remark 5.3. Conjecture 5.2 in the case $b=c=1$ yields the first part of Conjecture 1.1.

By Theorem 1.2(ii), if p is an odd prime then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{2^{2k}} \equiv \sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{6^{2k}} \equiv \left(\frac{-1}{p}\right) \pmod p.$$

Motivated by this and (1.22)–(1.24), we now give a further conjecture.

Conjecture 5.4. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}\frac{T_k(2,2)^2}{4^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k} \pmod{p^{(5+(\frac{-1}{p}))/2}}.$$

If $p>3$, then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k} \equiv \sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{36^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

Now, we raise a conjecture related to Theorem 1.2(iii).

Conjecture 5.5. *Let $b,c \in \mathbb{Z}$ and $d=b^2-4c$. For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}T_k(b,c)M_k(b,c)d^{n-1-k} \equiv 0 \pmod n.$$

If p is an odd prime not dividing cd, then

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k} \equiv \frac{pb^2}{2c}\left(\left(\frac{d}{p}\right)-1\right) \pmod{p^2}.$$

By Conjecture 5.5, for any prime $p>3$ we should have

$$\sum_{k=0}^{p-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}\equiv\frac{3p}{2}\left(\left(\frac{p}{3}\right)-1\right)\pmod{p^2}.$$

This can be further strengthened.

Conjecture 5.6. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}\equiv\begin{cases}2p^2\pmod{p^3}, & \text{if } p\equiv 1\pmod 3,\\ p^3-p^2-3p\pmod{p^4}, & \text{if } p\equiv 2\pmod 3.\end{cases}$$

In view of Theorem 1.2(ii), for $b,c\in\mathbb{Z}$ and a prime $p\nmid(b-2c)$, it is natural to investigate whether the sum $\sum_{k=0}^{p-1}T_k(b,c^2)^3/(b-2c)^{3k}$ mod p has a pattern. This leads us to raise the following two conjectures.

Conjecture 5.7. *Let $p>3$ be a prime. Then*

$$\begin{aligned}\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{8^k}&\equiv\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{(-64)^k}\\&\equiv\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{(-64)^k}\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{512^k}\\&\equiv\begin{cases}4x^2-2p\pmod{p^2}, & \text{if } p\equiv 1,7\pmod{24}\ \text{and}\ p=x^2+6y^2,\\ 2p-8x^2\pmod{p^2}, & \text{if } p\equiv 5,11\pmod{24}\ \text{and}\ p=2x^2+3y^2,\\ 0\pmod{p^2}, & \text{if } (\frac{-6}{p})=-1.\end{cases}\end{aligned}$$

And

$$\sum_{k=0}^{p-1}(3k+2)\frac{T_k(2,3)^3}{8^k}\equiv p\left(3\left(\frac{3}{p}\right)-1\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(3k+1)\frac{T_k(2,3)^3}{(-64)^k}\equiv p\left(\frac{-2}{p}\right)\pmod{p^3}.$$

When $(\frac{-6}{p})=1$, we have

$$\sum_{k=0}^{p-1}(72k+47)\frac{T_k(2,9)^3}{(-64)^k}\equiv 42p\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(72k+25)\frac{T_k(2,9)^3}{512^k}\equiv 12p\left(\frac{3}{p}\right)\pmod{p^2}.$$

Also,

$$\sum_{k=0}^{n-1}(3k+2)T_k(2,3)^3 8^{n-1-k}\equiv 0\pmod{2n},$$
$$\sum_{k=0}^{n-1}(3k+1)T_k(2,3)^3(-64)^{n-1-k}\equiv 0\pmod{n},$$

for every positive integer n.

Remark 5.8. Let $p>3$ be a prime. If $p\equiv 1,7\pmod{24}$ then $p=x^2+6y^2$ for some $x,y\in\mathbb{Z}$; if $p\equiv 5,11\pmod{24}$ then $p=2x^2+3y^2$ for some $x,y\in\mathbb{Z}$. The reader may consult [4,7] for such known facts.

Conjecture 5.9. *Let $p > 3$ be a prime. Then*

$$\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{8^{3k}} \equiv \sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{16^{3k}}$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2}, & \text{if } p\equiv 1 \pmod 4, \quad p=x^2+y^2\ (2\nmid x,\ 2\mid y),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

And

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{(-8)^{3k}} \equiv \left(\frac{6}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{12^{3k}}$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2}, & \text{if } p\equiv 1,3 \pmod 8, \quad p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2}, & \text{if } (\frac{-2}{p})=-1,\ i.e.,\ p\equiv 5,7 \pmod 8.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(7k+4)\frac{T_k(10,49)^3}{(-8)^{3k}} \equiv \frac{p}{14}\left(\frac{2}{p}\right)\left(65-9\left(\frac{p}{3}\right)\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(7k+3)\frac{T_k(10,49)^3}{12^{3k}} \equiv \frac{3p}{28}\left(13+15\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

For each $n = 1, 2, 3, \ldots,$ we have

$$\sum_{k=0}^{n-1}(7k+4)T_k(10,49)^3(-8^3)^{n-1-k} \equiv 0 \pmod{4n},$$
$$\sum_{k=0}^{n-1}(7k+3)T_k(10,49)^3(12^3)^{n-1-k} \equiv 0 \pmod{n}.$$

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 11171140) and the PAPD of Jiangsu Higher Education Institutions. The initial version of this paper was posted to arXiv in August 2010 as a preprint with the ID arXiv:1008.3887.

References

1 Andrews G E. Euler's "exemplum memorabile inductionis fallacis" and q-trinomial coefficients. J Amer Math Soc, 1990, 3: 653–669

2 Andrews G E, Askey R, Roy R. Special Functions. Cambridge: Cambridge University Press, 1999

3 Andrews G E, Baxter R J. Lattice gas generalization of the hard hexagon model (III): q-trinomial coefficients. J Statist Phys, 1987, 47: 297–330

4 Berndt B C, Evans R J, Williams K S. Gauss and Jacobi Sums. New York: John Wiley & Sons, 1998

5 Cao H Q, Pan H. Some congruences for trinomial coefficients. ArXiv:1006.3025v2, 2010

6 Caughman J S, Haithcock C R, Veerman J J P. A note on lattice chains and Delannoy numbers. Discrete Math, 2008, 308: 2623–2628

7 Cox D A. Primes of the Form $x^2 + ny^2$. New York: John Wiley & Sons, 1989

8 Gould H W. Combinatorial Identities. New York: Morgantown Printing and Binding Co., 1972

9 Graham R L, Knuth D E, Patashnik O. Concrete Mathematics, 2nd ed. New York: Addison-Wesley, 1994

10 Granville A. The square of the Fermat quotient. Integers, 2004, 4: #A22, 3pp (electronic)

11 Mattarei S, Tauraso R. Congruences for central binomial sums and finite polylogarithms. J Number Theory, 2013, 133: 131–157

12 Morley F. Note on the congruence $2^{4n} \equiv (-1)^n(2n)!/(n!)^2$, where $2n + 1$ is a prime. Ann Math, 1895, 9: 168–170

13 Noe T D. On the divisibility of generalized central trinomial coefficients. J Integer Seq, 2006, 9: Article 06.2.7, 12pp

14 Petkovšek M, Wilf H S, Zeilberger D. $A = B$. Wellesley: A K Peters, 1996
15 Sloane N J A. Sequences A001006, A001850, A002426 in OEIS (On-Line Encyclopedia of Integer Sequences). http://oeis.org
16 Sun Z H. Congruences concerning Legendre polynomials. Proc Amer Math Soc, 2011, 139: 1915–1929
17 Sun Z H. Congruences concerning Legendre polynomials (II). ArXiv:1012.3898v2, 2012
18 Sun Z W. Binomial coefficients, Catalan numbers and Lucas quotients. Sci China Math, 2010, 53: 2473–2488
19 Sun Z W. On congruences related to central binomial coefficients. J Number Theory, 2011, 131: 2219–2238
20 Sun Z W. On Delannoy numbers and Schröder numbers. J Number Theory, 2011, 131: 2387–2397
21 Sun Z W, Tauraso R. New congruences for central binomial coefficients. Adv Appl Math, 2010, 45: 125–148
22 Wilf H S. Generatingfunctionology. New York: Academic Press, 1990