The Gross conjecture over rational function fields

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Abstract We study the Gross conjecture for the cyclotomic function field extension $k(\Lambda_f)/k$ where $k = F_q(t)$ is the rational function field and f is a monic polynomial in $F_q[t]$. We prove the conjecture in the Fermat curve case(i.e., when $f = t(t-1)$) by a direct calculation. We also prove the case when f is irreducible, which is analogous to the Weil reciprocity law. In the general case, we manage to show the weak version of the Gross conjecture here.

Keywords: Gross conjecture, L-function, Theta-function, Weil reciprocity.

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1 Overview of this paper

Let k be a global field and K/k be a finite abelian extension with the Galois group G. Let S be a finite nonempty set of places of k, which contains all archimedean places and places ramified in K. Let T be a finite nonempty set of places disjoint from S. Let $U_{S,T}$ be the set of all S-units of k which are congruent to $1 \pmod{\mathfrak{p}}$ for all places $\mathfrak{p}$ in T. The Dirichlet unit theorem asserts that the unit group $U_{S,T}$ is a finitely generated abelian group with rank $n = |S| - 1$. In the function field case, $U_{S,T}$ is furthermore free. By a careful choice of T(for example, T contains the places of different characteristics) in the number field case, one can also assume that $U_{S,T}$ is free. Let Y be the free abelian group generated by S and let X be the kernel of the degree map $Y \to \mathbb{Z}$. Then X is also a free abelian group with rank n.

On the one hand, consider the following function:

$$\theta_{S,T}(s) = \sum_{\chi \in \hat{G}} L_{S,T}(s,\chi) e_{\bar{\chi}}$$

where $L_{S,T}(s,\chi)$ is the modified Hecke L-function of χ. The Stickelberger element $\theta_{S,T} = \theta_{S,T}(0)$ is shown to be in $\mathbb{Z}[G]$ and is uniquely determined by the relations $\theta_{S,T}(\chi) = L_{S,T}(0,\bar{\chi})$ for all characters $\chi \in \hat{G}$. On the other hand, for I the augmentation ideal of $\mathbb{Z}[G]$, Gross[1] defined a homomorphism $\lambda : U_{S,T} \to G \otimes X \cong (I/I^2) \otimes X$ such that

$$u \mapsto \sum_{v \in S} (\mathrm{rec}(u_v) - 1) \cdot v$$

where rec is the Artin reciprocity map and u_v is the idéle with entries u at a place v and 1 at other places. Choose $\mathbb{Z}$-bases for $U_{S,T}$ and X, then λ is given by an $n \times n$-matrix with entries in I/I^2. The determinant of this matrix, denoted by $\det_G\lambda$ and called the regulator of λ, is an element in I^n/I^{n+1} which is independent of the choices of the bases. The Gross conjecture then claims that

$$\theta_{S,T} \equiv \pm h_{S,T} \cdot \det_G\lambda \pmod{I^{n+1}},$$
$$\text{in particular } \theta_{S,T} \in I^n (\text{weak implication}).$$

where $h_{S,T} = h_S \cdot \prod\limits_{\mathfrak{p}\in T} \frac{(\mathbb{N}\mathfrak{p}-1)}{[U_S:U_{S,T}]}$ and h_S is the class number of the ring of S-integers $\mathcal{O}_S$. For a detailed description of the Gross conjecture, we recommend the Gross's original paper[1] or the papers by Yamagishi[2], Aoki[3] and Tan[4].

In this paper, we shall consider the Gross conjecture in the case where the base field k is $F_q(t)$. As is known in the function field case, $\theta_{S,T} = \theta_{S,T}(0) = \Theta_{S,T}(1)$ through the change of variables $\theta_{S,T}(s) = \Theta_{S,T}(q^{-s})$. We list here some general results concerning the Gross conjecture.

Theorem 1. (1) The Gross conjecture $\mathrm{Gr}(K/k, G, S, T)$ is true if and only if $\mathrm{Gr}(K_p/k, G_p, S, T)$ is true for all p, the Sylow quotient group G_p of G and the corresponding field extension K_p/k.

(2) For a fixed group G and a field extension K/k, we can always suppose S that contains only archimedean places and places ramified in K/k. In the function field extension case, we can always assume that T contains only one place.

(3) If the Gross conjecture is true for G and K/k, it is true for any quotient group of G and the related subfield extension.

Theorem 2 (Tan). The Gross Conjecture is true for any abelian p-group G where p is the characteristic of the function field.

Detailed proof of Theorem 1 could be found in refs. [2,3]. Theorem 2 is the main theorem of Tan[4].

From now on in this paper, we assume $k = F_q(t)$. By choosing the place ∞ properly (e.g. choose $\infty = t$ or t^{-1}), the Carlitz-Hayes theory claims that any finite abelian extension of k is some subfield of the cyclotomic function field $k_n(\Lambda_f)$ where f is a polynomial in k and $k_n = F_{q^n}(t)$. As the Gross conjecture in the constant field extension k_n/k is trivially satisfied, by the above two theorems, to prove the conjecture in the case $k = F_q(t)$, it suffices to prove the following case:

$$K = k(\Lambda_f), \quad G = F_q[t]/(f)^\times,$$

f is square free, S consists of all irreducible factors of f and the infinite place ∞ and T consists of only one place v which is not in S.

Following Aoki's line[3] of proof of the Gross conjecture in the case $k = \mathbb{Q}$(which is somehow ambiguous in some parts in that paper), we prove the weak version of the Gross conjecture here for the cyclotomic function field extension $k(\Lambda_f)/k$. We first verify the conjecture when G is two copies of the multiplicative group $F_q^\times$ and the corresponding field extension is $K = k(\Lambda_{t(t-1)})$ in Section 2. We then prove the full Gross conjecture in the case when f itself is irreducible, which turns out to be nothing but the Weil reciprocity law. We then show the weak Gross conjecture in Section 3.

One must note that in a recent preprint[1)], Burns vastly generalized the numerous refined Stark conjectures including the Gross refinement and conjectured a family of explicit congruences between derivatives of abelian L-functions at $s = 0$. As a result, he gave a proof of the Gross conjecture for $k = \mathbb{Q}$ or the global function field case.

2 The Gross conjecture in some simpler cases

2.1 The Fermat curve case

Let $x = \sqrt[q-1]{t}$, $y = \sqrt[q-1]{1-t}$ and $K = k(x,y) = k(\Lambda_{t(t-1)})$. The Galois group $G = \mathrm{Gal}(K/k) \cong F_q^{\times 2}$, where the isomorphism is given by sending the automorphism $(x \mapsto cx, y \mapsto dy) \in G$ to this element $[c,d] \in F_q^{\times 2}$. We shall identify these two groups by the isomorphism hereafter in this subsection. We let $S = \{0, 1, \infty\}$ be the set of places ramified in K/k, and let $T = \{v\}$ where v is a monic irreducible polynomial prime to $t(t-1)$, let $\sigma = \sigma_v$ be the element in G corresponding to v through the identification of G and $(F_q[t]/(t(t-1)))^\times$. Let $d = \deg v$ and $N = \frac{q^d-1}{q-1}$. The unit group $U_{S,T}$ is a free abelian group of rank 2. Suppose $\{\varepsilon_1, \varepsilon_2\}$ is a basis for $U_{S,T}$. Then one can write

$$\varepsilon_1 = c_1 t^{m_1}(1-t)^{n_1},\ \varepsilon_2 = c_2 t^{m_2}(1-t)^{n_2},\ \text{for some } c_1, c_2 \in F_q^\times.$$

We denote $\Delta = m_1 n_2 - m_2 n_1$, $e = N/|\Delta|$. Then $|\Delta| = |m_1 n_2 - m_2 n_1|$ is nothing but $[U_S : U_{S,T}]$. Note that $h_S = 1$, hence

$$h_{S,T} = \frac{h_S \cdot (q^d - 1)}{(q-1)\cdot|\Delta|} = \frac{N}{|\Delta|}.$$

Now since

$$\frac{t^N}{(-1)^d v(0)},\ \frac{(1-t)^N}{v(1)} \in U_{S,T},$$

we can write

$$\frac{t^N}{(-1)^d v(0)} = \varepsilon_1^{\alpha_1}\varepsilon_2^{\beta_1},\ \frac{(1-t)^N}{v(1)} = \varepsilon_1^{\alpha_2}\varepsilon_2^{\beta_2}.$$

Then we have

$$\begin{aligned}(-1)^d v(0) &= c_1^{-\alpha_1} c_2^{-\beta_1} = c_1^{-en_2} c_2^{en_1};\\ v(1) &= c_1^{-\alpha_2} c_2^{-\beta_2} = c_1^{em_2} c_2^{-em_1}.\end{aligned}$$

Recall that we have (see, for example, refs. [5, 6])

$$\Theta_S(u) = 1 + \sum [c, 1-c]u + \frac{u^2 \sum [a,b]}{1-qu},$$

1) Burns, D., Congruences between derivatives of abelian L-functions at $s = 0$.

where $a, b, c \in F_q^\times$ and $c \neq 1$. Using repeatedly the fact $I/I^2 \cong G$, as a consequence $q-1$ kills I/I^2, we have

$$\begin{aligned}\Theta_{S,T}(1) &= (1+\sum[c,1-c])(1-q^d\sigma) + N\sum[a,b] \\ &\equiv (q-[-1,-1])(1-q^d\sigma) + (q-[-1,-1])\cdot N\cdot(q-[-1,1]) \\ &\equiv (q-[-1,-1])(1-q^d\sigma) + (q-[-1,-1])(q^d-1+d(1-[-1,1])) \\ &\equiv (q-[-1,-1])(1-\sigma+d(1-[-1,1])) \\ &\equiv (q-[-1,-1])(1-\sigma[(-1)^d,1]) \equiv (q-[-1,-1])(1-[v(0),v(1)]) \\ &\quad \text{(for the case } q \text{ odd)} \\ &\equiv (1-[-1,-1][v(0)^{q-1/2}, v(1)^{q-1/2}])(1-[v(0),v(1)]) \pmod{I^3}.\end{aligned}$$

By the class field theory, the reciprocity map is

$$r(a) = \left[\prod_v N_{F_q}^{F_v}\left(\frac{t^{\mathrm{ord_v(a_v)}}}{a_v^{\mathrm{ord_v(t)}}}(-1)^{\mathrm{ord_v(a_v)ord_v(t)}} \bmod v\right),\right.$$
$$\left.\prod_v N_{F_q}^{F_v}\left(\frac{(1-t)^{\mathrm{ord_v(a_v)}}}{a_v^{\mathrm{ord_v(1-t)}}}(-1)^{\mathrm{ord_v(a_v)ord_v(1-t)}} \bmod v\right)\right].$$

Then

$$\det_G(\lambda) = \begin{pmatrix} 1-[(-1)^{m_1}c_1, 1] & 1-[1,(-1)^{n_1}c_1] \\ 1-[(-1)^{m_2}c_2, 1] & 1-[1,(-1)^{n_2}c_2] \end{pmatrix}.$$

Let g be a generator of $F_q^\times$ and let $g_1 = [g,1]$, $g_2 = [1,g]$. Suppose $c_1 = g^k$ and $c_2 = g^l$.

When q is odd(here and after, $\equiv$ means mod I^3),

$$\begin{aligned}\det_G(\lambda) \equiv &\left[\left(k+m_1\frac{q-1}{2}\right)\left(l+n_2\frac{q-1}{2}\right) - \left(k+n_1\frac{q-1}{2}\right)\left(l+m_2\frac{q-1}{2}\right)\right] \\ &\cdot(1-g_1)(1-g_2) \\ \equiv &\left(k(n_2-m_2)+l(m_1-n_1)+\Delta\frac{q-1}{2}\right)\frac{q-1}{2}(1-g_1)(1-g_2).\end{aligned}$$

If e is even, so is $h_{S,T}$, therefore $h_{S,T}\det_G(\lambda) \in I^3$ (again since $q-1$ kills I/I^2). But in this case,

$$\Theta_{S,T}(1) \equiv (1-[-v(0)^{q-1/2}, -v(1)^{q-1/2}])(1-[v(0),v(1)]) \equiv 0$$

since $[-v(0)^{q-1/2}, -v(1)^{q-1/2}]$ is of order 2 and $v(0)$ and $v(1)$ are both contained in $F_q^{\times 2}$. Now if e is odd, then $h_{S,T}$ is also odd, then $h_{S,T}\det_G(\lambda) \equiv \det_G(\lambda) \pmod{I^3}$, and

$$\begin{aligned}\Theta_{S,T}(1) &\equiv \left(1-g_1^{(1+d\cdot\frac{q-1}{2}-en_2k+en_1l)\frac{q-1}{2}} + 1 - g_2^{(1+em_2k-em_1l)\frac{q-1}{2}}\right) \\ &\quad\cdot\left(1-g_1^{(d\cdot\frac{q-1}{2}-en_2k+en_1l)} + 1 - g_2^{(em_2k-em_1l)}\right) \\ &\equiv \left(k(n_2-m_2)+l(m_1-n_1)+d\cdot\frac{q-1}{2}\right)\cdot\frac{q-1}{2}(1-g_1)(1-g_2).\end{aligned}$$

But in this case, $d \equiv \Delta \pmod 2$, therefore the Gross conjecture holds.

Now for the case that q is even, it is easy to see both $\Theta_{S,T}(1)$ and $\det_G(\lambda)$ are in I^3, thus the conjecture is also true. In a word, we have

Theorem 3. The Gross conjecture is true for the Fermat curve case.

2.2 The case when f is irreducible

In this subsection, we suppose that $f = P$ is a monic irreducible polynomial in $A = F_q[t]$. Let $K = k(\Lambda_P)$ and let $G = \mathrm{Gal}(K/k) \cong (A/(P))^\times$. Let $S = \{P, \infty\}$ and let $T = \{Q\}$. We suppose that $\deg P = m$ and $\deg Q = d$, furthermore we let $M = \frac{q^m-1}{q-1}$ and $N = \frac{q^d-1}{q-1}$. For our convenience, we let $A_+ = \{f : f \in A, f \text{ monic}\}$. By Hayes' proof of the refined Stark conjecture in the function field case, one knows the Gross conjecture is true in this case. Here we give an alternate proof of the Gross conjecture by explicit calculation.

First we know, in the cyclotomic function field extension K/k, the Artin reciprocity map is given by

$$\sigma_a : \exp_C\left(\frac{\bar{\pi}}{P}\right) \mapsto \exp_C\left(\frac{\bar{\pi}a}{P}\right),$$

where $\exp_C(\bar{\pi}/P)$ is a generator of the extension K/k. Through this map, we identify the Galois group G with $(A/(P))^\times$, hence any group element in G is corresponding to a polynomial with degree less than m. Now let $P - \infty$ be a fixed generator of X and let $\varepsilon = cP^i$ be a fixed generator of $U_{S,T}$, then we have $h_{S,T} = N/i$ and the Gross conjecture states that

$$\Theta_{S,T}(1) \equiv h_{S,T}\det_G(\lambda) \pmod{I^2},$$

By the natural isomorphism between I/I^2 and G, we can regard $\Theta_{S,T}(1)$ as an element in G, or in $(A/(P))^\times$. Recall

$$\begin{aligned}
\Theta_{S,T}(1) &= \left(\sum_{a\in A_+,\deg\ a<m} \sigma_a + \frac{1}{1-q}\sum_{a\in(A/(P))^\times} \sigma_a\right)(1-q^d\sigma_Q) \\
&= \sum_{a\in A_+,\deg\ a<m} \sigma_a(1-q^d\sigma_Q) + N\sum_{a\in(A/(P))^\times} \sigma_a \\
&\equiv \left(\prod_{a\in A_+,\deg\ a<m} \sigma_a\right)^{1-q^d}\left(\prod_{a\in(A/(P))^\times} \sigma_a\right)^N \sigma_Q^{-q^dM} - 1 \pmod{I^2} \\
&\mapsto \left(\prod_{a\in A_+,\deg\ a<m} a\right)^{1-q^d}\left(\prod_{a\in(A/(P))^\times} a\right)^N Q^{-q^dM} \pmod{P}
\end{aligned}$$

Note that

$$\prod_{a\in(A/(P))^\times} a = -1$$

and

$$\prod_{a\in(A/(P))^\times} a = \left(\prod_{c\in F_q^\times} c\right)^M \left(\prod_{a\in A_+,\deg\ a<m} a\right)^{q-1} = (-1)^m \left(\prod_{a\in A_+,\deg\ a<m} a\right)^{q-1}.$$

We have

$$\Theta_{S,T}(1) \mapsto (-1)^{md} Q^{-q^d M} = (-1)^{md} Q^{-M} \pmod P.$$

Note that $Q^M \pmod P \in F_q^\times$ is nothing but the power residue symbol $\left(\frac{Q}{P}\right)$.

Now for the right-hand side, by the reciprocity map,

$$\mathrm{rec}(1, \cdots, (c)_P, \cdots, 1) = c,$$

and

$$\mathrm{rec}(1, \cdots, (P)_P, \cdots, 1) = 1,$$

therefore

$$h_{S,T}\det_G(\lambda) \mapsto c^{h_{S,T}} \in F_q^\times.$$

Since $P^i \equiv c^{-1} \pmod Q$, we have $c^{h_{S,T}} \equiv P^{-N} \pmod Q$. Then

$$c^{h_{S,T}} = \left(\frac{P}{Q}\right)^{-1} \in F_q^\times.$$

By the Weil reciprocity law,

$$(-1)^{md}\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right)^{-1} = 1,$$

hence we get

Theorem 4. The Gross conjecture is true for $k(\Lambda_f)/k$ where k is the rational function field and f is irreducible. In this case, the relation proposed in the conjecture is nothing but the Weil reciprocity law.

3 The weak Gross conjecture in the general case

The Gross conjecture is a conjecture about the relationship between two group ring elements, therefore it is necessary to study the group ring $\mathbb{Z}[G]$. We first have an elementary lemma.

Lemma 1. Suppose that σ and τ are two elements in G and suppose that the orders of σ and of τ are prime to each other, then $(\sigma-1)(\tau-1) \in I^\infty$.

Proof. First without loss of generality, we may assume that G is generated by σ and τ. Let

$$\mathrm{ord}(\sigma) = m,\ \mathrm{ord}(\tau) = d,$$

Then there exist $k, l \in \mathbb{Z}$ such that $km + ld = 1$. Hence

$$\begin{aligned}(\sigma-1)(\tau-1) &= (\sigma^{ld}-1)(\tau^{km}-1)\\ &= (1+\sigma^l+\cdots+\sigma^{l(d-1)})(1+\tau^k+\cdots+\tau^{k(m-1)})(\sigma^l-1)(\tau^k-1).\end{aligned}$$

If $(\sigma-1)(\tau-1)\in I^r$, then $(\sigma^l-1)(\tau^k-1)\in I^r$, but $(1+\sigma^l+\cdots+\sigma^{l(n-1)})(1+\tau^k+\cdots+\tau^{k(m-1)})$ is of degree md, which annihilates the module I^r/I^{r+1}, therefore $(\sigma-1)(\tau-1)\in I^{r+1}$.

Immediately following the above lemma, we have

Proposition 1. Suppose that G is abelian and G_p is its Sylow p-quotient for any prime p. Let π_p be the projection from G to G_p, then for any element x of $\mathbb{Z}[G]$, $x\in I_G^m$ if and only if $\pi_p(x)\in I_{G_p}^m$ for all primes p dividing $|G|$.

Proof. The "only if" part is obvious. For the "if" part, since $\pi_p(x)\in I_{G_p}^m$, then we can write

$$x=x_p+x_p',\ x_p\in I_G^m,\ x_p'\in\ker(\pi_p:\mathbb{Z}[G]\to\mathbb{Z}[G_p]).$$

By the above lemma, we can furthermore suppose that $x_p'\in G_p'=\prod_{p_i\neq p}G_{p_i}$ (here we consider G_{p_i} as a subgroup of G through the natural isomorphism of G and $\prod_p G_p$). First we see immediately that $x\in I_G$ by counting its degree. Suppose $x\in I_G^k$ for some $1\leqslant k<m$, then $x_p'\in I_G^k$. Note that $I_{G_p'}/I_{G_p'}^2$ is killed by $\prod_{p_i\neq p}p_i$. Then we have $(\prod_{p_i\neq p}p_i)x_p'\in I_G^{k+1}$ and therefore $(\prod_{p_i\neq p}p_i)x\in I_G^{k+1}$. But the greatest common divisor of $(\prod_{p_i\neq p}p_i)$ for all p dividing $|G|$ is 1, so we have $x\in I_G^{k+1}$. By induction, $x\in I_G^m$. The "if" part is proved.

Now we suppose that G is an abelian group and has a cyclic decomposition

$$G=G_1\times\cdots\times G_s=\langle\sigma_1\rangle\times\cdots\times\langle\sigma_s\rangle.$$

We regard G_i for every i as a subgroup of G. Let

$$\phi_i:G\to G/G_i$$

be the natural quotient map of G to G/G_i. We also denote, by ϕ_i, the corresponding map from $\mathbb{Z}[G]$ to $\mathbb{Z}[G/G_i]$. Then we have the following proposition:

Proposition 2 (Aoki). Assume the above assumptions. Suppose $\alpha\in\mathbb{Z}[G]$. If for all $i\in\{1,\cdots,s\}$, we have $\phi_i(\alpha)\in I_{G/G_i}^{m+1}$ for a fixed nonnegative integer m, then we have

(1) If $s>m$, then $\alpha\in I^{m+1}$;

(2) if $s=m$, then $\alpha\equiv a(\sigma_1-1)\cdots(\sigma_m-1)(\text{mod } I^{m+1})$ where a is a nonnegative integer less than the greatest common divisor of the orders of G_i's.

Remark. This proposition was first showed in ref. [3]. However, his proof is rather unclear and probably even incorrect, so we give a new proof here.

Proof. Let

$$E_r=\{(e_1,\cdots,e_s):e_1+e_2+\cdots e_s=r,e_i\geqslant 0\}$$

and

$$E=\{(e_1,\cdots,e_s):e_1+e_2+\cdots e_s\leqslant m,e_i\geqslant 0\}.$$

For any $e=(e_1,\cdots,e_s)\in E$, we define the support of e to be

$$\text{Supp}(e)=\{i:e_i>0\}.$$

For any element $\prod \sigma_i^{n_i} \in G$, we know that

$$\prod(\sigma_i^{n_i} - 1) \equiv \sum n_i(\sigma_i - 1) \pmod{I^2},$$

hence for any element $\alpha \in \mathbb{Z}[G]$, we can write

$$\begin{aligned}\alpha &\equiv \sum_{r=0}^{m} \sum_{e\in E_r} a_e(\sigma_1 - 1)^{e_1} \cdots (\sigma_s - 1)^{e_s} \\ &\equiv \sum_{e\in E} a_e(\sigma_1 - 1)^{e_1} \cdots (\sigma_s - 1)^{e_s} \pmod{I^{m+1}}.\end{aligned}$$

Now for any subset J of $\{1, \cdots, s\}$, we define

$$\phi_J : \mathbb{Z}[G] \to \mathbb{Z}[G/G_J]$$

where $G_J = \prod_{i\in J} G_i$. The assumptions in the proposition assert that $\phi_J(\alpha) \in I_{G/G_J}^{m+1}$. Define

$$\alpha_J = \sum_{\mathrm{Supp}(e)\cap J=\emptyset} (\sigma_1 - 1)^{e_1} \cdots (\sigma_s - 1)^{e_s}.$$

By the definition, $\alpha \equiv \alpha_\emptyset$. Now the given condition just means that for any nonempty proper subset J, $\alpha_J \in I^{m+1}$. If $s > m$, the support of any element $e \in E$ is a proper subset of $\{1, \cdots, s\}$; If $s = m$, only the support of $\{1, \cdots, 1\}$ is the whole set. Now by the inclusion-exclusion principle, we have

$$\alpha \equiv \alpha_\emptyset = \sum_J (-1)^{|J|-1} \alpha_J.$$

The first claim follows immediately, so does the first part of the second claim. The second part follows from the order consideration.

Theorem 5. The weak version of the Gross conjecture holds for $k(\Lambda_f)/k$, i.e. $\Theta_{S,T}(1) \in I^n$ for $n = |S| - 1$.

Proof. First note that we may assume that f is square free by Tan's theorem. We proceed by induction on the number of monic irreducible factors of f. By Theorem 4, if f has only one prime factor P, the full Gross conjecture is true by the Weil reciprocity law, hence the weak version forms. Now suppose that the weak version of the Gross conjecture is true for any f up to n monic irreducible factors.

For any f with n monic irreducible factors, write $f = P_1 \cdots P_n$ and $f_i = f/P_i$. Let $S = \{P_1, \cdots, P_n, \infty\}$. There is a canonical cyclic decomposition

$$G = G_f = \mathrm{Gal}(k(\Lambda_f)/k) = \prod_{i=1}^{n} (A/P_i)^\times.$$

By induction, the weak Gross conjecture holds for $k(\Lambda_{f_i})/k$, in this case the Galois group is $G_{f_i} = G/G_{P_i}$, hence

$$\Theta_{k(\Lambda_{f_i})/k, G_{f_i}, S, T}(1) \in I_{G_{f_i}}^n.$$

Apply Proposition 2 (1) to $m = n - 1 < s = n$, note that

$$\phi_i(\Theta_{S,T}(1)) = \Theta_{k(\Lambda_{f_i})/k, G_{f_i}, S, T}(1),$$

we thus have the desired result $\Theta_{S,T}(1) \in I_G^{m+1} = I^n$ and hence the theorem is proved.

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