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等跨等截面连续梁的解析计算公式

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摘　要　求解等跨等截面连续梁的变形和内力是土木工程领域的典型问题. 基于 Euler–Bernoulli 梁理论，利用位移法和辅助数列推导出任意跨数的等跨等截面连续梁梁端转动刚度的解析表达式，进而得到连续梁支点转角、弯矩在跨中集中荷载、满跨均布荷载、竖向温差作用下的通用计算公式. 研究表明：当跨数趋于无穷大时，等跨等截面连续梁的梁端转动刚度上限值为单跨梁抗弯线刚度的$2\sqrt{3}$倍. 不同跨数的等跨等截面连续梁可采用形式统一的解析公式计算支点转角和弯矩，不同静力荷载作用结果的区别仅由单跨梁的固端弯矩决定. 所得公式形式简洁、通用性强、应用方便，能揭示跨数对连续梁力学特性的影响规律，亦可用于分析顶推施工导梁参数优化等实际工程问题.

关键词　连续梁；转角；弯矩；解析公式；结构力学
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Analytical formulas for prismatic continuous beams of equal spans

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ABSTRACT　Solving the deformation and internal force of a prismatic multispan continuous beam of equal spans is a fundamental and classic problem in the area of civil engineering. Based on the Euler–Bernoulli beam theory, this paper presents unified analytical formulas to calculate the member-end rotation and bending moment of prismatic continuous beams of equal spans. First, simple closed-form expressions to determine the beam-end rotational stiffness of an equal-span prismatic continuous beam comprising any number of spans are derived using the displacement method in structural mechanics and the auxiliary series in mathematics. Furthermore, the rotational stiffness formulas are used to derive the analytical formulas for determining the joint rotation and bending moment at the supports of continuous beams subjected to various types of static loads and actions, such as a single point load applied at mid-span, distributed load applied over the span length, and differential temperature change between the top and bottom surfaces of the beam. Moreover, the implications of the proposed formulas on some interesting academic problems are thoroughly discussed. It is observed that as the number of spans goes infinity, the beam-end rotational stiffness of an equal-span prismatic continuous beam approaches the upper limit of $2\sqrt{3}\, i_0$, where i_0 denotes the linear stiffness, which is the product of the modulus of elasticity (E) and the moment of inertia (I) divided by the length (l_0) of the member of single-span beams. For equal-span prismatic continuous beams with various spans, the analytical formulas of the joint rotation and bending moment at the supports have unified expressions, while the difference between formulas for different static loads and actions is solely dependent on the fixed-end bending moment of single-span beams. This set of formulas reveals the advantages of concise form, general applicability, and convenient calculation. They can reveal the influence of the number of spans on the mechanical characteristics of continuous beams and analyze real-world engineering problems, such as

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optimization of the launching noses for incrementally launched bridges. Additionally, the proposed formulas in this paper can serve as an important reference for course teaching in the area of structural mechanics.

KEY WORDS　continuous beams; rotation; bending moment; analytical formula; structural mechanics

连续梁是工程中广泛应用的结构形式[1−3], 其静力计算方法包括以有限元法为代表的数值解法和以力法、位移法、力矩分配法等为代表的理论解法[4−7]. 然而, 这些方法大多适用于跨数固定的情况, 若要研究跨数对连续梁力学特性的影响, 则需要针对不同的跨数分别计算[8−9].

许多研究人员尝试得到具有通用性的连续梁计算公式. 李瀛沧[10] 基于力矩分配的概念推导了任意跨数连续梁的支点弯矩计算公式, 公式中的弯矩一次传递系数需要从一端的边跨算起, 逐次向另一边跨计算. Dowell 等[11−12] 同样基于力矩分配原理提出连续梁支点弯矩的计算公式, 但公式中的参数涉及连乘运算, 对跨数较多的连续梁计算较繁琐. Hosseini-Tabatabaei 等[13−14] 利用转角/弯矩传递的概念, 提出了连续梁及刚架的静力计算方法, 但所得公式仍需递推计算, 不能直接反映跨数变化对结果的影响.

顶推连续梁桥通常采用等跨等截面的形式, 施工中需求解不同跨数的连续梁[2, 15]. 为掌握顶推施工主梁的内力变化规律, 周季湘[16] 应用三弯矩方程建立等跨等截面连续梁支点弯矩的递推方程, 并求得 1 ~ 10 跨连续梁的支点弯矩系数. 董创文与李传习[17] 通过递推法求解由力法建立的三弯矩方程组, 推导出自重作用下任意跨数连续梁的支点弯矩计算公式. Rosignoli[18]、王卫锋等[19]、Ji 等[20]、Lee 与 Jang[21] 也通过结构力学原理建立的计算公式, 分析导梁参数对主梁内力的影响, 但这些公式未将主梁跨数作为变量. 此外, 连续梁的理论求解方法还有三次样条梁函数法[22]、直写法[23]、等直梁置换法[24]、分段独立一体化积分法[25]、分布传递函数法[26] 等. 这些方法各有特点和针对性, 但都未能给出适用于等跨等截面连续梁的通用解析表达式.

本文基于 Euler–Bernoulli 梁理论, 推导出不同静力荷载作用下, 任意跨数的等跨等截面连续梁支点转角、弯矩的解析计算公式. 这些公式不仅提供了计算多跨连续梁的简便方法, 而且揭示了跨数对连续梁力学特性的影响规律, 并可用于解决连续梁合理跨径布置[27]、顶推施工导梁参数优化[28] 等实际工程问题.

1　公式推导

1.1　梁端作用集中弯矩时的解

首先考虑n跨等跨等截面连续梁在左端受到弯矩M_0作用时的解(图 1(a)), 弯矩以主梁下缘受拉为正, 不考虑剪切变形. 各跨主梁的抗弯线刚度为$i_0 = E\cdot I/l_0$, 其中E为材料弹性模量, I为截面的抗弯惯性矩, l_0为单跨跨度. 根据位移法, 可将主梁在每个支座处的结点转角设为未知量, 记为z_k, 其中$k = 1, 2, \cdots, n+1$分别对应从左至右的$n+1$个支座, 支点转角以顺时针转动为正.

列出各个支点处的弯矩平衡方程. 在z_1处:

$$4i_0\cdot z_1 + 2i_0\cdot z_2 = M_0 \tag{1}$$

在z_k($k = 2, 3, \cdots, n$)处:

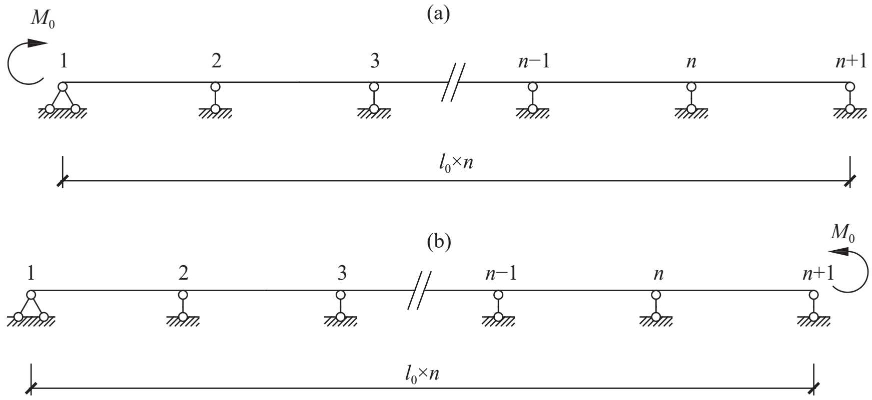


图 1　n 跨连续梁受到梁端弯矩作用的分析模型. (a) M_0 作用在最左端; (b) M_0 作用在最右端

Fig.1　Analytical model of an n-span continuous beam subjected to the beam-end moment: (a) M_0 applied at the leftmost end; (b) M_0 applied at the rightmost end

$$2i_0\cdot z_{k-1}+8i_0\cdot z_k+2i_0\cdot z_{k+1}=0 \tag{2}$$

在z_{n+1}处：

$$2i_0\cdot z_n+4i_0\cdot z_{n+1}=0 \tag{3}$$

由式 (3) 可得：

$$z_{n+1}=-\frac{1}{2}z_n \tag{4}$$

引入辅助数列$\{a_j\}$，令$a_1=1/2$，则$z_{n+1}=-a_1\cdot z_n$. 将此式代入式 (2)，并令k依次取$n, n-1, \cdots, 2$，经递推可得z_k与z_{k-1}之间的关系：

$$z_k=-a_{n+2-k}\cdot z_{k-1} \tag{5}$$

其中a_j($j=2, 3, \cdots, n$)满足：

$$a_j=\frac{1}{4-a_{j-1}} \tag{6}$$

为求出$\{a_j\}$的通项，将式 (6) 两边同时减去常数φ，即

$$a_j-\varphi=\frac{1}{4-a_{j-1}}-\varphi \tag{7}$$

对式 (7) 右边做恒等变形：

$$\begin{aligned}\frac{1}{4-a_{j-1}}-\varphi&=\frac{1-\varphi\left(4-a_{j-1}\right)}{4-a_{j-1}}=\frac{\varphi a_{j-1}+1-4\varphi}{4-a_{j-1}}\\&=\frac{\varphi\left(a_{j-1}-\varphi\right)+1-4\varphi+\varphi^2}{4-a_{j-1}}\end{aligned} \tag{8}$$

令分子中$1-4\varphi+\varphi^2=0$，可得$\varphi_1=2+\sqrt{3}$，$\varphi_2=2-\sqrt{3}$. 根据式 (7) 可得到如下两式：

$$a_j-\varphi_1=\frac{\varphi_1\left(a_{j-1}-\varphi_1\right)}{4-a_{j-1}} \tag{9}$$

$$a_j-\varphi_2=\frac{\varphi_2\left(a_{j-1}-\varphi_2\right)}{4-a_{j-1}} \tag{10}$$

以上两式相除可得：

$$\frac{a_j-\varphi_1}{a_j-\varphi_2}=\frac{\varphi_1}{\varphi_2}\cdot\frac{a_{j-1}-\varphi_1}{a_{j-1}-\varphi_2} \tag{11}$$

因此，数列$\{(a_j-\varphi_1)/(a_j-\varphi_2)\}$是以$\varphi_1/\varphi_2$为公比的等比数列，首项为$(a_1-\varphi_1)/(a_1-\varphi_2)$，故有：

$$\frac{a_j-\varphi_1}{a_j-\varphi_2}=\left(\frac{\varphi_1}{\varphi_2}\right)^{j-1}\cdot\frac{a_1-\varphi_1}{a_1-\varphi_2} \tag{12}$$

已知$a_1=1/2$，根据式 (12) 可求出a_j的表达式（$k=1, 2, \cdots, n$）：

$$a_j=\frac{\varphi_1^{j-1}+\varphi_2^{j-1}}{\varphi_1^{j}+\varphi_2^{j}} \tag{13}$$

将$z_2=-a_n\cdot z_1$(式 (5))代入式 (1) 可以解出z_1：

$$z_1=\frac{M_0}{2\sqrt{3}\cdot i_0}\cdot\frac{\varphi_1^n+\varphi_2^n}{\varphi_1^n-\varphi_2^n} \tag{14}$$

根据递推关系式 (5)，可得任意支点的转角计算公式：

$$z_k=(-1)^{k-1}\frac{M_0}{2\sqrt{3}\cdot i_0}\cdot\frac{\varphi_1^{n+1-k}+\varphi_2^{n+1-k}}{\varphi_1^n-\varphi_2^n} \tag{15}$$

式 (15) 适用于$k=1, 2, \cdots, n+1$. 需要注意，虽然式 (2) 中跨数$n\geqslant 2$，但式 (15) 适用于$n\geqslant 1$的情况.

根据镜像关系，不难得到n跨等截面连续梁在右端受到弯矩M_0作用时的支点转角（图 1(b)，$k=1, 2, \cdots, n+1$，n为正整数）：

$$z_k=(-1)^{n-k}\frac{M_0}{2\sqrt{3}\cdot i_0}\cdot\frac{\varphi_1^{k-1}+\varphi_2^{k-1}}{\varphi_1^n-\varphi_2^n} \tag{16}$$

1.2　转动刚度

将$k=1$，$z_k=1$代入式 (15)，可求得n跨等跨等截面连续梁在最左端发生单位转角时所需要施加的弯矩，即转动刚度S_n^{L}：

$$S_n^{\mathrm{L}}=2\sqrt{3}\cdot i_0\cdot\frac{\varphi_1^n-\varphi_2^n}{\varphi_1^n+\varphi_2^n} \tag{17}$$

同理，将$k=n+1$，$z_k=1$代入式 (15)，可得到对应于n跨等跨等截面连续梁最右端的转动刚度S_n^{R}. 因为支点转角以顺时针为正，而弯矩以截面下缘受拉为正，所以$S_n^{\mathrm{R}}=-S_n^{\mathrm{L}}$.

1.3　第 *u* 跨作用荷载时的解

考虑n跨等跨等截面连续梁在第u跨受到荷载作用时的解（图 2）.

根据位移法，可将支点u和$u+1$处的转角设为未知量x_1和x_2，由结点弯矩平衡可得：

$$\begin{cases}\left(4i_0-S_{u-1}^{\mathrm{R}}\right)x_1+2i_0\cdot x_2+M_1^{\mathrm{fix}}=0\\2i_0\cdot x_1+\left(4i_0+S_{n-u}^{\mathrm{L}}\right)\cdot x_2-M_2^{\mathrm{fix}}=0\end{cases} \tag{18}$$

其中，M_1^{fix}和M_2^{fix}分别是第u跨主梁在荷载作用下

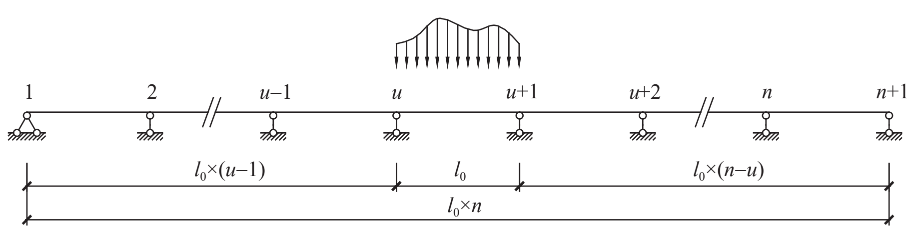


图 2　*n* 跨连续梁在第 *u* 跨受到荷载作用的分析模型

Fig.2　Analytical model of an *n*-span continuous beam with the *u*th span subjected to arbitrary loads

左、右支点的固端弯矩, 均以截面下缘受拉为正. S_{u-1}^{R}和S_{n-u}^{L}分别代表$u-1$跨连续梁最右端、$n-u$跨连续梁最左端的转动刚度, 可由第 1.2 节确定:

$$\begin{cases} S_{u-1}^{\mathrm{R}} = -2\sqrt{3}\cdot i_0\cdot\dfrac{\varphi_1^{u-1}-\varphi_2^{u-1}}{\varphi_1^{u-1}+\varphi_2^{u-1}} \\ S_{n-u}^{\mathrm{L}} = 2\sqrt{3}\cdot i_0\cdot\dfrac{\varphi_1^{n-u}-\varphi_2^{n-u}}{\varphi_1^{n-u}+\varphi_2^{n-u}} \end{cases} \tag{19}$$

将式 (19) 代入式 (18), 可得二元一次方程组:

$$\begin{cases} \dfrac{\varphi_1^{u}+\varphi_2^{u}}{\varphi_1^{u-1}+\varphi_2^{u-1}}\cdot x_1 + x_2 = -\dfrac{M_1^{\mathrm{fix}}}{2i_0} \\ x_1 + \dfrac{\varphi_1^{n-u+1}+\varphi_2^{n-u+1}}{\varphi_1^{n-u}+\varphi_2^{n-u}}\cdot x_2 = \dfrac{M_2^{\mathrm{fix}}}{2i_0} \end{cases} \tag{20}$$

由此解得:

$$\begin{cases} x_1 = -\dfrac{\varphi_1^{u-1}+\varphi_2^{u-1}}{4\sqrt{3}\cdot i_0\left(\varphi_1^{n}-\varphi_2^{n}\right)}\left[\left(\varphi_1^{n-u+1}+\varphi_2^{n-u+1}\right)M_1^{\mathrm{fix}}+\left(\varphi_1^{n-u}+\varphi_2^{n-u}\right)M_2^{\mathrm{fix}}\right] \\ x_2 = \dfrac{\varphi_1^{n-u}+\varphi_2^{n-u}}{4\sqrt{3}\cdot i_0\left(\varphi_1^{n}-\varphi_2^{n}\right)}\left[\left(\varphi_1^{u-1}+\varphi_2^{u-1}\right)M_1^{\mathrm{fix}}+\left(\varphi_1^{u}+\varphi_2^{u}\right)M_2^{\mathrm{fix}}\right] \end{cases} \tag{21}$$

根据转角之间的递推关系, 可得到所有支点的转角. 为简化表示, 引入数列$\{b_j^{(+)}\}$、$\{b_j^{(-)}\}$, 其中$b_j^{(+)}=\varphi_1^j+\varphi_2^j$, $b_j^{(-)}=\varphi_1^j-\varphi_2^j$, 则图 2(b) 中连续梁的支点转角可表示为:

$$z_k = \begin{cases} \dfrac{(-1)^{u-k+1}\cdot b_{k-1}^{(+)}}{4\sqrt{3}\cdot i_0\cdot b_n^{(-)}}\left[b_{n-u+1}^{(+)}\cdot M_1^{\mathrm{fix}}+b_{n-u}^{(+)}\cdot M_2^{\mathrm{fix}}\right] & 当k=1,2,\cdots,u \\ \dfrac{(-1)^{k-u-1}\cdot b_{n+1-k}^{(+)}}{4\sqrt{3}\cdot i_0\cdot b_n^{(-)}}\left[b_{u-1}^{(+)}\cdot M_1^{\mathrm{fix}}+b_{u}^{(+)}\cdot M_2^{\mathrm{fix}}\right] & 当k=u+1,u+2,\cdots,n+1 \end{cases} \tag{22}$$

式 (22) 给出了任意跨数连续梁支点转角的通用解析解(简支梁适用); 由于跨数n是变量, 因此能从理论上揭示跨数对连续梁力学特性的影响规律. 根据结构力学中的转角位移方程, 由式 (22) 还可进一步得到主梁各支点处的弯矩, 以及任意截面位置的位移和内力.

2　讨论

2.1　第 u 跨受常见荷载作用时的解

本节给出等跨等截面连续梁在第u跨受到跨中集中力F、均布荷载q、竖向温度差ΔT等常见荷载作用时的计算公式. 上述荷载对应的固端弯矩$M_1^{\mathrm{fix}}=M_2^{\mathrm{fix}}$, 因此式 (22) 中的支点转角$z_k$可简化为:

$$z_k = \begin{cases} \dfrac{(-1)^{u-k+1}\cdot b_{k-1}^{(+)}\cdot\left(\sqrt{3}\cdot b_{n-u}^{(+)}+b_{n-u}^{(-)}\right)}{4i_0\cdot b_n^{(-)}}\cdot M_1^{\mathrm{fix}} & 当k=1,2,\cdots,u \\ \dfrac{(-1)^{k-u+1}\cdot b_{n+1-k}^{(+)}\cdot\left(\sqrt{3}\cdot b_{u-1}^{(+)}+b_{u-1}^{(-)}\right)}{4i_0\cdot b_n^{(-)}}\cdot M_1^{\mathrm{fix}} & 当k=u+1,u+2,\cdots,n+1 \end{cases} \tag{23}$$

根据转角位移方程, 可得支点弯矩M_k(以截面下缘受拉为正)的通用解析解:

$$M_k = \begin{cases} \dfrac{(-1)^{u-k}\cdot b_{k-1}^{(-)}\cdot\left(3\cdot b_{n-u}^{(+)}+\sqrt{3}\cdot b_{n-u}^{(-)}\right)}{2\cdot b_n^{(-)}}\cdot M_1^{\mathrm{fix}} & 当k=1,2,\cdots,u \\ \dfrac{(-1)^{k-u+1}\cdot b_{n+1-k}^{(-)}\cdot\left(3\cdot b_{u-1}^{(+)}+\sqrt{3}\cdot b_{u-1}^{(-)}\right)}{2\cdot b_n^{(-)}}\cdot M_1^{\mathrm{fix}} & 当k=u+1,u+2,\cdots,n+1 \end{cases} \tag{24}$$

2.2　各跨受到相同荷载作用时的解

在第 2.1 节结果的基础上, 通过叠加原理可求得等跨等截面连续梁在各跨受到相同荷载作用时的解. 设连续梁有n跨, 由第u跨荷载引起的第k个支点的转角和弯矩分别记为z_{ku}和M_{ku}, 则考虑所有各跨荷载作用效应后的支点转角z'_k和支点弯矩M'_k分别为:

$$z'_k=\sum_{u=1}^{n}z_{ku}=\sum_{u=1}^{k-1}z_{ku}+\sum_{u=k}^{n}z_{ku}=\frac{(-1)^k\cdot b_{n+1-k}^{(+)}-(-1)^{n-k}\cdot b_{k-1}^{(+)}}{2\sqrt{3}\cdot i_0\cdot b_n^{(-)}}\cdot M_1^{\mathrm{fix}} \tag{25}$$

$$M'_k=\sum_{u=1}^{n}M_{ku}=\sum_{u=1}^{k-1}M_{ku}+\sum_{u=k}^{n}M_{ku}=\left[1+\frac{(-1)^k\cdot b_{n+1-k}^{(-)}+(-1)^{n-k}\cdot b_{k-1}^{(-)}}{b_n^{(-)}}\right]\cdot M_1^{\mathrm{fix}} \tag{26}$$

表 1 汇总了等跨等截面连续梁在常见荷载作用时的解（简支梁适用）.《建筑结构静力计算手册》[9] 给出了 1 ~ 5 跨连续梁的内力计算系数表，查表所得数据与表 1 中公式计算结果一致，读者可自行验证.

2.3　参数分析

对式 (23) ~ (26) 进行参数分析，可获得等跨等截面连续梁位移和内力随跨数n、支点号k以及荷载作用位置u的变化规律. 限于篇幅，这里只讨论式 (26) 中n和k对支点弯矩的影响.

记式 (26) 中M'_k与M_1^{fix}的比值为c_{odd}或c_{even}，下标表示跨数n为奇数或偶数的情况. 当n为奇数时，将$n=2\cdot m-1$（m取正整数）代入式 (26) 并化简（注意$\varphi_1\cdot\varphi_2=1$），可得：

$$c_{\mathrm{odd}}=\frac{M'_k}{M_1^{\mathrm{fix}}}=1+(-1)^k\cdot\frac{\left(\varphi_1^{m-k+1}-\varphi_2^{m-k}\right)\left(\varphi_1^{m-1}+\varphi_2^{m}\right)}{\left(\varphi_1^{m}-\varphi_2^{m-1}\right)\left(\varphi_1^{m-1}+\varphi_2^{m}\right)}=1+(-1)^k\cdot\frac{\varphi_1^{m-k+1}-\varphi_2^{m-k}}{\varphi_1^{m}-\varphi_2^{m-1}}\tag{27}$$

当n为偶数时，将$n=2\cdot m$（m取正整数）代入式 (26) 可得：

$$c_{\mathrm{even}}=\frac{M'_k}{M_1^{\mathrm{fix}}}=1+(-1)^k\cdot\frac{\left(\varphi_1^{m-k+1}+\varphi_2^{m-k+1}\right)\left(\varphi_1^{m}-\varphi_2^{m}\right)}{\left(\varphi_1^{m}+\varphi_2^{m}\right)\left(\varphi_1^{m}-\varphi_2^{m}\right)}=1+(-1)^k\cdot\frac{\varphi_1^{m-k+1}+\varphi_2^{m-k+1}}{\varphi_1^{m}+\varphi_2^{m}}\tag{28}$$

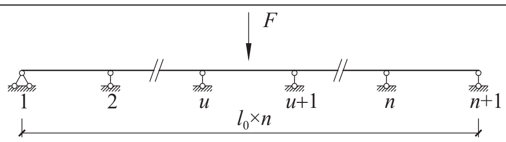
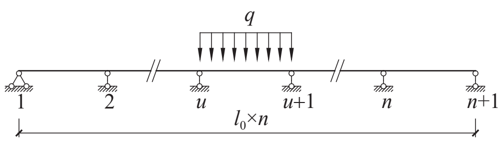
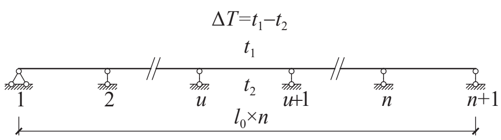
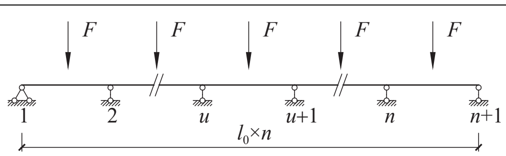
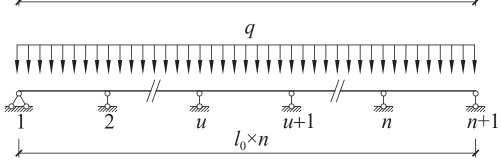
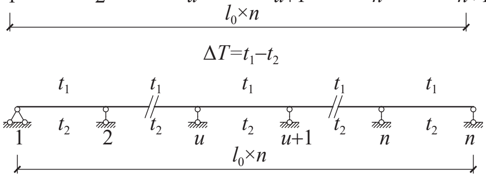
注意到$\varphi_1\approx3.732$，$\varphi_2\approx0.268$，因此当$m\geqslant3$时（即跨数$n\geqslant5$），式 (27) 和 (28) 中第 2 项的分母$\varphi_1^m-\varphi_2^{m-1}\approx\varphi_1^m$，$\varphi_1^m+\varphi_2^m\approx\varphi_1^m$. 此时，$c_{\mathrm{odd}}\approx1+(-1)^k(\varphi_2^{k-1}-\varphi_2^{n+1-k})$，$c_{\mathrm{even}}\approx1+(-1)^k(\varphi_2^{k-1}+\varphi_2^{n+1-k})$. 显然，当$k-1\geqslant3$且$n+1-k\geqslant3$时，$c_{\mathrm{odd}}\approx1$，$c_{\mathrm{even}}\approx1$. 由此可见，5 跨以上的等跨等截面连续梁在各跨受到相同作用的条件下，除最外侧的第 1、2 跨（或第$n-1$、n跨）外，其余中间各跨的内力十分接近，可以用 5 跨连续梁的第 3 跨代表. 结构计算手册中一般只给出 5 跨以内连续梁的计算结果[9].

2.4　转动刚度的性质

由第 1.2 节转动刚度的表达式可知，当跨数

表 1　等跨等截面梁的计算公式

Table 1　Formulas for a prismatic continuous beam of equal spans

Case No.	Analytical model	Rotation, z_k	Moment, M_k	Parameter, M_1^{fix}
1				$M_1^{\mathrm{fix}}=-Fl_0/8$
2		Eq. 23	Eq. 24	$M_1^{\mathrm{fix}}=-ql_0^2/12$
3				$M_1^{\mathrm{fix}}=EI\alpha\cdot\Delta T/h$
4				$M_1^{\mathrm{fix}}=-Fl_0/8$
5		Eq. 25	Eq. 26	$M_1^{\mathrm{fix}}=-ql_0^2/12$
6				$M_1^{\mathrm{fix}}=EI\alpha\cdot\Delta T/h$

Note: α and h are the linear expansion coefficient and the cross-sectional depth of the beam, respectively. The temperature changes in the top and bottom surfaces of the beam are denoted as t_1 and t_2, respectively.

n趋向于无穷大时, 等跨等截面连续梁最左端的转动刚度为:

$$S_{\infty}^{\mathrm{L}} = 2\sqrt{3} \cdot i_0 \approx 3.464 \cdot i_0 \tag{29}$$

其中, S_{∞}^{L}代表等跨等截面连续梁转动刚度的上限. 它比远端简支单跨梁的转动刚度$3 \cdot i_0$增大约 15%, 而比远端固定单跨梁的转动刚度$4 \cdot i_0$减小约 13%. 这一结论很难通过有限元分析等数值解法得到, 体现了理论解法的优越性.

另外, 虽然S_{∞}^{L}是主梁抗弯线刚度i_0的无理数倍($2\sqrt{3}$倍), 但是对于任意有限跨的连续梁, 式 (17) 总能写为$S_n^{\mathrm{L}} = \beta \cdot i_0$的形式, 其中$\beta$是有理数. 证明时可将式 (17) 重写为:

$$S_n^{\mathrm{L}} = i_0 \cdot \frac{(\varphi_1 - \varphi_2)\left(\varphi_1^n - \varphi_2^n\right)}{\varphi_1^n + \varphi_2^n} = i_0 \cdot \frac{\left(\varphi_1^{n+1} + \varphi_2^{n+1}\right) - \left(\varphi_1^{n-1} + \varphi_2^{n-1}\right)}{\varphi_1^n + \varphi_2^n} \tag{30}$$

对分母中的$\varphi_1^n + \varphi_2^n$应用二项展开:

$$\varphi_1^n + \varphi_2^n = \left(2 + \sqrt{3}\right)^n + \left(2 - \sqrt{3}\right)^n = \sum_{j=0}^{n} C_n^j \cdot 2^{n-j} \cdot \left(\sqrt{3}\right)^j + \sum_{j=0}^{n} C_n^j \cdot 2^{n-j} \cdot \left(-\sqrt{3}\right)^j \tag{31}$$

在式 (31) 中, 当j是奇数时, 与$(\sqrt{3})^j$和$(-\sqrt{3})^j$相关的项将相互抵消; 当j是偶数时, $(\sqrt{3})^j$和$(-\sqrt{3})^j$均等于$3^{j/2}$. 因此, $\varphi_1^n + \varphi_2^n$的结果是整数. 同理可知, 式 (30) 分子中的$\varphi_1^{n+1} + \varphi_2^{n+1}$和$\varphi_1^{n-1} + \varphi_2^{n-1}$也是整数, 所以$i_0$的系数$\beta$是两个整数之比, 即为有理数.

按类似方法可以证明, 对有限跨数的等跨等截面连续梁, 式 (22) ~ (26) 中固端弯矩前的系数均为有理数.

3　结论

本文通过解析计算公式揭示了跨数对连续梁力学特性的影响规律. 主要结论有:

(1) 不同跨数的等跨等截面连续梁可采用形式统一的解析公式计算支点转角和弯矩, 不同静力荷载作用结果的区别仅由单跨梁的固端弯矩决定. 表 1 给出了常见荷载作用时的解.

(2) 任意跨数的等跨等截面连续梁的梁端转动刚度的幅值可由式 (17) 计算; 当跨数趋于无穷大时, 其上限值为$2\sqrt{3} \cdot i_0$, 比简支梁仅增大约 15%.

(3) 超过 5 跨的等跨等截面连续梁在各跨受到相同荷载作用时, 距离梁端 2 跨以上的各跨内力近似相等, 可由 5 跨连续梁的第 3 跨代表.

后续研究将考虑剪切变形及非等跨对公式的修正, 以扩大公式的适用范围.

符 号 表

n: 连续梁的跨数;

l_0: 单跨跨度;

i_0: 抗弯线刚度;

z_k: 支座k处的结点转角, 以顺时针转动为正;

M_k: 支座k处的结点弯矩, 以截面下缘受拉为正;

$S_n^{\mathrm{L}}(S_n^{\mathrm{R}})$: n跨等跨等截面连续梁在左(右)端的转动刚度;

$M_1^{\mathrm{fix}}(M_2^{\mathrm{fix}})$: 单跨梁在左(右)端的固端弯矩;

$\varphi_1(\varphi_2)$: 常数, $\varphi_1 = 2 + \sqrt{3}$, $\varphi_2 = 2 - \sqrt{3}$;

$\{a_j\}$: 辅助数列, $a_j = (\varphi_1^{j-1} + \varphi_2^{j-1})/(\varphi_1^j + \varphi_2^j)$;

$\{b_j^{(+)}\}$: 辅助数列, $b_j^{(+)} = \varphi_1^j + \varphi_2^j$;

$\{b_j^{(-)}\}$: 辅助数列, $b_j^{(-)} = \varphi_1^j - \varphi_2^j$.

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