A matrix method for degree-raising of B-spline curves *

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Abstract A new identity is proved that represents the kth order B-splines as linear combinations of the $(k+1)$th order B-splines. A new method for degree-raising of B-spline curves is presented based on the identity. The new method can be used for all kinds of B-spline curves, that is, both uniform and arbitrarily nonuniform B-spline curves. When used for degree-raising of a segment of a uniform B-spline curve of degree $k-1$, it can help obtain a segment of curve of degree k that is still a uniform B-spline curve without raising the multiplicity of any knot. The method for degree-raising of Bezier curves can be regarded as the special case of the new method presented. Moreover, the conventional theory for degree-raising, whose shortcoming has been found, is discussed.

Keywords: B-spline curve, degree-raising, control polygon.

Nonuniform B-splines as well as uniform B-splines are widely used in computer-aided design (CAD), geometric modeling and numerical analysis and so on. Degree-raising of B-splines plays a very important role in CAD and geometric modeling. It is frequently used to link curves of different degrees together to form either surfaces or a composite curve. For instance, it can be used to find a common representation of two spline curves in skinning surfaces with cross-section curves of different degrees, patching two different spline surfaces, or transferring data between two incompatible CAD systems. Unlike Bézier curves, whose degree-raising is very easy[1], degree-raising of B-spline curves is difficult. All the algorithms for degree-raising of B-spline curves can be classified as two different classes: direct and indirect methods of degree raising.

In 1984, Prautzsch presented the first algorithm for degree-raising of B-splines[2]. One year later Cohen *et al*. proposed another method for the degree-raising[3,4], which is based on a recursive scheme. In 1991, Prautzsch and Piper improved the algorithm of the former[5] using its sophisticated organization. Piegl and Tiller used an approach for B-spline curves based on the degree-raising of Bézier curves in 1994[6]. Their algorithm consists of three steps: First, decompose the B-spline curve into piecewise Bézier curves by inserting knots. Then, degree-raise the Bézier curves. Finally, remove unnecessary knots, and transform the Bézier curves back to the B-spline one. Their method can be classified as an indirect one for degree-raising of B-splines, since Piegl and Tiller's method realized the degree-raising of B-spline curves by means of the degree elevation of Bezier curves. The other methods can be classified as a class of direct ones for degree-raising of B-spline curves. The indirect degree-raising algorithm for B-splines may be very efficient and competitive with direct algorithms for degree-raising of B-spline curves, but if rounding errorscould not be ignored, it would be difficult to transform back to the exact B-spline representation from the degree elevated Bézier representation. Thus only an approximative B-spline curve would be obtained.

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Note that, as shown in refs. [7] and footnotes 1 and 2, the methods of Prautzsch, Cohen *et al*. (see refs. [2—5]) will get incorrect results when used for degree-raising of the B-spline curves except endpoint-interpolating B-spline curves.

This paper discusses problems in degree-raising of B-spline curves. A matrix identity of B-splines is proved in the paper. Based on this identity, a new algorithm for degree-raising of B-spline curves is presented. The new degree-raising method can still get a segment of the uniform B-spline curve of degree k instead of a nonuniform B-spline one when used for degree-raising of a segment of uniform B-spline curve of degree $k-1$. It need insert fewer knots into uniform B-spline curves than the existing algorithms when used for degree-raising of cross-section curves for skinning surfaces[8] or patching two different spline surfaces.

1 B-spline basis functions

The normalized local support B-spline basis functions of degree $k-1$ are defined by the following deBoor-Cox recursive formula[9,10]

$$\begin{cases} B_{j,k}(t) = \dfrac{t-t_j}{t_{j+k-1}-t_j}B_{j,k-1}(t) + \dfrac{t_{j+k}-t}{t_{j+k}-t_{j+1}}B_{j+1,k-1}(t); \\ B_{i,1}(t) = \begin{cases} 1 & t \in [t_i, t_{i+1}), \\ 0 & t \notin [t_i, t_{i+1}), \end{cases} \end{cases} \tag{1}$$

with the convention that $0/0=0$, where t_j are the knots, $t_j \leqslant t_{j+1}$.

It is clear that B-spline basis functions can be represented as linear combinations of the B-splines of a lower degree. In practice, there is the reverse side of the coin; that is, B-spline basis functions can be represented as linear combinations of the B-splines of a higher degree.

Lemma 1. *Let* $N_{j,k}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}) \triangleq B_{j,k}(t)$. *Then*

$$N_{j,k}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}) = \frac{1}{k}\sum_{s=j}^{j+k} N_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{s-1}, t_s, t_s, t_{s+1}, \cdots, t_{j+k}). \tag{2}$$

Proof. Eq. (2) can be proved using the idea of multivariate polyhedral splines[2,11], B-spline recurrence relations[12], or a divided difference identity[13]. Q.E.D.

Theorem 1. *There is the following identity for B-spline functions of degree* k:

$$N_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{j+k+1}) = \frac{T-t_j}{t_{j+k}-t_j}N_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}, T)$$
$$+ \frac{t_{j+k+1}-T}{t_{j+k+1}-t_{j+1}}N_{j+1,k+1}(t \mid t_{j+1}, t_{j+2}, \cdots, t_{j+k+1}, T),$$

1) Qin, K., Two new algorithms for solutions to the problem in "Degree elevation of B-spline curves", to appear in *Journal of Tsinghua University*.

2) Qin, K., A note on the recursive degree raising algorithm for B-spline curves, to appear in *Journal of Tsinghua University*.

$$T \in [t_{j+1}, t_{j+k}]. \tag{3}$$

Proof. According to the definition of the kth divided differences, one can get

$$M_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{j+k+1})$$
$$= [M_{j,k}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}) - M_{j+1,k}(t \mid t_{j+1}, t_{j+2}, \cdots, t_{j+k+1})]/(t_j - t_{j+k+1}),$$
$$M_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}, T)$$
$$= [M_{j,k}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}) - M_{j+1,k}(t \mid t_{j+1}, t_{j+2}, \cdots, t_{j+k}, T)]/(t_j - T),$$
$$M_{j+1,k+1}(t \mid T, t_{j+1}, \cdots, t_{j+k+1})$$
$$= [M_{j+1,k}(t \mid T, t_{j+1}, t_{j+2}, \cdots, t_{j+k}) - M_{j+1,k}(t \mid t_{j+1}, \cdots, t_{j+k+1})]/(T - t_{j+k+1}),$$
$$M_{j+1,k}(t \mid T, t_{j+1}, t_{j+2}, \cdots, t_{j+k}) = M_{j+1,k}(t \mid t_{j+1}, t_{j+2}, \cdots, t_{j+k}, T).$$

Thus

$$(t_{j+k+1} - t_j)M_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{j+k+1}) = (T - t_j)M_{j,k+1}(t \mid t_j, t_{j+1}, \cdots, t_{j+k}, T)$$
$$+ (t_{j+k+1} - T)M_{j+1,k+1}(t \mid T, t_{j+1}, \cdots, t_{j+k+1}), \quad T \in [t_{j+1}, t_{j+k}]. \tag{4}$$

Using the relation between the normalized local support B-spline basis functions of degree $(k-1)$ and the divided differences, one can obtain eq. (3) from equation (4). Q.E.D.

Notice that T in eq. (3) can be equal to any value among t_{j+1} and t_{j+k}, that is, $T \in [t_{j+1}, t_{j+k}]$.

Theorem 2. *Nonuniform B-spline basis functions of degree $(k-1)$ can be represented as linear combinations of B-splines of degree k:*

$$\begin{bmatrix} B_{i-k+1,k}(t) \\ \vdots \\ B_{i-1,k}(t) \\ B_{i,k}(t) \end{bmatrix} = \frac{1}{k} \begin{bmatrix} a_{0,0}(i) & a_{0,1}(i) & \cdots & a_{0,k}(i) \\ a_{1,0}(i) & a_{1,1}(i) & \cdots & a_{1,k}(i) \\ \vdots & \vdots & \cdots & \vdots \\ a_{k-1,0}(i) & a_{k-1,1}(i) & \cdots & a_{k-1,k}(i) \end{bmatrix} \begin{bmatrix} B_{i-k,k+1}(t) \\ \vdots \\ B_{i-1,k+1}(t) \\ B_{i,k+1}(t) \end{bmatrix}, \tag{5}$$

where

$$a_{u,v}(i) = \begin{cases} (-1)^{u-v} \sum_{s=1}^{k-u} \dfrac{\alpha_{u,v}(i)}{1 - \gamma_{s-1}(i-k+u+1)}, & u \geqslant v; \\ (-1)^{v-u+1} \sum_{s=k-u-1}^{k-1} \dfrac{\beta_{u,v}(i)}{\gamma_{s+1}(i-k+u+1)}, & u+1 \leqslant v; \end{cases} \tag{6}$$

$$u = 0, 1, \cdots, k-1; \; v = 0, 1, \cdots, k.$$

$$\alpha_{u,v}(i) = \begin{cases} 1, & u = v; \\ \prod_{j=0}^{u-v-1} \dfrac{\gamma_{s+j}(i-k+u-j)}{1 - \gamma_{s+j}(i-k+u-j)}, & u > v; \end{cases}$$

$$\beta_{u,v}(i) = \begin{cases} 1, & u+1=v; \\ \prod\limits_{j=0}^{v-u-2} \dfrac{1-\gamma_{s-j}(i-k+u+j+2)}{\gamma_{s-j}(i-k+u+j+2)}, & u+1<v; \end{cases}$$

$$\gamma_s(j) = \frac{t_{j+s}-t_j}{t_{j+k}-t_j}, \quad s=0,1,\cdots,k.$$

Proof. Let $T = t_{j+1}, t_{j+2}, \cdots, t_{j+k}$ in eq. (3), respectively. One can obtain the following equations from Theorem 1.

$$A_{j,k+1}(t) = \left[\begin{array}{cccc|cccc} \gamma_1(j) & & & 0 & 1-\gamma_0(j+1) & & & 0 \\ & \gamma_2(j) & & & & 1-\gamma_1(j+1) & & \\ & & \ddots & & & & \ddots & \\ 0 & & & \gamma_k(j) & 0 & & & 1-\gamma_{k-1}(j+1) \end{array}\right]$$

$$\cdot \begin{bmatrix} N_{j,k+1}(t \mid t_j, t_{j+1}, t_{j+1}, t_{j+2}, \cdots, t_{j+k}) \\ N_{j,k+1}(t \mid t_j, t_{j+1}, t_{j+2}, t_{j+2}, \cdots, t_{j+k}) \\ \vdots \\ N_{j,k+1}(t \mid t_j, \cdots, t_{j+k-1}, t_{j+k}, t_{j+k}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ N_{j+1,k+1}(t \mid t_{j+1}, t_{j+1}, t_{j+2}, \cdots, t_{j+k+1}) \\ N_{j+1,k+1}(t \mid t_{j+1}, t_{j+2}, t_{j+2}, \cdots, t_{j+k+1}) \\ \vdots \\ N_{j+1,k+1}(t \mid t_{j+1}, \cdots, t_{j+k}, t_{j+k}, t_{j+k+1}) \end{bmatrix}, \quad j = i-k, i-k+1, \cdots, i, \tag{7}$$

where $A_{i,k+1}(t) = \begin{bmatrix} N_{j,k+1}(t \mid t_j, \cdots, t_{j+k+1}) \\ N_{j,k+1}(t \mid t_j, \cdots, t_{j+k+1}) \\ \vdots \\ N_{j,k+1}(t \mid t_j, \cdots, t_{j+k+1}) \end{bmatrix}$.

Using the local support property of B-splines, when $t \in [t_i, t_{i+1}]$, one can get the following identities:

$$\begin{aligned} N_{i+1,k+1}(t \mid t_{i+1}, t_{i+1}, t_{i+2}, \cdots, t_{i+k+1}) &= \cdots \\ &= N_{i+1,k+1}(t \mid t_{i+1}, t_{i+2}, \cdots, t_{i+k-1}, t_{i+k}, t_{i+k}, t_{i+k+1}) \equiv 0, \\ N_{i-k,k}(t \mid t_{i-k}, t_{i-k+1}, \cdots, t_i) &\equiv 0. \end{aligned}$$

From Lemma 1

$$N_{i-k,k}(t \mid t_{i-k}, t_{i-k+1}, \cdots, t_i) = \frac{1}{k}\sum_{s=i-k}^{i} N_{i-k,k+1}(t \mid t_{i-k}, t_{i-k+1}, \cdots, t_{s-1}, t_s, t_s, t_{s+1}, \cdots, t_i)$$

and the non-negative property of B-splines, one can obtain

$$\begin{aligned} N_{i-k,k+1}(t \mid t_{i-k}, t_{i-k}, t_{i-k+1}, t_{i-k+2}, \cdots, t_i) &= \cdots \\ &= N_{i-k,k+1}(t \mid t_{i-k}, t_{i-k+1}, \cdots, t_{i-1}, t_i, t_i) = 0, \quad t \in [t_i, t_{i+1}]. \end{aligned}$$

Thus, eq. (7) can be expanded to

$$
\begin{bmatrix}
\boldsymbol{A}_{i-k,k+1}(t) \\
\cdots\cdots \\
\boldsymbol{A}_{i-k+1,k+1}(t) \\
\cdots\cdots \\
\boldsymbol{A}_{i-k+2,k+1}(t) \\
\cdots\cdots \\
\vdots \\
\cdots\cdots \\
\boldsymbol{A}_{i-1,k+1}(t) \\
\cdots\cdots \\
\boldsymbol{A}_{i,k+1}(t)
\end{bmatrix}
= M
\begin{bmatrix}
N_{i-k+1,k+1}(t \mid t_{i-k+1}, t_{i-k+1}, t_{i-k+2}, \cdots, t_{i+1}) \\
N_{i-k+1,k+1}(t \mid t_{i-k+1}, t_{i-k+2}, t_{i-k+2}, \cdots, t_{i+1}) \\
\vdots \\
N_{i-k+1,k+1}(t \mid t_{i-k+1}, t_{i-k+2}, \cdots, t_i, t_{i+1}, t_{i+1}) \\
\cdots\cdots\cdots\cdots \\
\vdots \\
\cdots\cdots\cdots\cdots \\
N_{i-1,k+1}(t \mid t_{i-1}, t_{i-1}, t_i, t_{i+1}, \cdots, t_{i+k-1}) \\
N_{i-1,k+1}(t \mid t_{i-1}, t_i, t_i, t_{i+1}, \cdots, t_{i+k-1}) \\
\vdots \\
N_{i-1,k+1}(t \mid t_{i-1}, t_i, \cdots, t_{i+k-2}, t_{i+k-1}, t_{i+k-1}) \\
\cdots\cdots\cdots\cdots \\
N_{i,k+1}(t \mid t_i, t_i, t_{i+1}, t_{i+2}, \cdots, t_{i+k}) \\
N_{i,k+1}(t \mid t_i, t_{i+1}, t_{i+1}, t_{i+2}, \cdots, t_{i+k}) \\
\vdots \\
N_{i,k+1}(t \mid t_i, t_{i+1}, \cdots, t_{i+k-1}, t_{i+k}, t_{i+k})
\end{bmatrix}, \tag{8}
$$

where M is a $[k(k+1)] \times [k(k+1)]$ matrix, and

$$
M =
\begin{bmatrix}
\begin{matrix} 1-\gamma_0(i-k+1) & & & \\ & 1-\gamma_1(i-k+1) & & \\ & & \ddots & \\ & & & 1-\gamma_{k-1}(i-k+1) \end{matrix} & & & & \mathbf{0} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\begin{matrix} \gamma_1(i-k+1) & & \\ & \gamma_2(i-k+1) & \\ & \ddots & \\ & & \gamma_k(i-k+1) \end{matrix} & \begin{matrix} 1-\gamma_0(i-k+2) & & \\ & 1-\gamma_1(i-k+2) & \\ & \ddots & \\ & & 1-\gamma_{k-1}(i-k+2) \end{matrix} & & & \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
& \begin{matrix} \gamma_1(i-k+2) & & \\ & \gamma_2(i-k+2) & \\ & \ddots & \\ & & \gamma_k(i-k+2) \end{matrix} & \begin{matrix} 1-\gamma_0(i-k+3) & & \\ & 1-\gamma_1(i-k+3) & \\ & \ddots & \\ & & 1-\gamma_{k-1}(i-k+3) \end{matrix} & & \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
& & \ddots & \ddots & \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
& & & \begin{matrix} \gamma_1(i-1) & & \\ & \gamma_2(i-1) & \\ & \ddots & \\ & & \gamma_k(i-1) \end{matrix} & \begin{matrix} 1-\gamma_0(i) & & \\ & 1-\gamma_1(i) & \\ & \ddots & \\ & & 1-\gamma_{k-1}(i) \end{matrix} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\mathbf{0} & & & & \begin{matrix} \gamma_1(i) & & \\ & \gamma_2(i) & \\ & \ddots & \\ & & \gamma_k(i) \end{matrix}
\end{bmatrix}.
$$

The inverse matrix of M can easily be obtained as

$$M^{-1} = \begin{bmatrix} M_{0,0} & 0 & \cdots & 0 & 0 \\ M_{1,0} & M_{1,1} & \cdots & M_{1,k-1} & M_{1,k} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_{k-1,0} & M_{k-1,1} & \cdots & M_{k-1,k-1} & M_{k-1,k} \\ 0 & 0 & \cdots & 0 & M_{k,k} \end{bmatrix},$$

where $M_{i,j}(i, j=0, 1, \cdots, k)$ are block matrices:

$$M_{0,0} = \begin{bmatrix} \frac{1}{1-\gamma_0(i-k+1)} & & & 0 \\ & \frac{1}{1-\gamma_1(i-k+1)} & & \\ & & \ddots & \\ 0 & & & \frac{1}{1-\gamma_{k-1}(i-k+1)} \end{bmatrix},$$

$$M_{1,0} = \begin{bmatrix} 0 & & & & 0 \\ & \frac{-\gamma_1(i-k+1)}{[1-\gamma_1(i-k+1)][1-\gamma_0(i-k+2)]} & & & \\ & & \frac{-\gamma_2(i-k+1)}{[1-\gamma_2(i-k+1)][1-\gamma_1(i-k+2)]} & & \\ & & & \ddots & \\ 0 & & & & \frac{-\gamma_{k-1}(i-k+1)}{[1-\gamma_{k-1}(i-k+1)][1-\gamma_{k-2}(i-k+2)]} \end{bmatrix},$$

$$M_{1,1} = \begin{bmatrix} 1 & & & & \\ & \frac{1}{1-\gamma_0(i-k+2)} & & & 0 \\ & & \frac{1}{1-\gamma_1(i-k+2)} & & \\ & & & \ddots & \\ 0 & & & & \frac{1}{1-\gamma_{k-2}(i-k+2)} \end{bmatrix},$$

$\vdots$

$$M_{k,k} = \begin{bmatrix} \frac{1}{\gamma_1(i)} & & & 0 \\ & \frac{1}{\gamma_2(i)} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\gamma_k(i)} \end{bmatrix}.$$

Substituting eq. (8) multiplied by M^{-1} into eq. (2), one can get equation (5). Q.E.D.

2 Degree-raising of B-spline curves

Theorem 2 can be used for degree-raising of B-spline curves. Using identity (5), one can get a formula and an algorithm for degree-raising of B-spline curves easily.

Let a piecewise B-spline curve of degree $k-1$ be defined as follows:

$$\boldsymbol{C}^{i-k+1}(t) = [\boldsymbol{V}^i]^{\mathrm{T}}\boldsymbol{N}^i(t), \quad i = k-1, k, \cdots, n-k+1, \tag{9}$$

where $\boldsymbol{V}^i = [\boldsymbol{V}_{i-k+1}, \boldsymbol{V}_{i-k+2}, \cdots, \boldsymbol{V}_i]^{\mathrm{T}}$, $\boldsymbol{N}^i(t) = [B_{i-k+1,k}(t), B_{i-k+2,k}(t), \cdots, B_{i,k}(t)]^{\mathrm{T}}$, $\boldsymbol{V}_j(j = 0, 1, \cdots, n)$ are the control vertices. The knot vector is $\boldsymbol{U} = \{t_j\}_0^{n+k}$.

Substituting eq. (5) into eq. (9) yields

$$\boldsymbol{C}^{i-k+1}(t) = [\hat{\boldsymbol{V}}^i]^{\mathrm{T}}\hat{\boldsymbol{N}}^i(t),$$

where $\hat{\boldsymbol{N}}^i(t) = [B_{i-k,k+1}(t), B_{i-k+1,k+1}(t), \cdots, B_{i,k+1}(t)]^{\mathrm{T}}$,

$$[\hat{\boldsymbol{V}}^i]^{\mathrm{T}} = \frac{1}{k}[\boldsymbol{V}^i]^{\mathrm{T}}\begin{bmatrix} a_{0,0}(i) & a_{0,1}(i) & \cdots & a_{0,k}(i) \\ a_{1,0}(i) & a_{1,1}(i) & \cdots & a_{1,k}(i) \\ \vdots & \vdots & \cdots & \vdots \\ a_{k-1,0}(i) & a_{k-1,1}(i) & \cdots & a_{k-1,k}(i) \end{bmatrix}. \tag{10}$$

It is very easy to calculate $a_{u,v}(i)$ $(u=0,1,\cdots,k-1;\ v=0,1,\cdots,k)$ by eq. (6). $[\hat{\boldsymbol{V}}^i]^{\mathrm{T}}$ is the vector of the control vertices, which form the so-called control polygon, obtained by degree-raising. The new control polygon for a segment of a B-spline curve has one more control vertex than the old control polygon of the curve segment.

Equation (10) can be used for degree-raising of a segment of B-spline curves regardless of a segment of nonuniform or uniform curve. A new algorithm based on Theorem 2 and eq. (10) have been presented and used for degree-raising for all of nonuniform B-spline curves, which include uniform, endpoint-interpolating and other nonuniform B-spline curves[7].

3 Examples

Example 1. Degree-raising of a B-spline curve of order 2 defined by control vertices $\boldsymbol{V}_0$, $\boldsymbol{V}_1$, whose knot vector is defined by $\boldsymbol{U} = \{0,0,1,1\}$.

Using eq. (10), one can easily get the control vertices of the curve of order 3 as follows:

$$\begin{cases} \hat{\boldsymbol{V}}_{-1} = \boldsymbol{V}_0, \\ \hat{\boldsymbol{V}}_0 = \dfrac{1}{2}(\boldsymbol{V}_1 + \boldsymbol{V}_0), \\ \hat{\boldsymbol{V}}_1 = \boldsymbol{V}_1. \end{cases}$$

The refined knot vector is defined by $\widetilde{\boldsymbol{U}} = \{0,0,0,1,1,1\}$. This result is the same as that derived by the degree-raising of Bézier curves[1].

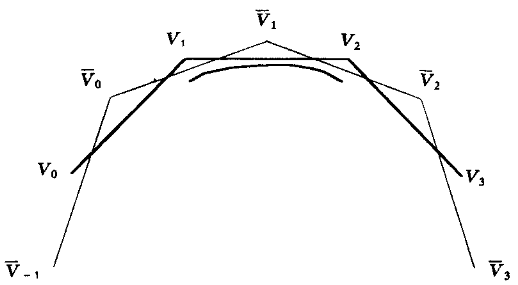

Fig. 1. Degree-raised polygon with original polygon and curve of order 4.

Example 2. Degree-raising of a uniform B-spline curve of order 4 (or degree 3) defined by control vertices $\boldsymbol{V}_0$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$. The knot vector is defined by $\boldsymbol{U} = \{1,2,3,4,5,6,7,8\}$.

Let the refined knot vector be

$$\widetilde{\boldsymbol{U}} = \{0,1,2,3,4,5,6,7,8,9\}.$$

According to eq. (10), one can obtain a curve of order 5, which is still a uniform B-spline curve as shown in figure 1.

4 Problem discussion

The conventional theory for degree-raising of B-spline curves[2—5] is well known in computer graphics and CAD. It includes[2—5]:

(i) If a B-spline curve needs degree-raising from degree $k-1$ to k, the multiplicities of all the knots must be elevated by one to form new refined knots for degree-raising.

(ii) The degree-raised control polygons will converge to the curve as the degree of the B-splines increases[3,14].

Unfortunately, we can find many counter examples like Example 2.

In general, if an algorithm works well for nonuniform B-splines, everybody may think that it must work well for uniform B-splines, too. But, when the conventional algorithms are used for degree-raising of uniform B-spline curves, they will fail to work although they work well for degree-raising of the nonuniform B-spline curves[2—5]. Why? What is wrong? Let us start discussing this problem from the conventional rationale of the degree-raising of B-splines.

A B-spline curve of degree $k-1$ is a piecewise polynomial curve defined as

$$\boldsymbol{C}(t) = \sum_{j=0}^{n-1} \boldsymbol{V}_j B_{j,k}(t),$$

where $\boldsymbol{V}_j$ are the control vertices, and $B_{j,k}(t)$ are the B-spline basis functions defined over a knot vector.

$$\boldsymbol{U} = \{t_0, t_1, \cdots, t_m\}, \quad t_j \leqslant t_{j+1}, \quad j = 0,1,\cdots, m-1, \quad m = n+k.$$

Assume that knot vector $\boldsymbol{U}$ has the following form:

$$\boldsymbol{U} = \left\{ \underbrace{u_1, \cdots, u_1}_{m_1} \quad \cdots, \quad \underbrace{u_s, \cdots, u_s}_{m_s} \right\},$$

where $\sum_{j=1}^{s} m_j = m+1$, and m_1, m_2, $\cdots$, and m_s denote the multiplicities of the knots. According to the conventional theory of the degree-raising of B-splines[2—5], the new knot vector for degree-raising must take the following form (see eq. (2.4) on p. 172 of ref. [4])

$$\hat{U} = \left\{ \underbrace{u_1, \cdots, u_1,}_{m_1 + 1} \quad \cdots, \quad \underbrace{u_s, \cdots, u_s}_{m_s + 1} \right\}$$

$$= \{\hat{t}_0, \hat{t}_1, \cdots, \hat{t}_{\hat{m}}\} \quad \hat{t}_j \leqslant \hat{t}_{j+1}, \quad j = 0, 1, \cdots, \hat{m} - 1,$$

where $\hat{m} = m + s$.

In practice, there are other alternative knot vectors for degree-raising unless the B-splines of degree $(k-1)$ are defined over a kind of the knot vector

$$\overline{U} = \left\{ \underbrace{a, \cdots, a,}_{k} \quad \underbrace{u_1, \cdots, u_1,}_{m_1} \quad \cdots, \quad \underbrace{u_s, \cdots, u_s}_{m_s} \quad \underbrace{b, \cdots, b}_{k} \right\}$$

such that the higher the degree of the B-splines is raised, the more the construction of the curve resembles that of Bezier curves. Clearly, if $m_1 \neq k$ and $m_s \neq k$ in knot vector $\boldsymbol{U}$, then knot vector $\boldsymbol{U}$ is one of the different forms of knot vectors from knot vector $\overline{\boldsymbol{U}}$. The following example can be used to explain this problem clearly.

Assume that a B-spline curve of degree 3, whose control vertices and knot vector are $\boldsymbol{V}_0$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$, $\boldsymbol{V}_3$ and $\{1,2,3,4,5,6,7,8\}$, needs degree-raising from degree 3 to 4.

First, if the degree of the curve is raised by Prautzsch's algorithms[2—5] from degree 3 to 4, then according to the conventional theory of degree-raising of B-spline curves, the new knot vector must be defined as

$$\{1,1,2,2,3,3,4,4,5,5,6,6,7,7,8,8\}.$$

Unfortunately, Boehm's algorithm[15] for inserting knots will fail to insert the knots 1, 2, 3, 4, 6, 7, 8 except the knot 5. Only the knot $\hat{t} \in (4,5]$ can be inserted into this curve by the algorithms for inserting new knots into B-spline curves. The Olso algorithms[16] will also fail to insert the boundary knots into the curve. Thus, Prautzsch's algorithms for degree-raising of B-spline curves cannot work unless the algorithms are improved. Neither can the recursive algorithm[3,4] for degree-raising of B-splines by Cohen *et al*. work without improving their algorithm for analogy reason.

Second, from a point of view of linear spaces, B-spline basis functions $B_{j,5}(t)$ $(4 = t_4 \leqslant t \leqslant t_5 = 5;\ j = 0, 1, \cdots, 5)$ defined over a knot vector $\boldsymbol{U}' = \{t_0, t_1, \cdots, t_9\}$, which are linearly independent, can be used for a set of bases of the linear space. On the other hand, the following polynomials

$$f_j(t) = t^j, \qquad j = 0, 1, \cdots, 4$$

can also be used for a set of bases of the space. According to the following equations[17]

$$x^{j-1} = \sum_{i=l+1-k}^{r} \xi_i(j) B_{i,k}(x), \qquad j = 1, 3, \cdots, k; \qquad t_l \leqslant x < t_{r+1},$$

where

$$\xi_i(j) = (-1)^{j-1}\frac{(j-1)!}{(k-1)!}D^{k-j}\varphi_{i,k}(0), \qquad i = l+1-k, \cdots, r,$$

$$\varphi_{i,k}(y) = \prod_{v=1}^{k-1}(y - t_{i+v}), \qquad y \in \mathbb{R},$$

$$l \leqslant r, \qquad t_l < t_{r+1},$$

it is clear that one of the two sets of bases can be represented as linear combinations of the basis functions of another one. Notice that $\boldsymbol{U}'$ can be either the form of knot vector $\hat{\boldsymbol{U}}$ or $\overline{\boldsymbol{U}}$, or other forms of knot vectors. In other words, the B-spline basis functions of a higher degree obtained by degree-raising can be those defined over different forms of knot vectors.

Third, as the degree of the splines increases, the control polygons associated with the given curve of degree 3 are not sure to converge to the curve. The counter situation may occur; that is, the higher the degree of the splines, the farther away the control polygons are from the curve. Such a couple of examples can be found in sec. 3 of this paper.

Finally, only a specific fraction of nonuniform B-splines are the splines defined over such a knot vector $\overline{\boldsymbol{U}}$. Thus, a method or theory for such a fraction of nonuniform B-spline curves cannot be used for all nonuniform B-spline curves and uniform B-spline curves without any limits to the method or theory.

From the above discussions, a main conclusion can be obtained as follows.

The conventional theory for degree-raising of B-splines is not always correct. It has to be revised.

5 Conclusions

In this paper, an efficient method, which makes a breakthrough at the conventional theory of degree-raising of B-spline curves, is presented for degree-raising of B-splines. It not only can raise the degree of any nonuniform B-spline curves, but also can get a segment of uniform B-spline curve of a higher degree after the degree of a given segment of uniform B-spline curve is raised.

In generation of skinning surfaces, cross-section curves are usually composed of conics and sculptured curves, which can be represented by NURBS[18]. It is necessary that the degrees of some of the cross-section curves be raised so that all the curves with different degrees have the same degree. The new method presented for degree-raising for uniform and other nonuniform B-spline curves except the endpoint-interpolating curves has more advantages than the conventional methods for degree-raising. It needs fewer knots to be inserted into the original curve, and generates fewer control vertices in the degree elevation process than the conventional algorithms, so that the computation of such surface modeling can significantly be reduced. For skinning surfaces with cross-section curves of uniform B-splines, the more the cross-section curves, the more the reduction for computation of the degree-raised control polygons, because the new algorithm needs fewer knots inserted into the curve than the conventional algorithms.

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