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Boundary blow-up analysis for approximate Dirac-harmonic maps into stationary Lorentzian manifolds

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Abstract For a sequence of approximate Dirac-harmonic maps from a Riemannian surface with a smooth boundary into a stationary Lorentzian manifold, we study the boundary blow-up analysis and prove the positive energy identity for spinors and the Lorentzian energy identity for maps. Moreover, the positive energy identity for maps holds when the target is a static Lorentzian manifold.

Keywords Dirac-harmonic maps, boundary regularity, blow-up, Lorentzian manifolds

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1 Introduction

Dirac-harmonic maps into Riemannian manifolds are a mathematical version of the nonlinear supersymmetric sigma model of quantum field theory [6, 9, 18]. The fields in this model correspond to spinors representing fermionic particles, which take values in Grassmann algebra and are anti-commuting. To overcome this anti-commuting technical problem for applying analysis tools, Chen et al. [6] proposed making all the fields real-valued. Although this does result in the loss of supersymmetry, it still retains a rich mathematical structure, especially the conformal invariance in two-dimensional domains. It is generally expected that Dirac-harmonic maps, as rooted in a deep structure in quantum field theory, have profound geometric applications.

In recent decades, the regularity, compactness, and existence of Dirac-harmonic maps into Riemannian manifolds have been extensively studied. From the viewpoint of general relativity and quantum field theory, it is natural to consider Dirac-harmonic maps into more general targets, particularly Lorentzian manifolds or pseudo-Riemannian manifolds. This generalization is of great interest due to its potential applications in theoretical physics and mathematics. Dirac-harmonic maps are natural generalizations

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of harmonic maps, and the recent development of harmonic maps into pseudo-Riemannian manifolds is related to minimal surfaces in the anti-de-Sitter space, as discussed by Alday and Maldacena [3]. Geometrically, harmonic maps into the Minkowski space $\mathbb{S}^4_1 \subset \mathbb{R}^5_1$ are linked to conformal Gauss maps of Willmore surfaces in $\mathbb{S}^3$, as studied by Bryant [4].

For harmonic maps into certain pseudo-Riemannian manifolds, especially static Lorentzian manifolds and more generally stationary Lorentzian manifolds, the regularity of weak solutions has been studied by Isobe [17] in the static case and later by Zhu [25] for the stationary case. The removability of isolated singularities was investigated by Hélein [16] for general pseudo-Riemannian targets. The energy identity for a sequence of harmonic maps with the uniformly bounded energy into stationary Lorentzian manifolds was established by Han et al. [15], and the neck property was studied by Han et al. [12] in the static case. The Dirichlet problem for harmonic maps from a disk into a Lorentzian warped product was investigated by Greco [11]. In the static case, the existence of Lorentzian harmonic maps has been studied via the heat flow method by Han et al. [13], which was motivated by the parabolic-elliptic system initiated in the Dirac-harmonic maps situation [7]. Under certain conditions of the initial energy or the curvature of the targets, Han et al. [13] proved the long-time existence of such a system. Furthermore, Han et al. [14] studied the existence of a global weak solution to this flow and proved the existence of static Lorentzian harmonic maps by studying the bubbling convergence of a time-slice sequence at infinity. For Dirac-harmonic maps into stationary Lorentzian manifolds, the interior regularity of weak solutions was proved in [1], and the interior bubbling convergence of a sequence of approximate Dirac-harmonic maps from a surface into static or stationary Lorentzian manifolds was studied in [2].

In this paper, we aim to investigate the blow-up behavior of Dirac-harmonic maps from a compact Riemannian surface with a boundary into Lorentzian manifolds. This is motivated by the application of solving the existence problem of Dirac-harmonic maps in Lorentzian manifolds, which will be studied in a forthcoming work. To this end, we generalize the new type of mixed parabolic-elliptic partial differential equations developed in [7,19,21] for the case of Riemannian manifolds as targets to the Lorentzian targets. This approach considers the Dirac-harmonic heat flow with an elliptic constraint on the spinor field, and under a proper boundary condition, the spinor field behaves as a constraint along the parabolic equation for the map. The boundary condition is necessary for such a method since it ensures the solvability of the Dirac-type equation of spinors, in contrast to the closed case, where the solution may have non-uniqueness. Furthermore, the heat flow may develop singularities at infinite times and we need to consider the convergence of a time-slice sequence of fields. In other words, we need to investigate the blow-up behavior of a sequence of approximate Dirac-harmonic maps with a properly chosen boundary condition, which will provide a full understanding of the behavior of the flow at infinite blow-up times and offer insights into the existence problem of Dirac-harmonic maps into Lorentzian manifolds.

The blow-up analysis is based on a sequence of analytic results of semi-linear partial differential equations, such as the small energy estimate or the small-energy regularity, the removal of isolated singularities, and the gap phenomenon. The small energy estimate can be obtained easily from the routine method, which can be traced back to Sacks and Uhlenbeck [23]. However, the removal of singularities is highly non-trivial, as it requires either a decay estimate of the energy near the isolated singularities or regularity of the weak solution. The regularity results of Dirac-harmonic maps into stationary Lorentzian manifolds (see [1]) provide full boundary regularity of such fields (see Lemma 3.1 and Corollary 3.2 in Section 3), and as an application, if we extend the fields delicately and properly across the boundary, then the removal of singularities is proved as a corollary, as shown in Corollary 3.4 in Section 3. Even in the case where the spinor vanishes, this removal of singularities theorem improved the corresponding result by Han et al. [12, Theorem 2.8], where a more restrictive condition than stationarity was assumed. This removal of singularities theorem will be used in the proof of a gap theorem over the upper half-space, as stated in Corollary 3.10 in Section 3. This result generalizes the corresponding theorem in [12, Theorem 2.7], which was originally formulated for Lorentzian harmonic maps and extended in this paper to Lorentzian Dirac-harmonic maps.

Before moving into the statement of our main results, let us first set up the notations. Let (M, g_M) be a Riemannian surface with a smooth boundary, equipped with a given spin structure, and ΣM be the spinor bundle over M. Let $(\mathcal{N}, g_{\mathcal{N}})$ be a *stationary* Lorentzian manifold, i.e., $\mathcal{N} = \mathbb{R} \times N$, where (N, g_N) is a closed n-dimensional Riemannian manifold, and the *stationary* Lorentzian metric over $\mathcal{N}$ is given by

$$g_{\mathcal{N}} := -\lambda(dr + \vartheta)^2 + g_N,$$

where $\lambda > 0$ is a smooth function and ϑ is a smooth 1-form over N, and dr is the length element of $\mathbb{R}^1$. Suppose that ϕ is a map from M to $\mathcal{N}$ and ψ is a section of the twisted bundle $\Sigma M \otimes \phi^{-1}T\mathcal{N}$. Note that there is a Riemannian metric $\langle \cdot, \cdot \rangle_{\Sigma M \otimes \phi^{-1}T\mathcal{N}}$ induced from the one of the spinor bundle ΣM and the pull-back bundle $\phi^{-1}T\mathcal{N}$. Locally, let $\{\partial_{y^j}\}_{j=0}^n$ be a basis of $\mathcal{N}$. Then the induced metric connection $\widetilde{\nabla}$ of the twisted bundle is defined as

$$\widetilde{\nabla}\psi = \nabla^{\Sigma M}\psi^i \otimes \partial_{y^i}(\phi) + \Gamma^i_{jk}(\phi) d\phi^j \psi^k \otimes \partial_{y^i}(\phi) \quad \text{for } \psi = \sum_{j=0}^n \psi^j \otimes \partial_{y^j}(\phi),$$

where $\{\psi^j\}_{j=0}^n$ are sections of ΣM, called *spinors*, and $\{\Gamma^i_{jk}\}_{i,j,k=0}^n$ are the Christoffel symbols of the pseudo-Riemannian manifold $(\mathcal{N}, g_{\mathcal{N}})$ with respect to the Levi-Civita connection in the pseudo-Riemannian setting. The *Dirac operator along the map* ϕ is defined by, for a local orthonormal frame $\{e_\alpha\}_{\alpha=1}^2$,

$$D\!\!\!/\,\psi := e_\alpha \cdot \widetilde{\nabla}_{e_\alpha} = \partial\!\!\!/\,\psi^i \otimes \partial_{y^i}(\phi) + \Gamma^i_{jk}(\phi) d\phi^j(e_\alpha) e_\alpha \cdot \psi^k \partial_{y^i}(\phi),$$

where $\cdot$ is the Clifford multiplication, and $\partial\!\!\!/ = e_\alpha \cdot \nabla^{\Sigma M}_{e_\alpha}$ is the usual Dirac operator. It is well known that $\partial\!\!\!/$ can be viewed as a generalization of the *Cauchy-Riemann operator* on $\mathbb{R}^2$.

A *Dirac-harmonic map* from M to $\mathcal{N}$ is a pair of smooth fields (ϕ, ψ) such that it is a critical point of the following Lagrangian:

$$\mathcal{L}(\phi, \psi) := \frac{1}{2} \int_M (|\nabla\phi|^2_{g_{\mathcal{N}}} + \langle \psi, D\!\!\!/\,\psi \rangle_{\Sigma M \otimes \phi^{-1}T\mathcal{N}}). \tag{1.1}$$

In order to define the notion of weakly Dirac-harmonic maps, we embed $(\mathcal{N}, g_{\mathcal{N}})$ isometrically to a model space $(\mathbb{R}^{K+1}, \bar{g})$ with the signature $(K, 1)$ and extend the definition of $\mathcal{L}$ to

$$(\phi, \psi) \in W^{1,2}(M, \mathcal{N}) \times W^{1,4/3}(M, \Sigma M \otimes \phi^{-1}T\mathcal{N}).$$

The boundary condition we consider is of the following *Dirichlet-chiral*-type:

$$\begin{cases} \phi(x) = \phi_\partial(x), \quad \phi_\partial \in C^{2,\alpha}(\partial M, \mathcal{N}), \quad x \in \partial M, \\ \mathcal{B}\psi(x) = \mathcal{B}\psi_\partial(x), \quad \psi_\partial \in C^{1,\alpha}(\partial M, \Sigma M \otimes \phi^{-1}T\mathcal{N}|_{\partial M}), \quad x \in \partial M, \end{cases} \tag{1.2}$$

where $\alpha \in (0,1)$, and $\mathcal{B} = \mathcal{B}^{\pm}_\phi$ is the *chiral boundary operator* defined by

$$\begin{aligned} \mathcal{B}^{\pm}_\phi : L^2(\partial M, \Sigma M \otimes \phi^{-1}T\mathcal{N}|_{\partial M}) &\to L^2(\partial M, \Sigma M \otimes \phi^{-1}T\mathcal{N}|_{\partial M}) \\ \psi &\mapsto \frac{1}{2}(\mathrm{Id} \pm \nu \cdot \chi) \cdot \psi, \end{aligned}$$

where ν is the outward unit normal vector field on ∂M with respect to the volume form on M, and

$$\chi = \sqrt{-1} e_1 \cdot e_2 : \Gamma(\Sigma M \otimes \phi^{-1}T\mathcal{N}) \to \Gamma(\Sigma M \otimes \phi^{-1}T\mathcal{N})$$

is the *chiral operator* defined by a local orthonormal frame $\{e_\alpha\}_{\alpha=1}^2$ on TM and satisfies

$$\chi \circ \chi = \mathrm{Id}, \quad \chi^* = \chi, \quad \nabla\chi = 0, \quad \chi \cdot X = -X \cdot \chi, \quad \forall\, X \in \Gamma(TM).$$

The chiral boundary condition generalizes the usual chirality boundary condition and mathematically interprets the supersymmetric nonlinear sigma model with boundaries and the D-branes in superstring theory [8].

The two spaces of *weak* Dirac-harmonic maps and *regular* Dirac-harmonic maps with Dirichlet-chiral boundary data

$$(\phi_\partial,\psi_\partial)\in C^{2,\alpha}(\partial M,\mathcal{N})\times C^{1,\alpha}(\partial M,\Sigma M\otimes\phi^{-1}T\mathcal{N}|_{\partial M})$$

are defined, respectively, as follows:

$$\begin{aligned}\mathcal{X}^w(M,\mathcal{N}):=\{(\phi,\psi):\ &\phi\in W^{1,2}(M,\mathcal{N})\text{ and }\psi\in W^{1,4/3}(M,\Sigma M\otimes\phi^{-1}T\mathcal{N}),\\ &\phi=\phi_\partial\text{ and }\mathcal{B}\psi=\mathcal{B}\psi_\partial\text{ for }x\in\partial M\},\\ \mathcal{X}(M,\mathcal{N}):=\{(\phi,\psi):\ &\phi\in W^{2,2}(M,\mathcal{N})\text{ and }\psi\in W^{1,2}(M,\Sigma M\otimes\phi^{-1}T\mathcal{N}),\\ &\phi=\phi_\partial\text{ and }\mathcal{B}\psi=\mathcal{B}\psi_\partial\text{ for }x\in\partial M\}.\end{aligned}$$

As usual, we also denote by $\mathcal{X}_{\mathrm{loc}}$, $\mathcal{X}^w_{\mathrm{loc}}$, and $\mathcal{X}_0$ the corresponding local Sobolev spaces and Sobolev spaces of functions with compact support, respectively.

With the help of the isometric embedding $(\mathcal{N},g_{\mathcal{N}})\hookrightarrow(\mathbb{R}^{K+1},\bar g)$, we can express the Euler-Lagrange equation of $\mathcal{L}$ extrinsically as

$$\tau(\phi,\psi)=\Delta_M\phi-\bar A(\nabla\phi,\nabla\phi)-\overline{\mathcal{P}}(\bar{\mathcal{A}}(d\phi(e_\alpha),e_\alpha\cdot\psi);\psi)-\overline{\mathcal{R}}(\phi,\psi), \tag{1.3}$$

$$\kappa(\phi,\psi)=\partial\!\!\!/\psi-\bar{\mathcal{A}}(d\phi(e_\alpha),e_\alpha\cdot\psi)+\overline{\Gamma}(d\phi(e_\alpha),e_\alpha\cdot\psi), \tag{1.4}$$

where $\overline{\mathcal{P}}$ and $\bar{\mathcal{A}}$ are defined via the shape operator $\overline{P}$ and the second fundamental form $\bar A$ of $\mathcal{N}\hookrightarrow\mathbb{R}^{K+1}$, respectively, $\overline{\mathcal{R}}$ is defined via the pseudo-Riemannian curvature $\overline{R}$, and $\overline{\Gamma}$ is the Christoffel symbol of $\bar g$. We refer to Section 2 for more details on the definition of notations.

Definition 1.1. A pair of fields $(\phi,\psi)\in\mathcal{X}^w(M,\mathcal{N})$ is called an (L^p,L^q)-*approximate Dirac-harmonic map with the Dirichlet-chiral boundary*, if they satisfy the boundary condition (1.2) on ∂M and satisfy (1.3) and (1.4) in the weak sense for some $\tau(\phi,\psi)\in L^p(M,\mathcal{N})$ and $\kappa(\phi,\psi)\in L^q(M,\Sigma M\otimes\phi^{-1}T\mathcal{N})$. If, in addition, there exists a constant $\Lambda>0$ such that

$$\|\nabla\phi\|^2_{L^2(M)}+\|\psi\|^4_{L^4(M)}\leqslant\Lambda,\quad \|\tau(\phi,\psi)\|^p_{L^p(M)}+\|\kappa(\phi,\psi)\|^q_{L^q(M)}\leqslant\Lambda,$$

then we call (ϕ,ψ) a $(\Lambda;L^p,L^q)$-*approximate Dirac-harmonic map*.

The aforementioned analysis tools are well-suited for the analysis of blow-up phenomena. With the small-energy regularity and standard blow-up argument, one can construct the bubble tree for a sequence of approximate Dirac-harmonic maps with the uniformly bounded energy into stationary Lorentzian manifolds. To study whether the energy identity holds in the bubbling convergence process, one needs to analyze the neck regions, which are regions that connect the base map (weak limit) and the bubbles or the regions connecting bubbles over bubbles.

We first consider a boundary case with only one bubble, i.e., the energy is concentrated on a point sitting on the boundary, and show that the neck region can be decomposed into three or four domains near each other, as illustrated in Figures 1 and 2 in Section 4. By a reduction argument, we prove that the (positive) energy on domains Ω_1 and Ω_3 can be made as small as desired. The domain Ω_4 is a whole annular region (the innermost hatch-filled region in Figure 2), which is a typical neck region considered for interior energy concentration points and can be handled as in [2]. The main results of the current paper concern the energy identity of approximate Dirac-harmonic maps on the region Ω_2.

Theorem 1.2. *Suppose that* $\{(\phi_n,\psi_n)\}$ *is a sequence of* $(\Lambda;L^2,L^4)$-*approximate Dirac-harmonic maps from the unit upper half-disc* D^+ (*with a given spin structure*) *into a stationary Lorentzian manifold* $(\mathcal{N},g_{\mathcal{N}})$ *satisfying the boundary condition* (1.2) *on the flat boundary* ∂^0D^+. *If* $0\in\partial^0D^+$ *is an isolated energy concentration point on* D^+ *of* $\{(\phi_n,\psi_n)\}$, *i.e.,*

$$\lim_{r\to0}\liminf_{n\to\infty}\int_{B_r\cap D^+}(|\nabla\phi_n|^2+|\psi_n|^4)\geqslant\max\{\epsilon_0^2,(\epsilon_0')^2\}=:\bar\epsilon_0^2,$$

where $\epsilon_0>0$ *and* $\epsilon_0'>0$ *are the constants as in Lemmas* 3.7 *and* 3.8 *in Section* 3, *respectively, then up to a subsequence, and without distinguishing between the subsequence and the original sequence, we* *see that there exist sequences of points* $\{x_n\}$ *and scaling radii* $\{r_n\}$ *such that as* $n \to \infty$, $x_n \to 0$ *and* $r_n \to 0$, *and the scaled fields* $\tilde{\phi}_n(x) := \phi_n(x_n + r_n x)$ *and* $\tilde{\psi}_n(x) = \sqrt{r_n}\psi_n(x_n + r_n x)$ *satisfy that for* $d_n := \mathrm{dist}\,(x_n, \partial^0 D^+) = \mathrm{dist}\,(x_n, x_n')$ *where* $x_n' \in \partial^0 D^+$, *the following hold*:

(1) (a) *If* $\lim_{n\to\infty} \frac{d_n}{r_n} < +\infty$, *then*

$$\tilde{\phi}_n(x) \rightharpoonup \bar{\sigma} \in W^{2,2}(B, \mathcal{N}), \quad \tilde{\psi}_n(x) \rightharpoonup \bar{\xi} \in W^{1,2}(B, \Sigma B \otimes \mathbb{R}^{K+1}),$$

where $(\bar{\sigma}, \bar{\xi})$ *is a non-trivial Lorentzian Dirac-harmonic disc with the Dirichlet-chiral boundary condition* $\bar{\sigma} = \mathrm{const.}$ *and* $\mathcal{B}\bar{\xi} = 0$ *on* ∂B, *i.e., a bubble with a boundary.*

(b) *If* $\lim_{n\to\infty} \frac{d_n}{r_n} = +\infty$, *then*

$$\tilde{\phi}_n(x) \rightharpoonup \sigma \in W^{2,2}(S^2, \mathcal{N}), \quad \tilde{\psi}_n(x) \rightharpoonup \xi \in W^{1,2}(S^2, \Sigma S^2 \otimes \mathbb{R}^{K+1}),$$

where (σ, ξ) *is a non-trivial Lorentzian Dirac-harmonic sphere, i.e., a bubble.*

In either case, we call the bubbles are corresponding to the blow-up data (x_n, r_n).

(2) *Assume that there is only one bubble at* 0, *either a Lorentzian Dirac-harmonic disc or a Lorentzian Dirac-harmonic sphere. Then for any* $\epsilon > 0$, *there exist* $\delta > 0$ *and* $R > 0$ *such that for large enough* n, *we have*

$$E(\phi_n, \psi_n; D^+_{2r}(x_n) \setminus D^+_r(x_n)) < \epsilon^2, \quad \forall\, r \in [r_n R/2, 2\delta].$$

(3) *Furthermore, in either case* (a) *or* (b), *there is no positive energy of* $\{\psi_n\}$ *on the neck* $P_n :\, = D^+_\delta \setminus D^+_{r_n R}(x_n)$ *in the limit sense, i.e.,*

$$\lim_{R\to\infty} \lim_{\delta\to 0} \lim_{n\to\infty} E(\psi_n; P_n) = 0. \tag{1.5}$$

(4) *If we define* $\Omega_2 :\, = D^+_{\delta/2}(x_n') \setminus D^+_{2d_n}(x_n') \subset P_n$ *in the case* $\lim_{n\to\infty} d_n/r_n = +\infty$ *and*

$$\Omega_2 = D^+_{\delta/2}(x_n') \setminus D^+_{2r_n R}(x_n')$$

otherwise, then there is no positive energy of $\{\phi_n\}$ *on* Ω_2 *of the neck region in the limit sense, i.e.,*

$$\lim_{\delta\to 0} \lim_{n\to\infty} E(\phi_n; \Omega_2) = 0. \tag{1.6}$$

We refer to Section 2 *for the definition of various energy.*

Remark 1.3. (1) In the last item, comparing [20, Theorem 1.1] and [12, Theorem 1.1], we can improve the argument of energy identities by directly controlling the total (positive) energy of the map on Ω_2, instead of just the tangent energy. This eliminates the need for the Pohozaev-type argument that relates the tangent energy and the radial energy and simplifies the argument significantly.

(2) The first item implies that the separation of different types of bubbles is based on the ratio of the distance of blow-up points to the boundary and the scaling factor.

(3) The second item is a reduction condition from which one can construct the bubble tree; it is also the starting point of proving the energy identities.

(4) The third item is the (positive) energy identity for spinors, the proof of which is similar to the interior case [2], except that we need a boundary-version elliptic estimate on the upper half annular region (see Lemma 3.11 in Section 3).

Combining Theorem 1.2 with the interior blow-up analysis presented in [2], which handles Ω_4, a typical annular region contained in the neck region, we obtain the following global results.

Corollary 1.4. *Let* M *be an oriented Riemannian surface with a smooth boundary* ∂M *and a given spin structure,* $(\mathcal{N}, g_{\mathcal{N}})$ *be a stationary Lorentzian manifold, and* $\{(\phi_n, \psi_n)\}$ *be a sequence of* $(\Lambda; L^2, L^4)$*-approximate Dirac-harmonic maps satisfying the boundary condition* (1.2) *on* ∂M *with the uniformly bounded positive energy, i.e.,*

$$E(\phi_n, \psi_n) = \int_M (|\nabla\phi_n|^2 + |\psi_n|^4) \leqslant \Lambda < +\infty$$

and

$$\|\tau(\phi_n,\psi_n)\|_{L^2(M)}+\|\kappa(\phi_n,\psi_n)\|_{L^4(M)}\leqslant\Lambda<+\infty.$$

Then, the energy can only be concentrated on a finite point set

$$\mathcal{S}:=\left\{x\in M:\lim_{r\to 0}\liminf_{n\to\infty}\int_{B_r(x)\cap M}(|\nabla\phi_n|^2+|\psi_n|^4)\geqslant\bar{\epsilon}_0^2\right\}$$

and up to a subsequence, $(\phi_n,\psi_n)\rightharpoonup(\phi_\infty,\psi_\infty)$ *in* $\mathcal{X}_{\mathrm{loc}}(M\setminus\mathcal{S},\mathcal{N})$*, where* $(\phi_\infty,\psi_\infty)$ *is an approximate Dirac-harmonic map satisfying the boundary condition* (1.2)*. Moreover, the following statements hold*:

(1) *For each* $p_i\in\mathcal{S}$ $(i=1,2,\ldots,I)$*, there are finitely many non-trivial Lorentzian Dirac-harmonic spheres* (σ_i^l,ξ_i^l) *and finitely many non-trivial Lorentzian Dirac-harmonic discs* $(\bar{\sigma}_i^k,\bar{\xi}_i^k)$ *with the Dirichlet-chiral boundary condition* $\bar{\sigma}_i^k=\mathrm{const.}$ *and* $\mathcal{B}\bar{\xi}_i^k=0$ *on* ∂B*, where* $l=1,2,\ldots,L_i$ *and* $k=1,2,\ldots,K_i$*, such that the following positive energy identity for the spinor fields* $\{\psi_n\}$ *holds*:

$$\lim_{n\to\infty}E(\psi_n)=E(\psi_\infty)+\sum_{i=1}^{I}\sum_{l=1}^{L_i}E(\xi_i^l)+\sum_{i=1}^{I}\sum_{k=1}^{K_i}E(\bar{\xi}_i^k),\tag{1.7}$$

and the Lorentzian energy identity holds, i.e., for the maps $\{\phi_n\}$,

$$\lim_{n\to\infty}E_g(\phi_n)=E_g(\phi_\infty)+\sum_{i=1}^{I}\sum_{l=1}^{L_i}E_g(\sigma_i^l)+\sum_{i=1}^{I}\sum_{k=1}^{K_i}E_g(\bar{\sigma}_i^k).\tag{1.8}$$

Moreover, for any $(i,l)\neq(j,k)$*, the corresponding blow-up data* $(x_{n,i}^l,r_{n,i}^l)$ *are related by*

$$\lim_{n\to\infty}\left(\frac{r_{n,i}^l}{r_{n,j}^k}+\frac{r_{n,j}^k}{r_{n,i}^l}+\frac{|x_{n,i}^l-x_{n,j}^k|}{r_{n,i}^l+r_{n,j}^k}\right)=+\infty.\tag{1.9}$$

(2) *If furthermore, we assume that* $(\mathcal{N},g_{\mathcal{N}})$ *is static, i.e.,* $\vartheta\equiv 0$*, then the positive energy identity for the maps* $\{\phi_n\}$ *also holds, i.e.,*

$$\lim_{n\to\infty}E(\phi_n)=E(\phi_\infty)+\sum_{i=1}^{I}\sum_{l=1}^{L_i}E(\sigma_i^l)+\sum_{i=1}^{I}\sum_{k=1}^{K_i}E(\bar{\sigma}_i^k).\tag{1.10}$$

Moreover, the image

$$\phi_\infty(M\setminus\partial M)\cup\bigcup_{i=1}^{I}\bigcup_{l=1}^{L_i}(\sigma_i^l(S^2))$$

is a connected set.

Remark 1.5. In order to prove Corollary 1.4, by Theorem 1.2, we only need to consider the region $\Omega_4\subset P_n$ (see the inner annular region with the vertical hatched pattern in Figure 2 in Section 4), which is a typical neck region for an interior energy concentration point. In [2], Ai and Zhu used a Pohozaev-type argument on this region to prove the metric energy identity. Furthermore, in the case of a static Lorentzian manifold as the target, the positive energy in Ω_4 for the map is also proved. This explains why, in the stationary case, we only obtain the Lorentzian energy identity as stated in (1.8), and the positive energy identity as stated in (1.10) is only proved in the static case.

The rest of this paper is organized as follows: In Section 2, we fix our notations and record some basic properties related to stationary Lorentzian manifolds and Dirac-harmonic maps. In Section 3, we prove the boundary regularity theorem, i.e., Lemma 3.1, and as an application, the boundary version of the removal of singularities theorem and the gap theorem are proved as corollaries. We also state the small-energy regularity theorem. Finally, we finish our proof of blow-up analysis in Section 4, and especially, Theorem 1.2 and Corollary 1.4 are proved.

2 Preliminaries

Let (M, g_M) be a Riemannian surface with the smooth boundary ∂M equipped with a fixed spin structure, and $(\mathcal{N}, g_{\mathcal{N}})$ be a *stationary* Lorentzian manifold. Let ϕ be a smooth map from M to $\mathcal{N}$ and ψ be a smooth section of $\Sigma M \otimes \phi^{-1}T\mathcal{N}$, the twisted bundle with the induced Riemannian metric $\langle\cdot,\cdot\rangle_{\Sigma M\otimes\phi^{-1}T\mathcal{N}}$ and the induced metric connection $\widetilde{\nabla} := \nabla^{\Sigma M\otimes\phi^{-1}T\mathcal{N}}$. The functional $\mathcal{L}$ is given in (1.1), and the Euler-Lagrange equation of $\mathcal{L}$ is given by

$$\begin{cases}\tau(\phi) = \mathcal{R}(\phi,\psi),\\ \not{D}\psi = 0,\end{cases}$$

where $\tau(\phi)$ is the tension field of ϕ and $\mathcal{R}$ is defined via the pseudo-Riemannian curvature R of $(\mathcal{N}, g_{\mathcal{N}})$, i.e.,

$$\mathcal{R}(\phi,\psi) := \frac{1}{2}R^s_{ijl}(\phi)\langle\psi^i, \nabla\phi^l\cdot\psi^j\rangle_{\Sigma M}\partial_{y^s}(\phi),$$

where $\{R^s_{ijl}\}$ are the components of the pseudo-Riemannian curvature tensor R of $(\mathcal{N}, g_{\mathcal{N}})$.

In order to define the weak solution, it is natural to take an extrinsic view. By our construction [1, Proposition 2.1], recall that $(\mathcal{N}, g_{\mathcal{N}})$ is isometrically embedded into another pseudo-Riemannian manifold $(\overline{\mathcal{N}}, g_{\overline{\mathcal{N}}}) = (\mathbb{R}^{K+1}, \bar{g})$, which is homeomorphic to the model space with the signature $(K, 1)$. In fact, since (N, g_N) is a closed Riemannian manifold, we can embed it isometrically into some Euclidean space $\mathbb{R}^K$. This allows us to construct a pseudo-Riemannian metric $\bar{g}$ over $\mathbb{R}^1\times\mathbb{R}^K \cong \mathbb{R}^{K+1}$ via the nearest projection such that $(\mathcal{N}, g_{\mathcal{N}}) \hookrightarrow (\mathbb{R}^{K+1}, \bar{g})$ is an isometric embedding. Moreover, by the compactness of N and the fact that both $\lambda > 0$ and ϑ are smooth, we have

$$\frac{1}{\Lambda'} \leqslant \lambda \leqslant \Lambda', \quad |\vartheta| + |\nabla\vartheta| + |\nabla\lambda| \leqslant \Lambda',$$

and by the construction of $(\mathbb{R}^{K+1}, \bar{g})$, if we denote by $\bar{A}$, $\overline{R}$, and $\overline{\Gamma}$ the second fundamental form, the pseudo-Riemannian curvature of $\mathcal{N} \hookrightarrow (\mathbb{R}^{K+1}, \bar{g})$, and the Christoffel symbol of $\bar{g}$, respectively, then the following *boundedness* condition holds:

$$|\bar{A}| + |\nabla\bar{A}| + |\bar{g}| + |\bar{R}| + |\bar{\Gamma}| \leqslant \Lambda' \tag{2.1}$$

for some positive constant $\Lambda' > 0$.

The Euler-Lagrange equation of $\mathcal{L}$ can be rewritten non-intrinsically as

$$\begin{cases}\Delta_M\phi - \bar{A}(\nabla\phi,\nabla\phi) = \overline{\mathcal{P}}(\bar{\mathcal{A}}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi) + \overline{\mathcal{R}}(\phi,\psi),\\ \not{\partial}\psi = \bar{\mathcal{A}}(d\phi(e_\alpha), e_\alpha\cdot\psi) - \overline{\Gamma}(d\phi(e_\alpha), e_\alpha\cdot\psi),\end{cases}$$

where $\overline{\mathcal{R}}$ is defined in a similar way to $\mathcal{R}$ by replacing the pseudo-Riemannian curvature of $\mathcal{N}$ to $\overline{R}$. Under a natural basis $\{\partial_{v^a}\}$ of $\overline{\mathcal{N}}$, with $\partial_{v^0} = \partial_{y^0}$, we have

$$\begin{aligned}&\bar{\mathcal{A}}(d\phi(e_\alpha), e_\alpha\cdot\psi) := \nabla\phi^l\cdot\psi^j\otimes\bar{A}_{jl}, \quad \bar{A}_{jl} := \bar{A}(\partial_{y^j},\partial_{y^l})\circ\phi,\\ &\overline{\Gamma}(d\phi(e_\alpha), e_\alpha\cdot\psi) := \overline{\Gamma}^a_{bc}(\phi)\nabla\phi^b\cdot\psi^c\otimes\partial_{v^a}(\phi),\end{aligned}$$

and

$$\overline{\mathcal{P}}(\bar{\mathcal{A}}(d\phi(e_\alpha), e_\alpha\cdot\psi);\psi) := g^{sk}_{\mathcal{N}}\langle\overline{P}(\bar{A}_{jl};\partial_{y^i}),\partial_{y^k}\rangle_{T\mathcal{N}}\langle\psi^i,\nabla\phi^l\cdot\psi^j\rangle_{\Sigma M}\partial_{y^s},$$

where $\overline{P}$ is the shape operator and $\bar{A}$ is the second fundamental form with respect to the isometric embedding. $\overline{\mathcal{R}}$ is defined in a similar way to $\mathcal{R}$ by replacing R with $\overline{R}$.

For an integer $l \geqslant 0$ and a real number $p \in (1, +\infty)$, define

$$\begin{aligned}&W^{l,p}(M,\mathcal{N}) := \{\phi\in W^{l,p}(M,\mathbb{R}^{K+1}) : \phi(x)\in\mathcal{N} \text{ for a.e. } x\in M\},\\ &W^{l,p}(M,\Sigma M\otimes\phi^{-1}T\mathcal{N}) := \{\psi\in W^{l,p}(M,\Sigma M\otimes\mathbb{R}^{K+1}) : \text{for any } \nu\in T^{\perp}_{\phi(x)}\mathcal{N},\\ &\qquad\qquad \langle\nu,\psi\rangle_{\bar{g}} = 0 \text{ for a.e. } x\in M\}.\end{aligned}$$

Here, for $\phi=(\varphi^0,\varphi)\in\mathbb{R}^1\times\mathbb{R}^K$, the norm is defined as

$$\|\phi\|^p_{W^{l,p}(M,\mathcal{N})}:=\sum_{k=0}^{l}\|\nabla^k\phi\|^p_{L^p(M,\mathcal{N})},\quad \|\nabla^k\phi\|^p_{L^p(M,\mathcal{N})}:=\|\nabla^k\varphi^0\|^p_{L^p(M,\mathbb{R}^1)}+\|\nabla^k\varphi\|^p_{L^p(M,\mathbb{R}^K)},$$

and for $\psi=\psi^a\otimes\partial_{v^a}(\phi)=:\psi^0\partial_{v^0}(\phi)+\psi'$, we have

$$\widetilde{\nabla}_{e_\alpha}\psi=\nabla^{\Sigma M}_{e_\alpha}\psi^a\otimes\partial_{v^a}+\Gamma^a_{bc}(\phi)d\phi^b(e_\alpha)e_\alpha\cdot\psi^c\partial_{v^a}=:\psi^a_{|\alpha}\otimes\partial_{v^a}=:\psi^0_{|\alpha}+\psi'_{|\alpha}$$

and

$$\begin{aligned}\|\psi\|^p_{W^{1,p}(M,\Sigma M\otimes\phi^{-1}T\mathcal{N})}&:=\|\psi\|^p_{L^p(M,\Sigma M\otimes\phi^{-1}T\mathcal{N})}+\|\widetilde{\nabla}\psi\|^p_{L^p(M,\Sigma M\otimes\phi^{-1}T\mathcal{N})},\\ \|\widetilde{\nabla}\psi\|^p_{L^p(M,\Sigma M\otimes\phi^{-1}T\mathcal{N})}&:=\|\psi^0_{|\alpha}\|^p_{L^p(M,\Sigma M\otimes\mathbb{R}^1)}+\|\psi'_{|\alpha}\|^p_{L^p(M,\Sigma M\otimes\mathbb{R}^K)},\end{aligned}$$

and define $\|\nabla^k\psi\|_{L^p(M,\Sigma M\otimes\phi^{-1}T\mathcal{N})}$ similarly.

Next, let us recall the definition of various energy. For a measurable subset $\Omega\subset M$, the *Lorentzian energy* of ϕ and ψ is defined by

$$E_g(\phi;\Omega):=\int_\Omega|\nabla\phi|^2_{g_\mathcal{N}},\quad E_g(\psi;\Omega):=\int_\Omega|\psi|^4_{\Sigma M\otimes\phi^{-1}T\mathcal{N}}=\int_\Omega\langle\psi,\psi\rangle^2_{\Sigma M\otimes\phi^{-1}T\mathcal{N}},$$

respectively. By employing the embedding $(\mathcal{N},g_\mathcal{N})\hookrightarrow(\mathbb{R}^{K+1},\bar{g})$, if we write $\phi=(\varphi^0,\varphi)\in\mathbb{R}^0\times\mathbb{R}^K$ and $\psi=(\psi^0,\psi')\hookrightarrow(\Sigma M\otimes\mathbb{R}^1)\times(\Sigma M\otimes\mathbb{R}^K)$, then the *positive energy* of ϕ and ψ is defined by

$$\begin{aligned}E(\phi;\Omega)&:=\int_\Omega|\nabla\phi|^2=\int_\Omega(|\nabla\varphi^0|^2+|\nabla\varphi|^2),\\ E(\psi;\Omega)&:=\int_\Omega|\psi|^4=\int_\Omega(\langle\psi^0,\psi^0\rangle^2_{\Sigma M\otimes\mathbb{R}^1}+\langle\psi',\psi'\rangle^2_{\Sigma M\otimes\mathbb{R}^K}),\end{aligned}$$

respectively.

Finally, let us recall some results of Dirac-harmonic maps into stationary Lorentzian manifolds. In [1], the equation of weak Dirac-harmonic maps into stationary Lorentzian manifolds was rewritten into a critical elliptic system with a potential. This potential is *a priori* in L^2 and has a certain hidden antisymmetric structure, but no divergence-free structure. This generalizes the case of harmonic maps [25], where the hidden divergence-free structure is assured and provides a new perspective on the interior regularity of weak Dirac-harmonic maps. More precisely, the following proposition is proved.

Proposition 2.1 (See [1, Proposition 4.1]). *Suppose that $(\phi,\psi)\in\mathcal{X}^w(B,\mathcal{N})$ is a weak Dirac-harmonic map from the unit ball $B\subset M$ into the stationary Lorentzian manifold. Then the equation of ϕ can be written as*

$$\begin{cases}-\mathrm{div}\,(Q\nabla\varphi)=\Theta\cdot Q\nabla\varphi+F\Omega\cdot Q\nabla\varphi+\upsilon,\\ -\mathrm{div}\,\Omega=W,\end{cases}\tag{2.2}$$

where locally we write $\phi=(\varphi^0,\varphi)=(\varphi^0,\varphi^1,\dots,\varphi^K)\in\mathbb{R}^1\times\mathbb{R}^K$ and

$$Q\equiv Q(\varphi):=\begin{pmatrix}\lambda(\varphi)&\lambda(\varphi)\vartheta\\0&I_K\end{pmatrix},\quad\vartheta:=(\vartheta_1,\dots,\vartheta_K),$$

$$\Theta:=\begin{pmatrix}0&0\\0&(\Theta^{ab})_{K\times K}\end{pmatrix},\quad F:=\begin{pmatrix}0&0&\cdots&0\\-\Upsilon_1&\Upsilon_{11}&\cdots&\Upsilon_{1K}\\\vdots&\vdots&\ddots&\vdots\\-\Upsilon_K&\Upsilon_{K1}&\cdots&\Upsilon_{KK}\end{pmatrix},$$

$$\Omega:=\lambda(\varphi)\mathrm{diag}\,(V^\sharp,\dots,V^\sharp),\quad V^\sharp:=\nabla\varphi^0+\vartheta_a\nabla\varphi^a,$$

$$W:=\mathrm{diag}(w,\dots,w),\quad w:=\mathcal{R}\mathcal{P}_0(\phi,\psi),$$

$$\upsilon := (w, v^1, \dots, v^K), \quad v^a := \mathcal{RP}^a(\phi,\psi) - \mathcal{Q}^a w,$$

$\Theta^{ab} \equiv \Theta^{ab}(\varphi)$, $\mathcal{RP}_0(\phi,\psi)$ and $\mathcal{RP}^a(\phi,\psi)$ are $\mathbb{R}^K$-valued vector fields on M, and $\Upsilon_a \equiv \Upsilon_a(\varphi)$, $\Upsilon_{ab} \equiv \Upsilon_{ab}(\varphi)$, and $\mathcal{Q}^a \equiv \mathcal{Q}^a(\varphi)$ are functions on M.

By imposing certain smallness of the norm on the coefficients in (2.2), we especially prove the following regularity results, which are a special case of the corresponding higher-dimensional one.

Lemma 2.2 (See [1, Theorem C])**.** *Suppose that $B \subset \mathbb{R}^2$ is the unit ball and $n > 0$ is an integer. Denote by $\mathrm{M}(n)$ the set of $n \times n$ real metrics. For any $\Lambda > 0$, there exists an $\epsilon = \epsilon(\Lambda) > 0$ such that for any $\Theta \in L^2(B, \mathfrak{so}(n) \otimes \wedge^1\mathbb{R}^2)$, $\Omega \in L^2(B, \mathrm{M}(n) \otimes \wedge^1\mathbb{R}^2)$, $F, G \in W^{1,2} \cap L^\infty(B, \mathrm{M}(n))$, $Q \in W^{1,2} \cap L^\infty(B, \mathrm{GL}(n))$, and*

$$W \in L^q(B, \mathrm{M}(n)), \quad \upsilon \in L^s(B, \mathbb{R}^n) \quad \textit{for some } 1 < q < 2 \textit{ and } 1 < s < 2,$$

if $u \in W^{1,2}(B, \mathbb{R}^n)$ is a weak solution of the following elliptic system

$$\begin{cases} -\mathrm{div}\,(Q\nabla u) = \Theta \cdot Q\nabla u + F\Omega \cdot G\nabla u + \upsilon, \\ -\mathrm{div}\,\Omega = W \end{cases} \tag{2.3}$$

with the coefficients satisfying the following conditions

$$\|\nabla u\|_{L^2} + \|\Theta\|_{L^2} + \|\Omega\|_{L^2} + \|W\|_{L^q} + \|\nabla Q\|_{L^2} + |\nabla F|_{L^2} + |\nabla G| \leqslant \epsilon$$

and

$$|Q| + |Q^{-1}| + |F| + |G| \leqslant \Lambda \quad \textit{a.e. in } B,$$

then for some $\alpha \in (0,1)$, we have

$$[u]_{C^\alpha(B_{1/2})} \leqslant C(\Lambda)(\epsilon + \|\upsilon\|_{L^s(B)}).$$

In particular, u is Hölder continuous in $B_{1/2}$.

Remark 2.3. One word about the notations: we denote by $\mathbb{R}^2_+ := \{x = (x^1, x^2) \in \mathbb{R}^2 : x^2 \geqslant 0\}$ the upper half-space, by $B_r(x)$ the ball in $\mathbb{R}^2$ with radius r and center x, and by $D_r^+(x) = B_r(x) \cap \mathbb{R}^2_+$ the part of $B_r(x)$ contained in the upper half-space; the two boundaries of $D_r^+(x)$ are denoted by

$$\partial^+ D_r^+(x) := \partial D_r(x) \cap \mathbb{R}^2_+, \quad \partial^0 D_r^+(x) := B_r(x) \cap \partial\mathbb{R}^2_+,$$

respectively.

3 Analysis lemmas for blow-up analysis

In this section, we recall and prove some basic lemmas for blow-up analysis, which include the small-energy regularity, the removal of singularities, and the gap theorem. The interior case is treated in [2] and we generalize it to the boundary case. The most important is the boundary regularity of weak Dirac-harmonic maps (see Corollary 3.2), which is proved as a corollary of the boundary regularity theorem for a class of partial differential equations, with a proper structure on the equation (see Lemma 3.1 for more details). This result is of fundamental importance, without which we cannot improve the removal of isolated singularities and the gap theorem compared with the corresponding results in [12].

3.1 Boundary regularity of weak Dirac-harmonic maps

We extend several quantities of the equation of ϕ in the proof of boundary regularity, and for the exact definition of these quantities, we refer to [1, Proposition 4.1]. Since the regularity question is local, we may choose coordinates x^i centered at a point $x_0 \in \partial M$ such that locally M is the upper half-disc D^+.

Since $\phi|_{\partial M} = \phi_\partial|_{\partial M}$ and $\phi_\partial \in C^{2,\alpha}(M)$, we may assume $\phi_\partial \in C^{2,\alpha}(B)$ and $\phi(0) = 0 \in \mathbb{R}^{K+1}$. Let us extend ϕ as

$$\hat{\phi}(x) = -(\phi(x^*) - \phi_\partial(x^*)), \quad x = (x^1, x^2) \in D^- := B \setminus D^+,$$

where $x^* = (x^1, -x^2)$, and we take $\hat{\phi} = \phi - \phi_\partial$ in D^+. It is clear that $\hat{\phi}|_{\partial^0 D^+} = -(\phi - \phi_\partial)|_{\partial^0 D^+} = 0$.

Let us write $\hat{\phi}$ locally as $\hat{u}$. Then the above extension implies $\hat{u}(x) = -\hat{u}(x^*)$, and we call it an *odd* extension. Let us also extend the Riemannian manifold (N, g_N) to $(\widehat{N}, \hat{g}_N)$ such that

$$\widehat{N} = \{-y : y \in N \hookrightarrow \mathbb{R}^K\}$$

and $\hat{g}_N = (\sigma^{-1})^* g_N$, where $\sigma : N \to \widehat{N}$ and $y \mapsto -y$. For the function $\lambda > 0$ and the 1-form ϑ on N, we extend them locally and evenly to $N \sqcup \widehat{N}$, i.e.,

$$\hat{\lambda}(\hat{u}(x)) = \hat{\lambda}(\hat{u}(x^*)) \Leftrightarrow \hat{\lambda}(\hat{u}(x)) = \begin{cases} \lambda(u(x)), & x \in D^+, \\ \lambda(u(x^*)), & x \in D^-, \end{cases}$$

and for $\rho : D^+ \to D^-$,

$$\hat{\vartheta}(\hat{u}) = \rho^*(\hat{\vartheta}(\hat{u})) \Leftrightarrow \hat{\vartheta}(\hat{u}(x)) = \begin{cases} \vartheta(u(x)), & x \in D^+, \\ \rho^*(\vartheta(u(x^*))), & x \in D^-. \end{cases}$$

Note that if we write $\vartheta(u(x)) = \vartheta_1 dx^1 + \vartheta_2 dx^2$ and $\hat{\vartheta}(\hat{u}(x)) = \hat{\vartheta}_1 dx^1 + \hat{\vartheta}_2 dx^2$, then

$$\hat{\vartheta}_1(x) = \begin{cases} \vartheta_1(x), & x \in D^+, \\ \vartheta_1(x^*), & x \in D^-, \end{cases} \quad \hat{\vartheta}_2(x) = \begin{cases} \vartheta_2(x), & x \in D^+, \\ -\vartheta_2(x^*), & x \in D^-. \end{cases}$$

Thus, $\hat{\vartheta}_1(x) = \hat{\vartheta}_1(x^*)$ and $\hat{\vartheta}_2(x) = -\hat{\vartheta}_2(x^*)$, from which the equivalence follows.

The above extension implies the quantity

$$Q \equiv Q(u) := \begin{pmatrix} \lambda(u) & \lambda(u)\vartheta(u) \\ 0 & I_K \end{pmatrix}$$

extends evenly, i.e.,

$$\hat{Q}(x) = \hat{Q}(x^*) \Leftrightarrow \hat{Q}(\hat{u}(x)) = \hat{Q}(\hat{u}(x^*)).$$

Since in the real application, $G = Q$, we extend G evenly as the same as Q.

Similarly, we can extend the quantity $\Omega = \lambda(u)\mathrm{diag}\,(V^\sharp, \dots, V^\sharp)$ oddly, where

$$V^\sharp = \nabla u^0 + \sum_{a=1}^{K} \vartheta_a \nabla u^a,$$

i.e., if we write $\Omega = \Omega_1 \partial_1 + \Omega_2 \partial_2$ and $\widehat{\Omega} = \widehat{\Omega}_1 \partial_1 + \widehat{\Omega}_2 \partial_2$, then

$$\widehat{\Omega}(x) = -\rho_* \widehat{\Omega}(x) \Leftrightarrow \begin{cases} \widehat{\Omega}_1(x) = -\widehat{\Omega}_1(x^*), \\ \widehat{\Omega}_2(x) = \widehat{\Omega}_2(x^*), \end{cases}$$

or equally,

$$\widehat{\Omega}_1(x) = \begin{cases} -\Omega_1(x), & x \in D^+, \\ \Omega_1(x^*), & x \in D^-, \end{cases} \quad \widehat{\Omega}_2(x) = \begin{cases} \Omega_2(x), & x \in D^+, \\ \Omega_2(x^*), & x \in D^-. \end{cases}$$

In particular, we see that if $\Omega \in W^{1,2}$, then $\widehat{\Omega} \in W^{1,2}$ too, since the odd extension $\widehat{\Omega}_1 \equiv 0$ on $\partial^0 D^+$, which follows from the definition of Ω and the fact that $u \equiv 0$ on $\partial^0 D^+$. For the same reason, we can extend $W = \mathrm{diag}\,(w, \dots, w)$ oddly, where $w = \mathcal{RP}_0(\phi, \psi)$.

For the matrix-valued 1-form $\Theta \in L^2$, we extend it evenly, i.e.,

$$\widehat{\Theta}(x) = \rho^* \widehat{\Theta}(x).$$

Finally, we need to extend the matrix $F \in W^{1,2}$ oddly, where

$$\begin{aligned} F &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\Upsilon_1 & \Upsilon_{11} & \cdots & \Upsilon_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ -\Upsilon_K & \Upsilon_{KK} & \cdots & \Upsilon_{KK} \end{pmatrix}, \\ \Upsilon_d &:= \frac{1}{2\lambda(u)}(\partial_d \lambda - \partial_e \lambda v_l^e v_l^d), \\ \Upsilon_{db} &:= \partial_b \vartheta_d - \partial_d \vartheta_b - (\partial_b \vartheta_e - \partial_e \vartheta_b) v_l^e v_l^d, \end{aligned}$$

and $\nu_l = (v_l^1, \dots, v_l^K)$, $l = n+1, \dots, K$ are the orthonormal frame of $T^\perp N$ in $\mathbb{R}^K$, and recall that λ and ϑ are extended to a tabular neighborhood of $N \hookrightarrow \mathbb{R}^K$ via the nearest projection. Note that we need to choose coordinates in $\mathbb{R}^K$ properly such that $\operatorname{grad} \lambda = 0$ at the origin and $\nabla_X^N \vartheta^\sharp = 0$ for any tangent vector X of N at the origin. These conditions ensure that $\Upsilon_d = \Upsilon_{db} = 0$ at the origin of $\mathbb{R}^K$, resulting in $\widehat{F} \in W^{1,2}(B)$ provided that $F \in W^{1,2}(D^+)$.

With these extensions in hand, we are now ready to prove the following regularity result, which serves as the starting point for blow-up analysis and plays a fundamental role in the development of a sequence of analysis lemmas that will be used repeatedly in blow-up analysis. It generalizes the regularity theorem of harmonic maps and Dirac-harmonic maps in pseudo-Riemannian manifolds to the boundary case.

Lemma 3.1. *Suppose that $D^+ \subset \mathbb{R}^2_+$ is a unit upper half-disc, and $n > 0$ is an integer. Denote by $\mathrm{M}(n)$ the set of $n \times n$ real matrices. For any $\Lambda > 0$, there exists an $\epsilon = \epsilon(\Lambda) > 0$ such that for every $\Theta \in L^2(D^+, \mathfrak{so}(n) \otimes \wedge^1 \mathbb{R}^2)$, $F, G \in W^{1,2} \cap L^\infty(D^+, \mathrm{M}(n))$, $Q \in W^{1,2} \cap L^\infty(D^+, \mathrm{GL}(n))$, and $W \in L^q(D^+, \mathrm{M}(n))$, $\upsilon \in L^s(D^+, \mathbb{R}^n)$ for some $1 < q < 2$ and $1 < s < 2$, if $u \in W^{1,2}(D^+, \mathbb{R}^n)$ is a weak solution of the following elliptic system*

$$\begin{cases} -\operatorname{div}(Q \nabla u) = \Theta \cdot Q \nabla u + F\Omega \cdot G \nabla u + \upsilon, & x \in D^+, \\ u = 0, & x \in \partial^0 D^+, \end{cases} \tag{3.1}$$

where $\partial^0 D^+$ is the flat boundary of D^+, and Ω satisfies

$$-\operatorname{div} \Omega = W, \quad x \in D^+ \tag{3.2}$$

in the weak sense, with the coefficients satisfying the following conditions

$$\begin{aligned} &\|\nabla u\|_{L^2(D^+)} + \|\Theta\|_{L^2(D^+)} + \|\Omega\|_{L^2(D^+)} + \|W\|_{L^q(D^+)} \\ &\quad + \|\nabla Q\|_{L^2(D^+)} + \|\nabla F\|_{L^2(D^+)} + \|\nabla G\|_{L^2(D^+)} \leqslant \epsilon \end{aligned} \tag{3.3}$$

and

$$|Q| + |Q^{-1}| + |F| + |G| \leqslant \Lambda \quad \text{a.e. in } D^+, \tag{3.4}$$

then for some $\alpha \in (0,1)$, we have

$$[u]_{C^\alpha(D^+_{1/2})} \leqslant C(\Lambda)(\epsilon + \|\upsilon\|_{L^s(D^+)}).$$

In particular, u is Hölder continuous in $D^+_{1/2}$.

Proof. Recall that we extend the objects appearing in the lemma as follows: For $x = (x^1, x^2)$, define $x^* = (x^1, -x^2)$,

$$\hat{u}(x) = \begin{cases} u(x), & x \in D^+, \\ -u(x^*), & x \in D^-, \end{cases} \quad \hat{\upsilon}(x) = \begin{cases} \upsilon(x), & x \in D^+, \\ -\upsilon(x^*), & x \in D^-, \end{cases}$$

and

$$\widehat{Q}(x) = \begin{cases} Q(x), & x \in D^+, \\ Q(x^*), & x \in D^-, \end{cases}$$

and similarly for the matrix G. For matrices W and F, we extend them oddly, and for Θ and Ω, we extend them as

$$\widehat{\Theta} = \begin{cases} \Theta(x), & x \in D^+, \\ (\rho^*\Theta)(x), & x \in D^-, \end{cases} \quad \widehat{\Omega} = \begin{cases} \Omega(x), & x \in D^+, \\ -(\rho^*\Omega)(x), & x \in D^-, \end{cases}$$

where $\rho : D^- \to D^+$ and $x \mapsto x^*$.

Let us first check the equation of $\widehat{\Omega}$. First, recall that the above extension makes $\widehat{\Omega} \in W^{1,2}(B)$ provided that $\Omega \in W^{1,2}(D^+)$, which is a consequence of $u \equiv 0$ on $\partial^0 D^+$. For any $\eta \in C_0^\infty(B, \mathbb{R}^n)$, we decompose it into odd and even parts as follows:

$$\eta(x) = \eta_o(x) + \eta_e(x), \quad \eta_o(x) = \frac{1}{2}(\eta(x) - \eta(x^*)), \quad \eta_e(x) = \frac{1}{2}(\eta(x) + \eta(x^*)).$$

Clearly, $\eta_o(x) = -\eta_o(x^*)$ and $\eta_e(x) = \eta_e(x^*)$.

Note that for $x \in D^-$,

$$\begin{aligned} &\langle \widehat{\Omega}(x), \nabla \eta_o(x) \rangle_{\bar{g}} = \langle \Omega(x^*), \nabla \eta_o(x^*) \rangle_{\bar{g}}, \quad \langle \widehat{\Omega}(x), \nabla \eta_e(x) \rangle_{\bar{g}} = -\langle \Omega(x^*), \nabla \eta_e(x^*) \rangle_{\bar{g}}, \\ &\langle \widehat{W}(x), \eta_o(x) \rangle_{\bar{g}} = \langle W(x^*), \eta_o(x^*) \rangle_{\bar{g}}, \quad \langle \widehat{W}(x), \eta_e(x) \rangle_{\bar{g}} = -\langle W(x^*), \eta_e(x^*) \rangle_{\bar{g}}, \end{aligned}$$

and since $\eta_o \equiv 0$ on $\partial^0 D^+$, test $-\operatorname{div} \Omega = W$ with η_o, and we obtain

$$0 = \int_{D^+} \langle \Omega, \nabla \eta_o \rangle_{\bar{g}} - \int_{D^+} \langle W, \eta_o \rangle_{\bar{g}} = \int_{D^-} \langle \Omega(x^*), \nabla \eta_o(x^*) \rangle_{\bar{g}} - \int_{D^-} \langle W(x^*), \eta_o(x^*) \rangle_{\bar{g}},$$

i.e.,

$$0 = \int_{D^+} \langle \widehat{\Omega}, \nabla \eta_o \rangle_{\bar{g}} - \int_{D^+} \langle \widehat{W}, \eta_o \rangle_{\bar{g}} = \int_{D^-} \langle \widehat{\Omega}, \nabla \eta_o \rangle_{\bar{g}} - \int_{D^-} \langle \widehat{W}, \eta_o \rangle_{\bar{g}}. \tag{3.5}$$

Similarly,

$$\int_{D^+} \langle \Omega, \nabla \eta_e \rangle_{\bar{g}} - \int_{D^+} \langle W, \eta_e \rangle_{\bar{g}} = \int_{D^-} \langle \Omega(x^*), \nabla \eta_e(x^*) \rangle_{\bar{g}} - \int_{D^-} \langle W(x^*), \eta_e(x^*) \rangle_{\bar{g}},$$

i.e.,

$$\int_{D^+} \langle \widehat{\Omega}, \nabla \eta_e \rangle_{\bar{g}} - \int_{D^+} \langle \widehat{W}, \eta_e \rangle_{\bar{g}} = -\int_{D^-} \langle \widehat{\Omega}, \nabla \eta_e \rangle_{\bar{g}} + \int_{D^-} \langle \widehat{W}, \eta_e \rangle_{\bar{g}}. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\int_{D^+} \langle \widehat{\Omega}, \nabla \eta \rangle_{\bar{g}} - \int_{D^+} \langle \widehat{W}, \eta \rangle_{\bar{g}} = -\int_{D^-} \langle \widehat{\Omega}, \nabla \eta \rangle_{\bar{g}} + \int_{D^-} \langle \widehat{W}, \eta \rangle_{\bar{g}},$$

which implies $-\operatorname{div} \widehat{\Omega} = \widehat{W}$ in the weak sense.

In the same vein, we can verify that $\hat{u}$ satisfies the following equation weakly:

$$-\operatorname{div}(\widehat{Q} \nabla \hat{u}) = \widehat{\Theta} \cdot \widehat{Q} \nabla \hat{u} + \widehat{F} \widehat{\Omega} \cdot \widehat{G} \nabla \hat{u}, \quad x \in B.$$

It can be verified directly that the extended quantities are still in the same Sobolev space as before, and the conditions (3.3) and (3.4) are satisfied with constants ϵ and Λ being replaced by a multiple, respectively. Then, we are ready to employ the interior regularity theorem (see [1, Theorem C] or Lemma 2.2, to conclude the result). $\square$

As a first application, the above regularity result implies the boundary regularity of *weak* Dirac-harmonic maps in stationary Lorentzian manifolds. Recall that for $V \subset \partial U$, the space $W^{k,p}_{\partial}(V,\mathbb{R}^n)$ is defined as

$$W^{k,p}_{\partial}(V,\mathbb{R}^n) := \{u_\partial \in L^1(V) : u_\partial = \tilde{u}|_V,\ \tilde{u} \in W^{k,p}(U)\}$$

with the norm

$$\|u_\partial\|_{W^{k,p}_{\partial}(V,\mathbb{R}^n)} := \inf_{\substack{\tilde{u}\in W^{k,p}(U),\\ \tilde{u}|_V = u_\partial}} \|\tilde{u}\|_{W^{k,p}(U)}.$$

Corollary 3.2. *Suppose that $u_\partial \in W^{2,2}_{\partial}(\partial^0 D^+,\mathbb{R}^n)$, and $u \in W^{1,2}(D^+,\mathbb{R}^n)$ is a weak solution of the following elliptic system with the non-homogeneous boundary condition*

$$\begin{cases} -\mathrm{div}\,(Q\nabla u) = \Theta\cdot Q\nabla u + F\Omega\cdot G\nabla u + \upsilon, & x \in D^+,\\ -\mathrm{div}\,\Omega = W, & x \in D^+,\\ u = u_\partial, & x \in \partial^0 D^+. \end{cases} \tag{3.7}$$

If the same condition in Lemma 3.1 *is satisfied, then for some $\alpha \in (0,1)$, we have*

$$[u]_{C^\alpha(D^+_{1/2})} \leqslant C(\Lambda)(\epsilon + \|\upsilon\|_{L^s(D^+)} + \|u_\partial\|_{W^{2,2}_{\partial}(\partial^0 D^+)}).$$

In particular, u is Hölder continuous in $D^+_{1/2}$.

Remark 3.3. Since the map ϕ of a weak Dirac-harmonic map $(\phi,\psi) \in \mathcal{X}^w(M,\mathcal{N})$ satisfies (3.7), Corollary 3.2 implies that ϕ is Hölder continuous on M up to the boundary.

Proof. Consider $v = u - \tilde{u}$, where $\tilde{u} \in W^{2,2}(D^+,\mathbb{R}^n)$, $\tilde{u} = u_\partial$ on $\partial^0 D^+$, and $u_\partial \in W^{2,2}_{\partial}(\partial^0 D^+,\mathbb{R}^n)$. The equation for v is given by

$$\begin{cases} -\mathrm{div}\,(Q\nabla v) = \Theta\cdot Q\nabla v + F\Omega\cdot G\nabla v + \tilde{\upsilon}, & x \in D^+,\\ -\mathrm{div}\,\Omega = W, & x \in D^+,\\ v = 0, & x \in \partial^0 D^+, \end{cases}$$

where

$$\tilde{\upsilon} := \upsilon + \nabla Q\cdot\nabla\tilde{u} + Q\cdot\Delta\tilde{u} + \Theta\cdot Q\nabla\tilde{u} + F\Omega\cdot G\nabla\tilde{u}.$$

Note that

$$\|\tilde{\upsilon}\|_{L^s} \leqslant C(\Lambda)(\epsilon + \|\upsilon\|_{L^s} + \|\tilde{u}\|_{W^{2,2}}),$$

and the results follow from the last inequality, Lemma 3.1, and taking the infimum of $\tilde{u}$. □

3.2 The removal of singularities

The following removal of singularities for weak Dirac-harmonic maps generalizes the corresponding interior results to the boundary case; we refer to [2, Theorem 2.6]. Thanks to the structure discovered in [1, Proposition 4.1], then the removal of the isolated singularities of Dirac-harmonic maps follows as a corollary of Lemma 3.1.

Corollary 3.4. *Suppose that $(\phi,\psi) \in \mathcal{X}(\mathring{D}^+,\mathcal{N})$ is a regular Dirac-harmonic map from the punctured upper half-disc $\mathring{D}^+ = D^+ \setminus \{0\}$ to a stationary Lorentzian manifold $(\mathcal{N}, g_{\mathcal{N}})$ satisfying the boundary condition* (1.2) *on $\partial^0 D^+$. Then (ϕ,ψ) can be extended to a regular Dirac-harmonic map over the whole upper half-disc D^+.*

Remark 3.5. The above result generalizes the corresponding result in [12, Theorem 2.8] by removing the static condition, where the authors addressed the removal of isolated energy of weak harmonic maps (vanishing spinor fields of Dirac-harmonic maps) into Lorentzian manifolds. The static condition is a strong one, which simplifies the structure of the equation and even allows for the no-neck property to hold for a uniformly bounded energy blow-up sequence.

Remark 3.6. The main improvement relies on the result that the equation of weak Dirac-harmonic maps from a surface into a static Lorentzian manifold satisfies a good structure (see [1, Proposition 4.1]), which allows for compensation regularity (see [1, Theorem C]) and generalizes many of the regularity theorems of harmonic maps and Dirac-harmonic maps. This structure provides a more comprehensive understanding of the regularity of these maps.

Proof of Corollary 3.4. As the same strategy in the interior case, we can verify that $(\phi,\psi)\in\mathcal{X}(\mathring{D}^+,\mathcal{N})$ can be extended to a weak Dirac-harmonic map over the whole upper half-disc D^+, and the map satisfies (3.7). Then, Corollary 3.2 of the regularity result Lemma 3.1 implies that ϕ is Hölder continuous. An easy modification of the argument in the interior case (see [1, Theorem 5.1]) to the boundary case implies that (ϕ,ψ) is actually regular over the whole upper half-disc D^+. $\square$

3.3 Small-energy regularity

In this subsection, we state the small-energy regularity, and the proof follows easily from [2, Theorem 2.2] and [19, Theorem 2.1]. We remark that there is a minor improvement in the range of index q compared with [19, Theorem 2.1].

Lemma 3.7 (See [2, Theorem 2.2]). *There is a small constant $\epsilon_0>0$ such that for any (L^p,L^q)-approximate Dirac-harmonic map $(\phi,\psi)\in\mathcal{X}(B,\mathcal{N})$, where $p,q\in(1,2]$, if*

$$E(\phi,\psi;B)=\int_B(|\nabla\phi|^2+|\psi|^4)<\epsilon_0^2, \tag{3.8}$$

then when $p,q\in(1,2)$,

$$\|\phi-\bar{\phi}\|_{W^{2,p}(B_{1/2})}\leqslant C(\|\nabla\phi\|_{L^p(B)}+\|\tau\|_{L^p(B)}), \tag{3.9}$$

$$\|\psi\|_{W^{1,q}(B_{1/2})}\leqslant C(\|\psi\|_{L^q(B)}+\|\kappa\|_{L^q(B)}), \tag{3.10}$$

where $\bar{\phi}$ is the integral mean over B and $C>0$ is a constant depending only on p,q, and the constant Λ' given in (2.1)*. Moreover, if we further assume that (ϕ,ψ) is a $(\Lambda;L^2,L^2)$-approximate Dirac-harmonic map, i.e.,*

$$\|\tau\|_{L^2(B)}+\|\kappa\|_{L_2(B)}\leqslant\Lambda<+\infty,$$

then the same results hold for $p=q=2$, except that the constant C also depends on Λ.

Let $\tilde{\phi}$ be a solution of

$$\begin{cases}\Delta\tilde{\phi}=0, & x\in M,\\ \tilde{\phi}=\phi_\partial, & x\in\partial M,\end{cases}$$

i.e., $\tilde{\phi}$ is the harmonic extension of ϕ_∂ on M. Since $\phi_\partial\in C^{2,\alpha}(\partial M)$, we have

$$\|\tilde{\phi}\|_{C^{2,\alpha}(M)}\leqslant C(\alpha,M)\|\phi_\partial\|_{C^{2,\alpha}(\partial M)}.$$

For the boundary case, we have the following lemma.

Lemma 3.8. *There is a small constant $\epsilon_0'>0$ such that for any (L^p,L^q)-approximate Dirac-harmonic map $(\phi,\psi)\in\mathcal{X}(D^+,\mathcal{N})$ satisfying the boundary condition* (1.2) *on ∂^0D^+, where $p,q\in(1,2]$, if*

$$E(\phi,\psi;D^+)=\int_{D^+}(|\nabla\phi|^2+|\psi|^4)<(\epsilon_0')^2, \tag{3.11}$$

then when $p,q\in(1,2)$,

$$\|\phi-\bar{\phi}_\partial\|_{W^{2,p}(D^+_{1/2})}\leqslant C(\|\nabla\phi\|_{L^p(D^+)}+\|\tau\|_{L^p(D^+)}+\|\nabla\tilde{\phi}\|_{W^{1,p}(D^+)}), \tag{3.12}$$

$$\|\psi\|_{W^{1,q}(D^+_{1/2})}\leqslant C(\|\psi\|_{L^q(D^+)}+\|\kappa\|_{L^q(D^+)}+\|\mathcal{B}\psi_\partial\|_{W^{1-1/q,q}(\partial^0D^+)}), \tag{3.13}$$

where $\bar{\phi}_\partial$ is the integral mean over $\partial^0 D^+$ and $C>0$ is a constant depending only on p, q, and the constant Λ' given in (2.1).

Moreover, if we further assume that (ϕ,ψ) is a $(\Lambda; L^2, L^2)$-approximate Dirac-harmonic map, i.e.,

$$\|\tau\|_{L^2(D^+)} + \|\kappa\|_{L^2(D^+)} \leqslant \Lambda < +\infty,$$

then the same results hold for $p=q=2$, except that the constant C also depends on Λ.

Corollary 3.9. *Suppose that (ϕ,ψ) satisfies the condition in Lemma* 3.7 *or Lemma* 3.8*. Then by the Sobolev embedding $W^{2,p}\subset C^0$, we have*

$$\|\phi\|_{\mathrm{osc}(B_{1/2})} := \sup_{x,y\in B_{1/2}} |\phi(x)-\phi(y)| \leqslant C(\|\nabla\phi\|_{L^p(B)} + \|\tau\|_{L^p(B)})$$

or

$$\|\phi\|_{\mathrm{osc}(D^+_{1/2})} := \sup_{x,y\in D^+_{1/2}} |\phi(x)-\phi(y)| \leqslant C(\|\nabla\phi\|_{L^p(D^+)} + \|\tau\|_{L^p(D^+)} + \|\phi_\partial\|_{C^{2,\alpha}(\partial^0 D^+)}),$$

respectively.

3.4 The gap theorem

We are now ready to prove the gap theorem of Dirac-harmonic maps from $\mathbb{R}^2_+$ to stationary Lorentzian manifolds. This proof is based on the removal of the singularities theorem (see Corollary 3.4), and extends the corresponding theorem in harmonic maps into Lorentzian harmonic maps to Dirac-harmonic maps, as seen in [12, Theorem 2.7].

Corollary 3.10. *Suppose that $(\phi,\psi)\in\mathcal{X}(\mathbb{R}^2_+,\mathcal{N})$ is a regular Dirac-harmonic map satisfying the boundary condition $\phi=\mathrm{const.}$ and $\mathcal{B}\psi=0$ on the boundary $\partial\mathbb{R}^2_+$. Then, there exists a constant $\underline{\epsilon}_0>0$ depending on $(\mathcal{N}, g_{\mathcal{N}})$ such that if*

$$\int_{\mathbb{R}^2_+} (|\nabla\phi|^2 + |\psi|^4) < \underline{\epsilon}_0^2,$$

then ϕ is trivial (*constant*) *and $\psi\equiv 0$.*

Proof. The proof is based on a modification of the argument for Dirac-harmonic maps in [19, Theorem 1.4]. Consider the conformal map

$$f:\mathbb{R}^2_+\to B,\quad (x,y)=z\mapsto \mathrm{i}\frac{z-\mathrm{i}}{z+\mathrm{i}}=(u,v),$$

and it is clear that $f(\mathrm{i})=0$ and $f(\partial\mathbb{R}^2_+)=\partial B\backslash\{\mathrm{i}\}$. Also, for $g_B:=du^2+dv^2$ and $g_{\mathbb{R}^2_+}:=dzd\bar{z}=dx^2+dy^2$, we have

$$(f^{-1})^* g_{\mathbb{R}^2_+} = \zeta^2 g_B,\quad \zeta=\frac{2}{u^2+(v-1)^2}.$$

Since $\mathcal{L}(\phi\circ f^{-1}, \xi^{-1/2}\psi\circ f^{-1}) = \mathcal{L}(\phi,\psi)$, we know $(\tilde{\phi},\tilde{\psi})$, where

$$\tilde{\phi}:=\phi\circ f^{-1},\quad \tilde{\psi}:=\xi^{-1/2}\psi\circ f^{-1},$$

is a regular Dirac-harmonic map on $B\setminus\{\mathrm{i}\}$. Since f^{-1} maps the boundary of $B\setminus\{\mathrm{i}\}$ to $\partial\mathbb{R}^2_+$, and the boundary condition involves taking no derivatives, we know that the boundary condition is transformed into $\tilde{\phi}=0$ and $\mathcal{B}\tilde{\psi}=0$ on $\partial B\setminus\{\mathrm{i}\}$.

By Corollary 3.4, we know that $(\tilde{\phi},\tilde{\psi})$ can be extended to a regular Dirac-harmonic map over the whole ball (B, g_B), and the boundary conditions are satisfied over ∂B. Since the positive energy is also conformal invariant, we have

$$\int_B |\nabla\tilde{\phi}|^2 = \int_B (|\nabla\tilde{\phi}|^2 + |\tilde{\psi}|^4) = \int_{\mathbb{R}^2_+} (|\nabla\phi|^2 + |\psi|^4) < \underline{\epsilon}_0^2.$$

Thus, by the elliptic estimates for Dirac equations (see [7, Theorem 1.1]), when ϵ_0 is small enough, we conclude that $\tilde{\psi} \equiv 0$ and $\tilde{\phi}$ is a harmonic map from B to $\mathcal{N}$ with the constant Dirichlet boundary condition. Now, note that $\tilde{\phi}|_{\partial B}$ is a constant. If we take $\epsilon_0 < \epsilon_0'$, then Lemma 3.8 implies

$$R\|\nabla\tilde{\phi}\|_{L^\infty(D^+_{R/2})} \leqslant C\|\nabla\tilde{\phi}\|_{L^2(D^+_R)} \leqslant C\epsilon_0, \quad \forall D^+_R \subset \mathbb{R}^2_+.$$

Taking $R \to +\infty$, we obtain $\tilde{\phi}$ is a constant. Since (ϕ, ψ) is conformal to $(\tilde{\phi}, \tilde{\psi})$, we know that ϕ is a constant and $\psi \equiv 0$. □

3.5 An elliptic estimate of spinors over the neck

In the proof of the positive energy identity of spinors, we need the following elliptic estimate of spinors over the neck. Define $A^+_{\rho_1,\rho_2} := D^+_{\rho_1} \setminus D^+_{\rho_2}$ for some $\rho_1 > \rho_2$. We refer to [2, Lemma 3.8] for the interior case.

Lemma 3.11. *Suppose that $(\phi, \psi) \in \mathcal{X}(A^+_{\rho_1,\rho_2}, \mathcal{N})$ is a $(\Lambda; L^2, L^2)$-approximate Dirac-harmonic map satisfying the boundary condition (1.2) on $\partial^0 A^+_{\rho_1,\rho_2} = \partial^0 D^+_{\rho_1} \setminus \partial^0 D^+_{\rho_2}$, where $0 < 4\rho_2 < \rho_1 < 1$. Then, we have*

$$\begin{aligned}
&\|\psi\|_{L^4(A^+_{\rho_1/2,2\rho_2})} + \|\nabla\psi\|_{L^{4/3}(A^+_{\rho_1/2,2\rho_2})} \\
&\quad\leqslant C_0(\|\nabla\phi\|_{L^2(A^+_{\rho_1,\rho_2})} \cdot \|\psi\|_{L^4(A^+_{\rho_1,\rho_2})} + \|\psi\|_{L^4(A^+_{\rho_1,\rho_2})} \\
&\qquad + \|\kappa\|_{L^{4/3}(A^+_{\rho_1,\rho_2})} + \|\rho_1^{1/4}\mathcal{B}\psi_\partial\|_{L^{4/3}(\partial^0 D^+_{\rho_1})} + [\mathcal{B}\psi_\partial]_{W^{1/4,4/3}(\partial^0 D^+_{\rho_1})}),
\end{aligned} \tag{3.14}$$

where C_0 is a constant independent of ρ_1 and ρ_2, and the Gagliardo semi norm $[u]_{W^{s,p}(\Omega)}$ is defined as

$$[u]_{W^{s,p}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{1/p}, \quad 0 < s < 1, \quad 1 \leqslant p < +\infty,$$

and the norm $\|u\|_{W^{s,p}(\Omega)}$ is defined as

$$\|u\|_{W^{s,p}(\Omega)} := (\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)})^{1/p}.$$

It is easy to verify that under the scaling $\tilde{\psi} = \sqrt{\rho}\psi(x_0 + \rho x)$, the semi norm $[\mathcal{B}\psi]_{W^{1/4,4/3}}$ is invariant.

Proof. Note that all the terms in (3.14) are invariant under the scalings $\tilde{\phi}(x) = \phi(x_0 + \rho x)$ and $\tilde{\psi}(x) = \sqrt{\rho}\psi(x_0 + \rho x)$. In particular, if we take $\rho = \rho_0 = \rho_2/\rho_1$, then we only need to prove (3.14) for $\rho_1 = 1$ and $\rho_2 = \rho_0$. Let η_0 be a cut-off function such that

$$\eta_0|_{A^+_{1/2,2\rho_0}} \equiv 1, \quad \operatorname{supp}\eta_0 \subset A^+_{1,\rho_0}, \quad |\nabla\eta_0| < \frac{4}{\rho_0}.$$

By (1.4), we know that

$$\partial\!\!\!/(\eta_0\psi) = \eta_0(\bar{\mathcal{A}}(d\phi(e_\alpha), e_\alpha\cdot\psi) - \overline{\Gamma}(d\phi(e_\alpha), e_\alpha\cdot\psi) + \kappa(\phi,\psi)) + \nabla\eta_0\cdot\psi.$$

Since

$$|\bar{\mathcal{A}}(d\phi(e_\alpha), e_\alpha\cdot\psi) - \overline{\Gamma}(d\phi(e_\alpha), e_\alpha\cdot\psi)| \leqslant C_0(\Lambda')|\nabla\phi||\psi|,$$

the elliptic estimate of the Dirac-operator implies

$$\begin{aligned}
\|\eta_0\psi\|_{W^{1,4/3}(D^+)} &\leqslant C_0(\Lambda')(\|\eta_0\cdot|\nabla\phi|\cdot|\psi|\|_{L^{4/3}(D^+)} + \|\eta_0\tilde{\kappa}\|_{L^{4/3}(D^+)} \\
&\quad + \||\nabla\eta_0|\cdot|\psi|\|_{L^{4/3}(D^+)} + \|\eta_0\cdot\mathcal{B}\psi_\partial\|_{W^{1/4,4/3}(\partial D^+)}) \\
&\leqslant C_0(\Lambda')(\|\nabla\phi\|_{L^2(A^+_{1,\rho_0})}\|\psi\|_{L^4(A^+_{1,\rho_0})} + \|\tilde{\kappa}\|_{L^{4/3}(A^+_{1,\rho_0})} \\
&\quad + \|\psi\|_{L^4(A_{2\rho_0,\rho_0})} + \|\mathcal{B}\psi_\partial\|_{W^{1/4,4/3}(\partial^0 D^+)}).
\end{aligned}$$

Now, note that

$$\|\mathcal{B}\psi_\partial\|_{W^{1/4,4/3}(\partial^0 D^+)} \leqslant 2(\|\mathcal{B}\psi_\partial\|_{L^{4/3}(\partial^0 D^+)} + [\mathcal{B}\psi_\partial]_{W^{1/4,4/3}(\partial^0 D^+)}).$$

Combining it with the Sobolev embedding $W^{1,4/3}(D^+) \hookrightarrow L^4(D^+)$, and scaling back, we obtain the result. □

4 The energy identity

In this section, we first prove Theorem 1.2 by applying the standard blow-up procedure to constructing the bubble tree and reduce the neck region to two types based on the ratio of the blow-up radius and the distance of the blow-up point to the boundary, as shown in Figures 1 and 2. We then prove the energy identity for spinors, which is based on the elliptic estimates in Lemma 3.11 over the neck. Finally, we prove the energy identity for maps, using a new observation which eliminates the need for a Pohozaev-type argument. Corollary 1.4 will then be proved as an easy consequence.

4.1 The construction of the bubble tree

Suppose that $\{(\phi_n,\psi_n)\}\subset\mathcal{X}^w(M,\mathcal{N})$ is a sequence of $(\Lambda;L^2,L^2)$-approximate Dirac-harmonic maps satisfying the boundary condition (1.2), i.e.,

$$E(\phi_n,\psi_n;M)\leqslant\Lambda,\quad \|\tau(\phi_n,\psi_n)\|_{L^2}+\|\kappa(\phi_n,\psi_n)\|_{L^2}\leqslant\Lambda,$$

and

$$\begin{cases}\phi_n=\phi_\partial, & x\in\partial M,\\ \mathcal{B}\psi_n=\mathcal{B}\psi_\partial, & x\in\partial M,\end{cases}$$

where $\phi_\partial\in C^{2,\alpha}(\partial M,\mathcal{N})$ and $\psi_\partial\in C^{1,\alpha}(\partial M,\Sigma M\otimes\phi_n^{-1}T\mathcal{N}|_{\partial M})$. The blow-up or energy concentration set of $\{(\phi_n,\psi_n)\}$ is defined by

$$\mathcal{S}:=\left\{x\in M:\lim_{r\to0}\liminf_{n\to\infty}\int_{B_r(x)\cap M}(|\nabla\phi_n|^2+|\psi_n|^4)\geqslant\bar{\epsilon}_0^2\right\}.$$

A direct consequence of the above definition is the finiteness of $\mathcal{S}$. In fact, suppose $\{p_i\}_{i=1}^I\subset\mathcal{S}$. Then we can take r small enough such that $\{B_r(p_i)\cap M\}_{i=1}^L$ are mutually disjoint, and

$$\Lambda\geqslant E(\phi_n,\psi_n;M)\geqslant\sum_{i=1}^I E(\phi_n,\psi_n;B_r(p_i)\cap M).$$

Taking limits, we obtain $I\leqslant\Lambda/\bar{\epsilon}_0^2<\infty$.

Since the energy identity and the no-neck property are proved for the closed Riemannian surface [2], we focus on the boundary case. Without loss of generality, assume that $0\in\partial M$ is an isolated energy concentration point in $D^+=B\cap M\subset M$, i.e., for any $0<r<1$,

$$\liminf_{n\to\infty}E(\phi_n,\psi_n;D_r^+)\geqslant\bar{\epsilon}_0^2.$$

Let $x_n\in D^+$ and $r_n>0$ such that

$$E(\phi_n,\psi_n;D_{r_n}^+(x_n))=\sup_{\substack{x\in D^+,r\leqslant r_n\\ D_r^+(x)\subset D^+}}E(\phi_n,\psi_n;D_r^+(x))=\bar{\epsilon}_0^2/4. \tag{4.1}$$

Clearly, by Lemma 3.8, $x_n\to0$ and $r_n\to0$.

We blow up the sequence $\{(\phi_n,\psi_n)\}$ at the scale r_n and obtain a bubble at the energy concentration point 0. Let

$$\tilde{\phi}_n(x):=\phi_n(x_n+r_nx),\quad \tilde{\psi}_n(x):=\sqrt{r_n}\psi_n(x_n+r_nx),\quad x\in B_n:=\{x\in\mathbb{R}^2:x_n+r_nx\in D^+\}.$$

Then the equations of $\tilde{\phi}_n$ and $\tilde{\psi}_n$ are given by

$$\tilde{\tau}(\tilde{\phi}_n,\tilde{\psi}_n)=r_n^2\tau(\phi_n,\psi_n)=\Delta_M\tilde{\phi}_n-\bar{A}(\nabla\tilde{\phi}_n,\nabla\tilde{\phi}_n)-\overline{\mathcal{P}}(\bar{\mathcal{A}}(d\tilde{\phi}_n(e_\alpha),e_\alpha\cdot\tilde{\psi}_n);\tilde{\psi}_n)-\overline{\mathcal{R}}(\tilde{\phi}_n,\tilde{\psi}_n) \tag{4.2}$$

and

$$\tilde{\kappa}(\tilde{\phi}_n,\tilde{\psi}_n)=r_n^{3/2}\kappa(\phi_n,\psi_n)=\partial\!\!\!/\tilde{\psi}_n-\bar{\mathcal{A}}(d\tilde{\phi}_n(e_\alpha),e_\alpha\cdot\tilde{\psi}_n)+\overline{\Gamma}(d\tilde{\phi}_n(e_\alpha),e_\alpha\cdot\tilde{\psi}_n) \tag{4.3}$$

with boundary data

$$\begin{cases}\tilde{\phi}_n(x) = \phi_\partial(x_n + r_n x), & x_n + r_n x \in \partial^0 D^+,\\ \mathcal{B}\tilde{\psi}_n(x) = \sqrt{r_n}\mathcal{B}\psi_\partial(x_n + r_n x), & x_n + r_n x \in \partial^0 D^+.\end{cases} \tag{4.4}$$

In order to consider the limit of $(\tilde{\phi}_n, \tilde{\psi}_n)$, we divide the discussion into two cases based on the limit of d_n/r_n. Recall that $d_n := \operatorname{dist}(x_n, \partial^0 D^+)$.

Case (a) $\limsup_{n\to\infty} \frac{d_n}{r_n} < +\infty$.

In this case, without loss of generality, we may assume that the limit is a real number $a \geqslant 0$. Note that $B_n \to \mathbb{R}^2_a := \{x = (x^1, x^2) \in \mathbb{R}^2 : x^2 \geqslant -a\}$ as $n \to \infty$. It is easy to see that for any $R > 0$, if we take n large enough, then

$$E(\tilde{\phi}_n, \tilde{\psi}_n; B_{4R} \cap B_n) \leqslant E(\phi_n, \psi_n; B_{1/2}(x_n) \cap D^+) \leqslant \Lambda < \infty,$$

and if we set $\tilde{\tau}_n := \tilde{\tau}(\tilde{\phi}_n, \tilde{\psi}_n)$ and similarly for the notations τ_n, $\tilde{\kappa}_n$, and κ_n, then

$$\begin{aligned}&\|\tilde{\tau}_n\|_{L^2(B_{4R}\cap B_n)} + \|\tilde{\kappa}_n\|_{L^2(B_{4R}\cap B_n)}\\ &\quad\leqslant r_n\|\tau_n\|_{L^2(B_{1/2}(x_n)\cap D^+)} + r_n^{1/2}\|\kappa_n\|_{L^2(B_{1/2}(x_n)\cap D^+)} \leqslant \Lambda < \infty.\end{aligned}$$

Thus, by scaling invariance of the energy $E(\tilde{\phi}_n, \tilde{\psi}_n)$, we may apply Lemma 3.8 to (4.2)–(4.4) to obtain

$$\|\tilde{\phi}_n\|_{W^{2,2}(B_{4R}\cap B_n)} + \|\tilde{\psi}_n\|_{W^{1,2}(B_{4R}\cap B_n)} \leqslant C(R, \Lambda, \Lambda').$$

Since $D^+_{3R} + (0, d_n/r_n) \subset B_{4R} \cap B_n$ for $R > 2|a|$, we obtain

$$\|\tilde{\phi}_n(x - (0, d_n/r_n))\|_{W^{2,2}(D^+_{3R})} + \|\tilde{\psi}_n(x - (0, d_n/r_n))\|_{W^{1,2}(D^+_{3R})} \leqslant C(R, \Lambda, \Lambda'),$$

and there exists a subsequence. We will not distinguish such that

$$(\tilde{\phi}_n(x - (0, d_n/r_n)), \tilde{\psi}_n(x - (0, d_n/r_n)))$$

converges strongly to some $(\bar{\sigma}, \bar{\xi})$ in $\mathcal{X}^w_{\mathrm{loc}}(\mathbb{R}^2_+, \mathcal{N})$ with boundary data $\bar{\sigma} = \phi_\partial(0)$ and $\mathcal{B}\bar{\xi} = 0$ on $\partial\mathbb{R}^2_+$. Note that for $R > 2|a|$, we have $B_{2R} \cap B_n \cap \mathbb{R}^2_a + (0, a) \subset D^+_{3R}$ and $B_{2R} \cap B_n \cap \mathbb{R}^2_a + (0, d_n/r_n) \subset D^+_{3R}$, and therefore, $(\tilde{\phi}_n, \tilde{\psi}_n)$ converges strongly to $(\bar{\sigma}(x + (0, a)), \bar{\xi}(x + (0, a)))$ in $\mathcal{X}^w_0(B_{2R} \cap B_n \cap \mathbb{R}^2_a, \mathcal{N})$. Now, since the measure of $B_{2R} \cap B_n \setminus \mathbb{R}^2_a$ converges to zero, and due to the $W^{2,2} \times W^{1,2}$ boundedness of $(\tilde{\phi}_n, \tilde{\psi}_n)$ on $B_{4R} \cap B_n$, we know that

$$\begin{aligned}&\lim_{n\to\infty} \|\tilde{\phi}_n\|_{W^{1,2}(B_R\cap B_n)} = \|\bar{\sigma}(x + (0, a))\|_{W^{1,2}(B_R\cap\mathbb{R}^2_a)},\\ &\lim_{n\to\infty} \|\tilde{\psi}_n\|_{L^4(B_R\cap B_n)} = \|\bar{\xi}(x + (0, a))\|_{L^4(B_R\cap\mathbb{R}^2_a)}.\end{aligned} \tag{4.5}$$

We also note that by (4.1), we have

$$E(\tilde{\phi}_n, \tilde{\psi}_n; B_1 \cap B_n) = E(\phi_n, \psi_n; D^+_{r_n}(x_n)) = \bar{\epsilon}_0^2/4. \tag{4.6}$$

Clearly, (4.5) and (4.6) imply $E(\bar{\sigma}, \bar{\xi}; \mathbb{R}^2_+) \geqslant \bar{\epsilon}_0^2/4$, and thus Corollary 3.10 implies that $(\bar{\sigma}, \bar{\xi})$ is a non-trivial Lorentzian Dirac-harmonic map over $\mathbb{R}^2_+$ with boundary data $\bar{\sigma} = \phi_\partial(0)$ and $\mathcal{B}\bar{\xi} = 0$. By the removal of singularities (see Corollary 3.4), we know that $(\bar{\sigma}, \bar{\xi})$ is a non-trivial Lorentzian Dirac-harmonic disc from D to $\mathcal{N}$ with the boundary condition $\bar{\sigma} = \text{const.}$ and $\mathcal{B}\bar{\xi} = 0$ on ∂B. In particular, this proves Theorem 1.2(1)(a).

Case (b) $\limsup_{n\to\infty} \frac{d_n}{r_n} = +\infty$.

For this case, the scaled fields $(\tilde{\phi}_n, \tilde{\psi}_n)$ defined on B_n tend to the whole plane $\mathbb{R}^2$ as $n \to \infty$. Now, for any $y \in \mathbb{R}^2$, when n is large enough, (4.1) implies that

$$E(\tilde{\phi}_n, \tilde{\psi}_n; B_1(y)) \leqslant \bar{\epsilon}_0^2/4.$$

By Lemma 3.7, there exists a subsequence, still denoted by $\{(\tilde{\phi}_n, \tilde{\psi}_n)\}$, such that $(\tilde{\phi}_n, \tilde{\psi}_n)$ converges strongly to some $(\sigma, \xi) \in \mathcal{X}^w_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{N})$. Moreover, from (4.1), we know that $E(\sigma, \xi; B_1) = \bar{\epsilon}_0^2/4$ and (σ, ξ) can be extended to a non-trivial Dirac-harmonic sphere [2, Subsection 3.4], i.e., a *bubble*. In particular, this proves Theorem 1.2(1)(b).

Next, we follow the scheme in [10] to construct the bubble tree (see also [22, Appendix] for more details on such a construction). We still focus on the boundary case, without loss of generality, assuming that $0 \in \partial^0 D^+$ is an isolated energy concentration point. For large enough n, set $P_n := D^+_\delta \setminus D^+_{r_n R}(x_n)$, where $R > 0$ is any large real number, and r_n and x_n are defined as in (4.1). Usually, P_n is called the *neck region* of the bubble corresponding to the data (x_n, r_n). Based on the limit of d_n/r_n being finite or not, we also divide the discussion into two cases.

Case (a′) $\limsup_{n\to\infty} \frac{d_n}{r_n} < +\infty$.

Note that in this case, for any fixed $R > 0$, when n is large enough, we have that the neck region can be decomposed as (see Figure 1)

$$P_n = (D^+_\delta \setminus D^+_{\delta/2}(x'_n)) \cup (D^+_{\delta/2}(x'_n) \setminus D^+_{2r_n R}(x'_n)) \cup (D^+_{2r_n R}(x'_n) \setminus D^+_{r_n R}(x_n)),$$

and we denote the three regions by Ω_i $(i = 1, 2, 3)$. $x'_n \in \partial^0 D^+$ is the projection of x_n onto $\partial^0 D^+$, i.e.,

$$d_n = \operatorname{dist}(x_n, \partial^0 D^+) = |x_n - x'_n|.$$

Since $\lim_{n\to\infty} d_n/r_n < +\infty$, we know that for any fixed large enough R, when n is large enough, we have

$$\Omega_1 \subset D^+_{2\delta}(x_n) \setminus D^+_{\delta/2}(x_n), \quad \Omega_3 \subset D^+_{4r_n R}(x_n) \setminus D^+_{r_n R}(x_n).$$

Case (b′) $\limsup_{n\to\infty} \frac{d_n}{r_n} = +\infty$.

Note that in this case, for any fixed $R > 0$, when n is large enough, we have

$$D^+_{r_n R}(x_n) = B_{r_n R}(x_n) \subset D^+_\delta,$$

and from Figure 2, it is clear that the neck region P_n can be decomposed as

$$\begin{aligned} P_n = {} & (D^+_\delta \setminus D^+_{\delta/2}(x'_n)) \cup (D^+_{\delta/2}(x'_n) \setminus D^+_{2d_n}(x'_n)) \\ & \cup (D^+_{2d_n}(x'_n) \setminus D^+_{d_n}(x_n)) \cup (D^+_{d_n}(x_n) \setminus D^+_{r_n R}(x_n)), \end{aligned}$$

and we denote the four regions by Ω_i $(i = 1, 2, 3, 4)$. Since $\lim_{n\to\infty} d_n = 0$ and $\lim_{n\to\infty} d_n/r_n = +\infty$, when n is large enough, we have

$$\Omega_1 \subset D^+_{2\delta}(x_n) \setminus D^+_{\delta/2}(x_n), \quad \Omega_3 \subset D^+_{4d_n}(x_n) \setminus D^+_{d_n}(x_n), \quad \Omega_4 = B_{d_n}(x_n) \setminus B_{r_n R}(x_n).$$

First, suppose that there is only *one bubble* at 0 (either a Lorentzian Dirac-harmonic sphere or a Lorentzian harmonic disc with the Dirichlet-chiral boundary condition), where 0 is the isolated boundary energy concentration point in D^+. Then, we make the following claim.

Claim 4.1. Under the one-bubble assumption, we have that for any $\epsilon > 0$, there exist $\delta = \delta(\epsilon) > 0$, $R > 0$, and $n_0 = n_0(\delta, R) > 0$ such that for any $n \geqslant n_0$,

$$E(\phi_n, \psi_n; D^+_{2r}(x_n) \setminus D^+_r(x_n)) < \epsilon^2, \quad \forall\, r \in [r_n R/2, 2\delta]. \tag{4.7}$$

In particular, this proves Theorem 1.2(2).

In fact, if (4.7) does not hold, then there would exist $\epsilon_1 > 0$, $\delta_n \to 0$, and $R_n \to \infty$ such that for some $r'_n \in [r_n R_n/2, 2\delta_n]$, we have

$$E(\phi_n, \psi_n; D^+_{2r'_n}(x_n) \setminus D^+_{r'_n}(x_n)) = \int_{D^+_{2r'_n}(x_n) \setminus D^+_{r'_n}(x_n)} (|\nabla \phi_n|^2 + |\psi_n|^4) \geqslant \epsilon_1. \tag{4.8}$$

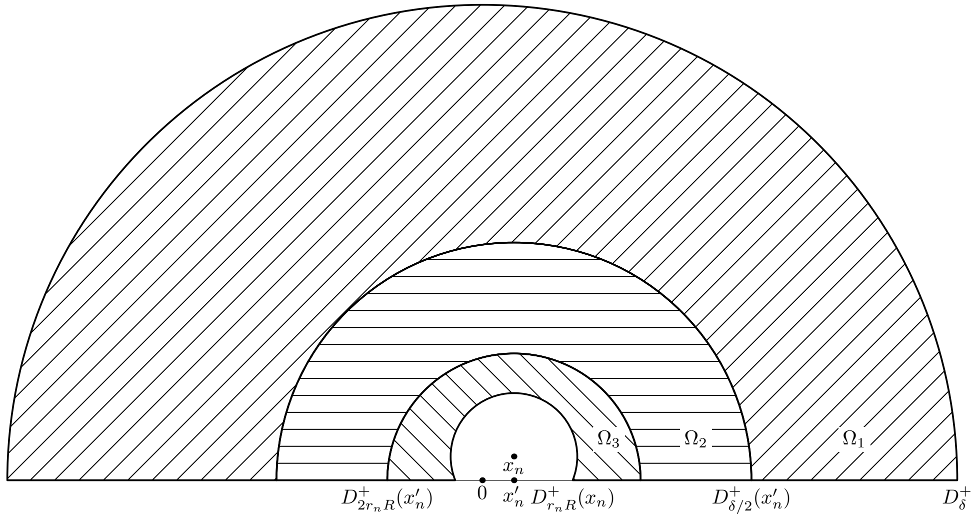

Figure 1 The neck region P_n is decomposed into three hatched regions in Case (a′)

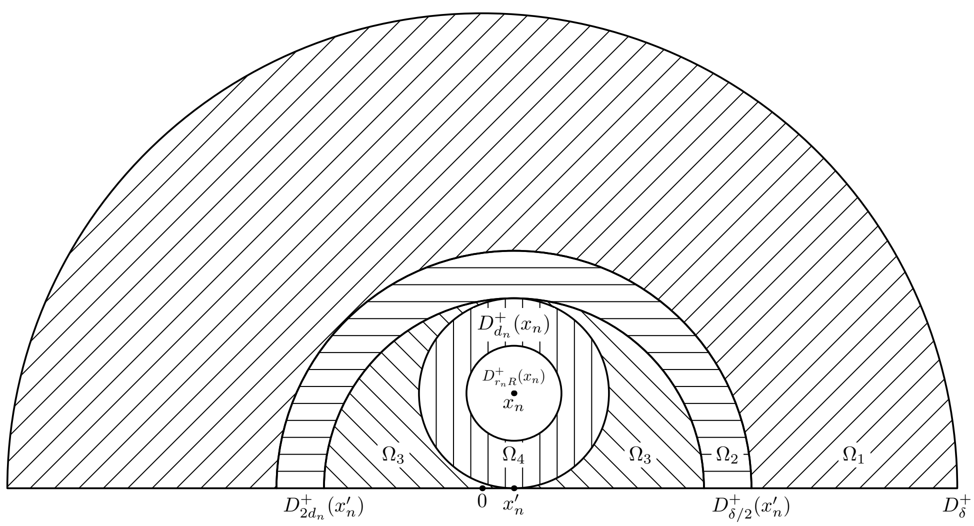

Figure 2 The neck region P_n is decomposed into four hatched regions in Case (b′)

We show that, in that case, there is another bubble. For that purpose, let

$$\phi'_n(x) := \phi_n(x_n + r'_n x), \quad \psi'_n(x) := \sqrt{r'_n}\psi_n(x_n + r'_n x), \quad x \in B'_n := \{x \in \mathbb{R}^2 : x_n + r'_n x \in D^+\}.$$

Clearly, (ϕ'_n, ψ'_n) is also an (L^2, L^2)-approximate Dirac-harmonic map with

$$\tau'_n := \tau(\phi'_n, \psi'_n) = {r'_n}^2 \tau(\phi_n, \psi_n), \quad \kappa'_n := \kappa(\phi'_n, \psi'_n) = {r'_n}^{3/2} \kappa(\phi_n, \psi_n),$$

which imply

$$E(\phi'_n, \psi'_n; B'_n) \leqslant E(\phi_n, \psi_n; D^+) \leqslant \Lambda < \infty$$

and

$$\|\tau'_n\|_{L^2(B'_n)} + \|\kappa'_n\|_{L^2(B'_n)} \leqslant r'_n \|\tau_n\|_{L^2(D^+)} + {r'_n}^{1/2} \|\kappa_n\|_{L^2(D^+)} \leqslant \Lambda < \infty.$$

By passing to a subsequence if necessary, we may assume

$$\lim_{n\to\infty} d_n / r'_n = b \in [0, +\infty].$$

We divide the discussion into two cases: (1) $b=+\infty$ and (2) $b<+\infty$. Clearly, when $b=+\infty$, $B_n'\to\mathbb{R}^2$, and when $b<+\infty$,

$$B_n'\to\mathbb{R}_b^+:=\{x=(x^1,x^2)\in\mathbb{R}^2:x^2>-b\}.$$

Note that in Case (a′), we must have $b=0$.

Claim 4.2. $0\in\mathcal{S}'$, where $\mathcal{S}'$ is the blow-up set of $\{(\phi_n',\psi_n')\}$.

In fact, by the definitions of r_n and x_n, we know that for any $r>0$, if n is large enough, then

$$\begin{aligned}\int_{B_r\cap B_n'}(|\nabla\phi_n'|^2+|\psi_n'|^4)&=\int_{B_{r_n\cdot r\cdot r_n'/r_n}(x_n)\cap D^+}(|\nabla\phi_n|^2+|\psi_n|^4)\\&\geqslant\int_{D^+_{r_n}(x_n)}(|\nabla\phi_n|^2+|\psi_n|^4)=\bar{\epsilon}_0^2/4,\end{aligned}\tag{4.9}$$

because $r_n'/r_n>R_n/2\to\infty$. This clearly implies that 0 is an energy concentration point for $\{(\phi_n',\psi_n')\}$. Otherwise, by the finiteness of energy concentration points, we can take r small enough such that $B_r\cap B_n'$ contains no energy concentration point. By Lemma 3.7, $(\phi_n',\psi_n')\to(\sigma',\xi')$ weakly in $\mathcal{X}(B_r,\mathcal{N})$ and $E(\phi_n',\psi_n';B_r)\to E(\sigma',\xi';B_r)$ when $b=+\infty$; similarly, by Lemma 3.8, $(\phi_n',\psi_n')\to(\sigma',\xi')$ weakly in $\mathcal{X}(B_r\cap\mathbb{R}_b^+,\mathcal{N})$ and $E(\phi_n',\psi_n';B_r\cap\mathbb{R}_b^+)\to E(\sigma',\xi';B_r\cap\mathbb{R}_b^+)$ when $b<+\infty$; in both cases, the energy $E(\phi_n',\psi_n';B_r\cap B_n')$ tends to 0 as $r\to0$, a contradiction to (4.9), and the claim is proved.

The proof of Claim 4.1 is based on the idea that if the claim is false, we can construct an additional bubble that contradicts the assumption of having only one bubble. To establish this, we divide the argument into the following cases:

(1) Case $b<+\infty$. In this case, $B_n'\to\mathbb{R}_b^2$ as $n\to\infty$.

To construct an alternative bubble, we further divide the argument into the following two cases:

(1a) $\{(\phi_n',\psi_n')\}$ has no other energy concentration point.

In this case, similar to the proof of Case (a), the small energy condition is satisfied on any $B_1(y)\cap B_n'$ for (ϕ_n',ψ_n'), and thus we can apply Lemma 3.8 to showing that there exists a Lorentzian Dirac-harmonic map $(\bar\sigma',\bar\xi')\in\mathcal{X}_{\rm loc}(\mathbb{R}^2_+,\mathcal{N})$ with boundary data $\bar\sigma'=\phi_\partial(0)$ and $\mathcal{B}\bar\xi'=0$ on $\partial\mathbb{R}^2_+$ such that by passing to a subsequence if necessary, we have

$$\begin{aligned}&\lim_{n\to\infty}\|\phi_n'\|_{W^{1,2}(B_R\cap B_n')}=\|\bar\sigma'(x+(0,b))\|_{W^{1,2}(B_R\cap\mathbb{R}_b^2)},\\&\lim_{n\to\infty}\|\psi_n'\|_{L^4(B_R\cap B_n')}=\|\bar\xi'(x+(0,b))\|_{L^4(B_R\cap\mathbb{R}_b^2)}.\end{aligned}$$

The above norm-convergence and (4.8) imply that

$$\begin{aligned}E(\bar\sigma',\bar\xi';\mathbb{R}^2_+)&\geqslant E(\bar\sigma'(x+(0,b)),\bar\xi'(x+(0,b));B_R\cap\mathbb{R}_b^2)\\&=\lim_{n\to\infty}E(\phi_n',\psi_n';B_R\cap B_n')\\&=\lim_{n\to\infty}E(\phi_n,\psi_n;B_{r_n'R}(x_n)\cap D^+)\\&\geqslant\liminf_{n\to\infty}E(\phi_n,\psi_n;D^+_{2r_n'}(x_n)\setminus D^+_{r_n'}(x_n))\geqslant\epsilon_1.\end{aligned}$$

By the conformal invariance of Lorentzian harmonic maps, the removal of singularities (see Corollary 3.4) and the gap theorem (see Corollary 3.10) imply that $(\bar\sigma',\bar\xi')$ can be extended to a non-trivial Lorentzian Dirac-harmonic disc, which can be viewed as another bubble with a boundary, and contradicts the "one bubble" assumption.

(1b) $\{(\phi_n',\psi_n')\}$ has another energy concentration point $p\neq0$.

Since the blow-up set of $\{(\phi_n',\psi_n')\}$ is a finite point set, we may assume that p is an isolated energy concentration point in $B(p)\cap B_n'$. Similar to the proceeding blow-up argument, there exist subsequences $x_n''\to p\in B(p)\cap B_n'$ and $r_n''\to0$ such that

$$E(\phi_n',\psi_n';B_{r_n''}(x_n'')\cap B_n')=\sup_{\substack{x\in B(p)\cap B_n',r\leqslant r_n''\\B_r(x)\cap B_n'\subset B(p)\cap B_n'}}E(\phi_n',\psi_n';B_r(x)\cap B_n')=\bar{\epsilon}_0^2/4.\tag{4.10}$$

For $x \in B''_n := \{x''_n + r''_n x \in B'_n\}$, define

$$\phi''_n(x) := \phi'_n(x''_n + r''_n x) = \phi_n(x_n + r'_n x''_n + r'_n r''_n x),$$
$$\psi''_n(x) := \sqrt{r''_n}\psi'_n(x''_n + r''_n x) = \sqrt{r'_n r''_n}\psi_n(x_n + r'_n x''_n + r'_n r''_n x).$$

Taking a subsequence if necessary, we may assume (note that by the definition of r''_n, we must have $r'_n r''_n > r_n$)

$$\lim_{n\to\infty} \frac{d_n + r'_n d'_n}{r''_n r'_n} = \lim_{n\to\infty} \frac{b + d'_n}{r''_n} = d \in [0, \bar{a}], \quad d'_n = \operatorname{dist}(x''_n, \partial B'_n), \quad \bar{a} = \limsup_{n\to\infty} \frac{d_n}{r_n}.$$

It is clear that if $d \leqslant \bar{a} < +\infty$, then $b = 0$ and $\lim_{n\to\infty} d'_n/r''_n = d$. As in the proof of Case (a), Lemmas 3.7 and 3.8 imply that (ϕ''_n, ψ''_n) sub-converges weakly to some Dirac-harmonic maps $(\bar{\sigma}''(\cdot + (0,d)), \bar{\xi}''(\cdot + (0,d)))$ in $\mathcal{X}_{\mathrm{loc}}(\mathbb{R}^2_d, \mathcal{N})$. The boundary condition for $(\bar{\sigma}'', \bar{\xi}'') \in \mathcal{X}_{\mathrm{loc}}(\mathbb{R}^2_+, \mathcal{N})$ is given by $\bar{\sigma}'' = \phi_\partial(0)$ and $\mathcal{B}\bar{\xi}'' = 0$ on $\partial\mathbb{R}^2_+$. By (4.10),

$$\begin{aligned}
E(\bar{\sigma}'', \bar{\xi}''; \mathbb{R}^2_+) &\geqslant E(\bar{\sigma}''(\cdot + (0,d)), \bar{\xi}''(\cdot + (0,d)); D^+ \cap \mathbb{R}^2_d)\\
&= \lim_{n\to\infty} E(\phi''_n, \psi''_n; D^+ \cap B''_n)\\
&= \lim_{n\to\infty} E(\phi'_n, \psi'_n; D^+_{r''_n}(x''_n) \cap B'_n) \geqslant \bar{\epsilon}_0^2/4.
\end{aligned}$$

Thus, similar to Case (1a), $(\bar{\sigma}'', \bar{\xi}'')$ can be extended to a non-trivial Lorentzian Dirac-harmonic disc, i.e., a bubble with a boundary.

If in the case $d = \bar{a} = +\infty$, $B''_n \to \mathbb{R}^2$, then similar to the construction of the bubble in Case (b), (ϕ''_n, ψ''_n) sub-converges to some Dirac-harmonic map $(\sigma'', \xi'') \in \mathcal{X}^w_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{N})$. Moreover, by the choices of x''_n and r''_n (see (4.10)), we know that (σ'', ξ'') is a non-trivial Dirac-harmonic map on $\mathbb{R}^2$, and the removal of singularities Corollary 3.4 implies that it is a bubble.

No matter whether d is finite or not, we can claim that the above-obtained bubble is a new one by verifying that the blow-up data $(x_n + r'_n x''_n, r'_n r''_n)$ of $(\bar{\sigma}'', \bar{\xi}'')$ and (σ'', ξ'') satisfy

$$\lim_{n\to\infty} \frac{|(x_n + r'_n x''_n) - x_n|}{r_n + r'_n r''_n} = \lim_{n\to\infty} \frac{|x''_n|}{r_n/r'_n + r''_n} = \infty,$$

as $r''_n \to 0$, $r'_n/r_n = R_n \to \infty$, and $x''_n \to p \neq 0$. In fact, [5, Lemma 3.6] shows two bubbles with blow-up data that violate (1.9) differ by an affine transformation. Thus, $(\bar{\sigma}'', \bar{\xi}'')$ is a bubble different from $(\bar{\sigma}, \bar{\xi})$ and (σ'', ξ'') is a bubble different from (σ, ξ).

(2) Case $b = +\infty$. In this case, $B'_n \to \mathbb{R}^2$ as $n \to \infty$.

As in the case $b < +\infty$ that we have already considered, to construct an alternative bubble, we further divide the argument into two cases.

(2a) $\{(\phi'_n, \psi'_n)\}$ has no other energy concentration point.

In this case, we know that by Lemma 3.7, $(\phi'_n, \psi'_n) \to (\sigma', \xi')$ weakly in $\mathcal{X}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}, \mathcal{N})$. By Corollary 3.4, (σ', ξ') is a Dirac-harmonic map over S^2. Since by (4.8),

$$E(\phi'_n, \psi'_n; B_2 \setminus B_1) = E(\phi_n, \psi_n; B_{2r'_n}(x_n) \setminus B_{r'_n}(x_n)) \geqslant \epsilon_1,$$

(σ', ξ') is a bubble. Note that the data for (σ', ξ') are (x_n, r'_n), which clearly satisfy (1.9), so (σ', ξ') and (σ, ξ) are distinct bubbles.

(2b) $\{(\phi'_n, \psi'_n)\}$ has another energy concentration point $p \neq 0$.

Since p is an energy concentration point of $\{(\phi'_n, \psi'_n)\}$ and the blow-up set is a finite point set, we may assume that p is an isolated energy concentration point in $B(p)$. We can do the same procedure as before to blow up (ϕ'_n, ψ'_n) such that for some $x''_n \to p \in B(p)$ and $r''_n \to 0$ satisfying (4.10), we have that $(\phi''_n(x), \psi''_n(x))$ converges to a bubble (σ'', ξ'') weakly in $\mathcal{X}_{\mathrm{loc}}(\mathbb{R}^2, \mathcal{N})$, where

$$\phi''_n(x) := \phi'_n(x''_n + r''_n x) = \phi_n(x_n + r'_n x''_n + r'_n r''_n x),$$

$$\psi_n''(x) := \sqrt{r_n''}\psi_n'(x_n'' + r_n''x) = \sqrt{r_n'r_n''}\psi_n(x_n + r_n'x_n'' + r_n'r_n''x).$$

Note that in this case, we must have $d = +\infty$, which is clear from $b = +\infty$ and the definition of d. Similar to Case (1b), we can verify that the blow-up data of (σ'', ξ'') satisfy (1.9). So (σ'', ξ'') is a new bubble distinct from (σ, ξ).

In conclusion, assuming that Claim 4.1 is false, then we can always construct a new bubble distinct from the original one, which contradicts the one-bubble assumption. This finishes the proof of Claim 4.1 under the one-bubble assumption.

4.2 The energy identity for spinors

Next, we prove the positive energy identity (1.5) for spinors in Theorem 1.2. We remark that the proof depends only on the elliptic estimates of Lemma 3.11. Let us decompose the neck region into finitely many parts, i.e.,

$$P_n = \bigcup_{k=1}^{Q_n'} P^k, \quad P^k = A^+_{r_{k+1}, r_k},$$

where $Q_n' \leqslant Q$ and Q is a uniform integer independent of n such that

$$E(\phi_n; P^k) \leqslant \frac{1}{4C^2}, \quad k = 1, \ldots, Q_n', \tag{4.11}$$

where C is the constant in Lemma 3.11. In fact, we can start from Claim 4.1 and increase the length of P^k until the energy of ϕ_n on P^k reaches $1/(4C^2)$, do this again and again, and we get the partition. As our energy $E(\phi_n, \psi_n)$ is uniformly bounded by Λ, the number of such divisions is uniformly bounded. We refer to [24, p. 134ff] for more details. Applying Lemma 3.11 on each P^k, we have

$$\begin{aligned}
\|\psi_n\|_{L^4(P^k)} &\leqslant C(\|\nabla\phi_n\|_{L^2(A^+_{2r_{k+1}, r_k/2})}\|\psi_n\|_{L^4(A^+_{2r_{k+1}, r_k/2})} + \|\psi_n\|_{L^4(A^+_{2r_{k+1}, r_k/2})} + \|\kappa_n\|_{L^{4/3}(A^+_{2r_{k+1}, r_k/2})} \\
&\quad + \|(2r_{k+1})^{1/4}\mathcal{B}\psi_\partial\|_{L^{4/3}(\partial^0 D^+_{2r_{k+1}})} + [\mathcal{B}\psi_\partial]_{W^{1/4,4/3}(\partial^0 D^+_{2r_{k+1}})}) \\
&\leqslant C(\|\nabla\phi_n\|_{L^2(P^k)}\|\psi_n\|_{L^4(P^k)} + \|\kappa_n\|_{L^{4/3}(D^+_{2r_{k+1}})}) \\
&\quad + C(\|(2r_{k+1})^{1/4}\mathcal{B}\psi_\partial\|_{L^{4/3}(\partial^0 D^+_{2r_{k+1}})} + [\mathcal{B}\psi_\partial]_{W^{1/4,4/3}(\partial^0 D^+_{2r_{k+1}})}) \\
&\quad + C(\|\nabla\phi_n\|_{L^2(A_{r_k, r_k/2})} + \|\nabla\phi_n\|_{L^2(A_{2r_{k+1}, r_{k+1}})})\|\psi_n\|_{L^4(P^k)} \\
&\quad + C\|\nabla\phi_n\|_{L^2(P^k)}(\|\psi_n\|_{L^4(A_{r_k, r_k/2})} + \|\psi_n\|_{L^4(A_{2r_{k+1}, r_{k+1}})}) \\
&\quad + C(\|\psi_n\|_{L^4(A_{r_k, r_k/2})} + \|\psi_n\|_{L^4(A_{2r_{k+1}, r_{k+1}})}) \\
&\quad \times (\|\nabla\phi_n\|_{L^2(A_{r_k, r_k/2})} + \|\nabla\phi_n\|_{L^2(A_{2r_{k+1}, r_{k+1}})}) + C\|\psi_n\|_{L^4(A_{r_k, r_k/2})}.
\end{aligned}$$

By Claim 4.1, we know that

$$\|\nabla\phi_n\|_{L^2(A_{r_k, r_k/2})} + \|\psi_n\|_{L^4(A_{r_k, r_k/2})} \leqslant 2\sqrt{\epsilon},$$

and combining it with (4.11) and the boundedness of the energy $E(\phi_n, \psi_n)$, we obtain

$$\|\psi_n\|_{L^4(P^k)} \leqslant C(\|\kappa_n\|_{L^{4/3}(D^+_{2r_{k+1}})} + \|(2r_{k+1})^{1/4}\mathcal{B}\psi_\partial\|_{L^{4/3}(\partial^0 D^+_{2r_{k+1}})} + [\mathcal{B}\psi_\partial]_{W^{1/4,4/3}(\partial^0 D^+_{2r_{k+1}})} + \sqrt{\epsilon}).$$

We also note that

$$\|\kappa_n\|_{L^{4/3}(D^+_{2r_{k+1}})} \leqslant \sqrt{2r_{k+1}}\|\kappa_n\|_{L^2(D^+_{2r_{k+1}})} \leqslant \Lambda\sqrt{2\delta}.$$

Finally, since $\psi_\partial \in C^{1,\alpha}(\partial M, \Sigma M \otimes \phi^{-1}T\mathcal{N})$ for some $\alpha \in (0,1)$, we obtain

$$\begin{aligned}
&\|(2r_{k+1})^{1/4}\mathcal{B}\psi_\partial\|_{L^4(\partial^0 D^+_{2r_{k+1}})} + \|\mathcal{B}\psi_\partial\|_{W^{1/4,4/3}(\partial^0 D^+_{r_{k+1}})} \\
&\quad \leqslant C(r_{k+1}^{1/4}\|\psi_\partial\|_{L^\infty(\partial^0 D^+_{2r_{k+1}})}|\partial^0 D^+_{2r_{k+1}}|^{1/4} + [\psi_\partial]_{C^{0,1}(\partial^0 D^+_{2r_{k+1}})}|\partial^0 D^+_{2r_{k+1}}|^{3/2})
\end{aligned}$$

$$\leqslant C\|\psi_\partial\|_{C^{1,\alpha}(\partial^0 M)}(r_{k+1}^{1/2}+r_{k+1}^{3/2}) \leqslant C(\delta^{1/2}+\delta^{3/2}).$$

In conclusion,

$$\|\psi_n\|_{L^4(P^k)} \leqslant C(\sqrt{\epsilon}+\sqrt{\delta}+\delta^{3/2}), \tag{4.12}$$

and the finiteness of the number of $\{P^k\}$ implies

$$\lim_{\delta\to 0}\lim_{R\to\infty}\lim_{n\to\infty}\|\psi_n\|_{L^4(D_\delta^+\setminus D_{Rr_n}^+(x_n))}=0, \tag{4.13}$$

which is equivalent to the positive energy identity (1.5) of spinors in Theorem 1.2. In particular, this proves Theorem 1.2(3).

4.3 The energy identity for maps

To prove Theorem 1.2(4), we only consider the case $\Omega_2 = D^+_{\delta/2}(x'_n)\setminus D^+_{2d_n}(x'_n)$. The argument is based on Claim 4.1, which holds in either case, so the case $\Omega_2 = D^+_{\delta/2}(x'_n)\setminus D^+_{2r_nR}(x'_n)$ can be proved without any further difficulty.

Firstly, we transform Claim 4.1 to regions centered at x'_n as follows: For any $2d_n \leqslant r \leqslant \delta$, we have

$$D^+_{2r}(x'_n)\setminus D^+_r(x'_n) \subset D^+_{4r}(x_n)\setminus D^+_{r/2}(x_n).$$

Thus, Claim 4.1 implies

$$E(\phi_n,\psi_n; D^+_{2r}(x'_n)\setminus D^+_r(x'_n)) \leqslant 3\epsilon^2, \quad \forall\, r\in[2d_n,\delta],$$

and applying the above small energy condition to Corollary 3.9, we see that the standard scaling argument induces

$$\begin{aligned}\operatorname{osc}_{D^+_{2r}(x'_n)\setminus D^+_r(x'_n)}\phi_n &\leqslant C(\|\nabla\phi\|_{L^2(D^+_{4r}(x'_n)\setminus D^+_{r/2}(x'_n))}+\|r\tau\|_{L^2(D^+_{4r}(x'_n)\setminus D^+_{r/2}(x'_n))}\\ &\quad+\sqrt{r}\|\phi_\partial\|_{C^{2,\alpha}(\partial^0 D^+_{4r}(x'_n)\setminus\partial^0 D^+_{r/2}(x'_n))})\\ &\leqslant C(\epsilon+\sqrt{\delta}).\end{aligned} \tag{4.14}$$

In order to estimate the energy of ϕ_n on Ω_2, let us define $\widehat{\Omega}_2 := B_{\delta/2}(x'_n)\setminus B_{2d_n}(x'_n)$ and $u_n(x) := \phi_n(x)-\tilde{\phi}(x)$ for $x\in\Omega_2$. Recalling that $\tilde{\phi}$ is the harmonic extension of ϕ_∂ over M and

$$\hat{u}_n := \begin{cases} u_n(x), & x\in\Omega_2,\\ -u_n(x^*), & x\in\widehat{\Omega}_2\setminus\Omega_2,\end{cases}$$

we see that $\hat{u}_n\in W^{2,2}(\widehat{\Omega}_2)$ satisfying the following equation:

$$\Delta\widehat{u}_n(x) = \begin{cases}\Delta\phi_n(x)-\Delta\tilde{\phi}, & x\in\Omega_2,\\ -(\Delta\phi_n(x^*)-\Delta\tilde{\phi}(x^*)), & x\in\widehat{\Omega}_2\setminus\Omega_2.\end{cases}$$

We also recall that

$$\Delta\phi_n(x) = \bar{A}(\nabla\phi_n,\nabla\phi_n)(x)+\overline{\mathcal{P}}(\bar{\mathcal{A}}(d\phi_n(e_\alpha),e_\alpha\cdot\psi_n);\psi_n)(x)+\overline{\mathcal{R}}(\phi_n,\psi_n)(x)+\tau(\phi_n,\psi_n)(x)$$

and

$$|\bar{A}(\nabla\phi_n,\nabla\phi_n)+\overline{\mathcal{P}}(\bar{\mathcal{A}}(d\phi_n(e_\alpha),e_\alpha\cdot\psi_n);\psi_n)+\overline{\mathcal{R}}(\phi_n,\psi_n)| \leqslant C(\Lambda')(|\nabla\phi_n|^2+|\nabla\phi_n|\cdot|\psi_n|^2).$$

Without loss of generality, we may assume $\delta/2 = 2^{T_n}(2d_n)$. Although the energy of ϕ_n is as small as that required on each $A^+_{2r,r}(x'_n)$, the main obstruction is that the number of that kind of annular regions tends to infinity as $n\to\infty$. Let us define

$$A^+_i := A^+_{2^{i+1}d_n,2^i d_n}(x'_n) := D^+_{2^{i+1}d_n}(x'_n)\setminus D^+_{2^i d_n}(x'_n),$$

$$A_i := A_{2^{i+1}d_n, 2^i d_n} = B_{2^{i+1}d_n}(x_n') \setminus B_{2^i d_n}(x_n'),$$

and $S_i := \partial B_{2^i d_n}(x_n')$. For simplicity, we drop the subscript n, and write $\hat{u}_n$ as $\hat{u}$ and so on, whenever there is no confusion. Now, let us consider a vector-valued piecewise linear function $h = h(|x|)$ on $\hat{\Omega}_2$ such that $h(2^i d_n)$ equals the integral mean value of $\hat{u}$ on S_i. In fact, by our construction of $\hat{u}$, we have $h \equiv 0$. It is clear that $\Delta(\hat{u} - h) = \Delta \hat{u}$, and in order to compare it with the argument of [12], we keep h for the sake of consistency, i.e.,

$$\begin{aligned}
\int_{A_i} |\nabla(\hat{u} - h)|^2 &= -\int_{A_i} \Delta\hat{u} \cdot (\hat{u} - h) + \left(\int_{S_i} - \int_{S_{i+1}}\right)\left((\hat{u} - h) \cdot \frac{\partial \hat{u}}{\partial r}\right) \\
&\leqslant C(\epsilon + \sqrt{\delta}) \int_{A_i} |\Delta \hat{u}| + \left(\int_{S_i} - \int_{S_{i+1}}\right)\left((\hat{u} - h) \cdot \frac{\partial \hat{u}}{\partial r}\right),
\end{aligned}$$

where we employ (4.14) to obtain

$$\begin{aligned}
\|\hat{u} - h\|_{L^\infty(\hat{\Omega}_2)} &= \max_{1 \leqslant i \leqslant T_n} \|\hat{u}\|_{L^\infty(A_i)} \leqslant \max_{1 \leqslant i \leqslant T_n} \|\hat{u}\|_{\mathrm{osc}(A_i)} \ (\text{since } \hat{u} = 0 \text{ on } \partial^0 D^+ \cap \Omega_2) \\
&\leqslant 2 \max_{1 \leqslant i \leqslant T_n} \|u\|_{\mathrm{osc}(A_i^+)} \leqslant C(\epsilon + \sqrt{\delta}).
\end{aligned} \tag{4.15}$$

Now, since

$$\begin{aligned}
\int_{A_i} |\Delta \hat{u}| &= \int_{A_i^+} |\Delta u| + \int_{A_i \setminus A_i^+} |-\Delta u(x^*)| = 2\int_{A_i^+} |\Delta u| \\
&\leqslant C(\Lambda') \int_{A_i^+} (|\nabla \phi_n|^2 + |\nabla \phi_n| \cdot |\psi_n|^2 + |\tau(\phi_n, \psi_n)| + |\Delta \tilde{\phi}|) \\
&\leqslant C(\Lambda') \left[\int_{A_i^+} (|\nabla \phi_n|^2 + |\nabla \phi_n| \cdot |\psi_n|^2) + 2^i d_n (\|\tau\|_{L^2(A_i^+)} + \|\tilde{\phi}\|_{C^{2,\alpha}(A_i^+)})\right] \\
&\leqslant C(\Lambda', \Lambda, \|\phi_\partial\|_{C^{2,\alpha}(\partial M)}) \left[\int_{A_i^+} (|\nabla \phi_n|^2 + |\psi_n|^4) + 2^i d_n\right],
\end{aligned}$$

which implies that, noting that[1] $h \equiv 0$, we drop the dependence on C for simplicity in what follows,

$$\int_{A_i} |\nabla \hat{u}|^2 \leqslant C(\epsilon + \sqrt{\delta}) \left[\int_{A_i^+} (|\nabla \phi_n|^2 + |\psi_n|^4) + 2^i d_n\right] + \left(\int_{S_i} - \int_{S_{i+1}}\right)\left((\hat{u} - h) \cdot \frac{\partial \hat{u}}{\partial r}\right).$$

We also note that

$$\int_{A_i} |\nabla \hat{u}|^2 = \int_{A_i^+} |\nabla u|^2 + \int_{A_i \setminus A_i^+} |-\nabla u(x^*)|^2 = 2\int_{A_i^+} |\nabla u|^2 \geqslant 2\int_{A_i^+} (|\nabla \phi_n|^2 - |\nabla \tilde{\phi}|^2),$$

and thus,

$$(2 - C(\epsilon + \sqrt{\delta})) \int_{A_i^+} |\nabla \phi_n|^2 \leqslant C(\epsilon + \sqrt{\delta}) \left[\int_{A_i^+} |\psi_n|^4 + 2^i d_n\right] + \left(\int_{S_i} - \int_{S_{i+1}}\right)\left((\hat{u} - h) \cdot \frac{\partial \hat{u}}{\partial r}\right).$$

Taking ϵ and δ small enough, we see that

$$\int_{A_i^+} |\nabla \phi_n|^2 \leqslant C(\epsilon + \sqrt{\delta}) \left[\int_{A_i^+} |\psi_n|^4 + 2^i d_n + \left(\int_{S_i} - \int_{S_{i+1}}\right)\left((\hat{u} - h) \cdot \frac{\partial \hat{u}}{\partial r}\right)\right]. \tag{4.16}$$

Summing over $i = 1, 2, \ldots, T_n$, we obtain

$$\int_{\Omega_2} |\nabla \phi_n|^2 \leqslant C(\epsilon + \sqrt{\delta}) \left[\int_{\Omega_2} |\psi_n|^4 + \delta\right] + \left(\int_{S_1} - \int_{S_{T_n+1}}\right)\left((\hat{u} - h) \cdot \frac{\partial \hat{u}}{\partial r}\right).$$

[1] In the case $h \not\equiv 0$, we only have the tangent energy of $\hat{u}$ controlled, and we need a Pohozaev-type argument to control the total energy by the tangent one.

Finally, let us estimate the boundary term. Applying the trace theorem $W^{1,2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ and noting (4.15), we have that for $r = 2d_n$ or $r = \delta/2$,

$$\begin{aligned}\int_{\partial B_r(x_n')} (\hat{u}-h)\cdot\frac{\partial \hat{u}}{\partial r} &\leqslant C(\epsilon+\sqrt{\delta})\int_{\partial B_r(x_n')}|\nabla \hat{u}| \\ &\leqslant C(\epsilon+\sqrt{\delta})\int_{\partial^+ D_r(x_n')}|\nabla u| \\ &\leqslant C(\epsilon+\sqrt{\delta})\int_{\partial^+ D_r(x_n')}(|\nabla\phi_n|+|\nabla\tilde{\phi}|) \\ &\leqslant C(\epsilon+\sqrt{\delta})\bigg[\sqrt{r}\bigg(\int_{\partial^+ D_r(x_n')}|\nabla\phi_n|^2\bigg)^{1/2}+1\bigg].\end{aligned}$$

Now, the scaling argument shows

$$r\int_{\partial^+ D_r(x_n')}|\nabla\phi_n|^2 \leqslant \int_{A^+_{3r/2,3r/4}(x_n')}|\nabla\phi_n|^2 + r^2\int_{A^+_{3r/2,3r/4}(x_n')}|\nabla^2\phi_n|^2,$$

and by Lemma 3.8, we have

$$\text{RHS} \leqslant C(\|\nabla\phi_n\|_{L^2(A^+_{2r,r/2}(x_n'))} + r\|\tau\|_{L^2(A^+_{2r,r/2}(x_n'))} + \sqrt{r}\|\phi_\partial\|_{C^{2,\alpha}(\partial^0 D^+_\delta)}) \leqslant C.$$

Plugging this back in, we obtain the boundary term

$$\bigg(\int_{S_1} - \int_{S_{T_n+1}}\bigg)\bigg((\hat{u}-h)\cdot\frac{\partial\hat{u}}{\partial r}\bigg) \leqslant C(\epsilon+\sqrt{\delta}).$$

In conclusion,

$$\int_{\Omega_2}|\nabla\phi_n|^2 \leqslant C(\epsilon+\sqrt{\delta})\bigg[\int_{\Omega_2}|\psi_n|^4+1\bigg],$$

and thus, (1.6) follows from the energy identity of ψ_n. In particular, Theorem 1.2(4) is proved, and the proof of Theorem 1.2 is finished.

Proof of Corollary 1.4. Firstly, we can repeat the steps in the proof of Claim 4.1 to obtain the whole bubble tree, and the relation of blow-up data holds for each bubble, as illustrated in the proof of Claim 4.1, and moreover on each neck (4.7) holds. By Lemmas 3.7 and 3.8, we have

- $(\phi_n, \psi_n) \to (\phi_\infty, \psi_\infty)$ weakly in $\mathcal{X}_0(M \setminus \mathcal{S}, \mathcal{N})$ and strongly in $\mathcal{X}_0^w(M \setminus \mathcal{S}, \mathcal{N})$ as $n \to \infty$;
- $(\phi^l_{n,i}, \psi^l_{n,i}) \to (\sigma^l_i, \xi^l_i)$ weakly in $\mathcal{X}(S^2, \mathcal{N})$ and strongly in $\mathcal{X}^w(S^2, \mathcal{N})$ as $n \to \infty$.

Clearly, with the above observation at hand, the energy identities (1.7), (1.8), and (1.10) are equivalent to there being no energy on the neck $P_n := D^+_\delta \setminus D^+_{r_n R}(x_n)$, i.e.,

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E(\psi_n; P_n) = 0, \tag{4.17}$$

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E_g(\phi_n; P_n) = 0, \tag{4.18}$$

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E(\phi_n; P_n) = 0, \tag{4.19}$$

respectively. In particular, (1.7) follows from (1.5).

By the neck region decomposition, we know that the neck region

$$P_n = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

in Case (a′) or

$$P_n = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$$

in Case (b′). By Claim 4.1, we know that in either case,

$$E(\phi_n, \psi_n; \Omega_1) + E(\phi_n, \psi_n; \Omega_3) \leqslant 3\epsilon^2.$$

Since $\Omega_4 = B_{d_n}(x_n) \setminus B_{r_n R}(x_n)$, the energy identity of approximate Dirac-harmonic maps with interior energy concentration points (see [2, Theorem 1.2]) implies

$$\lim_{R\to\infty}\lim_{n\to\infty} E_g(\phi_n, \psi_n; \Omega_4) = 0,$$

and in the static case,

$$\lim_{R\to\infty}\lim_{n\to\infty} E(\phi_n, \psi_n; \Omega_4) = 0.$$

Therefore, the energy identities for ϕ_n, i.e., (1.8) and (1.10), which are equivalent to (4.18) and (4.19) respectively, follow from (1.6), i.e., the energy is as small as possible on Ω_2. The connecting of images is proved in [2, Theorem 1.4] since we only consider interior blow-up points. This finishes the proof. □

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