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Higher-Dimensional Kaluza-Klein Theory in Weitzenböck Space*

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Abstract It is proved that the action of the higher-dimensional gravity in Weitzenböck space reduces to the sum of the action of gravity in four-dimensional space-time and that of gauge fields. In this sense we conclude that in Weitzenböck space the higher-dimensional Kaluza-Klein theory holds.

Keywords: Weitzenböck space, Kaluza-Klein theory, parallel vector fields, torsion tensor.

1 Introduction

It is well known that the Riemann-Cartan space U_4 has both nonvanishing curvature and torsion. From U_4, setting torsion tensor to be identically vanishing we get Riemann space V_4 on which the general relativity (GR) was established; setting curvature tensor to be identically vanishing, we get Weitzenböck space A_4, based on which Hayashi[1] formulated another theory of gravity, the so-called new general relativity (NGR), in 1979.

The Weitzenböck space-time is characterized by the existence of a quadruplet of linearly independent parallel vector fields $b=\{ba\}=\{b_a^\mu\}$. By definition the covariant derivative of parallel fields ∇b_a equals zero. Thus, in coordinate basis, the affine connection $\Gamma=\{\Gamma_{\mu\nu}^\lambda\}$ can be expressed as

$$\Gamma_{\mu\nu}^\lambda=b_a^\lambda\partial_\nu b_\mu^a. \qquad (1.1)$$

Here $b^*=\{b_\mu^a\}$ is also a quadruplet of vector fields, which is dual to b. It is easily seen from (1.1) that the curvature tensor $R(\Gamma)$ is identically vanishing, and the torsion tensor T reads

$$T_{\mu\nu}^\lambda=b_a^\lambda(\partial_\nu b_\mu^a-\partial_\mu b_\nu^a). \qquad (1.2)$$

In new general relativity the gravitation fields are completely described by the parallel vector fields $\{b\}$ or equivalently by the torsion tensor. New general relativity is one of the theories that are competitive with general relativity, and it predicts all the experimental facts that general relativity does. It seems impossible to detect the quantitative differences between their predictions at present. Besides the torsion $T=$

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0 for V_4 on which the GR is based and the curvature $R = 0$ for A_4 on which the NGR is based, there is another correspondence between these two models: the GR can be viewed as a gauge theory of frame rotation and the NGR as that of frame translation[2,3]. But in Riemann space there exists the Kaluza-Klein theory (K-K theory) of gravity which offers a possibility for unifying gravity with other interactions of nature[4-6]. This fact motivated us to raise a question as to whether the Kaluza-Klein theory also holds in Weitzenböck space. In this paper we shall show that the answer to this question is positive and a higher dimensional K—K theory of gravity in Weitzenböck space is established.

In Sec. 2 we review the coset geometry in Weitzenböck space. Sec. 3 is devoted to evaluate the higher dimensional parallel fields and torsion tensor. In Sec. 4 we show that the action of the higher dimensional gravity reduces to the sum of the action of gravity in four-dimensional space-time and that of gauge fields.

2 Coset Space

Let the internal space Y be taken as the coset space G/H of gauge group G, where H is the maximal subgroup of G. The map $\varphi: G \times Y \to Y$ realizes the left action from G to Y:

$$y^i = \varphi^i(g, y), \ i = 1, 2, \cdots, d, \tag{2.1}$$

where y^i are the local coordinates on Y, d is the dimension of Y. The generators of this left action are denoted by $K^i_\alpha(y)$:

$$K^i{}_\alpha(y) = \left.\frac{\partial \varphi^i(g, y)}{\partial g_\alpha}\right|_{g=e}, \ \alpha = 1, 2, \cdots, d_G, \tag{2.2}$$

which satisfies

$$[\boldsymbol{K}_\alpha(y), \ \boldsymbol{K}_\beta(y)] = f^\gamma_{\alpha\beta} \boldsymbol{K}_\gamma(y), \tag{2.3}$$

with

$$\boldsymbol{K}_\alpha(y) = K^i_\alpha(y) \frac{\partial}{\partial y^i}. \tag{2.4}$$

We endow the internal space with a symmetric metric $\gamma_{ij}(y)$. The symmetry of internal space is characterized by the existence of a special class of metrices, the forms of which are invariant under the action of G. The corresponding Killing equation for them reads

$$\gamma_{il}\partial_j K^l_\alpha + \gamma_{lj}\partial_i K^l_\alpha + \partial_l \gamma_{ij} K^l_\alpha = 0. \tag{2.5}$$

A solution of Eq. (2.5) is[6]

$$\gamma^{ij}(y) = L^{-2} K^i_\alpha(y) K^j_\alpha(y). \tag{2.6}$$

Here we introduce a length scale L into (2.6) in order to make γ^{ij} dimensionless.

Now we assume that the internal space is a Weitzenböck space. By definifion there exists a system of linearly independent orthogonal parallel vector fields $b = \{b_a\}$ such that

$$b_a^i \gamma_{ij} b_b^j = \delta_{ab}, \qquad a,\ b = 1,2,\cdots,d, \tag{2.7}$$

where b_a^i are the components of b in the coordinate basis $e_i = \partial/\partial y^i$. The components of dual vector fields will be denoted by b_i^a and satisfy

$$b_i^a b_b^i = \delta_b^a. \tag{2.8}$$

In the case of Weitzenböck space we should also find out a class of parallel vector fields which are invariant under the action of G. It is easy to get the Killing equation for parallel vector fields as follows:

$$\partial_j K_\alpha^k + b_j^a \partial_l b_a^k K_\alpha^l = 0. \tag{2.9}$$

This equation will play a key role in formulating K-K theory in A_n.

One may raise a question about whether it is possible to find a system of invariant vector fields which is consistent with the metric satisfying Eq. (2.5). To answer this question it is useful to split Eq. (2.8), after raising the index j, into symmetric and antisymmetric parts:

$$\partial^j K_\alpha^i + \partial^i K_\alpha^j + K_\alpha^l \partial_l \gamma^{ji} = 0, \tag{2.10a}$$

$$\partial^j K_\alpha^i - \partial^i K_\alpha^j + K_\alpha^l (b^{aj} \partial_l b_a^i - b^{ak} \partial_l b_a^j). \tag{2.10b}$$

We see that (2.10a) just coincides with Eq. (2.6). If we have a solution γ of Eq. (2.6) and find a system of vector fields b which are consistent with γ but may not satisfy (2.10b), we can get another system of vector fields b' through an orthogonal transformation O:

$$b_a^{\prime i} = O_a^c b_c^i, \tag{2.11}$$

such that the vector fields b' satisfy Eq. (2.10b). Thus we have a solution b' of the Killing equation (2.9) for parallel vector fields.

3 The Parallel Fields and Torsion Tensor in A_n

Now we turn to evaluate the parallel vector fields and torsion tensor in n-dimensional Weitzenböck space $A_n (n=4+d)$. First of all we present the notation as follows. The symbols with circumflexes refer to the geometrical objects in higher dimensional space A_n. The Greek letters with circumflexes ($\hat\mu$, $\hat\nu$, $\cdots$) denote the space indices in A_n (A_4), while the Latin letters (i, j, $\cdots$) refer to the coset space. The parallel vector fields in A_n are labeled by Latin letters with circumflexes ($\hat A$, $\hat B \cdots\cdots$), while the Latin letters without circumflex (A, B, $\cdots\cdots$) and (a, b, $\cdots$) are used to label the parallel vector fields in A_4 and in coset space, respectively. Then we may write

$$\begin{aligned} \hat\mu &= (\mu, i), \qquad \mu,\ A = 0,1,2,3 \\ \hat A &= (A,\ a), \quad\ i,\ a = 5,6,\cdots,n. \end{aligned} \tag{3.1}$$

The metric $\hat g$ of the higher demensional K-K theory in A_n is postulated to be the same as in Riemannian space V_n[6]:

$$\hat g_{\hat\mu\hat\nu} = \begin{pmatrix} g_{\mu\nu} - \kappa^2 L^{-2} \gamma_{ij}(y) K_\alpha^i(y) K_\beta^j(y) A_\mu^\alpha(x) A_\nu^\beta(x) & -\kappa L^{-1} \gamma_{ij}(y) K_\alpha^i(y) A_\mu^\alpha(x) \\ -\kappa L^{-1} \gamma_{ij}(y) K_\alpha^i(y) A_\mu^\alpha(x) & -\gamma_{ij}(y) \end{pmatrix}. \tag{3.2}$$

In order to make the metric block-diagonal we take the anholonomic basis $\hat{e}_{\hat{\mu}}$:

$$\hat{e}_{\mu}=\frac{\partial}{\partial x^{\mu}}-\kappa L^{-1}K_{a}^{i}(y)A_{\mu}^{a}(x)\frac{\partial}{\partial y^{i}}, \tag{3.3}$$

$$\hat{e}_{i}=\frac{\partial}{\partial y^{i}}, \tag{3.4}$$

which satisfies the commutative relations

$$[\hat{e}_{\mu},\ \hat{e}_{\nu}]=-\kappa L^{-1}F_{\mu\nu}^{a}(x)K_{a}^{i}(y)\frac{\partial}{\partial y^{i}}, \tag{3.5}$$

$$[\hat{e}_{\mu},\ \hat{e}_{i}]=\kappa L^{-1}A_{\mu}^{a}(x)\partial_{i}K_{a}^{j}(y)\frac{\partial}{\partial y^{j}}, \tag{3.6}$$

$$[\hat{e}_{i},\ \hat{e}_{j}]=0, \tag{3.7}$$

where

$$F_{\mu\nu}^{a}(x)=\partial_{\mu}A_{\nu}^{a}-\partial_{\nu}A_{\mu}^{a}+\kappa L^{-1}f_{\beta\gamma}^{a}A_{\mu}^{\beta}A_{\nu}^{\gamma}. \tag{3.8}$$

In this basis metric $\hat{g}$ is block-diagonal

$$\hat{g}_{\hat{\mu}\hat{\nu}}=\begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{ij}(y)\end{pmatrix}, \tag{3.9}$$

where $g_{ij}=-\gamma_{ij}$.

The parallel vector fields $\hat{b}=\{\hat{b}_{\hat{A}}\}$ and their dual $\hat{b}^{*}=\{\hat{b}^{\hat{A}}\}$ in A_{n} can be expressed in terms of the anholonomic basis $\hat{e}_{\hat{\mu}}$ with components $\hat{b}_{\hat{A}}^{\hat{\mu}}$ and $\hat{b}_{\hat{\mu}}^{\hat{A}}$ respectively. The orthogonality and the duality of the parallel vector fields are

$$\hat{b}_{\hat{A}}^{\hat{\mu}}\hat{b}_{\hat{\mu}}^{\hat{B}}=\delta_{\hat{A}}^{\hat{B}}, \qquad \hat{b}_{\hat{A}}^{\hat{\mu}}\hat{b}_{\hat{\nu}}^{\hat{A}}=\delta_{\hat{\nu}}^{\hat{\mu}}, \tag{3.10a}$$

$$\hat{b}_{\hat{\mu}}^{\hat{A}}\eta_{\hat{A}\hat{B}}\hat{b}_{\hat{\nu}}^{\hat{B}}=\hat{g}_{\hat{\mu}\hat{\nu}}, \qquad \hat{b}_{\hat{A}}^{\hat{\mu}}\hat{g}_{\hat{\mu}\hat{\nu}}\hat{b}_{\hat{B}}^{\hat{\nu}}=\hat{\eta}_{\hat{A}\hat{B}}, \tag{3.10b}$$

where $\hat{\eta}_{\hat{A}\hat{B}}$ is the higher-dimensional Minkowski metric: $\hat{\eta}_{\hat{A}\hat{B}}=\mathrm{diag}(+1,-1,-1,\cdots,-1)$. In the anholonomic basis (3.3) and (3.4), $\hat{b}$ and $\hat{b}^{*}$ can also be block-diagonalized:

$$\hat{b}_{\hat{A}}^{\hat{\mu}}=\begin{pmatrix} b_{A}^{\mu}(x) & b_{A}^{i}=0 \\ b_{a}^{\mu}=0 & b_{a}^{i}(y)\end{pmatrix}, \tag{3.11a}$$

$$\hat{b}_{\hat{\mu}}^{\hat{A}}=\begin{pmatrix} b_{\mu}^{A}(x) & b_{i}^{A}=0 \\ b_{\mu}^{a}=0 & b_{i}^{a}(y)\end{pmatrix}, \tag{3.11b}$$

such that

$$g_{\mu\nu}=b_{\mu}^{A}\eta_{AB}b_{\nu}^{B}, \qquad \eta_{AB}=(+1,\ -1,\ -1,\ -1) \tag{3.12a}$$

$$g_{ij}=b_{a}^{\mu}\eta_{ab}b_{j}^{b}=-\gamma_{ij}, \quad \eta_{ab}=-\delta_{ab}. \tag{3.12b}$$

By definition the covariant derivative of the parallel vector fields vanishes, namely,

$$\hat{\nabla}\hat{b}=\nabla_{\hat{e}_{\hat{\mu}}}(\hat{b}_{\hat{A}}^{\hat{\nu}}\hat{e}_{\hat{\nu}})=\nabla_{\hat{e}_{\hat{\mu}}}(\hat{b}_{\hat{A}}^{\hat{\nu}})\hat{e}_{\hat{\nu}}+\hat{b}_{\hat{A}}^{\hat{\nu}}\nabla_{\hat{e}_{\hat{\mu}}}\hat{e}_{\hat{\lambda}}=0. \tag{3.13}$$

Note that $\hat{\nabla}_{\hat{e}_{\hat{\mu}}}(b_{\hat{A}}^{\hat{\nu}})=\hat{e}\ (\hat{b}_{\hat{A}}^{\hat{\nu}})$ and $\hat{\nabla}_{\hat{e}_{\hat{\mu}}}\hat{e}_{\hat{\nu}}=\hat{\gamma}_{\hat{\nu}}^{\hat{\lambda}}\ \hat{e}_{\hat{\lambda}}$, here the $\hat{\gamma}_{\hat{\nu}}{}^{\hat{\lambda}}{}_{\hat{\mu}}$ are the affine connection coefficients of A_{n}. Thus we get from (3.13)

$$\hat{\gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}} = -\hat{b}^{\hat{A}}_{\hat{\mu}}\hat{e}_{\hat{\vartheta}}\hat{b}^{\hat{\lambda}}_{\hat{A}} = \hat{b}^{\hat{\lambda}}_{\hat{A}}\hat{e}_{\hat{\vartheta}}\hat{b}^{\hat{A}}_{\hat{\mu}}. \tag{3.14}$$

Using (3.3), (3.4) and (3.10a, b) the connection coefficients can be easily evaluated. We find that the nonvanishing coefficients are only

$$\hat{\gamma}^{\lambda}_{\mu\nu} = \gamma^{\lambda}_{\mu\nu}, \tag{3.15}$$

$$\hat{\gamma}^{i}_{j\mu} = -\kappa A^{a}_{\mu} K^{l}_{a} \gamma^{i}_{jl}, \tag{3.16}$$

$$\hat{\gamma}^{i}_{jk} = \gamma^{i}_{jk}, \tag{3.17}$$

where $\gamma^{\lambda}_{\mu\nu} = -b^{A}_{\mu}\partial_{\nu}b^{\lambda}_{A}$, $\gamma^{i}_{jk} = -b^{a}_{j}e_{k}b^{i}_{a}$ are the connection coefficients in the four-dimensional space-time and coset space respectively.

In an anholonomic basis the torsion tensor should be expressed as

$$\hat{T}^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}} = \hat{\gamma}^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}} - \hat{\gamma}^{\hat{\lambda}}_{\hat{\vartheta}\hat{\mu}} + \hat{C}^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}}, \tag{3.18}$$

where $\hat{C}^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}}$ are the structure coefficients of the basis $\hat{e}_{\hat{\mu}}$. Their values in basis (3.3) and (3.4) can be read readily from (3.5) and (3.6). We evaluate the components of torsion tensor in A_n by use of (3.15)—(3.17) and find the following results:

$$\hat{T}^{\lambda}_{\mu\nu} = T^{\lambda}_{\mu\nu}, \tag{3.19}$$

$$\hat{T}^{\lambda}_{\mu i} = 0, \tag{3.20}$$

$$\hat{T}^{\lambda}_{ij} = 0, \tag{3.21}$$

$$\hat{T}^{i}_{j\mu} = -\kappa L^{-1} A^{a}_{\mu}(\partial_j K^{i}_{a} + K^{l}_{a}\gamma^{i}_{jl}) = 0, \tag{3.22}$$

$$\hat{T}^{i}_{jk} = T^{i}_{jk}, \tag{3.23}$$

$$\hat{T}^{i}_{\mu\nu} = \hat{C}^{i}_{\mu\nu} = -\kappa L^{-1} F^{\beta}_{\mu\nu} K^{i}_{a}. \tag{3.24}$$

In (3.22) the Killing equation (2.9) has been used. It should be noted that the indices of a higher-dimensional tensor in anholonomic basis can be raised and lowered by $g_{\mu\nu}$, $g^{\mu\nu}$ or g_{ij}, g^{ij} because the metric $\hat{g}$ is blockd-iagonal.

4 Lagrangian and Action

The gravitational fields in higher-dimensional Weitzenböck space A_n is described by the parallel vector fields $\hat{b}^{\hat{\mu}}_{\hat{A}}$ or equivalently by the torsion tensor. The physical law of A_n should be invariant under general coordinate transformation and global higher-dimensional Lorentz transformation. Hence we will construct the Lagrangian density for higher-dimensional gravity in this space in the quadratic form of irreducible parts of the torsion tensor with respect to the global n-dimensional Lorentz transformation[1]:

$$\hat{b}'^{\hat{\mu}}_{\hat{A}} = \hat{\Lambda}^{\hat{B}}_{\hat{A}}\hat{b}^{\hat{\mu}}_{\hat{B}}, \tag{4.1}$$

where $\hat{\Lambda}^{\hat{B}}_{\hat{A}}$ is the element of proper orthochronous Lorentz transformation $\hat{\Lambda}(\hat{\tilde{\Lambda}}\hat{\eta}\hat{\Lambda} = \hat{\eta}$, $\det \hat{\Lambda} = 1$, $\hat{\Lambda}^{0}_{0} \geqslant 1)$. The torsion tensor $T^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}}$ (more precisely, the tensor $\hat{T}^{\hat{A}}_{\hat{B}\hat{C}} = \hat{b}^{\hat{A}}_{\hat{\lambda}}\hat{b}^{\hat{\mu}}_{\hat{B}}\hat{b}^{\hat{\vartheta}}_{\hat{C}}\hat{T}^{\hat{\lambda}}_{\hat{\mu}\hat{\vartheta}}$) is reducible with respect to the global Lorentz transformation. It could be decomposed into irreducible parts as

$$\hat{T}_{\hat{\lambda}\hat{\mu}\hat{\vartheta}} = \frac{2}{3}(\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\vartheta}} - \hat{t}_{\hat{\lambda}\hat{\mu}\hat{\vartheta}}) - \frac{1}{n-1}(\hat{g}_{\hat{\lambda}\hat{\mu}}\hat{v}_{\hat{\vartheta}} - \hat{g}_{\hat{\lambda}\hat{\vartheta}}\hat{v}_{\hat{\mu}})$$

$$+\frac{1}{(n-3)!}\operatorname{sgn}\hat{g}\,\hat{e}_{\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}\hat{a}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}, \tag{4.2}$$

where

$$\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}=\frac{1}{2}(\hat{T}_{\hat{\lambda}\hat{\mu}\hat{\nu}}+\hat{T}_{\hat{\mu}\hat{\lambda}\hat{\nu}})+\frac{1}{2(n-1)}(\hat{g}_{\hat{\nu}\hat{\lambda}}\hat{v}_{\hat{\mu}}+\hat{g}_{\hat{\nu}\hat{\mu}}\hat{v}_{\hat{\lambda}})-\frac{1}{n-1}\hat{g}_{\hat{\lambda}\hat{\mu}}\hat{v}_{\hat{\nu}}, \tag{4.3}$$

$$\hat{v}_{\hat{\mu}}=\hat{T}^{\hat{\lambda}}_{\hat{\lambda}\hat{\mu}}, \tag{4.4}$$

$$\hat{a}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}=\frac{1}{3!}\hat{e}^{\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}\hat{T}_{\hat{\lambda}\hat{\mu}\hat{\nu}} \tag{4.5}$$

and

$$\hat{e}_{\hat{\lambda}\hat{\mu}\hat{\nu}\ldots}=|g|^{1/2}\delta^{01235\cdots n}_{\hat{\lambda}\hat{\mu}\hat{\nu}\cdots\cdots},\ (g=\det(\hat{g}_{\hat{\mu}\hat{\nu}})), \tag{4.6}$$

$$\hat{e}^{\hat{\lambda}\hat{\mu}\hat{\nu}\cdots}=\operatorname{sgn}(\hat{g})|g|^{1/2}\delta^{\hat{\lambda}\hat{\mu}\hat{\nu}\cdots\cdots}_{01235\cdots n},\ \operatorname{sgn}(\hat{g})=g/|g| \tag{4.7}$$

denote the extended Levi-Civita tensors in n-dimensional space, and $\delta^{\cdots}_{\cdots}$ is the extended Kronecker delta symbol. Moreover, the tensor $\{\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}\}$ has the following properties derived from its definition (4.3):

$$\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}=\hat{t}_{\hat{\mu}\hat{\lambda}\hat{\nu}}, \tag{4.8}$$

$$\hat{t}^{\hat{\lambda}}_{\hat{\lambda}\hat{\mu}}=\hat{t}^{\hat{\lambda}}_{\hat{\mu}\hat{\lambda}}=0, \tag{4.9}$$

$$\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}+\hat{t}_{\hat{\mu}\hat{\nu}\hat{\lambda}}+\hat{t}_{\hat{\nu}\hat{\lambda}\hat{\mu}}=0. \tag{4.10}$$

Now we write the Lagrangian density for higher-dimensional gravity up to a constant factor as

$$\hat{L}=a_0+a_1\hat{t}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}+a_2\hat{v}^{\hat{\mu}}\hat{v}_{\hat{\mu}}+a_3\hat{a}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\cdots}\hat{a}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}, \tag{4.11}$$

where a_0, a_1, a_2 and a_3 are free parameters. From the definitions (4.3)—(4.7) and the properties of $\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}$(4.8) and (4.9), the terms in (4.11) can be espressed in terms of $\hat{T}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{T}_{\hat{\lambda}\hat{\mu}\hat{\nu}}$, $\hat{T}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{T}_{\hat{\mu}\hat{\lambda}\hat{\nu}}$, and $\hat{v}_{\hat{\mu}}\hat{v}_{\hat{\mu}}$:

$$\hat{t}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}=\frac{1}{2}(\hat{T}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{T}_{\hat{\lambda}\hat{\mu}\hat{\nu}}+\hat{T}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{T}_{\hat{\mu}\hat{\lambda}\hat{\nu}})-\frac{3}{2(n-1)}\hat{v}^{\hat{\mu}}\hat{v}_{\hat{\mu}}, \tag{4.12}$$

$$\hat{a}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\cdots}\hat{a}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}=\operatorname{sgn}(\hat{g})]\frac{(n-3)!}{(3!)^2}\hat{T}^{\hat{\lambda}\hat{\mu}\hat{\nu}}(\hat{T}_{\hat{\lambda}\hat{\mu}\hat{\nu}}-2\hat{T}_{\hat{\mu}\hat{\lambda}\hat{\nu}}). \tag{4.13}$$

Further using expressions (3.19)—(3.25) we evaluate these terms as follows:

$$\hat{t}^{\hat{\lambda}\hat{\mu}\hat{\nu}}\hat{t}_{\hat{\lambda}\hat{\mu}\hat{\nu}}=t^{\lambda\mu\nu}t_{\lambda\mu\nu}+t^{ijk}t_{ijk}+\frac{3}{2}\left(\frac{1}{3}-\frac{1}{n-1}\right)v^{\mu}v_{\mu}$$
$$+\frac{3}{2}\left(\frac{1}{d-1}-\frac{1}{n-1}\right)v^iv_i+\frac{1}{2}\kappa^2L^{-2}K^i_{\alpha}K_{\beta i}F^{\mu\nu\alpha}F^{\beta}_{\mu\nu}, \tag{4.14}$$

$$\hat{v}^{\hat{\mu}}\hat{v}_{\hat{\mu}}=v^{\mu}v_{\mu}+v^iv_i, \tag{4.15}$$

$$\hat{a}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\cdots}\hat{a}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\ldots}=\operatorname{sgn}(g_{\mu\nu})(n-3)!\,a^{\mu}a_{\mu}+\operatorname{sgn}(g_{ij})\frac{(n-3)!}{(d-3)!}a^{ij\cdots}a_{ij\ldots}$$
$$+\operatorname{sgn}(\hat{g})\frac{(n-3)!}{(3!)^2}\kappa^2L^{-2}K^i_{\alpha}K_{\beta i}F^{\mu\nu\alpha}F^{\beta}_{\mu\nu}, \tag{4.16}$$

where

$$a^{\mu}=\frac{1}{3!}\epsilon^{\rho\sigma\tau\mu}T_{\rho\sigma\tau},\tag{4.17}$$

$$a^{ij\cdots}=\frac{1}{3!}\epsilon^{lmnij\cdots}T_{lmn}.\tag{4.18}$$

With the help of (4.14)—(4.16) we see that the Lagrangian density $\hat{L}$ splits into three terms corresponding to the Lagrangian densities of gravity in 4-dimensional space-time, in coset space G/H and in gauge field, respectively

$$\hat{L}=L_4+L_{G/H}+a\kappa^2L^{-2}K^i_\alpha K_{\beta i}F^{\mu\nu\alpha}F^{\beta}_{\mu\nu}\tag{4.19}$$

with

$$L_4=a_0+a_1'\boldsymbol{t}^{\lambda\mu\nu}\boldsymbol{t}_{\lambda\mu\nu}+a_2'\boldsymbol{v}^{\mu}\boldsymbol{v}_{\mu}+a_3'\boldsymbol{a}^{\mu}\boldsymbol{a}_{\mu},\tag{4.20}$$

$$L_{G/H}=a_1''\boldsymbol{t}^{ijk}\boldsymbol{t}_{ijk}+a_2''\boldsymbol{v}^{i}\boldsymbol{v}_{i}+a_3''\boldsymbol{a}^{ij\cdots}\boldsymbol{a}_{ij\cdots},\tag{4.21}$$

where

$$a=a_1+\mathrm{sgn}(\hat{g})\frac{(n-3)!}{(3!)^2}a_3,\tag{4.22}$$

$$a_1'=a_1,\ a_2'=\left(\frac{1}{2}\frac{n-4}{n-1}a_1+a_2\right),\ a_3'=\mathrm{sgn}(\hat{g})\mathrm{sgn}(g_{\mu\nu})(n-3)!a_3,\tag{4.23}$$

$$a_1''=a_1,\ a_2''=\frac{3}{2}\left[\left(\frac{1}{d-1}-\frac{1}{n-1}\right)a_1+a_2\right],\ a_3''=\mathrm{sgn}\ (\hat{g})\mathrm{sgn}\ (g_{ij})\frac{(n-3)!}{(d-3)!}a_3.\tag{4.24}$$

If we set $a=-\frac{1}{4}$ as usual, then two of the three parameters a_1, a_2 and a_3 are still free. Now the action of gravity in A_n can be written as

$$I=\frac{1}{16\pi GV}\int dv_4dv_{G/H}(L_4+L_{G/H}-\frac{1}{4}\kappa^2L^{-2}K^i_\alpha K_{\beta i}F^{\mu\nu\alpha}F^{\beta}_{\mu\nu}),\tag{4.25}$$

where G is the Newtonian gravitational constant. dv_4 and $dv_{G/H}$ are the invariant volume elements of 4-dimensional space-time and coset space G/H respectively, and V is the volume of G/H. Performing the integral over coset space and setting

$$\kappa^2=16\pi G\left(\frac{d_G}{d}\right),\tag{4.26}$$

(d_G = dimension of the gauge group) we get[7]

$$I=\frac{1}{16\pi G}\int dv_4(L_4-2\Lambda)-\frac{1}{4}\int dv_4F^{a\mu\nu}F^{a}_{\mu\nu}.\tag{4.27}$$

Here $\Lambda=a_0+\int dv_{G/H}L_{G/H}$ may be regarded as the cosmological constant. From (4.27) we see that the action of higher dimensional gravity in Weitzenböck space reduces to the sum of the action of gravity in four-dimensional space-time and that of gauge field. In this sense we conclude that the Kaluza-Klein theory also holds in higher-dimensional Weitzenböck space.

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