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MATHEMATICS

THE LATTICE-POINTS IN A CIRCLE*

CHEN JING-RUN (陈景润)
(*Institute of Mathematics, Academia Sinica*)

1. Let $R(x)$ denote the number of lattice-points inside and on the circle $x^2 + y^2 = t$. It is easily proved that as $t \to \infty$, $R(t) \sim \pi t$, and in fact that

$$R(t) = \pi t + O(t^{\alpha}) \tag{1}$$

for some values of α less than 1. It is a question of finding the lower bound, $\mathscr{F}$, of the numbers α for which (1) is true. The best result hitherto obtained is that $\mathscr{F} \leqslant \frac{13}{40}$. This was proved by Loo-keng Hua in 1942[4]. It is the purpose of this paper to prove that $\mathscr{F} \leqslant \frac{12}{37}$. We have at the same time

$$\sum_{x^2+y^2+z^2 \leqslant a^2} 1 = \frac{4}{3}\pi a^3 + O(a^{\frac{4}{3}}).$$

2. Let

$$\Delta f(x, y) = f(x + m_1 + m_2 + m_3, y + n_1 + n_2 + n_3) - \\ - \sum f(x + m_1 + m_2, y + n_1 + n_2) + \sum f(x + m_1, y + n_1) - f(x, y),$$

and let

$$G(x, y) = \Delta(\sqrt{x^2 + y^2}), \quad X = 6m_1m_2m_3, \quad Y = 2\sum m_1m_2n_3, \\ Z = 2\sum m_1n_2n_3, \quad W = 6n_1n_2n_3.$$

Let $m_i = O(\eta)$, $n_i = O(\eta)$ and $\max(x, y) \geqslant L$. We have[1]

$$G_{xx}G_{yy} - G_{xy}^2 = -\frac{3}{4(x^2 + y^2)^7} Q(X, Y, Z, W) + \\ + O\left(\frac{(X^2 + Y^2 + Z^2 + W^2)}{L^9}\right),$$

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where

$$Q(X,Y,Z,W) = (8x^4 + 6x^2y^2 + 3y^4)y^2X^2 +$$
$$+ 3(4x^6 + 4x^4y^2 + 21x^2y^4 + 6y^6)Y^2 +$$
$$+ 3(6x^6 + 21x^4y^2 + 4x^2y^4 + 4y^6)Z^2 + (3x^4 + 6x^2y^2 + 8y^4)x^2W^2 -$$
$$- 2xy(8x^4 - 4x^2y^2 + 3y^4)XY - 2xy(3x^4 - 4x^2y^2 + 8y^4)ZW -$$
$$- 2(2x^6 + 20x^4y^2 + 9x^2y^4 + 6y^6)XZ -$$
$$- 2(6x^6 + 9x^4y^2 + 20x^2y^4 + 2y^6)YW +$$
$$+ 2(4x^4 + 3x^2y^2 + 4y^4)xyXW - 90x^3y^3YZ.$$

We begin proving three lemmas.

Lemma 1. *If* $x \geqslant y$, *then*

$$Q(X,Y,Z,W) \geqslant 10^{-5}\{x^4(yX - xY)^2 +$$
$$+ x^2y^2(yY - xZ)^2 + x^4(yZ - xW)^2\}.$$

Proof. We begin proving

$$Y^2 \geqslant XZ, \quad Z^2 \geqslant YW.$$

The first inequality follows immediately from the following identity

$$Y^2 = 4[2m_1m_2m_3(m_1n_2n_3 + m_2n_1n_3 + m_3n_1n_2) +$$
$$+ \frac{1}{2}(m_1m_2n_3 - m_1m_3n_2)^2 + \frac{1}{2}(m_1m_2n_3 - m_2m_3n_1)^2 +$$
$$+ \frac{1}{2}(m_1m_3n_2 - m_2m_3n_1)^2 + m_1^2m_2m_3n_2n_3 +$$
$$+ m_1m_2^2m_3n_1n_3 + m_1m_3^2m_2n_1n_2] = XZ +$$
$$+ \frac{1}{2}(m_1m_2n_3 - m_1m_3n_2)^2 + \frac{1}{2}(m_1m_2n_3 - m_2m_3n_1)^2 +$$
$$+ \frac{1}{2}(m_1m_3n_2 - m_2m_3n_1)^2$$

and an analogous identity proves the second inequality.

Case I. Let $yX - xY = \alpha xY$, $|\alpha| \leqslant \frac{1}{2}$, and let

$$Q_1(X,Y,Z,W) = (8x^4 - 4x^2y^2 + 3y^4)(yX - xY)^2 +$$
$$+ (3x^4 - 4x^2y^2 + 8y^4)(yZ - xW)^2 + 42x^2y^2(yY - xZ)^2 +$$
$$+ 10x^2(y^2X - x^2Z)^2 + 6y^2(y^2Y - x^2W)^2 +$$
$$+ (9x^4y^2 + 8x^2y^4 + 4y^6)(Z^2 - YW) +$$
$$+ (4x^6 + 12x^4y^2 + 12x^2y^4 + 4y^6)(Y^2 - XZ),$$

then we obtain

$$
\begin{aligned}
&Q(X, Y, Z, W) = Q_1(X, Y, Z, W) + \\
&+ \{4\,x^4y^2W^2 - 8\,x^2y^4YW + 8\,x^2y^4Z^2 + 8\,y^6(Y^2 - XZ) - 12\,x^2y^4YW + \\
&+ 8\,xy^5XW\} + \{-6\,x^3y^3YZ + 9\,x^4y^2Z^2 - 9\,x^4y^2YW + 6\,x^3y^3XW + \\
&+ 6\,x^2y^4(Y^2 - XZ)\} + \{4\,x^4y^2Y^2 - 8\,x^4y^2XZ + 8\,x^6Z^2 - 12\,x^6YW + \\
&+ 8\,x^5yXW\} = Q_1(X, Y, Z, W) + \{4\,y^2(x^2W - y^2Y)^2 + \\
&+ [(4 + 8\alpha) + (4 - 8\alpha)]x^2y^4Z^2 + 4\,y^6Y^2 - 8\,y^6XZ - \\
&- (4 - 8\alpha)x^2y^4YW\} + \{3\,x^2y^2(yY - xZ)^2 + [(3 + 6\alpha) + \\
&+ (3 - 6\alpha)]x^4y^2Z^2 - (3 - 6\alpha)x^4y^2YW + 3\,x^2y^4Y^2 - 6\,x^2y^4XZ\} + \\
&+ \{4\,x^4y^2Y^2 - 8\,x^4y^2XZ + [(4 + 8\alpha) + (4 - 8\alpha)]x^6Z^2 - \\
&- (4 - 8\alpha)x^6YW\} \geqslant Q_1(X, Y, Z, W) + \{4\,y^6Y^2 + (4 + 8\alpha)x^2y^4Z^2 - \\
&- 8\,y^6XZ\} + \{3\,x^2y^4Y^2 + (3 + 6\alpha)x^4y^2Z^2 - 6\,x^2y^4XZ\} + \\
&+ \{4\,x^4y^2Y^2 + (4 + 8\alpha)x^6Z^2 - 8\,x^4y^2XZ\}. \qquad (2)
\end{aligned}
$$

Then on introducing

$$xZ - yY = \beta\alpha(xZ),$$

we have

$$y^2X = (1 + \alpha)(1 - \beta\alpha)x^2Z, \quad (y^2X - x^2Z)^2 = \alpha^2(1 - \beta\alpha - \beta)^2x^4Z^2,$$

$$Y^2 = \frac{1 - \beta\alpha}{1 + \alpha}XZ, \qquad (3)$$

$$Y^2 - XZ = \frac{-\alpha - \beta\alpha}{1 + \alpha}XZ. \qquad (4)$$

We may suppose $Z > 0$, otherwise from (2) Lemma 1 follows clearly for $|\alpha| \leqslant \frac{1}{2}$.

CASE Ia. If $|\beta| \geqslant 1$ then $x^2y^2(xZ - yY)^2 \geqslant \alpha^2x^4y^2Z^2$,

CASE Ib. If $|\beta| < 1$, it follows from (4) that $\alpha \leqslant 0$, (otherwise $Y^2 - XZ < 0$),

CASE Ic. If $-\frac{1}{2} \leqslant \alpha \leqslant 0$ and $\beta \geqslant 0$, then $Y^2 - XZ \geqslant -\frac{\alpha}{1 + \alpha}XZ$,

CASE Id. If $-\frac{1}{2} \leqslant \alpha \leqslant 0$ and $-\frac{1}{2} \leqslant \beta \leqslant 0$, then

$$(y^2X - x^2Z)^2 \geqslant \alpha^2x^4Z^2, \quad Y^2 - XZ \geqslant -\frac{\alpha}{2(1 + \alpha)}XZ,$$

and if $-\frac{1}{2} \leqslant \alpha \leqslant 0$, $\beta \leqslant -\frac{1}{2}$ then $(y^2X - x^2Z)^2x^2 \geqslant \frac{25}{16}x^6Z^2 \cdot \alpha^2$.

Case IA. If $\frac{1}{2} \geqslant \alpha \geqslant 0$, then we have

$$\{4\, y^6 Y^2 + (4 + 8\alpha) x^2 y^4 Z^2 - 8\, y^6 XZ\} +$$
$$+ \{3\, x^2 y^4 Y^2 + (3 + 6\alpha) x^4 y^2 Z^2 - 6\, x^2 y^4 XZ\} +$$
$$+ 7\, x^2 y^2 (yY - xZ)^2 \geqslant \{4\, y^6 Y^2 + (4 + 8\alpha + 4\alpha^2) x^2 y^4 Z^2 - 8\, y^6 XZ\} +$$
$$+ \{3\, x^2 y^4 Y^2 + (3 + 6\alpha + 3\alpha^2) x^4 y^2 Z^2 - 6\, x^2 y^4 XZ\} =$$
$$= \left\{ 4 \frac{y^8}{x^2} \frac{X^2}{(1+\alpha)^2} - 8\, y^6 XZ + 4(1+\alpha)^2 x^2 y^4 Z^2 \right\} +$$
$$+ \left\{ 3 \frac{y^6}{(1+\alpha)^2} X^2 - 6\, x^2 y^4 XZ + 3(1+\alpha)^2 x^4 y^2 Z^2 \right\} \geqslant 0.$$

$$\{4\, x^4 y^2 Y^2 + (4 + 8\alpha) x^6 Z^2 - 8\, x^4 y^2 XZ\} +$$
$$+ 6\, x^4 (yX - xY)^2 \geqslant (4 + 6\alpha^2) x^4 y^2 Y^2 + (4 + 8\alpha) x^6 Z^2 - 8\, x^4 y^2 XZ =$$
$$= \frac{(4 + 6\alpha^2)}{(1+\alpha)^2} x^2 y^4 X^2 - 8\, x^4 y^2 XZ + (4 + 8\alpha) x^6 Z^2 \geqslant 0.$$

Case IB. If $-\frac{1}{2} \leqslant \alpha \leqslant 0$, $\beta \geqslant 0$ then

$$\{4\, y^6 Y^2 + (4 + 8\alpha) x^2 y^4 Z^2 - 8\, y^6 XZ\} +$$
$$+ \{3\, x^2 y^4 Y^2 + (3 + 6\alpha) x^4 y^2 Z^2 - 6\, x^2 y^4 XZ\} +$$
$$+ \{4\, x^4 y^2 Y^2 + (4 + 8\alpha) x^6 Z^2 - 8\, x^4 y^2 XZ\} +$$
$$+ (8\, y^6 + 6\, x^2 y^4 + 8\, x^4 y^2)(Y^2 - XZ) \geqslant$$
$$\geqslant \left\{ 4 \frac{y^8}{x^2 (1+\alpha)^2} X^2 + (4 + 8\alpha) x^2 y^4 Z^2 - 8 \left(1 + \frac{\alpha}{1+\alpha}\right) y^6 XZ \right\} +$$
$$+ \left\{ 3 y^6 \frac{X^2}{(1+\alpha)^2} - 6 \left(1 + \frac{\alpha}{1+\alpha}\right) x^2 y^4 XZ + (3 + 6\alpha) x^4 y^2 Z^2 \right\} +$$
$$+ \left\{ 4 x^2 y^4 \frac{X^2}{(1+\alpha)^2} - 8 \left(1 + \frac{\alpha}{1+\alpha}\right) x^4 y^2 XZ + (4 + 8\alpha) x^6 Z^2 \right\} \geqslant 0.$$

Case IC. If $-\frac{1}{2} \leqslant \alpha \leqslant 0$, $\beta \leqslant 0$ then

$$\{4\, y^6 Y^2 + (4 + 8\alpha) x^2 y^4 Z^2 - 8\, y^6 XZ\} + 4\, x^2 (y^2 X - x^2 Z)^2 +$$
$$+ \{3\, x^2 y^4 Y^2 - 6\, x^2 y^4 XZ + (3 + 6\alpha) x^4 y^2 Z^2\} + 3\, x^2 (y^2 X - x^2 Z)^2 \geqslant$$
$$\geqslant \left\{ 4 \frac{y^8}{x^2 (1+\alpha)^2} X^2 + 4(1+\alpha)^2 x^2 y^4 Z^2 - 8\, y^6 XZ \right\} +$$

$$+ \left\{3\, y^6 \frac{X^2}{(1+\alpha)^2} - 6\, x^2y^4XZ + 3(1+\alpha)^2x^4y^2Z^2\right\} \geqslant 0.$$

If $-\frac{1}{2} \leqslant \beta \leqslant 0$ then

$$\{4\, x^4y^2Y^2 + (4+8\alpha)x^6Z^2 - 8\, x^4y^2XZ\} + (4\, x^6 + 12\, x^4y^2)(Y^2 - XZ) \geqslant$$
$$\geqslant 4\, x^2y^4 \frac{X^2}{(1+\alpha)^2} - 8\left(1 + \frac{\alpha}{1+\alpha}\right) x^4y^2XZ + (4+8\alpha)x^6Z^2 \geqslant 0.$$

If $\beta \leqslant -\frac{1}{2}$ then

$$\{4\, x^4y^2Y^2 + (4+8\alpha)x^6Z^2 - 8\, x^4y^2XZ\} +$$
$$+ 3\, x^2(y^2X - x^2Z)^2 \geqslant 4x^2y^4 \frac{X^2}{(1+\alpha)^2} - 8\, x^4y^2XZ +$$
$$+ \left(4 + 8\alpha + \frac{75}{16}\alpha^2\right) x^6Z^2 \geqslant 0.$$

Hence, for $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$ Lemma 1 follows clearly.

Case II. Suppose $yX - xY = \alpha xY$ and $\alpha \geqslant \frac{1}{2}$ then we have $Y \geqslant 0$.

Case IIa. If $W \geqslant 0$, then we have

$$Q(X, Y, Z, W) = \{10\, x^2y^4X^2 + 18\, x^6Z^2 - 26\, x^4y^2XZ\} +$$
$$+ \{(Y^2 - XZ)(4\, x^6 + 14\, x^4y^2 + 18\, x^2y^4 + 12\, y^6) +$$
$$+ 45\, x^2y^2(yY - xZ)^2 + (8\, x^4 - 4\, x^2y^2 + 3\, y^4)(yX - xY)^2 +$$
$$+ (2\, x^4y^2 - 3\, x^2y^4 + 4\, y^6)Y^2 + (3\, x^4 - 4\, x^2y^2 + 8\, y^4)(yZ - xW)^2\} +$$
$$+ \{(15\, x^4y^2 + 16\, x^2y^4 + 4\, y^6)Z^2 - (9\, x^4y^2 + 22\, x^2y^4 + 4\, y^6)YW\} +$$
$$+ \{2\, y^6Y^2 + 10\, x^4y^2W^2 - 6\, x^2y^4YW\} +$$
$$+ \left\{(8\, x^5 + 6\, x^3y^2 + 8\, xy^4)W\left(yX - \frac{3}{2}\, xY\right)\right\}.$$

From $yX \geqslant \frac{3}{2}\, xY$ the result follows clearly.

Case IIb. If $W < 0$, then we have

$$Q(X, Y, Z, W) = \{(1.5\, x^4 - 4\, x^2y^2 + 3\, y^4)(yX - xY)^2 +$$
$$+ 6\, y^6Y^2 + 10\, x^2y^4Z^2 + 12\, y^6Z^2 + 45\, x^2y^2(yY - xZ)^2 +$$
$$+ (2\, x^2y^4X^2 - 3\, x^2y^4XZ + 2\, x^2y^4Z^2) + (4\, x^6 + 16\, x^4y^2 + 15\, x^2y^4 + 12\, y^6)\cdot$$
$$\cdot\, (Y^2 - XZ)\} + \{6.5\, x^4(yX - xY)^2 + 2.5\, x^6|W|^2 + 12\, x^6Y|W| -$$
$$- 8\, x^5yX|W|\} + \{(0.5\, x^6 + 6\, x^4y^2 + 8\, x^2y^4)W^2 +$$

$$+ (8x^2y^4X^2 - 24x^4y^2XZ + 18x^6Z^2 + 18x^4y^2Z^2) +$$
$$+ (18x^4y^2 + 40x^2y^4 + 4y^6)Y|W| - (6x^3y^2 + 8xy^4)\cdot yX|W| -$$
$$- 2xy(3x^4 - 4x^2y^2 + 8y^4)ZW\}.$$

Also we have

$$6.5x^4(yX - xY)^2 + 2.5x^6|W|^2 + 12x^6Y|W| - 8x^5yX|W| =$$
$$= 6.5x^4y^2\frac{\alpha^2}{(1+\alpha)^2}X^2 - \frac{8\alpha - 4}{1+\alpha}x^5y|W|X + 2.5x^6|W|^2 \geqslant 0.$$

CASE IIb$_1$. Suppose that $W \leqslant 0$, $Z \geqslant 0$ then $-ZW \geqslant 0$. Our lemma is also obvious if $\alpha \leqslant 2$. Suppose that $\alpha \geqslant 2$ then

$$3x^2y^4X^2 \geqslant 3x^2y^2(9x^2Y^2) \geqslant 27x^4y^2XZ,$$

and

$$5x^2y^4X^2 + (6x^4y^2 + 8x^2y^4)W^2 - (6x^3y^2 + 8xy^4)X|W|y \geqslant 0.$$

Hence the lemma follows.

CASE IIb$_2$. Suppose that $W \leqslant 0$, $Z \leqslant 0$ then $-XZ \geqslant 0$. From

$$8x^2y^4X^2 + (5x^4y^2 + 3x^2y^4)W^2 - (6x^3y^2 + 8xy^4)yX|W| \geqslant 0,$$
$$(x^4y^2 + 5x^2y^4)W^2 + (10x^6 + 18x^4y^2)Z^2 - 2xy(3x^4 + 8y^4)ZW \geqslant 0,$$

the lemma follows.

CASE III. Suppose that $yX - xY = \alpha xY$, where $-1 < \alpha \leqslant -\frac{1}{2}$. It is easily seen that $Y \geqslant 0$.

CASE IIIa. Suppose that $W \geqslant 0$, $Z \geqslant 0$ then $xY = \dfrac{yX}{1+\alpha} \geqslant 2yX$.

CASE IIIa$_1$. Suppose that $3yX \geqslant xY \geqslant 2yX$ then we have

$$Q(X, Y, Z, W) = \{(1.5x^4 - 4x^2y^2 + 3y^4)(yX - xY)^2 +$$
$$+ 4x^6(Y^2 - XZ) + (3x^4 - 4x^2y^2 + 8y^4)(yZ - xW)^2 +$$
$$+ 45x^2y^2(yY - xZ)^2\} + \left\{2(4x^4 + 3x^2y^2 + 4y^4)xW\left(yX - \frac{1}{3}xY\right) +\right.$$
$$\left. + \left(9\frac{1}{3}x^6 + 9x^4y^2 + 16x^2y^4 + 4y^6\right)(Z^2 - YW)\right\} +$$
$$+ \left\{6.5x^4(yX - xY)^2 + 10x^2y^4X^2 + 9x^4y^2Y^2 + 8\frac{2}{3}x^6Z^2 - 40x^4y^2XZ\right\} +$$
$$+ \{7x^4y^2Y^2 - 7x^4y^2YW + 2x^4y^2W^2\} + \{15x^2y^4Y^2 - 18x^2y^4XZ + 3x^4y^2Z^2\} +$$
$$+ \{3.5y^6Y^2 - 12y^6XZ + 3x^4y^2Z^2\} +$$
$$+ \left\{14.5y^6Y^2 + 8x^4y^2W^2 - 21\frac{1}{3}x^2y^4YW\right\}. \tag{5}$$

By using $yX \geqslant \frac{1}{3}xY$, $x^2Y^2 \geqslant 4\,y^2X^2$ it is easily seen that the Lemma 1 follows.

CASE IIIa$_2$. Suppose that $xY \geqslant 3yX$ then

$$
\begin{aligned}
Q(X, Y, Z, W) &= \{(1.5\,x^4 - 4\,x^2y^2 + 3\,y^4)(yX - xY)^2 + \\
&+ (12\,x^6 + 10\,x^4y^2 + 16\,x^2y^4 + 4\,y^6)(Z^2 - YW) + (3\,x^4 - 4\,x^2y^2 + 8\,y^4)\cdot \\
&\cdot (yZ - xW)^2 + 45\,x^2y^2(yY - xZ)^2 + 4\,x^6(Y^2 - XZ)\} + \\
&+ \{6.5\,x^4(yX - xY)^2 + 10\,x^2y^4X^2 + 6\,x^4y^2Y^2 - 40\,x^4y^2XZ + 6\,x^6Z^2\} + \\
&+ \{15\,x^2y^4Y^2 + 5\,x^4y^2Z^2 - 18\,x^2y^4XZ - 12\,y^6XZ\} + \\
&+ \{2(4\,x^4 + 3\,x^2y^2 + 4\,y^4)xyXW\} + \{18\,y^6Y^2 + 8\,x^4y^2W^2 - 24\,x^2y^4YW\} + \\
&+ \{10\,x^4y^2Y^2 - 8\,x^4y^2YW + 2\,x^4y^2W^2\}.
\end{aligned}
$$

By using $(xY)^2 \geqslant 9\,y^2X^2$ it is easily seen that Lemma 1 follows.

CASE IIIb. Suppose that $Y \geqslant 0$, $W \geqslant 0$, $Z \leqslant 0$. It is easily seen that the Lemma 1 follows.

CASE IIIc. Suppose that $W < 0$, then from (5) and the following inequality

$$
\begin{aligned}
&2(4\,x^4 + 3\,x^2y^2 + 4\,y^4)xW\left(yX - \frac{1}{3}\,xY\right) + \\
&\qquad + \left(9\frac{1}{3}\,x^6 + 9\,x^4y^2 + 16\,x^2y^4 + 4\,y^6\right)(Z^2 - YW) \geqslant 0;
\end{aligned}
$$

our lemma follows.

CASE IV. Suppose $\alpha = -1$, then we have $X = 0$. It is easily seen that Lemma 1 follows.

CASE V. Suppose $\alpha < -1$, then we have $Y \leqslant 0$. If $W \geqslant 0$, then $-XY$, $-YW$, XW are positive. It is easily seen that the Lemma 1 follows.

CASE Va. If $W \leqslant 0$, $Z \leqslant 0$, then by using $Y \leqslant 0$,

$$
\begin{aligned}
Q(X, Y, Z, W) &= 2\,xy(8\,x^4 - 4\,x^2y^2 + 3\,y^4)X|Y| + \\
&+ (0.5\,x^4 - 4\,x^2y^2 + 8\,y^4)(yZ - xW)^2 + \{(4\,x^6Y^2 + 4\,y^6Z^2 - \\
&- 8\,x^3y^3YZ) + [(4\,x^6 + 12\,x^4y^2)Y^2 + 16\,x^2y^4Z^2 - 32\,x^3y^3YZ] + \\
&+ (63\,x^2y^4Y^2 + 10\,x^4y^2Z^2 - 50\,x^3y^3YZ)\} + \\
&+ \{(18\,x^6 + 50\,x^4y^2)Z^2 - (12\,x^6 + 18\,x^4y^2 + 38\,x^2y^4)YW\} + \\
&+ \{(3\,x^6 + 6\,y^6)Y^2 - (2\,x^2y^4 + 4\,y^6)YW + x^4y^2W^2\} + \\
&+ \{(6\,x^2y^2 + 3\,y^4)y^2X^2 + 9\,x^4y^2W^2 - 2(3\,x^2y^2 + 4\,y^4)xyX|W|\} + \\
&+ \{(18\,x^2y^4 + 12\,y^6)X|Z| + 0.001\,x^4y^2X^2 + x^6Y^2 + 12\,y^6Y^2\} + \\
&+ \{2.5\,x^4(yZ - xW)^2 + (4\,x^6 + 40\,x^4y^2)X|Z| + \\
&+ 7.999\,x^4y^2X^2 - 8\,x^5yX|W|\}.
\end{aligned}
$$

Suppose that $x|W| - y|Z| = \beta y|Z|$. If $\beta \leqslant 4.5$ then

$(4x^6 + 40x^4y^2)X|Z| \geqslant 8x^5yX|W|$ and the result follows clearly.

Suppose that $x|W| - y|Z| = \beta y|Z|$. If $4.5 \leqslant \beta \leqslant 9$ then

$$7.999x^4y^2X^2 + 2.5x^4(yZ - xW)^2 + (4x^6 + 40x^4y^2)X|Z| - 8x^5yX|W| \geqslant$$
$$\geqslant 7.999x^4y^2X^2 + 2.5x^6\frac{\beta^2W^2}{(1+\beta)^2} - \left(8 - \frac{44}{1+\beta}\right)x^5 \cdot yX|W| >$$
$$> 7.999x^4y^2X^2 + 2.5x^6\left(\frac{4.5}{5.5}\right)^2W^2 - 3.6x^5yX|W| > 0.$$

If $\beta \geqslant 9$ then

$$7.999x^4y^2X^2 + 2.5x^6\left(\frac{\beta}{\beta+1}\right)^2W^2 - \left(8 - \frac{44}{1+\beta}\right)x^5yX|W| \geqslant$$
$$\geqslant 7.999x^4y^2X^2 + (2.5)(0.81)x^6W^2 - 8x^5yX|W| \geqslant 0.$$

Case Vb. If $W \leqslant 0$, $Z \geqslant 0$ then on noticing $Y \leqslant 0$ we see that $-YZ$, $-WZ$, and $-XY$ are positive and we have

$$\{(2x^2y^2 + 3y^4)y^2X^2 + (6x^2y^2 + 8y^4)x^2W^2 - 2(3x^2y^2 + 4y^4)xyX|W|\} +$$
$$+ \{7.5x^4y^2X^2 - 8x^5yX|W| + 2.5x^6|W|^2\} \geqslant 0$$

and

$$(7x^6 + 45x^2y^4)Y^2 = \{7x^6 - 2\sqrt{7\cdot 45}\,x^4y^2 + 45x^2y^4\}Y^2 + 2\sqrt{7\cdot 45}\,x^4y^2Y^2.$$

Now our lemma follows from $Y^2 \geqslant XZ$ and $Z^2 \geqslant YW$.

Hence in any case, our lemma is correct.

Lemma 2. *We have*[1]

$$G_{xx} = (x^2+y^2)^{-\frac{9}{2}}\Big\{(-60x^3y^2 + 45xy^4)\left(\frac{1}{6}X\right) +$$
$$+ (24x^4y - 72x^2y^3 + 9y^5)\left(\frac{1}{2}Y\right) + (-6x^5 + 63x^3y^2 - 36xy^4)\left(\frac{1}{2}Z\right) +$$
$$+ (-36x^4y + 63x^2y^3 - 6y^5)\left(\frac{1}{6}W\right)\Big\}.$$

Lemma 3. *We assume that there is one or more than one m_i such that $m_i \geqslant t^\varepsilon n_i$ where ε is an arbitrary positive number, then we have, as $t \to \infty$:*

$$Q(X, Y, Z, W) \geqslant 10^{-5}x^6(Y^2 + Z^2 + W^2).$$

Prove. Using

$$Z^2 - \frac{2}{3}YW = 4\{(n_1n_2m_3)^2 + (n_2n_3m_1)^2 + (n_1n_3m_2)^2\} \geqslant t^\varepsilon W^2, \tag{6}$$

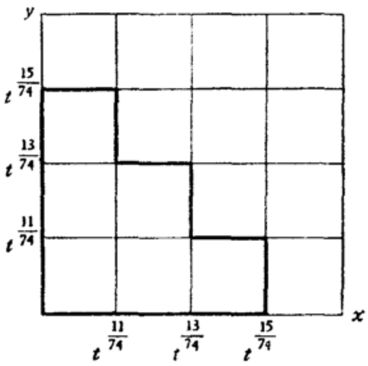
we have

$$Q(X, Y, Z, W) \geqslant (8x^4 + 6x^2y^2 + 3y^4)y^2X^2 +$$
$$+ 3(4x^6 + 4x^4y^2 + 21x^2y^4 + 6y^6)Y^2 + (3 - 10^{-10})(6x^6 + 21x^4y^2 +$$
$$+ 4x^2y^4 + 4y^6)Z^2 + \{(3 + 10^{-10}t^\varepsilon)x^6 + (6 + 10^{-10}t^\varepsilon)x^4y^2 +$$
$$+ (8 + 10^{-10}t^\varepsilon)x^2y^4 + 10^{-10}t^\varepsilon y^6\}W^2 - 2xy(8x^4 - 4x^2y^2 + 3y^4)XY -$$
$$- 2xy(3x^4 - 4x^2y^2 + 8y^4)ZW - 2(2x^6 + 20x^4y^2 + 9x^2y^4 + 6y^6)XZ -$$
$$- \left\{\left(12 - \frac{12}{3}\cdot 10^{-10}\right)x^6 + \left(18 - \frac{42}{3}\cdot 10^{-10}\right)x^4y^2 +\right.$$
$$\left. + \left(40 - \frac{8}{3}\cdot 10^{-10}\right)x^2y^4 + \left(4 - \frac{8}{3}\cdot 10^{-10}\right)y^6\right\} YW +$$
$$+ 2(4x^4 + 3x^2y^2 + 4y^4)xyXW - 90x^3y^3YZ.$$

Since $t^\varepsilon \to \infty$ and $(8x^4 - 4x^2y^2 + 3y^4)(yX - xY)^2 \geqslant 0$, we have

$$10x^2y^4X^2 + 17x^6Z^2 - 26x^4y^2XZ \geqslant 0,$$

$45x^2y^2(yY - xZ)^2 \geqslant 0$ and $Y^2 \geqslant XZ$. It is easily seen that Lemma 3 follows.



3. It is known that

$$\int_0^t \{R(n) - \pi n\}dn = \frac{t}{\pi}\sum_{v=1}^{\infty}\frac{r(v)}{v}J_2\{2\pi\sqrt{vt}\},$$

where $r(v)$ is the number of solutions of the Diophantion equation

$$m^2 + n^2 = v.$$

Evidently, we have

$$\int_0^t \{R(n) - \pi n\}dn = \frac{4t}{\pi}\sum_{x=1}^{\infty}\sum_{y=0}^{\infty}\frac{J_2\{2\pi\sqrt{(x^2+y^2)t}\}}{x^2+y^2}.$$

Let C denote the region bounded by heavy lines in the figure and C' denote the remaining part in the first quadrant. It is easy to deduce that, if $0 < \alpha < 1$ $\left(\text{here } \alpha = \frac{12}{37}\right)$

$$\int_t^{t\pm t^\alpha}\{R(n) - \pi n\}dn = 4\int_t^{t\pm t^\alpha}\sum_C\sum\frac{\sqrt{n}\,J_1\{2\pi\sqrt{(x^2+y^2)n}\}}{(x^2+y^2)^{1/2}}\,dn +$$
$$+ \frac{4}{\pi}\left\{\sum_{C'}\sum\frac{nJ_2\{2\pi\sqrt{(x^2+y^2)n}\}}{x^2+y^2}\right\}_t^{t\pm t^\alpha} = \Sigma_1 + \{\Sigma_2\}_t^{t\pm t^\alpha}.$$

Here we have

$$J_1\{2\pi\sqrt{vn}\} = \frac{\sin\left\{2\pi\sqrt{vn} - \frac{1}{4}\pi\right\}}{\pi(vn)^{1/4}} + O\left(\frac{1}{(vn)^{3/4}}\right).$$

Hence

$$\sum\nolimits_1 = O\left(\int_t^{t\pm t^a} |\phi(n)| n^{\frac{3}{4}} dn\right) + O(t^{a+\frac{1}{4}}),$$

where

$$\phi(n) = \sum_C\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)n}}}{(x^2+y^2)^{3/4}},$$

$$t - t^a \leqslant n \leqslant t + t^a.$$

Similarly we have

$$\sum\nolimits_2 = O\{n^{\frac{3}{4}}|\psi(n)|\} + O(t^{\frac{1}{4}}),$$

where

$$\psi(n) = \sum_{C'}\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)n}}}{(x^2+y^2)^{5/4}},$$

$$t - t^a \leqslant n \leqslant t + t^a.$$

If $x \leqslant t^{\frac{11}{74}}$, then

$$n^{\frac{1}{4}}\left|\sum_{\substack{C\\ x\leqslant t^{11/74}}}\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)n}}}{(x^2+y^2)^{3/4}}\right| \leqslant n^{\frac{1}{4}} \sum_{x\leqslant t^{11/74}} \sum_{y=1}^{\infty} \frac{1}{(x^2+y^2)^{3/4}} \leqslant$$

$$\leqslant n^{\frac{1}{4}} \sum_{x\leqslant t^{11/74}} \left(\sum_{y=1}^{x} \frac{1}{x^{3/2}} + \sum_{y=x+1}^{\infty} \frac{1}{y^{3/2}}\right) =$$

$$= O\left(n^{\frac{1}{4}} \sum_{x\leqslant t^{11/74}} x^{-\frac{1}{2}}\right) = O\left(t^{\frac{1}{4}+\frac{11}{148}}\right) = O\left(t^{\frac{12}{37}}\right).$$

The same result holds for $x \leqslant t^{\frac{11}{74}}$.

If $x \geqslant t^{\frac{15}{74}}$ then

$$n^{\frac{3}{4}}\left|\sum_{\substack{C'\\ x\geqslant t^{15/74}}}\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)n}}}{(x^2+y^2)^{5/4}}\right| \leqslant n^{\frac{3}{4}} \sum_{x\geqslant t^{15/74}} O\left(\frac{1}{x^{3/2}}\right) =$$

$$= O\left(t^{\frac{3}{4}-\frac{15}{148}}\right) = O\left(t^{2\cdot\frac{12}{37}}\right) = O\left(t^{\frac{24}{37}}\right).$$

The same result holds for $y \geqslant t^{\frac{15}{74}}$. Let D be the region common to C and the square $t^{\frac{11}{74}} \leqslant x,\ y \leqslant t^{\frac{15}{74}}$, and D' be the remaining part of the square. Then

$$n^{\frac{1}{4}} |\phi(n)| = n^{\frac{1}{4}} \left| \sum_D \sum \frac{e^{2\pi i \sqrt{(x^2+y^2)n}}}{(x^2+y^2)^{3/4}} \right| + O(t^{\frac{12}{37}}),$$

and

$$n^{\frac{3}{4}} |\psi(n)| = n^{\frac{3}{4}} \left| \sum_{D'} \sum \frac{e^{2\pi i \sqrt{(x^2+y^2)n}}}{(x^2+y^2)^{5/4}} \right| + O(t^{\frac{12}{37}}).$$

4. Now we consider the sum of the form

$$S = \sum_{x=N}^{2N} \sum_{y=M}^{2M} e^{2\pi i f(x,y)},$$

where

$$f(x,\ y) = \sqrt{(x^2+y^2)t}.$$

Write

$$S = \sum_{\substack{x=N \\ x \geqslant y}}^{2N} \sum_{y=M}^{2M} e^{2\pi i f(x,\ y)} + \sum_{\substack{x=N \\ x \leqslant y}}^{2N} \sum_{y=M}^{2M} e^{2\pi i f(x,\ y)}.$$

The estimation of the first sum is similar to that of the second. Now we estimate the first sum. Let $L = \max(x,\ y)$ satisfy the condition

$$t^{\frac{11}{74}} \leqslant L \leqslant t^{\frac{15}{74}}.$$

When $L \geqslant t^{\frac{2}{11}-2.5\varepsilon}$, from Formula (23) of [1] we have

$$S = O\{L^{\frac{7}{4}} t^{\frac{1}{32}} (\log t)^{\frac{1}{8}}\}.$$

Now we only need to consider $t^{\frac{11}{74}} \leqslant L \leqslant t^{\frac{2}{11}-2.5\varepsilon}$. Using Lemma 2 of [1] three times, we have

$$S = O(L^2 \rho^{-\frac{1}{2}}) + O\left[L^{\frac{3}{2}} \rho^{-1} \sum_{i=1}^{4} \left\{ \sum_{m_1=1}^{\rho^{\frac{1}{2}}-1} \sum_{n_1=0}^{\rho^{\frac{1}{2}}-1} \left(\sum_{m_2=0}^{\rho-1} \sum_{n_2=0}^{\rho-1} |S_2^{(i)}| \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}} \right] =$$

$$= O(L^2 \rho^{-\frac{1}{2}}) + O\Bigg[L^{\frac{7}{4}} \rho^{-\frac{3}{2}} \sum_{i=1}^{8} \Bigg\{ \sum_{m_1=1}^{\rho^{\frac{1}{2}}-1} \sum_{n_1=0}^{\rho^{\frac{1}{2}}-1} \times$$

$$\times \left[\sum_{m_2=1}^{\rho-1} \sum_{n_2=0}^{\rho-1} \left(\sum_{m_3=1}^{\rho^2-1} \sum_{n_3=0}^{\rho^2-1} |S_3^{(i)}| \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \Bigg\}^{\frac{1}{2}} \Bigg].$$

Here

$$S_3^{(1)} = S_3 = \sum\sum e^{2\pi i\psi(x,y)}, \quad \rho = t^{-\frac{1}{15}-\varepsilon} L^{\frac{8}{15}},$$
$$\psi(x, y) = \sqrt{t}\,\Delta\{\sqrt{x^2+y^2}\},$$

and $S_3^{(i)}$, $i = 2, \cdots, 8$, are similar sums. In §5 we shall prove that if $W \neq 0$ and $L \leqslant t^{\frac{2}{11}-2.5\varepsilon}$, then we have

$$|S_3^{(i)}| \ll L^{-2}\rho^7 t^{\frac{1}{2}} W^{-1}.$$

If $W = 0$, then we have[1]

$$|S_3^{(i)}| \ll L^{-2}\rho^7 t^{\frac{1}{2}}[X^2+Y^2+Z^2]^{-\frac{1}{2}}.$$

Hence we have

$$S = O(L^2\rho^{-\frac{1}{2}}) + O[L^{\frac{7}{4}}(L^{-2}t^{\frac{1}{2}}\rho^{3.5})^{\frac{1}{8}}] =$$
$$= O[L^2(L^{\frac{8}{15}}t^{-\frac{1}{15}-\varepsilon})^{-\frac{1}{2}}] + O(t^{\frac{1}{16}}L^{\frac{3}{2}}\rho^{\frac{3.5}{8}}) = O(L^{\frac{26}{15}}t^{\frac{1}{30}+\frac{1}{2}\varepsilon}). \tag{7}$$

5. We consider the sums of the form

$$S = \sum_{M<y\leqslant 2M}\ \sum_{N<x\leqslant 2N} e^{2\pi i\psi(x,y)}$$

and divide the sum into two parts, namely,

$$S = \sum_{M<y\leqslant 2M}\ \sum_{\substack{N<x\leqslant 2N\\ x>y}} e^{2\pi i\psi(x,y)} + \sum_{M<y\leqslant 2M}\ \sum_{\substack{N<x\leqslant 2N\\ x\leqslant y}} e^{2\pi i\psi(x,y)}.$$

We shall give a method to estimate the first sum and the same method can also be applied to the second. Here we shall suppose that $L \leqslant t^{\frac{2}{11}-2.5\varepsilon}$. We divide the domain $D(N \leqslant x \leqslant 2N,\ M \leqslant y \leqslant 2M,\ x \geqslant y)$ into two subdomains. We define a basic domain consisting of the points (x, y), $N \leqslant x \leqslant 2N$, $M \leqslant y \leqslant 2M$ given by

$$|yX - xY| \leqslant \alpha xY, \quad |yY - xZ| \leqslant \beta xZ, \quad |yZ - xW| \leqslant \gamma xW,$$

where α, β, γ denote arbitrarily small positive numbers. The domains which remain after the removal from the domain $(N \leqslant x \leqslant 2N,\ M \leqslant y \leqslant 2M)$ of all the basic domains will be called the supplementary domains. If (x, y) belongs to the supplementary domains and $W \neq 0$ then we have

$$Q(X, Y, Z, W)t \gg L^6W^2t.$$

If $W = 0$, we have on using the results of [1]

$$Q(X, Y, Z, W)t \gg L^6(X^2+Y^2+Z^2+W^2)t.$$

Hence we may apply Lemma ε and Lemma ζ of [2] and also use the method of [1] to estimate the sum. This will prove our result. Hence we need only to estimate the sum in the basic domain. In the basic domain we can see easily that

$$L^{-4}Wt^{\frac{1}{2}} \ll |\psi_{xx}| \ll L^{-4}(X+Y+Z+W)t^{\frac{1}{2}}.$$

Let S_1 be that part of the sum S which corresponds to the basic domain. Now we estimate the sum S_1. We have the inequalities

$$L^{-4}Wt^{\frac{1}{2}} \ll |\psi_{xx}| \ll L^{-4}(X+Y+Z+W)t^{1/2},$$
$$|\psi_{xxx}| \ll t^{\frac{1}{2}}L^{-5}(X+Y+Z+W).$$

The domain D_1 can be divided into $O((\log t)^2)$ subdomains, such that in each subdomain we have

$$\frac{1}{R} \ll |\psi_{xx}| \ll \frac{1}{R},$$

where

$$t^{-\frac{1}{2}}L^4(X+Y+Z+W)^{-1} \ll R \ll t^{-\frac{1}{2}}L^4W^{-1}.$$

Let D_{11} be any subdomain. We divide the subdomain D_{11} into two subdomains D_{111} and D_{112}, such that D_{111} contains all the points (x, y) of D_{11} satisfying

$$|yZ - xW| \ll L^{1-a}W,$$

and D_{112} contains the points not belonging to D_{111}. We use the notation S_{11} to denote the partial sum of S_1 which corresponds to the domain D_{111}. The notation S_{12} denotes the partial sum of S_1 which corresponds to the domain D_{112}. Here we take $L^a = L^{\frac{1}{2}}W^{\frac{1}{2}}\rho^{-\frac{3.5}{2}}R^{-\frac{1}{4}}$, $\rho = t^{-\frac{1}{15}-\varepsilon}L^{\frac{8}{15}}$.

We may use ordinary method to estimate the sum S_1, namely, letting $L^{1-a}\psi_{xx} \gg 1$ and using the Lemma 7 of [3], we get

$$S_{11} \ll L \cdot L^{1-a}\psi_{xx}^{1/2} \ll L^2L^{-\frac{1}{2}}W^{-\frac{1}{2}}\rho^{\frac{3.5}{2}}(\rho^{3.5}L^{-4}t^{\frac{1}{2}})^{\frac{1}{4}} \ll L^{\frac{1}{2}}W^{-\frac{1}{2}}\rho^{\frac{10.5}{4}}t^{\frac{1}{8}}.$$

Also, when $L^{1-a}\psi_{xx} \ll 1$, we have

$$S_{11} \ll L \cdot \psi_{xx}^{-1/2} \ll L(L^{-4}Wt^{\frac{1}{2}})^{-\frac{1}{2}} \ll L^3W^{-\frac{1}{2}}t^{-\frac{1}{4}}.$$

Now let us estimate the sum S_{12}:

$$S_{12} = \sum_{y} \sum_{C_1(y) \leqslant x \leqslant C_2(y)} e^{2\pi i\psi(x, y)},$$

where $C_1(y)$ and $C_2(y)$ are linear functions of y, i.e., $C_1(y) = \alpha_1 y + \beta_1$, $C_2(y) = \alpha_2 y + \beta_2$, and we have $y \leqslant C_1(y)$, $C_2(y) \leqslant 2L$. For any x

belonging to the interval $C_1(y) \leqslant x \leqslant C_2(y)$, it must satisfy the condition $|yZ - xW| \gg L^{1-a}W$. Using Lemma 6 of [3], we have

$$S_{12} = S_{12\mathrm{I}} + S_{12\mathrm{II}} + S_{12\mathrm{III}},$$

where

$$S_{12\mathrm{I}} = e^{\frac{\pi}{4}i} \sum_y \sum_{v_1(y) \leqslant v \leqslant v_2(y)} e^{2\pi i \eta(y)} / \sqrt{\phi_{xx}(n_v(y), y)} =$$

$$= e^{\frac{\pi}{4}i} \sum_y \sum_{v_1(y) \leqslant v \leqslant v_2(y)} \frac{e^{2\pi i \phi(n_v(y), y) - v n_v(y)}}{\sqrt{\phi_{xx}(n_v(y), y)}},$$

$$S_{12\mathrm{II}} \ll \sum_y (L + R) L^{-1} \ll (L + R),$$

$$S_{12\mathrm{III}} \ll \sum_y \sqrt{R} \ll L \sqrt{R},$$

$$v_1(y) = \phi_x(C_2(y), y), \quad v_2(y) = \phi_x(C_1(y), y).$$

Here we put also $\phi_{xx} \leqslant 0$. For a given v, let y run through all positive integers such that $v_1(y) \leqslant v$ and $v_2(y) \geqslant v$. We shall interchange the order of summation in question. Since the number of solutions of $v_1(y) = v$ or $v_2(y) = v$ is at most finite, and $v_1(y)$ and $v_2(y)$ are continued functions of y, it follows that the solutions of y satisfying both $v_1(y) \leqslant v$ and $v_2(y) \geqslant v$ consist of a finite number of intervals on the y-axis whose end-points are positive integers (whose length $\leqslant L$). Considering now the values of y belonging to one of the above intervals, we have

$$\sum_y e^{2\pi i \eta(y)} \cdot \frac{1}{\sqrt{\phi_{xx}(n_v(y), y)}},$$

$$\eta(y) = \phi(n_v(y), y) - v n_v(y),$$

$$\eta'(y) = \phi_y(n_v(y), y),$$

$$\eta''(y) = (\phi_{xx}\phi_{yy} - \phi_{xy}^2)\phi_{xx}^{-1} = H(n_v(y), y)\phi_{xx}^{-1}(n_v(y), y).$$

For these values of $v, n_v(y)$ satisfy the equation $\phi_x(x, y) = v$. For any given y, it must also satisfy

$$v_1(y) \leqslant v \leqslant v_2(y).$$

When $v = v_2(y) = \phi_x(C_1(y), y)$, we have $n_v(y) = C_1(y)$. When $v = \phi_x(x, y) \leqslant v_2(y) = \phi_x(C_1(y), y)$, we have $n_v(y) \geqslant C_1(y)$, where $\phi_{xx} \leqslant 0$. Similarly when $v \geqslant v_1(y)$, we have $n_v(y) \leqslant C_2(y)$. Hence we have

$$C_1(y) \leqslant n_v(y) \leqslant C_2(y),$$

namely,

$$|yZ - n_v(y)W| \gg L^{1-a}W.$$

From Lemmas 1 and 3 we have

$$H(n_v(y), y) \gg L^{-8}W^2tL^{-2\alpha},$$

where we put

$$L^{-2\alpha} = L^{-1}W^{-1}\rho^{3.5}R^{\frac{1}{2}} \gg L^{-1}W^{-1}\rho^{3.5}(t^{\frac{1}{2}}\rho^{3.5}L^{-4})^{-\frac{1}{2}} \gg$$
$$\gg L\rho^{\frac{3.5}{2}}t^{-\frac{1}{4}}W^{-1} \gg L\rho^{-\frac{3.5}{2}}t^{-\frac{1}{4}} \gg (\rho^2/L)t^{2.5\varepsilon},$$

and

$$\rho^{3.75} = (t^{-\frac{1}{15}}L^{\frac{8}{15}})^{3.75} \ll t^{-\frac{1}{4}}L^2t^{-3\varepsilon},$$
$$W^2 \gg t^{-2\varepsilon}(X^2 + Y^2 + Z^2 + W^2).$$

Also we have

$$H(n_v(y), y) \ll L^{-8}t(X^2 + Y^2 + Z^2 + W^2).$$

Hence from Lemma 7 of [3], we have

$$\sum_y e^{2\pi i\eta(y)} \frac{1}{\sqrt{\phi_{xx}(n_v(y), y)}} \ll L(H\phi_{xx}^{-1})^{\frac{1}{2}}\phi_{xx}^{-\frac{3}{2}} + (H\phi_{xx}^{-1})^{-\frac{1}{2}}\phi_{xx}^{-\frac{3}{2}} \ll$$
$$\ll L(L^{-8}\rho^7t)^{\frac{1}{2}}R + (L^{-8-2\alpha}W^2t)^{-\frac{1}{2}}.$$

And the number of v does not exceed

$$L(|\phi_{xx}| + |\phi_{xy}|) \ll L^{-3}\rho^{3.5}t^{\frac{1}{2}}.$$

Hence we have

$$S_{111} \ll (L(L^{-8}\rho^7t)^{\frac{1}{2}})L^{-3}\rho^{3.5}t^{\frac{1}{2}}(L^{-4}Wt^{\frac{1}{2}})^{-1} +$$
$$+ (L^{-8-2\alpha}W^2t)^{-\frac{1}{2}}L^{-3}\rho^{3.5}t^{\frac{1}{2}} \ll L^{-2}\rho^7t^{\frac{1}{2}}W^{-1} +$$
$$+ L^{1+\alpha}W^{-1}\rho^{3.5} \ll L^{-2}\rho^7t^{\frac{1}{2}}W^{-1} +$$
$$+ L^{\frac{3}{2}}W^{-\frac{1}{2}}\rho^{\frac{3.5}{2}}(t^{\frac{1}{2}}\rho^{3.5}L^{-4})^{\frac{1}{4}} \ll$$
$$\ll L^{-2}\rho^7t^{\frac{1}{2}}W^{-1} + t^{\frac{1}{8}}L^{\frac{1}{2}}W^{-\frac{1}{2}}\rho^{\frac{10.5}{4}} \ll L^{-2}\rho^7t^{\frac{1}{2}}W^{-1},$$

by using

$$L^{-2}\rho^7t^{\frac{1}{2}}W^{-1} \gg t^{\frac{1}{8}}L^{\frac{1}{2}}W^{-\frac{1}{2}}\rho^{\frac{10.5}{4}},$$
$$L^{-\frac{5}{2}}t^{\frac{3}{8}}W^{-\frac{1}{2}}\rho^{7-\frac{10.5}{4}} \gg 1,$$
$$L^{-\frac{5}{2}}t^{\frac{3}{8}}\rho^{\frac{28-10.5-7}{4}} = L^{-\frac{5}{2}}t^{\frac{3}{8}}\rho^{\frac{10.5}{4}} \gg 1,$$
$$(t^{-\frac{1}{15}-\varepsilon}L^{\frac{8}{15}})^{\frac{10.5}{4}}t^{\frac{3}{8}}L^{-\frac{5}{2}} \gg 1,$$
$$t^{\frac{45-21}{120}-\frac{10.5}{4}\varepsilon} \gg L^{\frac{300-168}{120}}, \text{ i.e., } L \ll t^{\frac{2}{11}-2.5\varepsilon}.$$

Hence we have

$$S_{12II} \ll (L+R) \ll L + L^4W^{-1}t^{-\frac{1}{2}} \ll L^{-2}\rho^7 t^{\frac{1}{2}}W^{-1},$$

$$S_{12III} \ll L\sqrt{R} \ll L(L^{-4}Wt^{\frac{1}{2}})^{-\frac{1}{2}} \ll L^3W^{-\frac{1}{2}}t^{-\frac{1}{4}} \ll L^{-2}\rho^7 t^{\frac{1}{2}}W^{-1},$$

by using

$$L^{-2}\rho^7 t^{+\frac{1}{2}}W^{-1} \gg L, \quad \rho^7 t^{\frac{1}{2}} \gg L^3W, \quad \rho^{3.5}t^{\frac{1}{2}} \gg L^3,$$

$$(t^{-\frac{1}{15}-\varepsilon} L^{\frac{8}{15}})^{3.5}t^{\frac{1}{2}} = t^{\frac{15-7}{30}} L^{\frac{28}{15}} t^{-3.5\varepsilon} \gg L^3,$$

$$t^{\frac{8}{30}-3.5\varepsilon} \gg L^{\frac{17}{15}}, \text{ i.e., } L \ll t^{\frac{8}{34}-5\varepsilon}.$$

It is easily seen that the above expressions are satisfied when $L \ll t^{\frac{8}{34}-5\varepsilon}$, and

$$L^{-2}\rho^7 t^{\frac{1}{2}}W^{-1} \gg L^4W^{-1}t^{-\frac{1}{2}}, \quad \rho^7 t \gg L^6,$$

$$t(t^{-\frac{1}{15}-\varepsilon} L^{\frac{8}{15}})^7 \gg L^6, \quad t^{\frac{8}{15}-7\varepsilon} \gg L^{\frac{90-56}{15}} = L^{\frac{34}{15}}, \quad \text{i.e.,} \quad L \ll t^{\frac{8}{34}-7\varepsilon}.$$

It is also easily seen that the above expressions are satisfied when $L \ll t^{\frac{8}{34}-5\varepsilon}$, and

$$L^{-2}\rho^7 t^{\frac{1}{2}}W^{-1} \gg L^3W^{-\frac{1}{2}}t^{-\frac{1}{4}},$$

$$(t^{-\frac{1}{15}-\varepsilon})^{\frac{14-3.5}{2}} (L^{\frac{8}{15}})^{\frac{14-3.5}{2}} t^{\frac{3}{4}} \gg L^5,$$

$$t^{\frac{45-21}{60}-\frac{10.5}{2}\varepsilon} L^{\frac{84}{30}} \gg L^5,$$

$$t^{\frac{24}{60}-\frac{10.5}{2}\varepsilon} \gg L^{\frac{300-168}{60}}, \text{ i.e., } L \ll t^{\frac{2}{11}-2.5\varepsilon}.$$

In any case, when $L \ll t^{\frac{2}{11}-2.5\varepsilon}$ and $W \neq 0$, we have

$$S \ll L^{-2}\rho^7 t^{\frac{1}{2}}W^{-1}.$$

When $W = 0$, it is known from [1] that

$$S \ll L^{-2}\rho^7 t^{\frac{1}{2}}[X^2 + Y^2 + Z^2]^{-\frac{1}{2}}.$$

6. *The proof of our result.* By (7) and Lemma 1 of [1] we have

$$\sum_R\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)t}}}{(x^2+y^2)^{3/4}} = O(L^{\frac{26}{15}-\frac{3}{2}}t^{\frac{1}{30}+\frac{1}{2}\varepsilon}) = O(L^{\frac{7}{30}}t^{\frac{1}{30}+\frac{1}{2}\varepsilon}),$$

and

$$\sum_R\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)t}}}{(x^2+y^2)^{5/4}} = O(L^{\frac{26}{15}-\frac{5}{2}}t^{\frac{1}{30}+\frac{1}{2}\varepsilon}) = O(L^{-\frac{23}{30}}t^{\frac{1}{30}+\frac{1}{2}\varepsilon}),$$

if $\max(x, y) \leqslant t^{\frac{2}{11}-2.5\varepsilon}$. By (25) of [1] we have

$$\sum_R\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)t}}}{(x^2+y^2)^{5/4}} = O(L^{-\frac{3}{4}} t^{\frac{1}{32}+\varepsilon}),$$

if $\max(x, y) \geqslant t^{\frac{2}{11}-2.5\varepsilon}$.

Now we divide the sum into some parts, i.e.,

$$\sum_D\sum \frac{e^{2\pi i\sqrt{(x^2+y^2)t}}}{(x^2+y^2)^{3/4}} = \sum_{p=1}^{P}\sum_{q=1}^{Q}\left\{\sum_{2^{p-1}}^{2^p}\sum_{2^{q-1}}^{2^q}\frac{e^{2\pi i\sqrt{(x^2+y^2)t}}}{(x^2+y^2)^{3/4}}\right\}.$$

We have from the figure that

$$n^{\frac{1}{4}}|\phi(n)| = O\left(t^{\frac{1}{4}}\sum_{p=1}^{P}\sum_{q=1}^{Q}\{\max(2^p, 2^q)\}^{\frac{7}{30}} t^{\frac{1}{30}+\frac{\varepsilon}{2}}\right) +$$

$$+ O(t^{\frac{12}{37}}) = O(t^{\frac{1}{4}+\frac{13}{74}\cdot\frac{7}{30}+\frac{1}{30}+\frac{1}{2}\varepsilon}) + O(t^{\frac{12}{37}}) = O(t^{\frac{12}{37}+\varepsilon}).$$

Similarly we have from the figure

$$n^{\frac{3}{4}}|\phi(n)| = O\Bigg[t^{\frac{3}{4}}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left\{\begin{matrix}\max\ (2^p, 2^q) \\ \text{here}\ \max\ (2^q, 2^p) \leqslant t^{\frac{2}{11}-2.5\varepsilon}\end{matrix}\right\}^{-\frac{23}{30}} t^{\frac{1}{30}+\varepsilon} +$$

$$+ t^{\frac{3}{4}}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left\{\begin{matrix}\max\ (2^p, 2^q) \\ \text{here}\ \max\ (2^p, 2^q) \geqslant t^{\frac{2}{11}-2.5\varepsilon}\end{matrix}\right\}^{-\frac{3}{4}} t^{\frac{1}{32}+\varepsilon} + t^{\frac{12}{37}+\frac{1}{4}}\Bigg] =$$

$$= O[t^{\frac{3}{4}-\frac{13}{74}\cdot\frac{23}{30}+\frac{1}{30}+\frac{1}{2}\varepsilon} + t^{\frac{3}{4}-\frac{2}{11}\cdot\frac{3}{4}+\frac{1}{32}} + t^{\frac{12}{37}+\frac{1}{4}}] = O(t^{\frac{24}{37}}).$$

Thus we have

$$\int_t^{t\pm t^{\frac{12}{37}}}\{R(n) - \pi n\}dn = O(t^{\frac{24}{37}+\varepsilon}).$$

Hence, in the usual way, we deduce easily

$$R(t) = \pi t + O(t^{\frac{12}{37}+\varepsilon}).$$

References

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