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Gerbes and twisted orbifold quantum cohomology

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Abstract In this paper, we construct an orbifold quantum cohomology twisted by a flat gerbe. Then we compute these invariants in the case of a smooth manifold and a discrete torsion on a global quotient orbifold.

Keywords: gerbe, orbifold, quantum cohomology

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1 Introduction

An important part of the stringy orbifold theory is the various twistings the theory possesses. Unfortunately, it is also the part of the stringy orbifold theory that we understand least. For example, for the untwisted theory, we have a rather complete conjectural answer to its structure and its relation to the structure of its crepant resolution. On the other hand, both the structure of the twisted theory and its relation to the desingularization are still mysterious at this moment. This article fills one piece of the puzzle.

Recall that for any almost complex orbifold X, we can associate the Chen-Ruan orbifold cohomology ring $H^*_{\mathrm{CR}}(X;\mathbb{C})$ by [1] as the summation of the ordinary cohomology of all the sectors with an appropriate degree shifting. There are two important factors of this ring: (i) there is a K-theoretic counterpart $K_{\mathrm{orb}}(X)$ due to Adem-Ruan[2], (ii) a precise relation between the Chen-Ruan orbifold cohomology ring and the cohomology ring of its crepant resolution has been proposed[3]. These two key aspects of the theory will always serve as the benchmarks to our future constructions. Namely, any theory we shall construct should have two properties: (i) it should be compatible with the K-theory; (ii) it should describe the ring structure of its crepant resolution or more generally its desingularizations. Here, a desingularization is obtained by first deforming the equation of a Gorenstein orbifold and then taking a crepant resolution. The miracle is that the right answer for one is often automatically the right answer for the other one. This gives us two powerful approaches to the stringy orbifolds.

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Historically, the earliest twisting is due to Vafa[4,5] in the case of a global quotient orbifold $X = Y/G$. Vafa's twisting is a group cohomology class $\alpha \in H^2(G; S^1)$ called the discrete torsion. The notion of the discrete torsion was generalized to the arbitrary orbifolds as a class in $H^2(\pi_1^{\mathrm{orb}}(X); S^1)$ by [6]. One can construct a twisted orbifold cohomology using the discrete torsion[5,6]. Its K-theoretic counterpart was constructed by Adem-Ruan[2]. However, it fails badly in describing the cohomology of the desingularization. To remedy the situation, a more general twisting was proposed in [6]. This new twisting is a flat line bundle $\mathcal{L}$ over the inertia orbifold satisfying the certain compatibility conditions (see Definition 3.1). $\mathcal{L}$ is called an inner local system. The inner local system works well for the second task, i.e., in describing the cohomology group of a desingularization. But Adem-Ruan's construction of the twisted orbifold K-theory fails to cover the case of an inner local system.

A more important problem is to twist orbifold quantum cohomology, which is unknown even for the discrete torsion. Recall that the Cohomological Crepant Resolution Conjecture[3] can be phrased as follows: the cup product of a crepant resolution Y of a Gorenstein orbifold X is the Chen-Ruan product of X plus the quantum corrections coming from the Gromov-Witten invariants of the exceptional rational curves. This conjecture was obtained by understanding the behavior of the quantum cohomology when it was deformed from a crepant resolution to an orbifold, even though our initial goal was only to understand the cohomology. Therefore, to even understand the ordinary ring structure of the desingularization, one has to understand the quantum cohomology and its twisting. This is the main goal of this paper.

However, both the problems seemed to be hopeless. The situation changed when Lupercio-Uribe introduced the notion of gerbes to orbifolds in [7] (see also [8]). Lupercio-Uribe-Tu-Xu-Laurent-Gengoux constructed a twisted K-theory over a groupoid using a gerbe on the groupoid, which is much more general than an orbifold. Their twisted K-theory generalizes Adem-Ruan's twisted K-theory on the orbifolds and the twisted K-theory on the smooth manifolds studied by Witten[9], Bouwknegt-Mathai[10], Freed-Hopkins-Teleman[11] and others. The beauty of gerbes is that one can easily do differential geometry, which is precisely what we were doing for the quantum cohomology. In this context, Lupercio-Uribe interpreted an inner local system as the holonomy line bundle on the inertia groupoid of a gerbe. In this article, we would like to go one step further to use the gerbe to twist the orbifold quantum cohomology. During the course of this work, some subtleties arose. In the theory of gerbes, there is a distinction between flat gerbes and non-flat gerbes. A flat gerbe has a torsion characteristic class and is often referred as a torsion gerbe. It has a rather long history in classical geometry under the name of Brauer group. The flat gerbe, or the element of the Brauer group, is precisely the data we are able to use to twist the orbifold quantum cohomology. On a smooth manifold, our twisted orbifold quantum cohomology did not give any new information (see Corollary 6.2). However, the orbifold discrete torsion is a particular case of a flat gerbe. It gives an abundance of new invariants. Our construction does not work for the non-flat gerbes. In many ways, the non-flat gerbes seem to fall into the realm of the non-commutative geometry. A further understanding of the twisted orbifold quantum cohomology may require a full-fledged theory of the non-commutative quantum cohomology. The on-going development of geometry with B-field by Hitchin and others may provide another approach to this type of question.

Since a gerbe and its twisted K-theory can be constructed over a singular space much more general than an orbifold, a natural question is: Can we construct a (twisted) orbifold (quantum) cohomology for a general groupoid such that (i) it agrees with the twisted K-theory rationally; (ii) it describes the cohomology of its desingularization?

The main results of this paper were announced by the second author in 2002 at ICM Satellite conference on Stringy Orbifolds in Chengdu. For various reasons, we were distracted by other projects. We apologize for such a long delay. During the preparation of this paper, we received an article of Lupercio and Uribe where there is some overlap between our Section 4 and their paper[12].

The paper is organized as follows. In Section 2, we will review the basic definitions of orbifolds and groupoids. In Sections 3 and 4, we will review the definition of gerbes and their holonomy. In Section 5, we will show how to use the holonomy of a gerbe to twist the orbifold GW-invariants. Some examples will be computed in the last section.

2 A review of orbifold Gromov-Witten invariants

We will review the construction of the ordinary orbifold Gromov-Witten invariants due to [13]. We will only sketch the main construction and refer the detail to [13]. But we take this opportunity to streamline the definition.

From now on, we will use X_o to denote a connected component of X. We will also assume that all intersections $U_{i_1,\ldots,i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$ are connected. Otherwise, we work component by component.

An orbifold atlas is defined in the same way.

Definition 2.1 (Orbifold Atlas). *An n-dimensional smooth orbifold atlas on a connected open cover $\{U_i\}_{i\in I}$ of X is given by the following data*:

(1) *Each U_i is covered by an uniformizing system $(\tilde{U}_i, G_{\tilde{U}_i}, \pi_i)$ in the following sense. $\tilde{U}_i$ is smooth. $G_{\tilde{U}_i}$ is a finite group acting smoothly on $\tilde{U}_i$ and $\pi_i : \tilde{U}_i \to U_i$ is invariant under $G_{\tilde{U}_i}$ such that it induces a homeomorphism $\tilde{U}_i/G_{\tilde{U}_i} \cong U_i$. We call U_i an orbifold chart.*

Choose a component $\pi_i^{-1}(U_{ij})_o$ and let $G_{\pi_i^{-1}(U_{ij})_o} \subset G_{\tilde{U}_i}$ be the subgroup fixing $\pi_i^{-1}(U_{ij})_o$. Then $(\pi_i^{-1}(U_{ij})_o, G_{\pi_i^{-1}(U_{ij})_o}, \pi_i)$ is a uniformizing system of U_{ij} (called an induced uniformizing system). Other induced uniformizing systems are of the form $(g\pi_i^{-1}(U_{ij})_o, gG_{\pi_i^{-1}(U_{ij})_o}g^{-1}, \pi_i)$ for $g \in G_{\tilde{U}_i}$. Namely, $G_{\tilde{U}_i}$ acts transitively on the collection of the induced uniformizing systems. In the same way, $(\tilde{U}_j, G_{\tilde{U}_j}, \pi_j)$ induces a collection of the uniformizing systems over U_{ij} acting transitively by $G_{\tilde{U}_j}$.

(2) Tran(U_i, U_j) *is a collection of the isomorphisms from $(\pi_i^{-1}(U_{ij})_o, G_{\pi_i^{-1}(U_{ij})_o}, \pi_i)$ to $(\pi_j^{-1}(U_{ij})_o, G_{\pi_j^{-1}(U_{ij})_o}, \pi_j)$. Here, the isomorphism is an isomorphism $\lambda_{ij} : G_{\pi_i^{-1}(U_{ij})_o} \to G_{\pi_j^{-1}(U_{ij})_o}$ and an equivariant diffeomorphism $\phi_{ij} : \pi_i^{-1}(U_{ij})_o \to \pi_j^{-1}(U_{ij})_o$. Moreover, all such isomorphisms are generated from a fixed one by the action of $G_{\tilde{U}_i} \times G_{\tilde{U}_j}$ in an obvious way. Each isomorphism is called a transition map.*

(3) Tran(U_i, U_i) *is generated by the identity.* Tran$(U_j, U_i) =$ Tran$(U_i, U_j)^{-1}$ *in the sense that each transition in* Tran(U_j, U_i) *is the inverse of some transition of* Tran(U_i, U_j).

Over the triple intersection U_{ijk}, each of U_i, U_j, U_k induces a uniformizing system and the transitions restrict to them as well. Then, we require

(4) *There is a multiplication such that $(\phi_{jk}, \lambda_{jk}) \circ (\phi_{ij}, \lambda_{ij})$ is the restriction of an element*

of $\mathrm{Tran}\,(U_i, U_k)$.

Note that we do not require $\tilde{U}_i$ to be connected.

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$, then there is an induced orbifold atlas over $\mathcal{U}'$ in an obvious fashion. Two orbifold atlases are considered to be equivalent if their induced orbifold atlases are equivalent over a common refinement in an obvious fashion. Such an equivalence class is called an orbifold structure over X. So we may choose $\mathcal{U}$ to be arbitrarily fine.

Let $p \in X$. By choosing a small neighborhood $V_p \in \mathcal{U}$, we may assume that its uniformizing system $\mathcal{V}(V_p) = (U_p, G_p)$ has the property that U_p is an n-ball centered at the origin o and $\pi_p^{-1}(p) = o$ where $\pi_p : U_p \to V_p = U_p/G_p$ is the projection map. In particular, the origin o is fixed by G_p. We called G_p the local group at p. If G_p acts effectively for every p, we call X an effective orbifold.

Recall Satake's definition of an orbifold map. A map $f : X \to Y$ is an orbifold map iff locally $f : U_i \to V_i$ can be lifted to an equivariant map $\tilde{f}_i : \tilde{U}_i \to \tilde{V}_i$ with a homomorphism $\lambda_i : G_{\tilde{U}_i} \to G_{\tilde{V}_i}$. Suppose that we want to pull back an orbifold vector bundle from Y. We can use a local lifting $\tilde{f}_i$ to construct the local pull-back. But there is no reason why the local pull-backs can be glued together. In order to glue them together, we have to impose a condition on the transitions. Then, we obtain the notion of the orbifold morphism. Now it is clear how we should impose our condition called a compatible system.

Definition 2.2. *Fix an underlying map $f : X \to Y$. A compatible system consists of an orbifold atlas $\{(\tilde{U}_i, G_{\tilde{U}_i}), \mathrm{Tran}(U_i, U_j)\}$ of X and an orbifold atlas $\{(\tilde{V}_i, G_{\tilde{V}_i}), \mathrm{Tran}\,(V_i, V_j)\}$ of Y with the following additional properties*:

(i) *f maps a member of one atlas to a member of the other atlas, i.e., $f : U_i \to V_{\kappa(i)}$*;

(ii) *The local map in* (i) *can be lifted to $\lambda_i : G_{\tilde{U}_i} \to G_{\tilde{V}_{\kappa(i)}}$ and an equivariant map $\tilde{f}_i : \tilde{U}_i \to \tilde{V}_{\kappa(i)}$*;

(iii) *There is a map $\lambda_{ij} : \mathrm{Tran}(U_i, U_j) \to \mathrm{Tran}(V_{\kappa(i)}, V_{\kappa(j)})$ preserving the identity, inverse and multiplication*;

(iv) $\lambda_{ij}(g) \circ \tilde{f}_i = \tilde{f}_j \circ g$.

Suppose that $\{V_\beta\}$ is a refinement of $\{V_j\}$. Then, $\{f^{-1}(V_\beta)\}$ is a refinement of $\{U_i\}$. We can take a further refinement $\{U_\alpha\}$ of $\{f^{-1}(V_\beta)\}$. Then we still have property (i). Furthermore, the original compatible system induces compatible systems over $\{U_\alpha\}$, $\{V_\beta\}$. We call this a refinement of compatible system.

Definition 2.3 (Isomorphism of compatible systems)**.** *Two compatible systems given by $(\tilde{f}_i, \lambda_i, \lambda_{ij}), (\tilde{f}'_i, \lambda'_i, \lambda'_{ij})$ over the same orbifold atlas $(\tilde{U}_i, G_{\tilde{U}_i}), (\tilde{V}_j, G_{\tilde{V}_j})$ are said to be isomorphic if they differ by an automorphism of the orbifold structure $(\tilde{V}_j, G_{\tilde{V}_j})$. Namely, there is an element $\delta_i \in \mathrm{Tran}\,(V_i, V_i)$ such that*

$$\tilde{f}' = \delta_i \circ \tilde{f}_i, \quad \lambda'_i = \delta_i \lambda_i \delta_i^{-1}, \quad \lambda'_{ij} = \delta_j \lambda_{ij} \delta_i^{-1}.$$

For two arbitrary compatible systems over isomorphic orbifold atlases, by taking refinements and components if necessary, we can induce two compatible systems over the same orbifold atlas. Then the original ones are isomorphic if the induced ones are isomorphic in the above sense. An orbifold morphism (good map) is a map with an isomorphism class of compatible systems.

Chen and Ruan also developed a machinery to classify good maps. The key is an invariant they called the characteristic. The case we will use is the global quotient orbifold denoted by the stack notation $[X/G]$. The characteristic can be interpreted as follows.

Suppose that $f : Y \to [X/G]$ is a good map. We can pull back the G-bundle $X \to X/G$ to obtain a G-bundle $p : E \to Y$ and a G-map $F : E \to X$. In fact, the equivalence class of a good map f is equivalent to the pair (p, F) modulo bundle isomorphisms. Namely, $(p, F) \simeq (p', F')$ iff $p' = ph, F' = Fh$ for a bundle isomorphism $h : E' \to E$.

Since G is a finite group, $p : E \to Y$ is an orbifold cover. By the covering space theory, E is determined by the conjugacy class of a homomorphism $\rho : \pi_1^{\mathrm{orb}}(Y, x_0) \to G$. We call ρ and its conjugacy class the Chen-Ruan characteristic.

Consider the pairs: $\wedge X = \{(p, (g)_{G_p});\ p \in X, g \in G_p\}$, where $(g)_{G_p}$ is the conjugacy class of g in G_p. If there is no confusion, we will omit the subscript G_p to simplify the notation. $\wedge X$ has a natural orbifold structure (Proposition 2.4) and is called the inertia orbifold. More generally, we can define the multisector

$$\widetilde{X}_k = \{(p, (g_1, \ldots, g_k)_{G_p});\ p \in X, g_i \in G_p\}.$$

It is clear that $\wedge X = \widetilde{X}_1$. There are two classes of maps between the multisectors. $I : \widetilde{X}_k \to \widetilde{X}_k$ by $I(p, (g_1, \ldots, g_k)_{G_p}) = (p, (g_1^{-1}, \ldots, g_k^{-1})_{G_p})$, and $e_{i_1,\ldots,i_l} : \widetilde{X}_k \to \widetilde{X}_l$ by

$$e_{i_1,\ldots,i_l}(p, (g_1, \ldots, g_k)_{G_p}) = (p, (g_{i_1}, \ldots, g_{i_l})_{G_p}).$$

Suppose that X is an orbifold with an orbifold atlas $\{(\tilde{U}_i, G_{\tilde{U}_i}), \mathrm{Tran}(U_i, U_j)\}$.

Proposition 2.4 (Lemma 3.1.1 in [14]). *$\widetilde{X}_k$ is naturally an orbifold, with the orbifold atlas given by $(\bigsqcup_{\boldsymbol{g} \in G^k_{\tilde{U}}} \tilde{U}^{\boldsymbol{g}}, G_{\tilde{U}})$, where $\tilde{U}^{\boldsymbol{g}} = \tilde{U}^{g_1} \cap \tilde{U}^{g_2} \cap \cdots \cap \tilde{U}^{g_k}$. Here $\boldsymbol{g} = (g_1, \ldots, g_k)$, $\tilde{U}^g$ stands for the fixed-point set of g in $\tilde{U}$. When X is almost complex, $\widetilde{X}_k$ inherits an almost complex structure from X, and when X is closed, $\widetilde{X}_k$ is a finite disjoint union of closed orbifolds.*

Proposition 2.5 (Lemma 3.1.3 in [14]). *Both the evaluation maps $e_{i_1,\ldots,i_l}$ and I are orbifold morphisms.*

Next, we would like to describe the connected components of $\widetilde{X}_k$. Suppose that p, q are in the same orbifold chart U_i uniformized by $(\tilde{U}_i, G_{\tilde{U}_i}, \pi_i)$. Then G_p and G_q are both subgroups of $G_{\tilde{U}_i}$. We say that $(\boldsymbol{g}_1)_{G_p} \cong (\boldsymbol{g}_2)_{G_q}$ if $\boldsymbol{g}_1 = h\boldsymbol{g}_2h^{-1}$ for some element $h \in G_{\tilde{U}_i}$. For two arbitrary points $p, q \in X$, we say $(\boldsymbol{g})_{G_p} \cong (\boldsymbol{g}')_{G_q}$ if there is a sequence $(p_0, (\boldsymbol{g}_0)_{G_{p_0}}), \ldots, (p_k, (\boldsymbol{g}_k)_{G_{p_k}})$ such that $(p_0, (\boldsymbol{g}_0)_{G_{p_0}}) = (p, (\boldsymbol{g})_{G_p}), (p_k, (\boldsymbol{g}_k)_{G_{p_k}}) = (q, (\boldsymbol{g}')_{G_q})$ and p_i, p_{i+1} are in the same orbifold chart and $(\boldsymbol{g}_i)_{G_{p_i}} \cong (\boldsymbol{g}_{i+1})_{G_{p_{i+1}}}$ for $i = 0, \ldots, k-1$. This defines an equivalence relation on $\{(\boldsymbol{g})_{G_p}\}$. In particular, it is possible that $(\boldsymbol{g})_{G_p} \cong (\boldsymbol{g}')_{G_p}$ while $(\boldsymbol{g})_{G_p} \neq (\boldsymbol{g}')_{G_p}$. Let T_k be the set of equivalence classes. By abuse of the notation, we often use $(\boldsymbol{g})$ to denote the equivalence class which $(\boldsymbol{g})_{G_q}$ belongs to. It is clear that $\widetilde{X}_k$ decomposes as a disjoint union of connected components $\widetilde{X}_k = \bigsqcup_{(\boldsymbol{g}) \in T_k} X_{(\boldsymbol{g})}$, where $X_{(\boldsymbol{g})} = \{(p, (\boldsymbol{g}')_{G_p});\ p \in X, \boldsymbol{g}' \in G_p^k, (\boldsymbol{g}')_{G_p} \in (\boldsymbol{g})\}$. Let $T_k^o \subset T_k$ be such that $(g_1, \ldots, g_k) \in T_k^o$ has the property $g_1 \cdots g_k = 1$. $\overline{\mathcal{M}}_k(X) = \bigsqcup_{(\boldsymbol{g}) \in T_k^o} X_{(\boldsymbol{g})}$.

Definition 2.6. *$X_{(g)}$ for $g \neq 1$ is called a twisted sector. $X_{(\boldsymbol{g})}$ is called a k-multi-sector or a k-sector. Furthermore, we call $X_{(1)} = X$ the non-twisted sector.*

Once we define the sectors, we diagonalize the action of g on the normal space of $T_pX_{(g)}$ in T_pX for each $p \in X_{(g)}$. Let $\operatorname{codim}_{\mathbf{C}} X_{(g)} = k$, then the g action can be diagonalized as

$$\operatorname{diag}(1, \ldots, 1, e^{2\pi i \frac{m_{1,g}}{m_g}}, \ldots, e^{2\pi i \frac{m_{k,g}}{m_g}}),$$

where m_g is the order of g in G_p and $0 < m_{i,g} < m_g$. Then we define the degree shifting number $\iota_{(g)} = \sum_{i=1}^{k} \frac{m_{i,g}}{m_g}$. One can show that $\iota_{(g)}$ is independent of $p \in X_{(g)}$.

Example 2.7. Suppose that $X = Y/G$ is a global quotient. By the definition, $\wedge X = (\bigsqcup_{g\in G} Y^g)/G$, where Y^g is the fixed-point set of elements $g \in G$. Equivalently, $\wedge X = \bigsqcup_{\{(g);\, g\in G\}} Y^g/C(g)$, here, $C(g)$ is the centralizer of g in G.

Next, we extend the notion of the orbifold morphism to the case in which the domain is a nodal orbifold Riemann surface.

Recall that a nodal curve with k marked points is a pair $(\Sigma, \boldsymbol{z})$ consisting of a connected topological space $\Sigma = \bigcup \pi_\nu(\Sigma_\nu)$, where Σ_ν is a smooth complex curve and $\pi_\nu : \Sigma_\nu \to \Sigma$ is a continuous map, and $\boldsymbol{z} = (z_1, \ldots, z_k)$ consists of k distinct points in Σ with the following properties:

- For each $z \in \Sigma_\nu$, there is a neighborhood of z such that the restriction of $\pi_\nu : \Sigma_\nu \to \Sigma$ to this neighborhood is a homeomorphism to its image.
- For each $z \in \Sigma$, we have $\sum_\nu \sharp\pi_\nu^{-1}(z) \leqslant 2$.
- $\sum_\nu \sharp\pi_\nu^{-1}(z_i) = 1$ for each $z_i \in \boldsymbol{z}$.
- The number of complex curves Σ_ν is finite.
- The set of nodal points $\{z \mid \sum_\nu \sharp\pi_\nu^{-1}(z) = 2\}$ is finite.

A point $z \in \Sigma_\nu$ is called singular (or a node) if $\sum_\omega \sharp\pi_\omega^{-1}(\pi_\nu(z)) = 2$. A point $z \in \Sigma_\nu$ is said to be a marked point if $\pi_\nu(z) = z_i \in \boldsymbol{z}$. Each Σ_ν is called a component of Σ. Let k_ν be the number of points on Σ_ν which are either singular or marked, and g_ν be the genus of Σ_ν, a nodal curve $(\Sigma, \boldsymbol{z})$ is called stable if $k_\nu + 2g_\nu \geqslant 3$ holds for each component Σ_ν of Σ.

Definition 2.8. *A nodal orbicurve is a nodal marked curve $(\Sigma, \boldsymbol{z})$ with an orbifold structure as follows*:

- *The set $\boldsymbol{z}_\nu$ of orbifold points of each component Σ_ν is contained in the set of marked points and nodal points.*
- *A neighborhood of a marked point is uniformized by a branched covering map $z \to z^{m_i}$ with $m_i \geqslant 1$.*
- *A neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{xy = 0\} \subset \mathbb{C}^2$) is uniformized by a branched covering map $(x, y) \to (x^{n_j}, y^{n_j})$, with $n_j \geqslant 1$, and with group action $e^{2\pi i/n_j}(x, y) = (e^{2\pi i/n_j} x, e^{-2\pi i/n_j} y)$.*

Here m_i and n_j are allowed to be equal to one, i.e., the corresponding orbifold structure is trivial there. We denote the corresponding nodal orbicurve by $(\Sigma, \boldsymbol{z}, \boldsymbol{m}, \boldsymbol{n})$ where $\boldsymbol{m} = (m_1, \ldots, m_k)$ and $\boldsymbol{n} = (n_j)$.

Once we have the definition of nodal orbicurve, we can extend the definition of the compatible system and the orbifold morphism word by word to the case where the domain is a nodal orbicurve.

First, recall that for every point $p \in \Sigma$, an orbifold morphism $f : \Sigma \to X$ induces a homomorphism $G_p \to G_{f(p)}$.

Definition 2.9. *Let (X, J) be an almost complex orbifold. An orbifold stable map into (X, J) is a triple $(f, (\Sigma, \boldsymbol{z}, \boldsymbol{m}, \boldsymbol{n}), \xi)$ described as follows:*

(1) *f is a continuous map from the nodal orbicurve $(\Sigma, \boldsymbol{z}, \boldsymbol{m}, \boldsymbol{n})$ into X such that each $f_\nu = f \circ \pi_\nu$ is a pseudo-holomorphic map from Σ_ν into X;*

(2) *ξ is an isomorphism class of compatible structures;*

(3) *let k_ν be the order of the set $\boldsymbol{z}_\nu$, namely the number of points on Σ_ν which are special (i.e. nodal or marked), if f_ν is a constant map, then $2g_\nu - 2 + k_\nu > 0$;*

(4) *at any marked or nodal point p the induced homomorphism on the local group $\lambda_p : G_p \to G_{f(p)}$ is injective.*

Finally we observe that each C^∞ orbifold morphism from an orbifold nodal Riemann surface with k marked points into an orbifold X determines a point in the product of inertia orbifolds $(\wedge X)^k$ as follows: let the underlying continuous map be f and for each marked point z_i, $i = 1, \ldots, k$, let x_i be the positive generator of the cyclic local group at z_i, and λ_{z_i} be the homomorphism determined by the given compatible system, then the determined point in $(\wedge X)^k$ is

$$((f(z_1), (\lambda_{z_1}(x_1))_{G_{f(z_1)}}), \ldots, (f(z_k), (\lambda_{z_k}(x_k))_{G_{f(z_k)}})).$$

Let $\boldsymbol{x} = (X_{(g_1)}, \ldots, X_{(g_k)})$ be a connected component in $(\wedge X)^k$. We say that a good map with a compatible system is of type $\boldsymbol{x}$ if the above point determined in $(\wedge X)^k$ lies in the component $\boldsymbol{x}$.

Remark 2.10. If $f : \Sigma \to X$ is a pseudo-holomorphic map whose image intersects the singular locus of X at only finitely many points, then there is a unique choice of the orbifold structure on Σ together with a unique $(\tilde{f}, \xi)$, where $\tilde{f}$ is a good map with an isomorphism class of compatible systems ξ whose underlying continuous map is f. If the image of f lies completely inside the singular locus, there could be different choices, and they are regarded as different points in the moduli space.

Definition 2.11. (1) *An orbifold X is symplectic if there is a closed 2-form ω on X whose local liftings are non-degenerate.*

(2) *A projective orbifold is a complex orbifold which is a projective variety as an analytic space.*

Proposition 2.12 (Proposition 2.3.8 in [13]). *Suppose that X is a symplectic or projective orbifold. Then the moduli space of orbifold stable maps $\overline{\mathcal{M}}_{g,k}(X, J, A, \boldsymbol{x})$ is a compact metrizable space under a natural topology, whose "virtual dimension" is $2d$, where*

$$d = c_1(TX) \cdot A + (\dim_{\mathbb{C}} X - 3)(1 - g) + k - \iota(\boldsymbol{x}).$$

Here $\iota(\boldsymbol{x}) = \sum_{i=1}^k \iota_{(g_i)}$ for $\boldsymbol{x} = (X_{(g_1)}, \ldots, X_{(g_k)})$.

For any component $\boldsymbol{x} = (X_{(g_1)}, \ldots, X_{(g_k)})$, there are k evaluation maps $e_i : \overline{\mathcal{M}}_{g,k}(X, J, A, \boldsymbol{x}) \to X_{(g_i)}$, $i = 1, \ldots, k$. e_i has a natural compatible system to make it a good map. For any set of cohomology classes $\alpha_i \in H^*(X_{(g_i)}; \mathbb{C}) \subset H^*_{\mathrm{CR}}(X; \mathbb{C})$, $i = 1, \ldots, k$, the orbifold Gromov-Witten invariant is defined as

$$\Psi^{X,J}_{(g,k,A,\boldsymbol{x})}(\alpha_1^{l_1}, \ldots, \alpha_k^{l_k}) = \prod_{i=1}^k c_1(L_i)^{l_i} e_i^* \alpha_i [\overline{\mathcal{M}}_{g,k}(X, J, A, \boldsymbol{x})]^{\mathrm{vir}},$$

where L_i is the line bundle generated by the cotangent space of the i-th marked point. The virtual fundamental cycle $[\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})]^{\mathrm{vir}}$ is defined as the fundamental cycle of a certain orbifold X[13].

The inertia orbifold admits another interpretation as the space of constant loops. Then it is naturally a subset of the free loop space. We shall sketch this construction due to [15] (see [16] for a groupoid description).

Let LX be the space of orbifold morphisms from S^1 with a trivial orbifold structure to X. LX is the analog of the free loop space of a smooth manifold. We need the following important

Lemma 2.13 (Lemma 3.15 in [15]). *Let $X = Y/G$ be a global quotient. Then, $LX = P(Y,G)/G$, where*

$$P(Y,G) = \{(\gamma, g);\ \gamma : [0,1] \to Y, g \in G, \gamma(1) = g\gamma(0)\}.$$

Here, G acts on $P(Y,G)$ by $h(\gamma,g) = (h \circ \gamma, h^{-1}gh)$.

We call τ a constant loop if the underlying map is constant. Suppose that the image is $p \in X$. Let U_p/G_p be the orbifold chart at p. By the lemma, $L(U_p/G_p) = P(U_p,G_p)/G_p$. In particular, τ is an equivalence class of a pair (γ, g) where im $(\gamma) = p$. Under the action of G_p, we naturally identify it as $(p, (g)_{G_p})$. Therefore, the space of constant loops is precisely the inertia orbifold $\wedge X$.

Suppose that $f : \Sigma \to X$ is an orbifold stable map. We take a real blow-up of Σ at all the marked points to obtain a Riemann surface with the boundary $\Sigma^\dagger$. $\Sigma^\dagger$ can be understood as follows. We remove the marked point x_i. A neighborhood of a puncture point x_i is biholomorphic to $S^1 \times [0,\infty)$. Hence, we can view $\Sigma^\dagger$ as a manifold with a cylindrical end and x_i is replaced by a circle S_∞ attached at ∞. Another way to interpret the evaluation map is that $e_i(f) = f(S_\infty)$. This description is important later in our construction.

3 Gerbes and their holonomy

After reviewing the construction of the orbifold quantum cohomology in the last section, we are ready to touch upon the main topic of this article, the twisting. The earliest twisting from physics is discrete torsion by Vafa[4]. However, the discrete torsion is too restrictive to describe the interesting examples. Therefore, a more general twisting is needed. For this purpose, the second author introduced the notion of the inner local system. Roughly speaking, an inner local system is a flat orbifold line bundle over the inertia orbifold $\wedge X$ satisfying certain compatibility conditions. Later, Lupercio and Uribe introduced the concept of gerbe to orbifolds. The holonomy line bundle of a gerbe with connection is naturally an inner local system. However, not all inner local systems are induced in this way[17]. In this section, we will study the relation between a gerbe and its holonomy in detail.

3.1 Inner local system

Recall that for $(g_1,\dots,g_k) \in T_k$, there are $k+1$ evaluation maps $e_i : X_{(g_1,\dots,g_k)} \to X_{(g_i)},\ i \leqslant k$, and $e_{k+1} : X_{(g_1,\dots,g_k)} \to X_{(g_1\cdots g_k)}$.

Now we introduce the notion of the inner local system for an orbifold.

Definition 3.1. *Suppose that X is an orbifold* (*almost complex or not*). *An inner local system $\mathcal{L} = \{L_{(g)}\}_{(g)\in T_1}$ is an assignment of a flat complex orbifold line bundle $L_{(g)} \to X_{(g)}$ to each* *sector $X_{(g)}$ satisfying the following compatibility conditions:*

(1) *$L_{(1)}$ is a trivial orbifold line bundle with a fixed trivialization;*

(2) *There is a non-degenerate pairing $L_{(g)} \otimes I^* L_{(g^{-1})} \to \mathbb{C} \cong L_{(1)}$;*

(3) *There is a multiplication $e_1^* L_{(g_1)} \otimes e_2^* L_{(g_2)} \xrightarrow{\theta} e_3^* L_{(g_1 g_2)}$ over $X_{(g_1,g_2)}$ for $(g_1,g_2) \in T_2$;*

(4) *The multiplication θ is associative in the following sense. For $(g_1,g_2,g_3) \in T_3$, the evaluation maps $e_i : X_{(g_1,g_2,g_3)} \to X_{(g_i)}$ factor through $P = (P_1,P_2) : X_{(g_1,g_2,g_3)} \to X_{(g_1,g_2)} \times X_{(g_1 g_2,g_3)}$.*

Let $e_{12} : X_{(g_1,g_2,g_3)} \to X_{(g_1 g_2)}$. We first use P_1 to define $\theta : e_1^* L_{(g_1)} \otimes e_2^* L_{(g_2)} \to e_{12}^* L_{(g_1 g_2)}$. Then, we can use P_2 to define a product $\theta : e_{12}^* L_{(g_1 g_2)} \otimes e_3^* L_{(g_3)} \to e_4^* L_{(g_1 g_2 g_3)}$. Taking the composition, we define

$$\theta(\theta(e_1^* L_{(g_1)}, e_2^* L_{(g_2)}), e_3^* L_{(g_3)}) : e_1^* L_{(g_1)} \otimes e_2^* L_{(g_2)} \otimes e_3^* L_{(g_3)} \to e_4^* L_{(g_1 g_2 g_3)}.$$

On the other hand, the evaluation maps e_i also factor through $P' : X_{(g_1,g_2,g_3)} \to X_{(g_1,g_2 g_3)} \times X_{(g_2,g_3)}$. In the same way, we can define another triple product

$$\theta(e_1^* L_{(g_1)}, \theta(e_2^* L_{(g_2)}, e_3^* L_{(g_3)})) : e_1^* L_{(g_1)} \otimes e_2^* L_{(g_2)} \otimes e_3^* L_{(g_3)} \to e_4^* L_{(g_1 g_2 g_3)}.$$

Then, we require the associativity

$$\theta(\theta(e_1^* L_{(g_1)}, e_2^* L_{(g_2)}), e_3^* L_{(g_3)}) = \theta(e_1^* L_{(g_1)}, \theta(e_2^* L_{(g_2)}, e_3^* L_{(g_3)})).$$

If X is a complex orbifold, we assume that $L_{(g)}$ is holomorphic.

Definition 3.2. *Given an inner local system $\mathcal{L}$, we define the twisted orbifold cohomology $H^*_{CR}(X;\mathcal{L}) = \oplus_{(g)} H^{*-2\iota_{(g)}}(X_{(g)}; L_{(g)})$.*

Definition 3.3. *Suppose that X is a closed complex orbifold and $\mathcal{L}$ is an inner local system. We define the Dolbeault cohomology groups $H^{p,q}_{\mathrm{CR}}(X;\mathcal{L}) = \oplus_{(g)} H^{p-\iota_{(g)}, q-\iota_{(g)}}(X_{(g)}; L_{(g)})$.*

Proposition 3.4. *If X is a Kähler orbifold, we have the Hodge decomposition $H^k_{\mathrm{CR}}(X;\mathcal{L}) = \oplus_{k=p+q} H^{p,q}_{\mathrm{CR}}(X;\mathcal{L})$.*

Proof. Note that each sector $X_{(g)}$ is a Kähler orbifold. The proposition follows by applying the ordinary Hodge theorem with the twisted coefficients to each sector $X_{(g)}$.

3.2 Basics on gerbes and connections

The original motivation for the introduction of gerbes to orbifolds by Lupercio-Uribe is to understand the inner local systems conceptually. Let us start from the definition of a gerbe on a smooth manifold. We follow closely the exposition of [18].

Let us suppose X is a smooth manifold and $\mathcal{U} = \{U_\alpha\}$ an open cover. Recall the definition of the line bundle. It can be described by transition functions $g_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to S^1$ satisfying the conditions $g_{\alpha\alpha} = 1, g_{\beta\alpha} = g^{-1}_{\alpha\beta}, (\delta g)_{\alpha\beta\gamma} = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$. In terms of cohomological language, $g_{\alpha\beta}$ is a Čech 1-cocycle of the sheaf of S^1-valued functions $C^\infty(S^1)$. Two sets of transition functions induce isomorphic line bundles iff they induce the same class in $H^1(X; C^\infty(S^1))$.

A gerbe is a generalization of a line bundle. It is defined as a Čech 2-cocycle of the sheaf of S^1-valued function $C^\infty(S^1)$ over some open cover $\mathcal{U}$. Two gerbes are equivalent if they induce the same cocycle over a common refinement. They are isomorphic if they induce the same cohomology class in $H^2(X, C^\infty(S^1))$. In terms of local data, they are the functions $g_{\alpha\beta\gamma}$: $U_\alpha \cap U_\beta \cap U_\gamma \to S^1$ defined on the threefold intersections satisfying $g_{\alpha\beta\gamma} = g^{-1}_{\alpha\gamma\beta} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\gamma\beta\alpha}$ and the cocycle condition

$$(\delta g)_{\alpha\beta\gamma\eta} = g_{\beta\gamma\eta} g^{-1}_{\alpha\gamma\eta} g_{\alpha\beta\eta} g^{-1}_{\alpha\beta\gamma} = 1$$

on the four-fold intersections $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\eta$. It also defines a class in $H^3(X; \mathbb{Z})$. Consider the long exact sequence of cohomology

$$\cdots \to H^i(X; C^\infty(\mathbb{R})) \to H^i(X; C^\infty(S^1)) \xrightarrow{\tau_i} H^{i+1}(X; \mathbb{Z}) \to \cdots$$

derived from the exact sequence of sheaves $0 \to \mathbb{Z} \to C^\infty(\mathbb{R}) \to C^\infty(S^1) \to 1$. Recall that $\tau_1([g_{\alpha\beta}]) \in H^2(X; \mathbb{Z})$ is the first Chern class of the corresponding line bundle. In the same way, the characteristic class of a gerbe is $\tau_2([g_{\alpha\beta\gamma}])$. As the sheaf $C^\infty(\mathbb{R})$ is a fine sheaf, we get that $H^2(X; C^\infty(S^1)) \cong H^3(X; \mathbb{Z})$. We might say that a gerbe is determined topologically by its characteristic class. Furthermore, we can tensor them using the product of cocycles.

We call a gerbe $g = \{g_{\alpha\beta\gamma}\}$ a trivial gerbe if $g = \delta f$ is a coboundary for some 1-cochain f. f is called a trivialization of g. In terms of local data, f is defined by the functions $f_{\alpha\beta} = f^{-1}_{\beta\alpha} : U_\alpha \cap U_\beta \to S^1$ on the twofold intersections such that $g_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha}$. Hence, g is represented as a coboundary $\delta f = g$.

Suppose that f_1, f_2 are two different trivializations of g. Then $\delta(f_1 f_2^{-1}) = 1$. Hence $h = f_1 f_2^{-1}$ is a 1-cocycle and hence defines a line bundle.

A connection will consist of a pair $(A_{\alpha\beta}, F_\alpha)$ where $A_{\alpha\beta}$ are 1-forms over the double intersections $U_\alpha \cap U_\beta$, such that $iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$ and the 2-forms F_α are defined over U_α such that $F_\beta - F_\alpha = dA_{\alpha\beta}$. Note that we define a global 3-form G such that $G|_{U_\alpha} = dF_\alpha$. This 3-form G is called the curvature of the gerbe connection.

When the curvature G vanishes we say that the connection on the gerbe is flat. Therefore, $dF_\alpha = 0$. Since U_α is contractible, we can find B_α such that $F_\alpha = dB_\alpha$. Then, on $U_\alpha \cap U_\beta$, $F_\beta - F_\alpha = dA_{\alpha\beta} = d(B_\beta - B_\alpha)$. This implies that $A_{\alpha\beta} - B_\beta + B_\alpha = df_{\alpha\beta}$. From the definition of connection $iA_{\alpha\beta} + iA_{\beta\gamma} + iA_{\gamma\alpha} = g^{-1}_{\alpha\beta\gamma} dg_{\alpha\beta\gamma}$, hence, $d(if_{\alpha\beta} + if_{\beta\gamma} + if_{\gamma\alpha} - \log g_{\alpha\beta\gamma}) = 0$. Let $c_{\alpha\beta\gamma} = e^{if_{\alpha\beta}} e^{if_{\beta\gamma}} e^{if_{\gamma\alpha}} g_{\alpha\beta\gamma}$. $c_{\alpha\beta\gamma}$ is constant. It is clear that $c_{\alpha\beta\gamma}$ is a 2-cocycle differing from $g_{\alpha\beta\gamma}$ by a coboundary $e^{if_{\alpha\beta}} e^{if_{\beta\gamma}} e^{if_{\gamma\alpha}}$. Since it is constant, $c_{\alpha\beta\gamma}$ represents a Čech class in $H^2(X; S^1)$ which we call the holonomy of the connection.

Next, we check that $\{c_{\alpha\beta\gamma}\}$ is independent of the choice of B_α and $f_{\alpha\beta}$ as a Čech cohomology class. This is the analogue of the fact that one can use a flat connection on a line bundle to change the transition function to be a constant. If we have different B'_α, then $d(B_\alpha - B'_\alpha) = 0$ and hence we can write $B_\alpha - B'_\alpha = df_\alpha$. Let

$$\begin{aligned} f'_{\alpha\beta} &= f_{\alpha\beta} - f_\alpha + f_\beta, \\ df'_{\alpha\beta} &= A_{\alpha\beta} - B_\beta + B_\alpha - B_\alpha + B'_\alpha + B_\beta - B'_\beta = A_{\alpha\beta} - B'_\beta + B'_\alpha. \end{aligned}$$

Then $c'_{\alpha\beta\gamma} = e^{if'_{\alpha\beta}} e^{if'_{\beta\gamma}} e^{if'_{\gamma\alpha}} g_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}$. If we have a different choice $A_{\alpha\beta} - B_\beta + B_\alpha = d\tilde{f}_{\alpha\beta}$, $\tilde{f}_{\alpha\beta} = f_{\alpha\beta} + \lambda_{\alpha\beta}$, where $\lambda_{\alpha\beta}$ is a constant function. Then,

$$\tilde{c}_{\alpha\beta\gamma} = e^{i\tilde{f}_{\alpha\beta}} e^{i\tilde{f}_{\beta\gamma}} e^{i\tilde{f}_{\gamma\alpha}} g_{\alpha\beta\gamma} = c_{\alpha\beta\gamma} e^{i\lambda_{\alpha\beta}} e^{i\lambda_{\beta\gamma}} e^{i\lambda_{\gamma\alpha}}.$$

Namely, it differs by a coboundary in the constant sheaf S^1.

For a line bundle, when the holonomy is trivial we get a covariant constant trivialization of the bundle. If the holonomy of a gerbe is trivial, then $c_{\alpha\beta\gamma}$ is a coboundary, so that there are constants $k_{\alpha\beta} \in S^1$ such that $c_{\alpha\beta\gamma} = k_{\alpha\beta}k_{\beta\gamma}k_{\gamma\alpha}$. Let $h_{\alpha\beta} = k_{\alpha\beta}e^{-if_{\alpha\beta}}$, then $h_{\alpha\beta}h_{\beta\gamma}h_{\gamma\alpha} = g_{\alpha\beta\gamma}$ and so we have a trivialization of the gerbe, which we call a flat trivialization.

Suppose that the line bundle is given by a Čech cocycle $g_{\alpha\beta}$. Recall that a connection is a 1-form A_α on U_α such that $iA_\beta - iA_\alpha = g_{\alpha\beta}^{-1}dg_{\alpha\beta}$. A section is $f_\alpha : U_\alpha \to S^1$ such that $f_\alpha = g_{\alpha\beta}f_\beta$. It is covariant constant iff it satisfies the equation $df_\alpha = iA_\alpha f_\alpha$. If we write $f_\alpha = e^{ip_\alpha}$, then $dp_\alpha = A_\alpha$. Therefore, a necessary condition is $F_\alpha = dA_\alpha = 0$, i.e., the connection is flat. In the case of a gerbe, $dA_{\alpha\beta} \neq 0$ in general. We have to allow the freedom to choose B_α such that $d(A_{\alpha\beta} - B_\beta + B_\alpha) = 0$. Hence, the trivialization $h_{\alpha\beta}$ satisfies a modified equation $dh_{\alpha\beta} = i(A_{\alpha\beta} - B_\beta + B_\alpha)h_{\alpha\beta}$.

Suppose that we have a second flat trivialization $h'_{\alpha\beta}$, then $g_{\alpha\beta} = h'_{\alpha\beta}/h_{\alpha\beta}$ defines a line bundle L. Moreover,

$$iB_\beta - iB_\alpha - iA_{\alpha\beta} = d\log h_{\alpha\beta}, \quad iB'_\beta - iB'_\alpha - iA_{\alpha\beta} = d\log h'_{\alpha\beta}.$$

Hence, $i(B' - B)_\beta - i(B' - B)_\alpha = d\log g_{\alpha\beta}$ and $A_\alpha = (B' - B)_\alpha$ defines a connection on L. By the definition of B_α and B'_α, $F_\alpha = dB_\alpha = dB'_\alpha$. Hence the curvature $dA_\alpha = 0$. Thus, the difference of two flat trivializations of a gerbe is a flat line bundle. One can show that the converse is also true.

3.3 String connection

Recall that a connection on a line bundle induces a holonomy map $\mathrm{Hol} : LX \to S^1$. The holonomy of a connection on a gerbe has a similar property. One way to understand this is via its analogy to topological quantum field theory. Recall that topological quantum field theory can be described as follows. For any oriented d-dimensional manifold D, we associate a Hilbert space $\mathcal{H}_D$. For any cobordism W such that $\partial W = D_1 \cup \bar{D}_2$, where $\bar{D}_2$ denotes D_2 with the opposite orientation, we associate a homomorphism $\theta_W : \mathcal{H}_{D_1} \to \mathcal{H}_{D_2}$. The map θ_W satisfies the gluing axiom. Suppose that $\partial W_{12} = D_1 \cup \bar{D}_2, \partial W_{23} = D_2 \cup \bar{D}_3$. We can glue W_{12}, W_{23} along D_2 to obtain W_{13}. Then the gluing axiom is $\theta_{13} = \theta_{23} \circ \theta_{12}$. The analogy for a gerbe is called a string connection. It contains the following ingredients:

(i) Let $l : S^1 \to X$ be a smooth map. Since S^1 is one-dimensional, the pull-back of a gerbe with connection to the circle is flat and has trivial holonomy. Thus we have flat trivializations. For each l, we associate the moduli space of flat trivializations $\mathcal{L}_l$ ($\mathcal{L}_l$ is analogous to $\mathcal{H}_D$). Recall that we identify flat trivializations if they differ by a flat line bundle with trivial holonomy. Then, for each loop we have a space which is acted on freely and transitively by the moduli space of flat line bundles $H^1(S^1; S^1) \cong S^1$. Hence, $\mathcal{L}_l$ is isomorphic to S^1. In other words we have a principal S^1 bundle $\mathcal{L}$ over the free loop space LX.

We will pay special attention to the space of constant loops. Since X is embedded in LX as the space of constant loops, it is interesting to compute the restriction of $\mathcal{L}$ over X. Suppose that $f : S^1 \to X$ is a constant map. Then f is the composition of $p : S^1 \to pt$ and $i_f : pt \to X$. The pull-back gerbe $i_f^* g_{\alpha\beta\gamma}$ is obviously trivial. Furthermore, $i_f^* F_\alpha = 0, i_f^* A_{\alpha\beta} = 0$. Any trivialization of $i_f^* g_{\alpha\beta\gamma}$ is a flat trivialization. A key observation is that the flat line bundle over a point is trivial as well. Therefore, the pull-back gerbe with connection by i_f fixes a unique flat trivialization. Its pull-back by the projection map $p : S^1 \to pt$ defines a canonical element $s_f \in \mathcal{L}_f$ and hence a canonical section of $\mathcal{L}|_X$. Hence, $\mathcal{L}|_X$ is trivial with a canonical trivialization. Therefore, we obtain

Lemma 3.5. *$\mathcal{L}|_X$ is independent of the connection of the gerbe. Furthermore, it is trivial with a canonical trivialization.*

(ii) Suppose that $f : \Sigma \to X$, where Σ is a closed Riemann surface. Then the pull-back connection of the gerbe $(f^*F_\alpha, f^*A_{\alpha\beta})$ is flat. Its holonomy $\mathrm{Hol}_f = \{c_{\alpha\beta\gamma}\}$ is a cohomology class in $H^2(\Sigma; S^1)$. Since $H^2(\Sigma; S^1) = \mathrm{Hom}\,(H_2(\Sigma;\mathbb{Z}), S^1)$, its evaluation on the fundamental class of Σ naturally identifies it as a complex number.

A more interesting case is the case of a Riemann surface with boundaries. Suppose that $f : \Sigma \to X$, where Σ is a Riemann surface with boundary t_i with a fixed orientation-preserving parameterization $\delta_i : S^1 \to t_i$. Let $l_i = f \circ \delta_i$. Since $H^2(\Sigma; S^1) = H^3(\Sigma;\mathbb{Z}) = 0$, the pull-back gerbe is trivial and its holonomy is trivial as well. A flat trivialization restricts to a flat trivialization on each t_i. Namely, it induces an element σ in $\prod_i \mathcal{L}_{l_i}$. A different flat trivialization of Σ differs by a flat line bundle τ of Σ. It restricts to a flat line bundle τ_i over each boundary circle viewed as a standard S^1 via δ_i. Recall that

$$\pi_1(\Sigma) = \left\{ \lambda_1, \dots, \lambda_{2g}, l_1, \dots, l_k \,\middle|\, \prod_{i=1}^{g} [\lambda_{2i-1}, \lambda_{2i}] l_1 \cdots l_k = 1 \right\}.$$

Hence, $\tau_1 \cdots \tau_k = 1$. Therefore, different flat trivializations induce elements differing by a multiplication of $(\tau_1, \dots, \tau_k)$ of $\tau_i \in S^1$ with $\tau_1 \cdots \tau_k = 1$. Suppose that $L_1, \dots, L_k$ are k-many circle bundles. Then $L_1 \otimes \cdots \otimes L_k$ can be constructed as follows. Let H be the $(S^1)^k$-bundle with fiber $\prod_i (L_i)_x$. Then $\otimes_i L_i = H \times S^1/(S^1)^k$ via the product homomorphism $(S^1)^k \to S^1$. From the previous construction, we have

Lemma 3.6 (Theorem 6.2.4 in [19])**.** *The pull-back gerbe on Σ induces a canonical element $\theta_\Sigma \in \otimes \mathcal{L}_{l_i}$, or a trivialization $\theta_\Sigma : \otimes \mathcal{L}_{l_i} \to S^1$.*

Note that if we reverse the orientation of a boundary circle l_i to obtain $\bar{l}_i$, then $\mathcal{L}_{\bar{l}_i} = \mathcal{L}^*_{l_i}$.

(iii) θ has a decomposition property as follows. We decompose Σ along a circle $\Sigma = \Sigma_1 \cup_{S^1} \Sigma_2$ where Σ_1, Σ_2 are glued along boundary circles $l, \bar{l}$. Let $l_l, l_{\bar{l}}$ be the corresponding loops. Then $l_{\bar{l}} = \bar{l}_l$ and there is a canonical isomorphism $\mathcal{L}_{l_l} \otimes \mathcal{L}_{l_{\bar{l}}} \cong S^1$. Hence $\otimes_i \mathcal{L}_{l_i} \cong \otimes_i \mathcal{L}_{l_i} \otimes \mathcal{L}_{l_l} \otimes \mathcal{L}_{l_{\bar{l}}}$. Under this identification, it is clear that

Gluing axiom: $\theta_\Sigma = \theta_{\Sigma_1} \otimes \theta_{\Sigma_2}$.

θ admits another interpretation closely analogous to the topological quantum field theory. We can view Σ as a cobordism between incoming circles l_i (with an opposite orientation from the boundary orientation) and outgoing circles l_j. Then, θ_Σ can also be interpreted as an element of $\mathrm{Hom}\,(\otimes_i \mathcal{L}_{l_i}, \otimes_j \mathcal{L}_{l_j})$. Then the gluing axiom corresponds to the usual gluing.

One application of θ_Σ is to define a connection on $\mathcal{L} \to LX$. Take a path in the loop space $F : [0,1] \times S^1 \to X$. Applying the above discussion, we obtain a canonical isomorphism between $\mathcal{L}_{\{0\}\times S^1}$ and $\mathcal{L}_{\{1\}\times S^1}$. This defines the notion of the parallel transport over $\mathcal{L}$, and hence a connection on the line bundle $\mathcal{L}$ over LX. Recall that a section generated by the parallel transport from a point is precisely a covariant constant section. From our construction, the restriction of a flat trivialization on $[0,1] \times S^1$ to each $\{t\} \times S^1$ gives a covariant constant section.

Lemma 3.7. *The canonical section s of $\mathcal{L}|_X$ is a covariant constant section.*

Proof. Suppose that $F : [0,1]\times S^1 \to X$ is a path of constant loops. Then F is the composition of the projection to the first factor $p_1 : [0,1] \times S^1 \to [0,1]$ and a path of X, $i_F : [0,1] \to X$. We first use i_F to pull back a gerbe with its connection to $[0,1]$. Such a pull-back is flat with trivial holonomy. Then, we construct a flat trivialization on $[0,1]$. Now, we pull it back to $[0,1] \times S^1$ to obtain a flat trivialization s_F over $[0,1] \times S^1$. It is clear that the restriction of s_F to $\{t\} \times S^1$ is s_{l_t} for $l_t = F(t,\cdot)$. By the definition, $s_t = s_{l_t}$ is a covariant constant section along the path.

4 Gerbes on orbifolds

The previous construction has been generalized to the orbifolds by [7, 16]. It is amazing that $\mathcal{L} \to X$ starts to become nontrivial on an orbifold! Therefore, it is more interesting to study the gerbes on orbifolds than on smooth manifolds!

4.1 Basics

Lupercio-Uribe's construction is carried out for an arbitrary groupoid. The precise definition of the gerbe over an orbifold is not important for us. Therefore, instead of giving a long technically correct definition, let us motivate the definition of a groupoid from an orbifold. We first start from a smooth manifold where one can view a groupoid as a language to formalize the construction of an open cover. Consider the open cover $\{U_\alpha\}$ of a manifold and let $G_0 = \sqcup_\alpha U_\alpha, G_1 = \sqcup_{\alpha,\beta} U_{\alpha\beta}$. In the language of groupoids, G_0 is called the space of objects and G_1 is called the space of arrows. There are two maps $s : U_{\alpha\beta} \to U_\alpha, t : U_{\alpha\beta} \to U_\beta$. s,t are called the source map and the target map. Consider the fiber product

$$G_2 = G_1 \,{}_t\times_s G_1 = \{(x,y); t(x) = s(y)\}.$$

Using an open cover, it is not hard to see that $G_2 = \sqcup_{\alpha,\beta,\gamma} U_{\alpha\beta\gamma}$. There is an additional multiplication map $m : G_2 \to G_1$ corresponding to the inclusion $U_{\alpha\beta\gamma} \to U_{\alpha\gamma}$. To complete the definition of groupoid, we also need an identity $e : G_0 \to G_1$ and an inverse $I : G_1 \to G_1$. In our set-up, $e : U_\alpha \to U_{\alpha\alpha}, I : U_{\alpha\beta} \to U_{\beta\alpha}$ are identity maps. These structure maps satisfy several obvious compatibilility conditions for which we refer to [7]. We often use $\mathcal{G} = \{G_1 \overset{s,t}{\rightrightarrows} G_0\}$ to denote the groupoid. The process of taking a refinement of an open cover is called the Morita equivalence in groupoid language.

One can go on to construct $G_n = G_{n-1} \,{}_t\times_s G_1$. It corresponds to the disjoint union of $(n+1)$-fold intersections.

With the above correspondence, we can state Lupercio-Uribe's definition of the gerbes over a groupoid as a function $g : G_2 \to S^1$ satisfying the obvious cocycle condition generalizing the condition on smooth manifolds. If two gerbes g_1, g_2 differ by a coboundary, we call them equivalent. By a result due to Moerdijk[20], if $\mathcal{G}$ is an étale groupoid then the cohomology of the chain complex is the Čech cohomology for the sheaf $C^\infty(S^1)$ over the so-called classifying space $B\mathcal{G}$ of the groupoid. It is clear that an equivalence class of the gerbes is a Čech cohomology class of the sheaf $C^\infty(S^1)$ over $B\mathcal{G}$. Furthermore, we have a long exact sequence

$$\cdots \to H^2(B\mathcal{G}; C^\infty(\mathbb{R})) \to H^2(B\mathcal{G}; C^\infty(S^1)) \xrightarrow{\tau} H^3(B\mathcal{G};\mathbb{Z}) \to \cdots.$$

It is different from the smooth case that the characteristic class $\tau([g])$ is an integral cohomology class of the classifying space $B\mathcal{G}$ instead of its space of orbits $|\mathcal{G}|$. For an orbifold, we can always choose its groupoid representative $\mathcal{G}$ with the property that the components of G_0 are contractible. Such a kind of groupoids is called a fine groupoid. Over a fine groupoid, $C^\infty(\mathbb{R})$ is a fine sheaf. In this case, the equivalence class of the gerbes is still classified by its characteristic class.

Over a groupoid a connection on a gerbe is a pair (A, F) where A is a one-form on G_1 and F is a two-form on G_0 satisfying the condition:

$$t^*F - s^*F = dA, \quad i\pi_1^*A + i\pi_2^*A + im^*I^*A = g^{-1}dg.$$

Now, to extend the definition of a gerbe to an orbifold, we just have to associate a groupoid to an orbifold, this was done by Moerdijk-Pronk[21]. To this purpose, we just have to construct G_0, G_1 and s, t.

Let X be an orbifold and $\{(\tilde{U}_i, G_{\tilde{U}_i}), \mathrm{Tran}(U_i, U_j)\}$ be an orbifold atlas. We simply define $G_0 = \bigsqcup_i \tilde{U}_i, G_1 = \bigsqcup_{i,j} \mathrm{Tran}(U_i, U_j)$. s, t are the natural projections $s : \mathrm{Tran}(U_i, U_j) \to \tilde{U}_i,\ t : \mathrm{Tran}(U_i, U_j) \to \tilde{U}_j$. We call the above groupoid an orbifold groupoid.

An orbifold morphism corresponds to a Morita equivalence of morphisms between orbifold groupoids. An obvious and important fact is

Remark 4.1. An orbifold morphism between the orbifolds pulls back a gerbe with connection to a gerbe with connection.

In particular, if $f : Y \to X$ is a smooth orbifold morphism from a smooth manifold Y (viewed with a trivial orbifold structure) and g is a gerbe with connection (A, F) on X, then $(f^*g,\ f^*A, f^*F)$ is a gerbe with connection on a smooth manifold Y even though we start from an orbifold X. Therefore, the previous construction on the holonomy line bundle $\mathcal{L}$ goes through trivially. However, its restriction to the inertia orbifold is no longer trivial.

We first look at the case of the discrete torsion for a global quotient orbifold. The inner local system has been constructed in [6]. We would like to show that it agrees with the holonomy line bundle from the gerbe induced by the discrete torsion. Recall that discrete torsion is a 2-cocycle $\alpha : G \times G \to S^1$. Being a cocyle means $\alpha_{g,1} = \alpha_{1,g} = 1, (\delta\alpha)_{g,h,k} = \alpha_{h,k}\alpha^{-1}_{gh,k}\alpha_{g,hk}\alpha^{-1}_{g,h} = 1$. The groupoid presentation of the global quotient orbifold X/G is a translation groupoid with $G_0 = X, G_1 = X \times G$ and $s(x, g) = x, t(x, g) = gx$. We will use the stack notation $[X/G]$ to denote this groupoid structure. One can check that $G_2 = X \times G \times G$. And the map α induces a gerbe on $\mathcal{G}$ in an obvious way. Furthermore, we can choose a flat connection with $F = 0, A = 0$. Recall that the inertia orbifold is $[(\bigsqcup_{g\in G} X^g)/G]$ and let $\gamma_{g,h} = \alpha_{g,h}\alpha^{-1}_{ghg^{-1},g}$. Recall from [6] that we can define an inner local system on $[X/G]$ as follows. Consider the trivial bundle $\bigsqcup_g X^g \times \mathbb{C}_g$ where we use $\mathbb{C}_g$ to denote the fiber $\mathbb{C}$ associated to X^g and 1_g to denote the element 1 in $\mathbb{C}_g$. Then, we define an action of $g : \mathbb{C}_h \to \mathbb{C}_{ghg^{-1}}$ by $g(1_h) = \gamma_{g,h}1_{ghg^{-1}}$. Using the fact that α is a 2-cocycle, we can get $\gamma_{g_2,g_1hg_1^{-1}}\gamma_{g_1,h} = \gamma_{g_2g_1,h}$. Let $\mathcal{L}_\alpha$ be the quotient of the trivial bundle under the above action.

Theorem 4.2. $\mathcal{L}|_{\wedge[X/G]} = \mathcal{L}_\alpha$.

Proof. We start with some algebraic preliminaries. If a 2-cocycle α can be expressed as a coboundary $\alpha_{g,h} = \rho_g\rho_h\rho^{-1}_{gh}$, we call ρ a flat trivialization of α.

A 2-cocycle α corresponds to an equivalence class of group extensions $1 \to S^1 \to \widetilde{G}_\alpha \to G \to 1$. The group $\widetilde{G}_\alpha$ can be given the structure of a compact Lie group, where $S^1 \to \widetilde{G}_\alpha$ is the inclusion of a closed subgroup. The elements in the extension group can be represented by the pairs $\{(g,a);\ g \in G, a \in S^1\}$ with the product $(g_1,a_1)(g_2,a_2) = (g_1g_2, \alpha_{g_1,g_2}a_1a_2)$.

Lemma 4.3. *There is a* 1-1 *correspondence between the set of characters of* $\widetilde{G}_\alpha$ *restricted to the scalar multiplication on the central* S^1 *and the set of flat trivializations of* α.

Proof. If $\phi : \widetilde{G}_\alpha \to S^1$ is such a character then we define an associated trivialization of α via $\rho(g) = \phi(g,1)$. Note that $\rho(gh) = \phi(gh,1) = \alpha^{-1}_{g,h}\phi(gh,\alpha_{g,h}) = \alpha^{-1}_{g,h}\phi((g,1)(h,1)) = \alpha^{-1}_{g,h}\rho(g)\rho(h)$. Conversely, given a flat trivialization $\rho : G \to S^1$, we simply define $\phi(g,a) = a\rho(g)$. Note that

$$\phi((g,a)(h,b)) = \phi(gh,\alpha_{g,h}ab) = ab\rho(g)\rho(h) = a\rho(g)b\rho(h) = \phi(g,a)\phi(h,b).$$

We have proved the lemma.

Now, we come back to the proof of Theorem 4.2. Any element $g \in G$ generates an abelian subgroup $\langle g\rangle$. It pulls back the 2-cocycle α and we can define the corresponding group extension $\widetilde{\langle g\rangle}_\alpha$. It is well known that any 2-cocycle of a finite cyclic abelian group is a coboundary. Hence, we have a nontrivial set of flat trivializations and hence a set of characters of $\widetilde{\langle g\rangle}_\alpha$. For any $h \in G$, h sends $\langle g\rangle$ to $\langle hgh^{-1}\rangle$. We would like to calculate the action of h on the set of characters (hence flat trivializations). Given a character ϕ for $\widetilde{\langle g\rangle}_\alpha$ and a lifting (h,a) of h, the action is defined by the formula

$$\begin{aligned}
&(h,a)\phi(k,b) = \phi((h,a)(k,b)(h,a)^{-1}).\\
&(h,a)(k,b)(h,a)^{-1} = (hk,\alpha_{h,k}ab)(h^{-1},\alpha^{-1}_{h,h^{-1}}a^{-1}) = (hkh^{-1},\alpha_{h,k}\alpha_{hk,h^{-1}}\alpha^{-1}_{h,h^{-1}}b)\\
&\qquad\qquad\qquad\quad = (hkh^{-1},\alpha_{h,k}\alpha^{-1}_{hkh^{-1},h}b) = (hkh^{-1},\gamma_{h,k}b).
\end{aligned}$$

Recall that $\wedge[X/G] = (\bigsqcup_g X^g \times \{g\})/G$. It can be interpreted as follows. Let $f : S^1 \to [X/G]$ be a constant good map with the Chen-Ruan characteristic (g). Then, it factors through the constant morphism to $[X/\langle g\rangle]$ which is represented by (x,g) for $x \in X^g$. One can also factor through the abelian orbifold $[X/\langle hgh^{-1}\rangle]$ which is equivalent to the previous one. This is the action of G on $\bigsqcup_g X^g \times \{g\}$. Now we consider the space of constant morphisms to $[X/\langle g\rangle]$, which is parameterized by $X^g \times \{g\}$. It admits a flat gerbe from G through the embedding $\langle g\rangle \to G$ (denoted by $\alpha_{\langle g\rangle}$), since $\langle g\rangle$ is a cyclic abelian group and such a gerbe is trivial. Pick a flat trivialization ρ of $\alpha_{\langle g\rangle}$. ρ defines a section s_ρ of $\mathcal{L}_{X^g\times\{g\}}$. We can use the same argument as in the smooth case to show that s_ρ is covariant constant. Therefore, $\mathcal{L}_{X^g\times\{g\}}$ is trivial as a flat line bundle. However, it does not have a canonical flat trivialization since a different flat trivialization of $\alpha_{\langle g\rangle}$ will define a different flat trivialization of $\mathcal{L}_{X^g\times\{g\}}$. Using the calculation above, we conclude that the action of G on $\mathcal{L}_{X^g\times\{g\}}$ is via the character $\gamma_{h,g}$. We have proved the theorem.

Theorem 4.4 (Lupercio-Uribe)**.** $\mathcal{L}|_{\wedge X}$ *is flat.*

Proof. Lupercio-Uribe proved their theorem using a generalization of Brylinski's relevant formula in the smooth case. Here, we use our previous analysis to give a direct, geometric proof. The question is local. Therefore, we can assume that our orbifold is a global quotient $[\mathbb{R}^n/G]$ with the gerbe and the connection given by (α^0, F, A). It follows that $\alpha^0 = \alpha\beta$ where $\alpha : G\times G \to S^1$ is a 2-cocyle and β is a coboundary over $[\mathbb{R}^n/G]$ and $(\alpha^0, F, A) = (\alpha, 0, 0)+(\beta, F, A)$.

Let $\mathcal{L}^0$, $\mathcal{L}$ and $\mathcal{L}'$ represent the holonomy line bundles of (α^0, F, A) , $(\alpha, 0, 0)$ and (β, F, A) restricted to the inertia orbifoid respectively. It is clear that $\mathcal{L}^0 = \mathcal{L} \otimes \mathcal{L}'$ and, fixing any path in the inertia orbifold, the parallel transport on $\mathcal{L}^0$ is the tensor product of those on $\mathcal{L}$ and $\mathcal{L}'$. Since β is a coboundary , there is a flat trivialization of $\mathcal{L}'$. The fact that $\mathcal{L}$ is flat implies that $\mathcal{L}^0$ is flat by Theorem 4.2.

4.2 Holonomy on an orbifold Riemann surface

If we only consider the maps from a smooth Riemann surface, the construction in the smooth case can be readily generalized to the case of orbifolds. A more interesting case is the case when $f : \Sigma \to X$ is a good map from an orbifold Riemann surface Σ. In orbifold quantum cohomology, we have to consider its generalization where Σ is a nodal orbifold Riemann surface.

Unfortunately, many things go wrong and we do not have a straightforward generalization of a string connection. One of the critical facts for an oriented smooth Riemann surface is that $H_2(\Sigma; \mathbb{Z}) = \mathbb{Z}$ is generated by its fundamental class σ. We use this fact to interpret the holonomy of a gerbe with a connection as a number in S^1. We start our discussion from the following computation of $H_2(B\Sigma; \mathbb{Z})$ for an orbifold Riemann surface. Indeed, a certain subtlety arises.

Let Σ be an orientable orbifold Riemann surface, with marked points $\{z_1, \ldots, z_k\}$ and corresponding multiplicities $\{m_1, \ldots, m_k\}$. Note that the underlying topological space $|\Sigma|$ of the orbifold Σ is a topological surface. Let $B\Sigma$ be the corresponding classifying space of the orbifold.

Given an action of the group G on a space X, let EG be a free G space which is contractible. One can utilize the Borel construction $X_G = EG \times_G X$ to define $H_*^G(X; \mathbb{Z}) = H_*(EG \times_G X; \mathbb{Z})$. It is well known (see [22, Corollary 1.24]) that there is an action of S^1 on a 3-manifold M such that $[M/S^1]$ is the given orbifold Σ. It is well-known that up to a weak homotopy equivalence $B\Sigma \cong M_{S^1}$. Hence, $H_2(B\Sigma; \mathbb{Z}) = H_2^{S^1}(M; \mathbb{Z})$.

There is a canonical map $\pi_X : X_G \to X/G$ and thus a homomorphism $\pi_{X*} : H_*^G(X; \mathbb{Z}) \to H_*(X/G; \mathbb{Z})$ which is known to be an isomorphism if the action of G on X is free.

Theorem 4.5. *Let Σ be an oriented orbifold Riemann surface, then $H_2(B\Sigma; \mathbb{Z}) = \mathbb{Z}$.*

Proof. Choose small open discs D_i centered around x_i such that their closures B_i are disjoint. Let $V_i = f^{-1}(B_i)$, $V = \bigcup_i V_i$, $V_i^* = f^{-1}(D_i)$, $V^* = \bigcup_i V_i^*$ and $U = M - V^*$, here $f : M \to \Sigma$ is the quotient map. Note that the actions of S^1 on U and $U \cap V$ are free.

Now $H_2^{S^1}(U; \mathbb{Z}) \cong H_2(U/S^1; \mathbb{Z}) = 0$ since the action is free and U/S^1 is a smooth surface with boundary. $H_j(V/S^1; \mathbb{Z}) = 0$ for $j > 0$ since V/S^1 is the disjoint union of closed discs. It is well known that the local model of a marked point of an orbifold Riemann surface is the quotient of a disc by the action of a cyclic group. It follows that $H_j^{S^1}(V; \mathbb{Z}) = \bigoplus_i H_j(B\mathbb{Z}_{m_i}; \mathbb{Z})$ and $H_2^{S^1}(V; \mathbb{Z}) = 0$, $H_1^{S^1}(V; \mathbb{Z}) = \bigoplus_i \mathbb{Z}_{m_i}$ since $H_2(B\mathbb{Z}_m; \mathbb{Z}) = 0$ for all positive integers.

One has the following Mayer-Vietoris sequence:

$$0 \to H_2^{S^1}(M; \mathbb{Z}) \to H_1^{S^1}(U \cap V; \mathbb{Z}) \to H_1^{S^1}(U; \mathbb{Z}) \oplus H_1^{S^1}(V; \mathbb{Z}) \to .$$

Now $H_1^{S^1}(U \cap V; \mathbb{Z}) = \mathbb{Z}^k$, thus $H_2^{S^1}(M; \mathbb{Z})$ is free. On the other hand, $H^2(B\Sigma; \mathbb{Q}) = H^2(|\Sigma|; \mathbb{Q}) = \mathbb{Q}$. Therefore, we get $H_2^{S^1}(M; \mathbb{Z}) = \mathbb{Z}$.

It is clear that any gerbe connection (F, A) on an orbifold Riemann surface is flat for dimension reasons. Therefore, we can define its holonomy $\mathrm{Hol}\,(F, A)$ as a class in $H^2(B\Sigma; S^1)$

where S^1 means the constant sheaf. Since $H^2(B\Sigma; S^1) = S^1$, one can evaluate $\mathrm{Hol}\,(F, A)$ on the generator $e \in H_2(B\Sigma; \mathbb{Z})$ to obtain a number $\mathrm{Hol}\,(F, A)(e) \in S^1$. However, if we view this from the point of trivializations, there associate more than one numbers in S^1. Consider the gerbe connection over the local 2-dimensional orbifold disc $[D/\mathbb{Z}_m]$. The connection as well as its holonomy is trivial. A choice of flat trivialization is restricted to a flat trivialization on the boundary circle $\partial(D/\mathbb{Z}_m)$. Namely, we obtain an element of $\mathcal{L}_{\partial(D/\mathbb{Z}_m)}$. However, the space of flat line bundles on $[D/\mathbb{Z}_m]$ is non-trivial. In fact, it is parameterized by $\mathbb{Z}_m$. They induce a set of $\mathbb{Z}_m$-points on $\mathcal{L}_{\partial(D/\mathbb{Z}_m)}$. Now, we go back to the closed orbifold Riemann surface Σ with orbifold points at $(x_1, \ldots, x_k)$ of multiplicity $m_1, \ldots, m_k$ and $f : \Sigma \to X$. We decompose Σ as the disjoint union of orbifold discs $[D_i/\mathbb{Z}_{m_i}]$ and $V = \Sigma - \bigsqcup_i [D_i/\mathbb{Z}_{m_i}]$. Then the flat trivialization at V specifies an element on $\mathcal{L}_{l_i}$ of each boundary circle l_i. The gluing law indicates that we should associate a set of $m_1 \cdots m_k$-numbers in S^1.

Remark 4.6. On an orbifold Riemann surface, the numbers given by the trivializations are not unique, hence they cannot be interpreted as the holonomy of a gerbe with a connection.

5 Orbifold quantum cohomology twisted by a flat gerbe

5.1 String connection on an orbifold and orbifold stable maps

Recall the compatibility condition of an inner local system over each $X_{(g_1,g_2)}$; $\theta_{(g_1,g_2)} : e_1^* L_{(g_1)} \otimes e_2^* L_{(g_2)} \otimes e_3^* L_{((g_1g_2)^{-1})} \cong 1$. The purpose of this condition is as follows. $X_{(g_1,g_2)} = X_{(g_1,g_2,g_3)}$ with $g_3 = (g_1g_2)^{-1}$ can be identified as the moduli space of degree zero genus zero maps with three marked points—$\overline{\mathcal{M}}_{0,3}(X, J, 0, \boldsymbol{x})$. The evaluation maps at the marked points are e_i. Let $\alpha_i \in H^*(X_{(g_i)}; L_{(g_i)})$. The trivialization $\theta_{(g_1,g_2)}$ maps $e_1^*\alpha_1 \wedge e_2^*\alpha_2 \wedge e_3^*\alpha_3$ to an ordinary cohomology class of $X_{(g_1,g_2,g_3)}$ and hence can be integrated. The latter property allows us to define the twisted orbifold product. To carry out the same construction for orbifold quantum cohomology, we have to construct the trivialization θ over $\overline{\mathcal{M}}_{g,k}(X, J, A, \boldsymbol{x})$ for general g, k, A. This can be accomplished by Theorem 5.1.

Suppose that $f : \Sigma \to X$ is an orbifold morphism. We take a real blow-up at the marked points to obtain a Riemann surface with boundary $\Sigma^\dagger$. Let $l_{i\infty}$ be the corresponding boundary circle. It is clear that each $f : \Sigma \to X$ induces a morphism $f^\dagger : \Sigma^\dagger \to X$. It is clear that $f_{i\infty} = f^\dagger(l_{i\infty})$ is a constant loop. Moreover, we have an identification $f_{i\infty} = e_i(f)$. Next, the holonomy $\theta_\Sigma = \theta_{\Sigma^\dagger}$ is interpreted as $\theta_{\Sigma^\dagger} : \otimes_i e_i^* \mathcal{L} \to S^1$.

Next, we extend the above discussion to orbifold stable maps. Suppose that $f : \Sigma \to X$ is an orbifold stable map where Σ is a marked orbifold nodal Riemann surface. For simplicity, we assume that $\Sigma = \Sigma_1 \wedge \Sigma_2$ joined at the point $p \in \Sigma_1, q \in \Sigma_2$. We observe that $f_{p\infty}$ is the same as $f_{q\infty}$ with the reversed orientation. Hence, $\mathcal{L}_{f_{p\infty}} = \mathcal{L}^*_{f_{q\infty}}$. Suppose that the marked points on Σ_1 are $x_1, \ldots, x_l$ and the marked points on Σ_2 are $x_{l+1}, \ldots, x_k$. We have

$$\theta_{\Sigma_1} : \otimes_{1 \leqslant i \leqslant l} e_i^* \mathcal{L} \otimes e_p^* \mathcal{L} \to S^1; \qquad \theta_{\Sigma_2} : \otimes_{l+1 \leqslant j \leqslant k} e_j^* \mathcal{L} \otimes e_q^* \mathcal{L} \to S^1.$$

Using the canonical isomorphism $e_p^* \mathcal{L} \otimes e_q^* \mathcal{L} \to S^1$, we obtain $\theta_\Sigma : \otimes_{1 \leqslant i \leqslant k} e_i^* \mathcal{L} \to S^1$. The analogue of the gluing law is the following

Theorem 5.1 (Gluing law)**.** *The map $\theta_{\Sigma^\dagger}$ is continuous with respect to the degeneration of orbifold stable maps, i.e., it induces a continuous trivialization of $\otimes_i e_i^* \mathcal{L}$ over $\overline{\mathcal{M}}_{g,k}(X, J, A, \boldsymbol{x})$.*

Proof. The map θ is clearly continuous over each stratum of $\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})$. We only have to check that it is continuous with respect to the degeneration of orbifold stable maps. It is enough to discuss the case of creation of a new nodal point. Suppose that $(f_n,(\Sigma_n,\boldsymbol{z}_n),\xi_n)$ converges to $(f_0,(\Sigma_0,\boldsymbol{z}_0),\xi_0)$ and $z_0\in\Sigma_0$ is the nodal point. It is instructive to see the degeneration of ξ_n to ξ_0. Recall the construction in [13].

Locally, $f_n: W_{t_n}\to(V_p-\{p\})/G_p$, where $W_t=\{xy=t;|x|,|y|<\epsilon\}$ and (V_p,G_p,π_p) is a uniformizing system of $p=f_0(z_0)\in X$. The key is to construct the lifting $\tilde{f}_n$ mapping into V_p.

By Lemma 2.2.4 of [13], ξ_n determines a characteristic $\theta_n:\pi_1(W_{t_n})\to G_p$. Suppose that g is the image of a generator and m is the order of g. Then θ_n determines a covering $\widetilde{W}^m_{t_n}\to W_{t_n}$. The argument of Lemma 2.2.6 in [13] constructs a lifting $\tilde{f}_n:\widetilde{W}^m_{t_n}\to V_p$. Then the convergence of f_n as a good map is interpreted as the convergence of ordinary maps $\tilde{f}_n$ to $\tilde{f}_0:\widetilde{W}^m_0\to V_p$, which gives a natural compatible system ξ_0 at p. Note that Σ_0 acquires a natural orbifold structure at the nodal point z_0, whose uniformizing system is given by $(\widetilde{W}^m_0,\mathbb{Z}_m)$, where the action of $\mathbb{Z}_m$ on $\widetilde{W}^m_0$ is the limit of the action on $\widetilde{W}^m_{t_n}$.

Let $S^1_n\subset W_{t_n}$ be the circle given by $|x|=\frac{|t_n|}{2\epsilon}$ with a complex orientation of x. S^1_n converges to a constant loop supported at p. On the other hand, the same S^1 with the opposite orientation can be expressed as $|y|=\frac{|t_n|}{2\epsilon}$ (denoted by S^{1*}_n). S^{1*}_n converges to the constant loop supported at q. We decompose Σ_n along S^1_n as $\Sigma_n=\Sigma^1_n\cup_{S^1_n}\Sigma^2_n$. Then Σ^1_n converges to Σ_1 and Σ^2_n converges to Σ_2. The above construction implies that $f_n|_{S^1_n}$ converges to $(f_0)_{p\infty}$. Moreover, $f_n|_{S^{1*}_n}$ as a good map converges to $(f_0)_{q\infty}$. By the gluing axiom, $\theta_{\Sigma_n}:\otimes_i e^*_i\mathcal{L}\to S^1$ can be decomposed as the product of $\theta_{\Sigma^1_n}:\otimes_{1\leqslant i\leqslant l}e^*_i\mathcal{L}\otimes\mathcal{L}_{f_n|_{S^1_n}}\to S^1$ and $\theta_{\Sigma^2_n}:\otimes_{l+1\leqslant j\leqslant k}e^*_j\mathcal{L}\otimes\mathcal{L}_{f_n|_{S^{1*}_n}}\to S^1$ using canonical trivialization $\mathcal{L}_{f_n|_{S^1_n}}\otimes\mathcal{L}_{f_n|_{S^{1*}_n}}\cong S^1$. It is clear that θ_{Σ_n} converges to θ_{Σ_0}.

5.2 Twisted orbifold Gromov-Witten invariants

So far, we have not yet brought in the flatness condition. Recall that $\otimes_i e^*_i\mathcal{L}$ in our construction is used as the coefficient system or the flat line bundle. Therefore, we also need to construct a flat trivialization. This requires the assumption of flatness of the gerbe.

Theorem 5.2. *For a flat gerbe, $\theta_{\Sigma^\dagger}$ is a flat trivialization.*

Proof. A flat bundle is completely determined by its holonomy around a loop. Even though $\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})$ is not a smooth manifold in general, we can still discuss the flat trivialization of a flat bundle. Namely, it is enough to determine if the trivialization is flat around each loop. Since the flatness is a local condition, it is enough to prove that it is flat along a curve $f:[0,1]\to\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})$.

Recall that there is a universal family $\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})\to\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})$ as an orbifold fiber bundle whose fiber is the domain of orbifold stable maps modulo automorphisms. The pullback $\pi:f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})\to[0,1]$ is an orbifold Riemann surface bundle. The total space $f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})$ is an orbifold 3-manifold with boundary. Each marked point defines a section s_i. For simplicity, we first assume that there is no nodal point. Now, we take a real blow-up along the image of s_i to obtain $\pi_\dagger:f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})^\dagger\to[0,1]$. Then we replace each s_i by $S^1\times[0,1]$ (denoted by $S^1_i\times[0,1]$). It is clear that $\pi^{-1}_\dagger(t)$ is the real blow-up of $\pi^{-1}(t)$ along the marked points, where $S^1_i\times\{t\}$ is precisely the infinity circle associated to the i-th marked point of $\Sigma_t=\pi^{-1}(t)$. A moment of thought tells that $f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})^\dagger$ is an oriented 3-manifold with the boundary given by

$$\partial f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})^\dagger = \pi_\dagger^{-1}(0)\bigcup \cup_i S_i^1\times[0,1]\bigcup \pi_\dagger^{-1}(1).$$

Furthermore, the identification happens precisely at the infinity circles corresponding to the marked points on $\pi_\dagger^{-1}(0), \pi_\dagger^{-1}(1)$. Furthermore, there is an evaluation $e:\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})\to X$ whose restriction to the i-th marked point defines e_i. It induces an evaluation map $e_\dagger : f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})^\dagger\to X$ whose restriction to each infinity circle defines the corresponding evaluation map $e_{i\dagger}=f_{i\infty}$. Since the gerbe is flat, the holonomy around the boundary $\partial f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})^\dagger$ is zero. Recall that the restriction of a flat trivialization of $S_i^1\times[0,1]$ to its boundary defines parallel transport from $\mathcal{L}_{f_{i\infty 0}}$ to $\mathcal{L}_{f_{i\infty 1}}$. By our definition, the restriction of the flat trivialization to the boundary of $\pi_\dagger^{-1}(0), \pi_\dagger^{-1}(1)$ defines elements $\theta_0\in\otimes\mathcal{L}_{f_{i\infty 0}}, \theta_1\in\otimes\mathcal{L}_{f_{i\infty 1}}$ respectively. The property that the total holonomy around $\partial f^*\overline{\mathcal{U}}_{g,k}(X,J,A,\boldsymbol{x})^\dagger$ is zero can be interpreted as the statement that the parallel transport maps θ_0 to θ_1. Then we prove that θ is flat.

If we have the nodal points, we do a real blow-up along the nodal point. It creates an additional boundary component. However, it is clear that the total holonomy of the additional components cancel each other. The above argument still applies.

Remark 5.3. The proof of the above theorem depends critically on the flatness of the gerbe.

Now, we are ready to construct the twisted orbifold GW-invariants. Using e_i, we can pull back $\mathcal{L}$ (now as an orbifold vector bundle) to define the tensor product $\otimes_i e_i^*\mathcal{L}$. Then θ provides a flat trivialization $\theta:\otimes_i e_i^*\mathcal{L}\to\mathbb{C}$, continuous with respect to the topology of $\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})$. Suppose that $\boldsymbol{x}=\prod_i X_{(g_i)}$. It induces a homomorphism $\otimes_i H^*(X_{(g_i)};\mathcal{L}_{(g_i)})\to H^*(\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x});\mathbb{C})$ by $(\alpha_1,\dots,\alpha_k)\to\theta_*(\wedge_i e_i^*\alpha_i)$, where

$$\theta_*: H^*(\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x});\otimes e_i^*\mathcal{L}_{(g_i)})\to H^*(\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x});\mathbb{C})$$

is the isomorphism induced by θ and $\mathcal{L}_{(g_i)}=\mathcal{L}|_{X_{(g_i)}}$.

Definition 5.4. *The orbifold GW-invariants twisted by a flat gerbe are defined as*

$$\Psi^{X,J}_{(g,k,A,\boldsymbol{x},(A,F))}(\alpha_1^{l_1},\dots,\alpha_k^{l_k})=\theta_*\left(\prod_{i=1}^k c_1(L_i)^{l_i}e_i^*\alpha_i\right)[\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})]^{\mathrm{vir}},$$

where L_i is the line bundle generated by the cotangent space of the i-th marked point and $[\overline{\mathcal{M}}_{g,k}(X,J,A,\boldsymbol{x})]^{\mathrm{vir}}$ is the virtual fundamental cycle constructed in [13].

A standard argument in the Gromov-Witten theory will show that our twisted orbifold GW-invariants satisfy the standard axioms[13]. In particular, it implies that there is an associative quantum multiplication on $H^*_{\mathrm{CR}}(X;\mathcal{L})$.

6 Computations

The current treatment of the gerbes in literature is usually abstract. One of the authors' goals for this article is to be as concrete as possible. In this section, we will try to figure out how far we can go to compute the holonomy inner local system explicitly. We will divide this section into the smooth versus orbifold cases.

6.1 Smooth case

When X is a smooth manifold, the holonomy line bundle $\mathcal{L}|_X$ is canonically trivial. Hence $\otimes_i e_i^*\mathcal{L}|_X$ is canonically trivial. Recall that for a stable map $f:\Sigma\to X$, θ_f is another trivialization $\otimes_i e_i^*\mathcal{L}|_X\cong S^1$. Therefore, the difference of two trivializations is a number in S^1. By abuse of notation, we still denote it by θ_f. On the other hand, we can also associate the holonomy $\mathrm{Hol}_f\in S^1$. The key theorem in the smooth case is

Theorem 6.1. *Suppose that X is a smooth manifold and $\mathcal{L}|_X$ is trivialized by its canonical trivialization. Then $\theta_f=\mathrm{Hol}_f$.*

Proof. Let g be the cocycle representing the gerbe. First, we assume that the stable map $f:\Sigma\to X$ has domain Σ as an irreducible Riemann surface. We take a real blow-up to obtain $\Sigma^\dagger$.There is an obvious map $\pi:\Sigma^\dagger\to\Sigma$ by contracting $l_{i\infty}$ to x_i. Let $f^\dagger=f\circ\pi$. f^*g with its connection is flat on Σ. We can define its holonomy $\mathrm{Hol}_f\in H^2(\Sigma;S^1)$. Since $H^2(\Sigma;S^1)=S^1$. We use Hol_f to denote its Čech cocycle or the number through the above isomorphism without any confusion. From the construction of Hol_f, $f^*g=\mathrm{Hol}_f\delta r$. Namely, they differ by a coboundary. Furthermore, we are allowed to change r by a constant cochain. Therefore, we can choose r in such fashion that $r|_{\boldsymbol{z}}$ is a flat trivialization of $f^*g|_{\boldsymbol{z}}$. Therefore, $s_{f_{i\infty}}=\pi^*r|_{x_i}$, where $f_{i\infty}$ is the restriction of $f^\dagger$ to the boundary circle $l_{i\infty}$. We use $f^\dagger$ to pull back the gerbe represented by the cocycle g. Since $H^2(\Sigma^\dagger;S^1)=0$, $\pi^*\mathrm{Hol}_f$ is trivial. Choose a trivialization $\pi^*\mathrm{Hol}_f=\delta h$ where h is constant. A flat trivialization of $(f^\dagger)^*g$ is of the form $\delta(h\pi^*r)$. Then θ_f is the image of $((h\pi^*r)|_{l_{1\infty}},\dots,(h\pi^*r)|_{l_{k\infty}})$ in $\otimes_i\mathcal{L}_{f_{i\infty}}$ and $\theta_f=\prod_i h_{l_{i\infty}}s_{f_{i\infty}}$. We claim that $\prod_i h_{l_{i\infty}}$ is Hol_f through the canonical isomorphism $P:H^2(\Sigma;S^1)\cong S^1$. This canonical isomorphism is defined by the evaluation on the fundamental class of Σ. Let us consider the isomorphism induced by π, $\pi^*:H^2(\Sigma,\boldsymbol{z};S^1)\to H^2(\Sigma^\dagger,\partial\Sigma^\dagger;S^1)$ and the isomorphism induced by the inclusion $(\Sigma,\emptyset)\subseteq(\Sigma,\boldsymbol{z})$, $H^2(\Sigma,\boldsymbol{z};S^1)\to H^2(\Sigma;S^1)$. Note that $\mathrm{Hol}_f|_{\boldsymbol{z}}=1$ and hence can be viewed as an element of $H^2(\Sigma,\boldsymbol{z};S^1)$ as well as $H^2(\Sigma^\dagger,\partial\Sigma^\dagger;S^1)$ as its pull-back by π. Therefore, the image of Hol_f under P can also be obtained by the evaluation of $\pi^*\mathrm{Hol}_f$ on the relative fundamental class of $(\Sigma^\dagger,\partial\Sigma^\dagger)$ which is precisely $\prod_i h_{l_{i\infty}}$ since $\langle\pi^*\mathrm{Hol}_f,[\Sigma^\dagger,\partial\Sigma^\dagger]\rangle=\langle\delta h,[\Sigma^\dagger,\partial\Sigma^\dagger]\rangle=\langle h,[\partial\Sigma^\dagger]\rangle=\prod_i h_{l_{i\infty}}$.

Now, we consider the general case in which Σ may have more than one component. Then, we apply the previous argument for each component. By the gluing axiom, θ_f is multiplicative. Hol_f is obviously multiplicative. Hence, the theorem is true for multi-components Σ as well.

Suppose that we have a flat connection (A,F). Then we have the global holonomy $\mathrm{Hol}\in H^2(X;S^1)$. It is clear that $\mathrm{Hol}_f=\mathrm{Hol}(f_*[\Sigma])$. Then we have proved

Corollary 6.2. *Suppose that (g,F,A) is a flat gerbe. Under the canonical trivialization of $\mathcal{L}|_X$, the twisted GW-invariant $\Psi^{X,J}_{(g,k,A,\boldsymbol{x},(A,F))}(\alpha_1^{l_1},\dots,\alpha_k^{l_k})=\mathrm{Hol}(A)\Psi^{X,J}_{(g,k,A,\boldsymbol{x})}(\alpha_1^{l_1},\dots,\alpha_k^{l_k})$.*

6.2 Discrete torsion

Suppose that we have a global quotient orbifold $[X/G]$ and a discrete torsion $\alpha:G\times G\to S^1$. Recall that we can express the inertia orbifold as a global quotient $\wedge[X/G]=[(\bigsqcup_g X^g\times\{g\})/G]$. Furthermore, the holonomy line bundle $\mathcal{L}|_{\wedge[X/G]}=(\bigsqcup_g X^g\times\{g\})\times_\gamma\mathbb{C}$. We would like to express the moduli space of orbifold stable maps as a G-global quotient as well. This has already been done in [23]. Let us briefly review their construction. As we mentioned previously, by pulling back via the G-bundle $X \to X/G$, an orbifold stable map is equivalent to a G-orbifold cover $E \to \Sigma$ and a G-map $\phi : E \to X$. A G-stable map has additional data $\tilde{z}_i$—a lifting of the marked point $z_i \in \Sigma$. Consider the isotropy subgroup $G_{\tilde{z}_i} \subset G$. For dimensional reasons $G_{\tilde{z}_i} \cong \mathbb{Z}_m$ for some m, $\tilde{z}_i$ uniquely determines an element g_i as the generator of $G_{\tilde{z}_i}$. g_i has a geometric interpretation as the monodromy of a small loop around z_i. It is independent of the lifting of small loop because a different lifting will conjugate g_i by an element of $G_{\tilde{z}_i}$, which is abelian. Now, the evaluation map e_i is lifted to $\tilde{e}_i(E \to \Sigma, \phi, \tilde{z}_1, \ldots, \tilde{z}_k) = (\phi(\tilde{z}_i), g_i)$. G acts on G-stable maps by its action on $\tilde{z}_i$ and $\tilde{e}_i$ is G-equivariant. Let $\overline{\mathcal{M}}^G_{g,k}(X, J, A)$ be the moduli space of G-stable maps. We can apply virtual fundamental cycle techniques for $\overline{\mathcal{M}}^G_{g,k}(X, J, A)$ to obtain a G-orbifold GW-invariant. The orbifold GW-invariant is the invariant part of the G-orbifold GW-invariant.

We can use $\tilde{e}_i$ to pull back $\mathcal{L}|_{\wedge[X/G]}$ to form a flat line bundle $\overline{\mathcal{M}}^G_{g,k}(X, J, A) \times_{\gamma_{g_1} \cdots \gamma_{g_k}} (\otimes_i \mathbb{C}_{g_i})$. By our construction the holonomy of Σ should give a flat trivialization of this flat line bundle and we would like to write it down explicitly.

Without loss of generality, we assume that Σ is irreducible. We take a real blow-up $\Sigma^\dagger$ at all the marked points. There results a real blow-up $E^\dagger$ of E at all the preimage points of z_i. G acts on $E^\dagger$ freely and $\Sigma^\dagger = E^\dagger/G$. Using the translation groupoid representation $[E^\dagger/G]$ of $\Sigma^\dagger$, we can pull back α to a flat gerbe on $\Sigma^\dagger$. The general theory tells us that this flat gerbe is trivial. Furthermore, a flat trivialization is restricted to a flat trivialization on each boundary circle. Let us study the induced gerbe on $\Sigma^\dagger$ from the point of view of the Chen-Ruan characteristic. Fix a base point x_0 in the interior of $\Sigma^\dagger$ and choose a lifting $\tilde{x}_0 \in E^\dagger$. It defines the CR-characteristic $\rho : \pi_1(\Sigma^\dagger, x_0) \to G$. In fact, $E^\dagger = E_{\mathrm{univ}} \times_\rho G$ where E_{univ} is the universal cover. We can use ρ to pull back α to a 2-cocycle $\tilde{\alpha}$ of $\pi_1(\Sigma^\dagger, x_0)$. The induced gerbe on $\Sigma^\dagger$ is given by $\tilde{\alpha}$. Since E_{univ} is contractible,

$$H^2(\pi_1(\Sigma^\dagger, x_0); S^1) \cong H^2(\Sigma^\dagger; S^1) = 0.$$

Therefore, $\tilde{\alpha}$ is a coboundary. Choose a cobundary $\tilde{\alpha} = \delta h$.

Let us go back to the local monodromy g_i at $\tilde{z}_i$. Indeed, it is the monodromy along the boundary circle associated with z_i. We would like to embed $\langle g_i \rangle$ in $\pi_1(\Sigma^\dagger, x_0)$ as a subgroup. This can be done as follows. Choose a path d_i from $\tilde{x}_0$ to the boundary circle associated with $\tilde{z}_i$ with the end point x_i, then go around the boundary circle to $g_i x_i$ and go back to $g_i \tilde{x}_0$ along $g_i d_i^{-1}$. Its projection l_i on $\Sigma^\dagger$ is a loop at x_0 whose lifting defines $\rho(l_i) = g_i$. Then, we map g_i to l_i. It can be shown that a different path d'_i conjugates l_i by an element of the image of $\pi_1(E^\dagger, \tilde{x}_0)$. The image of $\pi_1(E^\dagger, \tilde{x}_0)$ is precisely the kernel of ρ. It is clear that $\tilde{\alpha}_{\langle g_i \rangle} = \alpha_{\langle g_i \rangle} = \delta h_{\langle g_i \rangle}$. Therefore, we can use $h_{\langle g_i \rangle}$ to trivialize $\mathcal{L}_{X^{g_i} \times \{g_i\}}$. Hence, the trivialization given by holonomy on $\Sigma^\dagger$ corresponds to the pull-back of the trivial bundle $\mathcal{L}_{X^{g_i} \times \{g_i\}}$. We still have to show that $\gamma_{g_1} \cdots \gamma_{g_k} = 1$ in order to descend to a trivial bundle over $\overline{\mathcal{M}}_{g,k}([X/A], J, A)$. This follows directly from the CR-characteristic ρ. Recall that $\pi_1(\Sigma^\dagger, x_0)$ has generators $l_1, \ldots, l_k, \mu_1, \ldots, \mu_{2g}$ with relation $\prod_j [\mu_{2j-1}, \mu_{2j}] \, l_1 \cdots l_k = 1$, then $\rho(l_i) = g_i$. Therefore,

$$1 = \gamma_{\rho(\prod_j [\mu_{2j-1}, \mu_{2j}] l_1 \cdots l_k)} = \prod_j [\gamma_{\rho(\mu_{2j-1})}, \gamma_{\rho(\mu_{2j})}] \gamma_{g_1} \cdots \gamma_{g_k} = \gamma_{g_1} \cdots \gamma_{g_k}.$$

Then we can apply the previous construction to the G-orbifold Gromov-Witten invariants constructed in [23]. Thus, we proved

Theorem 6.3. *The twisted orbifold Gromov-Witten invariant of a discrete torsion is the G-invariant part of the G-orbifold Gromov-Witten invariant, under the action twisted by the discrete torsion.*

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References

1 Chen W, Ruan Y. A new cohomology theory for orbifold. *Comm Math Phys*, **248**(1): 1–31 (2004)
2 Adem A, Ruan Y. Twisted orbifold K-theory. *Comm Math Phy*, **237**(3): 533–556 (2003)
3 Ruan Y. Cohomology ring of crepant resolutions of orbifolds. Gromov-Witten theory of spin curves and orbifolds. Contemp Math, Vol 403. Providence: Amer Math Soc, 2006, 117–126
4 Vafa C. Modular Invariance and discrete torsion on orbifolds. *Nuclear Phys B*, **273**: 592–606 (1986)
5 Vafa C, Witten E. On orbifolds with discrete torsion. *J Geom Phys*, **15**(3): 189–214 (1995)
6 Ruan Y. Discrete torsion and twisted orbifold cohomology. *J Symplectic Geom*, **2**(1): 1–24 (2003)
7 Lupercio E, Uribe B. Gerbes over orbifolds and twisted K-theory. *Comm Math Phys*, **245**(3): 449–489 (2004)
8 Tu J, Xu P, Laurent-Gengoux C. Twisted K-theory of differentiable stacks. *Ann Sci École Norm Sup*, **37**(6): 841–910 (2004)
9 Witten E. D-Branes And K-Theory. *J High Energy Phys*, **12**(12): 19–41 (1998)
10 Bouwknegt P, Mathai V. D-brane, B-field and twisted K-theory. *J High Energy Phys*, **3**: 7–11 (2000)
11 Freed D, Hopkins M, Teleman C. Twisted equivariant K-theory with complex coefficients. *J Topology*, **1**(1): 16–44 (2008)
12 Lupercio E, Uribe B. Differential characters on orbifolds and string connections I. Gromov-Witten theory of spin curves and orbifolds. In: Contemp Math, Vol 403. Providence: Amer Math Soc, 2006, 127–142
13 Chen W, Ruan Y. Orbifold Gromov-Witten theory, Orbifolds in mathematics and physics. Contemp Math, Vol 310. Providence: Amer Math Soc, 2002, 25–85
14 Ruan Y. String geometry and topology of orbifolds. Symposium in Honor of C H Clemens. In: Contemp Math, Vol 312. Providence: Amer Math Soc, 2002, 187–233
15 Chen W. A homotopy theory of orbispaces. arXiv:math.AT/0102020
16 Lupercio E, Uribe B. Loop groupoids, gerbes and twisted sectors on orbifolds. Orbifolds in mathematics and physics. In: Contemp Math, Vol 310. Providence, RI: Amer Math Soc, 2002, 163–184
17 Adem A, Pan J. Toroidal orbifolds, gerbes and group cohomology. *Trans Amer Math Soc*, **358**(9): 3969–3983 (2006)
18 Hitchin N. Lectures on special Lagrangian submanifolds. Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds. Stud Adv Math, Vol 23. Providence: Amer Math Soc, 2001, 151–182
19 Brylinski J L. Loop group, Characteristic classes and geometric quantization. In: Progress in Mathematics, Vol 107. Boston: Birkhäuser Boston Inc, 1993
20 Moerdijk I. Proof of a conjecture of A. Haefliger. *Topology*, **37**(4): 735–741 (1998)
21 Moerdijk I, Pronk D. Orbifolds, sheaves and groupoids. *K-theory*, **12**(1): 3–21 (1997)
22 Adem A, Leida J, Ruan Y. Orbifolds and stringy topology. In: Cambridge Tracts in Mathematics, Vol 171. Cambridge: Cambridge University Press, 2007
23 Jarvis T J, Kaufmann R, Kimura T. Pointed admissible G-covers and G-equivariant cohomological field theories. *Compos Math*, **141**(4): 926–978 (2005)