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Indecomposable representations of the Lie algebra of derivations for d-torus

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Abstract Let Der$A$ be the Lie algebra of derivations of the d-torus $A = \mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]$. By applying Shen-Larsson's functors we get a class of indecomposable Der$A$-modules from finite-dimensional indecomposable gl_d-modules. We also give a complete description of the submodules of these indecomposable Der$A$-modules. Our results generalize those obtained by Rao.

Keywords Lie algebra, indecomposable representation, torus

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0 Introduction

Let Der$A$ be the Lie algebra of derivations of the d-torus $A = \mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]$. Der$A$ is also the Lie algebra of diffeomorphisms of torus T^d by [9]. When $d = 1$, Der$A$ is the Witt algebra and its universal central extension is called the Virasoro algebra. The representation theory of Virasoro algebra has been studied extensively (see [1, 2, 5–7]). When $d \geqslant 2$, Der$A$ has no nontrivial central extension [9]. Let $A \rtimes \mathrm{Der}A$ be the Lie algebra by defining

$$[D, a] = D(a), \quad \forall\, a \in A,\ D \in \mathrm{Der}A.$$

The authors in [13] classified the irreducible integrable modules of the full toroidal Lie algebras and characterized explicitly such modules by means of the modules $F^\alpha(\psi, b)$ for the Lie algebra $A \rtimes \mathrm{Der}A$, except the extreme case that the multi-loop algebra acts as zero and the modules of the full toroidal Lie algebras thus degenerate to the modules of Der$A$.

Larsson [3] constructed a functor F^α from gl_d-modules to Der$A$-modules, which is a special case of the modules constructed by Shen [14]. So we call this functor Shen-Larsson's functor in this paper (see Definition 1.1). Rao [11] generalized F^α and studied the image of finite-dimensional irreducible gl_d-modules. These modules are weight modules for a certain maximal abelian subalgebra $\mathcal{H}$ of Der$A$ and the dimension of each $\mathcal{H}$-weight space are the dimensions of the gl_d-module. Rao proved that these modules are most often irreducible. Moreover, Rao [12] proved that these modules are irreducible as $A \rtimes \mathrm{Der}A$ modules and any irreducible $A \rtimes \mathrm{Der}A$ which have finite-dimensional weight spaces has to come from Shen-Larsson's construction.

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Recall that gl_d is the Lie algebra of $d \times d$ matrices over the field of complex numbers $\mathbb{C}$. Let $\mathrm{gl}_d = \mathrm{sl}_d \oplus \mathbb{C}I_d$ where sl_d is the finite-dimensional simple Lie algebra of trace zero matrices and I_d is the identity matrix which is central. A Lie algebra L is called reductive if $\mathrm{Rad}L = Z(L)$, where $\mathrm{Rad}\, L$ is the maximal sovable ideal of L and $Z(L)$ is the center of L. It is well known that a finite-dimensional module of a reductive Lie algebra L is completely reducible if and only if $Z(L)$ is represented by semisimple endomorphisms. Since gl_d is a reductive Lie algebra, a finite-dimensional gl_d-module is completely reducible if and only if I_d is represented by a semisimple endomorphism. The author in [11] (see also [4]) studied the modules arising from irreducible finite-dimensional gl_d-modules. But, finite-dimensional gl_d-modules are not necessarily completely reducible. It is easy to see that every finite-dimensional gl_d-module is a direct sum of indecomposable submodules. Therefore, it is interesting to study the image of indecomposable finite-dimensional gl_d-modules under the Shen-Larsson's functor.

Recall that a module is called uniserial if its submodules are linearly ordered by inclusion. It is clear that every uniserial module is indecomposable. In this work, we study the $\mathrm{Der}A$-modules arising from indecomposable finite-dimensional gl_d-modules. We prove that these $\mathrm{Der}A$-modules are most often uniserial and always indecomposable. The paper is arranged as follows. In Section 1, we give some notations and recall the facts on the Lie algebra of derivations of d-torus; we prove that any nonzero finite-dimensional indecomposable gl_d-module is uniserial and isomorphic to $\mathcal{V}^m(\psi, b)$ for some dominant integral weight ψ, complex number b and positive integer m (see Proposition 1.5). In Section 2, we prove that for $(\psi, b) \neq (\delta_k, k), 1 \leqslant k \leqslant d-1$, $F^\alpha(\mathcal{V}^m(\psi, b))$ is uniserial $\mathrm{Der}A$-module (see Theorems 2.4 and 2.5). Finally, in Section 3, we prove that $F^\alpha(\mathcal{V}^m(\delta_k, k))$ $(1 \leqslant k \leqslant d-1)$ is indecomposable and $F^\alpha(\mathcal{V}^m(\delta_k, k))$ is uniserial for $\alpha \notin \Gamma$. Our results generalize the results obtained by Rao [11]. Moreover, our construction of modules in the case $d = 1$ corresponds to the indecomposable weight Virasoro modules with weight spaces of dimension m.

1 Shen-Larsson's functor and indecomposable gl_d-modules

Throughout this work we fix a positive integer $d \geqslant 2$ and a Laurent polynomial ring $A = \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ in d commuting variables. Let $\mathcal{U}$ be a vector space over $\mathbb{C}$ with a basis $e_1, e_2, \ldots, e_d$. Let $(\cdot, \cdot)$ be a bilinear form on $\mathcal{U}$ such that $(e_i, e_j) = \delta_{ij}$. Let $\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_d$ be a lattice in $\mathcal{U}$.

Let $\mathcal{G}$ be a Lie algebra with Cartan subalgebra $\mathfrak{H}$. Let $\mathfrak{H}^*$ be the dual space of $\mathfrak{H}$. A $\mathcal{G}$-module W is called a weight module if $W = \bigoplus_{\lambda \in \mathfrak{H}^*} W_\lambda$, where $W_\lambda = \{v \in W | hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}$. Any nonzero vector in W_λ is called a weight vector of weight λ.

Let E_{ij} be the elementary $d \times d$ matrix with (i, j)-th entry 1 and 0 elsewhere. Then E_{ij} $(1 \leqslant i, j \leqslant d)$ forms a basis of gl_d. Let $\mathrm{gl}_d = \mathrm{sl}_d \oplus \mathbb{C}I_d$, where sl_d is the subspace of trace zero matrices and I_d is the identity matrix. Let E be the subspace of diagonal matrices of gl_d. Let $\mathfrak{h} = E \cap \mathrm{sl}_d$ with a standard basis $\check{\alpha_i} = E_{ii} - E_{i+1,i+1}, 1 \leqslant i \leqslant d-1$, be a Cartan subalgebra of sl_d. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$ and $\delta_1, \ldots, \delta_{d-1}$ be the *fundamental weights* in $\mathfrak{h}^*$ defined by $\delta_i(\check{\alpha_j}) = \delta_{ij}$. An element ψ in $\mathfrak{h}^*$ is called dominant integral weight if $\psi(\check{\alpha_i})$ are nonnegative integers for all i. It is well known that finite-dimensional irreducible sl_d-modules and dominant integral weights are in 1-1 correspondence.

Let $\mathrm{Der}A$ be the Lie algebra of derivations of A. For $\boldsymbol{n} = \sum n_i e_i \in \Gamma$ let $t^{\boldsymbol{n}} = t_1^{n_1} t_2^{n_2} \cdots t_d^{n_d}$. For $\boldsymbol{u} = \sum u_i e_i \in \mathcal{U}$ and $\boldsymbol{r} = \sum r_i e_i \in \Gamma$, let $D(\boldsymbol{u}, \boldsymbol{r}) = \sum u_i t^{\boldsymbol{r}} t_i \frac{\partial}{\partial t_i}$. Then $\mathrm{Der}A = \mathrm{span}_{\mathbb{C}}\{D(\boldsymbol{u}, \boldsymbol{r}) | \boldsymbol{u} \in \mathcal{U}, \boldsymbol{r} \in \Gamma\}$ with the following Lie structure:

$$[D(\boldsymbol{u}, \boldsymbol{r}), D(\boldsymbol{v}, \boldsymbol{s})] = D((\boldsymbol{u}, \boldsymbol{s})\boldsymbol{v} - (\boldsymbol{v}, \boldsymbol{r})\boldsymbol{u}, \boldsymbol{r} + \boldsymbol{s}), \ \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}, \ \boldsymbol{r}, \boldsymbol{s} \in \Gamma.$$

Let $\mathcal{H}$ be the subspace spanned by $D(e_i, \boldsymbol{0}) = t_i \frac{\partial}{\partial t_i}, 1 \leqslant i \leqslant d$, which is a Cartan subalgebra of $\mathrm{Der}A$.

Definition 1.1 (Shen-Larsson's functor [11, 14]). *For $\alpha = \sum \alpha_i e_i \in \mathcal{U}$, define a map*

$$\begin{aligned} F^\alpha: \ & \mathrm{gl}_d\text{-}\textit{modules} \longrightarrow \mathrm{Der}A\text{-}\textit{modules}, \\ & V \longmapsto F^\alpha(V) := V \otimes A = \textstyle\bigoplus_{\boldsymbol{n} \in \Gamma} V(\boldsymbol{n}), \end{aligned}$$

where $V(\boldsymbol{n}) = V \otimes t^{\boldsymbol{n}}$ *for each* $\boldsymbol{n} \in \Gamma$*, and the action of* $\mathrm{Der}A$ *on* $F^\alpha(V)$ *is defined as follows:*

$$D(\boldsymbol{u},\boldsymbol{r})v(\boldsymbol{n}) = (\boldsymbol{u},\boldsymbol{n}+\alpha)v(\boldsymbol{n}+\boldsymbol{r}) + \bigg(\sum_{i,j} u_i r_j E_{ji} v\bigg)(\boldsymbol{n}+\boldsymbol{r}), \tag{1.1}$$

where $v(\boldsymbol{n}) := v \otimes t^{\boldsymbol{n}} \in V(\boldsymbol{n})$*,* $\boldsymbol{n},\boldsymbol{r} \in \Gamma$ *and* $\boldsymbol{u} \in \mathcal{U}$*.*

Remark 1.2. According to the definition, we have the following results (cf. [11]):

(i) $F^\alpha(V)$ is a $\mathrm{Der}A$-module and F^α is a functor from gl_d-modules to $\mathrm{Der}A$-modules;

(ii) For any gl_d-module V, $F^\alpha(V)$ is an $\mathcal{H}$ weight module and $F^\alpha(V) = \bigoplus_{\boldsymbol{n}\in\Gamma} V(\boldsymbol{n})$ is the direct sum of weight spaces;

(iii) $F^\alpha(V_1 \oplus V_2) = F^\alpha(V_1) \oplus F^\alpha(V_2)$, where V_1, V_2 are gl_d-modules;

(iv) If $\alpha - \beta \in \Gamma$, then $F^\alpha(V) \cong F^\beta(V)$ for gl_d-module V.

For any dominant integral weight ψ, let $\mathcal{V}(\psi)$ be the finite-dimensional irreducible sl_d-module with respect to ψ. For any positive integer i, let $\mathcal{V}(\psi)_{(i)} = \{u_{(i)} | u \in \mathcal{V}(\psi)\}$ be a linear copy of sl_d-module $\mathcal{V}(\psi)$, that is,

$$x \cdot u_{(i)} = (x \cdot u)_{(i)}, \quad \forall\, x \in \mathrm{sl}_d, u \in \mathcal{V}(\psi). \tag{1.2}$$

Given any complex number b and positive integer m, let $\mathcal{V}^m(\psi,b) = \bigoplus_{i=1}^m \mathcal{V}(\psi)_{(i)}$ be a direct sum of sl_d-module $\mathcal{V}(\psi)_{(i)}$. For convenience, let $\mathcal{V}^0(\psi,b) = \{0\}$. The following results about the finite-dimensional indecomposable gl_d-modules should be known, but we have not found them in any reference.

Lemma 1.3. $\mathcal{V}^m(\psi,b)$ *is a* gl_d*-module with the action of* I_d *defined by*

$$I_d(v_{(1)}) = bv_{(1)}, \qquad I_d(v_{(i)}) = bv_{(i)} + v_{(i-1)}, \quad i = 2,\dots,m, \qquad \forall\, v \in \mathcal{V}(\psi). \tag{1.3}$$

Proof. Checking directly by the definition, the action of I_d commutes with that of sl_d. This completes the proof.

Remark 1.4. If $m = 1$, then $\mathcal{V}^1(\psi,b)$ is an irreducible gl_d-module and is denoted by $\mathcal{V}(\psi,b)$ in [11]. Using (1.2) and (1.3), one can see that for $1 \leqslant i \leqslant m$, $\mathcal{V}^i(\psi,b)$ is a submodule of the gl_d-module $\mathcal{V}^m(\psi,b)$ and the quotient module $\mathcal{V}^i(\psi,b)/\mathcal{V}^{i-1}(\psi,b)$ is isomorphic to $\mathcal{V}(\psi,b)$.

For simplicity, we write $\mathcal{V}^i(\psi,b) = \mathcal{V}^i$ and $\mathcal{V}(\psi)_{(i)} = \mathcal{V}_{(i)}$ for $i = 1,\dots,m$.

Proposition 1.5. (i) *For any dominant integral weight* ψ*, complex number* b *and positive integer* m*,* $\mathcal{V}^m(\psi,b)$ *is a uniserial* gl_d*-module with the chain of submodules given as follows:*

$$\{0\} \subset \mathcal{V}^1(\psi,b) \subset \mathcal{V}^2(\psi,b) \subset \cdots \subset \mathcal{V}^m(\psi,b).$$

(ii) *Let* M *be a nonzero finite-dimensional indecomposable* gl_d*-module. Then* M *is isomorphic to* $\mathcal{V}^m(\psi,b)$ *for some dominant integral weight* ψ*, complex number* b *and positive integer* m*.*

Proof. For (i), we use induction on m. The case $m = 1$ is true because $\mathcal{V}^1$ is an irreducible gl_d-module. Assume that $\mathcal{V}^{m-1}$ ($m \geqslant 2$) is a uniserial gl_d-module and its submodules are $\{0\} \subset \mathcal{V}^1 \subset \mathcal{V}^2 \subset \cdots \subset \mathcal{V}^{m-1}$. By Remark 1.4 and induction, it is sufficient to show that if U is a gl_d-submodule of $\mathcal{V}^m$ and $U \nsubseteq \mathcal{V}^{m-1}$ then $U = \mathcal{V}^m$. Let U be such a submodule, then the quotient module $(U+\mathcal{V}^{m-1})/\mathcal{V}^{m-1}$ is a nonzero submodule of $\mathcal{V}^m/\mathcal{V}^{m-1}$. Since $\mathcal{V}^m/\mathcal{V}^{m-1}$ is irreducible (Remark 1.4), we have $U+\mathcal{V}^{m-1} = \mathcal{V}^m$. We claim that $\mathcal{V}^{m-1} \cap U = \mathcal{V}^{m-1}$. Indeed, by induction, it is sufficient to find an element $u \in (U \cap \mathcal{V}^{m-1}) \backslash \mathcal{V}^{m-2}$. Since $U+\mathcal{V}^{m-1} = \mathcal{V}^m$, for a given nonzero vector $v \in \mathcal{V}(\psi)$, there exists $x \in \mathcal{V}^{m-1}$ such that $x+v_{(m)} \in U$. By simply computing we have

$$I_d \cdot (x + v_{(m)}) = b(x + v_{(m)}) + v_{(m-1)} + w$$

for some $w \in \mathcal{V}^{m-2}$. Thus we have $v_{(m-1)} + w \in (U \cap \mathcal{V}^{m-1}) \backslash \mathcal{V}^{m-2}$. So $\mathcal{V}^{m-1} \cap U = \mathcal{V}^{m-1}$. Since $U + \mathcal{V}^{m-1} = \mathcal{V}^m$, we have $U = \mathcal{V}^m$ as required.

For (ii), by Weyl's theorem, M can be written as a direct sum of irreducible sl_d-modules, that is $M = \bigoplus_{i=1}^k n_i M_i$ for some $k, n_i \geqslant 1$, where $n_i M_i := M_i \oplus \cdots \oplus M_i$ (n_i times) and $M_i \ncong M_j$ if $i \neq j$.

We claim that $k=1$. In fact, let ρ_j be the canonical projection from M to n_jM_j, we have $\rho_j I_d|_{n_iM_i} \in \mathrm{Hom}_{\mathrm{sl}_d}(n_iM_i, n_jM_j)$. For $j \neq i$, since $\mathrm{Hom}_{\mathrm{sl}_d}(n_iM_i, n_jM_j) \cong n_in_j\mathrm{Hom}_{\mathrm{sl}_d}(M_i, M_j) = \{0\}$, we have $\rho_j I_d|_{n_iM_i} = 0$. Thus each n_iM_i is a gl_d-module. The indecomposability of M yields $k=1$. So we may assume that $M = \bigoplus_{i=1}^m M_i$, where M_i is a linear copy of $\mathcal{V}(\psi)$ for some dominant integral weight ψ and positive integer m. We shall prove that M is isomorphic to $\mathcal{V}^m(\psi, b)$ for some $b \in \mathbb{C}$.

For $i = 1, \ldots, m$, let w_i be a highest weight vector of sl_d-module M_i. Let M_ψ be the highest weight space of M with respect to the Cartan subalgebra $\mathfrak{h}$, then we have $M_\psi = \mathrm{span}_{\mathbb{C}}\{w_i : i = 1, 2, \ldots, m\}$ and $I_d|_{M_\psi} \in \mathrm{End}M_\psi$. Choose a basis $v_1, v_2, \ldots, v_m$ of M_ψ so that the corresponding matrix of $I_d|_{M_\psi}$ is

$$J = \begin{pmatrix} J(b_1, n_1) & & \\ & \ddots & \\ & & J(b_k, n_k) \end{pmatrix},$$

where

$$J(b_i, n_i) = \begin{pmatrix} b_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & b_i & 1 \\ 0 & \cdots & 0 & b_i \end{pmatrix}$$

is an $n_i \times n_i$ Jordan block and $n_1 + \cdots + n_k = m$. We claim that $k = 1$. In fact, let $V_i = U(\mathrm{sl}_d)v_i$, where $U(\mathrm{sl}_d)$ is the universal enveloping algebra of sl_d. By the choice of v_i, one can see that each V_i is isomorphic to the irreducible sl_d-module $\mathcal{V}(\psi)$ and $V_i \cap \sum_{j\neq i} V_j = \{0\}$. So M has a decomposition $M = \bigoplus_{i=1}^k V_i$ as sl_d-module. Let $U_1 = V_1 \oplus \cdots \oplus V_{n_1}$ and $U_i = V_{n_1+\cdots+n_{i-1}+1} \oplus \cdots \oplus V_{n_1+\cdots+n_i}$ $(i = 2, \ldots, k)$. By the choice of v_i, we have $M = \bigoplus_{i=1}^k U_i$ which is a direct sum of gl_d-modules. The indecomposability of M yields $k = 1$. So we have $M = U_1 = \bigoplus_{i=1}^m V_i$ (a direct sum of m linear copies of $\mathcal{V}(\psi)$ as sl_d-module) and

$$\begin{aligned} &I_d \cdot (x \cdot v_1) = x \cdot (I_d \cdot v_1) = b_1(x \cdot v_1), \\ &I_d \cdot (x \cdot v_i) = x \cdot (I_d \cdot v_i) = b_1(x \cdot v_i) + x \cdot v_{i-1}, \quad i = 2, \ldots, m, \end{aligned}$$

for any $x \in U(\mathrm{sl}_d)$. By the definition of the gl_d-module $\mathcal{V}^m(\psi, b_1)$, one can see that $M \cong \mathcal{V}^m(\psi, b_1)$. This completes the proof.

By Remark 1.2, Proposition 1.5 and the fact that every finite-dimensional gl_d-module is a direct sum of a finite number of indecomposable submodules, it is sufficient to consider $F^\alpha(\mathcal{V}^m(\psi, b))$ for dominant integral weight ψ, complex number b and positive integer m, where $\alpha = \sum \alpha_i e_i \in \mathcal{U}$. We end this section with the following lemma, which can be checked easily.

Lemma 1.6. *For $1 \leqslant i \leqslant m$, $F^\alpha(\mathcal{V}^i(\psi, b))$ is a* Der$A$*-submodule of $F^\alpha(\mathcal{V}^m(\psi, b))$ and the quotient module $F^\alpha(\mathcal{V}^i(\psi, b))/F^\alpha(\mathcal{V}^{i-1}(\psi, b))$ is isomorphic to $F^\alpha(\mathcal{V}(\psi, b))$.*

2 Uniserial Der$\boldsymbol{A}$-module $F^\alpha(\mathcal{V}^m(\psi, b))$

Let V be any nonzero gl_d-module, then by Remark 1.2, $F^\alpha(V)$ is a Γ-graded weight module of Der$A$. Let W be any nonzero submodule of $F^\alpha(V)$, then W is also Γ-graded. Let $W(\boldsymbol{n})$ be the $\boldsymbol{n}$-th space of W, then $W(\boldsymbol{n}) = W \cap V(\boldsymbol{n})$ and $W = \bigoplus_{\boldsymbol{n}\in\Gamma} W(\boldsymbol{n})$.

In this section, we will describe the structure of the Der$A$-module $F^\alpha(\mathcal{V}^m(\psi, b))$ for $(\psi, b) \neq (\delta_k, k)$, $1 \leqslant k \leqslant d-1$. In [11], Rao described the structure of $F^\alpha(\mathcal{V})$, where $\mathcal{V} = \mathcal{V}(\psi, b)$. We firstly list some Rao's results.

Lemma 2.1 [11]. *Assume that $(\psi, b) \neq (\delta_k, k), (0, b)$, $1 \leqslant k \leqslant d-1$. Then $F^\alpha(\mathcal{V})$ is an irreducible* Der$A$*-module.*

Lemma 2.2 [10, 11]. *Assume that $\psi = 0$.*

(i) *If* $\alpha \notin \Gamma$ *or* $b \notin \{0, d\}$, *then* $F^\alpha(\mathcal{V})$ *is an irreducible* $\mathrm{Der}A$*-module;*
(ii) *If* $\alpha \in \Gamma$ *and* $b = 0$, *then* $\mathcal{V}(-\alpha)$ *is the only proper submodule;*
(iii) *If* $\alpha \in \Gamma$ *and* $b = d$, *then* $\sum_{\boldsymbol{n} \neq -\alpha} \mathcal{V}(\boldsymbol{n})$ *is the only proper submodule.*

Lemma 2.3 [11]**.** *Let* V *be a nonzero* gl_d*-module and* W *be a nonzero submodule of* $F^\alpha(V)$. *Let* $v(\boldsymbol{n}) \in W(\boldsymbol{n})$ *then* $W(\boldsymbol{n})$ *contains the following vectors:*
(V1) $(E_{ii} - E_{ii}^2)v(\boldsymbol{n})$,
(V2) $E_{ji}^2 v(\boldsymbol{n})$ $(i \neq j)$,
(V3) $E_{ij}E_{ii}v(\boldsymbol{n})$ $(i \neq j)$,
(V4) $((n_i + \alpha_i)E_{jk} - (n_k + \alpha_k)E_{ji})v(\boldsymbol{n})$, *where* $\boldsymbol{n} = \sum n_i e_i, \alpha = \sum \alpha_i e_i$.

Now we consider $F^\alpha(\mathcal{V}^m(\psi, b))$. Let v_ψ be a highest weight vector of the irreducible sl_d-module $\mathcal{V}(\psi)$ with respect to the Cartan subalgebra $\mathfrak{h}$, then $v_{\psi(i)}$ is the highest weight vector of sl_d-module $\mathcal{V}(\psi)_{(i)}$ for all $1 \leqslant i \leqslant m$.

The following two theorems describe the structure of $\mathrm{Der}A$-module $F^\alpha(\mathcal{V}^m(\psi, b))$ for $(\psi, b) \neq (\delta_k, k)$, $1 \leqslant k \leqslant d - 1$.

Theorem 2.4. *Suppose that* $(\psi, b) \neq (\delta_k, k), (0, b)$, $1 \leqslant k \leqslant d - 1$. *Then for any positive integer* m, $F^\alpha(\mathcal{V}^m)$ *is a uniserial* $\mathrm{Der}A$*-module with the chain of submodules given as follows:*

$$\{0\} \subset F^\alpha(\mathcal{V}^1) \subset F^\alpha(\mathcal{V}^2) \subset \cdots \subset F^\alpha(\mathcal{V}^m).$$

Proof. We use induction on m. The case $m = 1$ is true by Lemma 2.1. Assume that $m \geqslant 2$ and the theorem holds for $m - 1$. By induction, it is sufficient to show that if W is a submodule of $F^\alpha(\mathcal{V}^m)$ and $W \nsubseteq F^\alpha(\mathcal{V}^{m-1})$ then $W = F^\alpha(\mathcal{V}^m)$. Let W be any such module. By Lemmas 1.6 and 2.1, the quotient module $F^\alpha(\mathcal{V}^m)/F^\alpha(\mathcal{V}^{m-1})$ is irreducible. So we have $W + F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^m)$. We need to show $W \cap F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^{m-1})$. Since

$$W = \bigoplus_{\boldsymbol{n} \in \Gamma} W(\boldsymbol{n}) \quad \text{and} \quad F^\alpha(\mathcal{V}^m) = F^\alpha(\mathcal{V}^{m-1}) + \sum_{\boldsymbol{n} \in \Gamma} \mathcal{V}_{(m)}(\boldsymbol{n}),$$

for any $\boldsymbol{n} \in \Gamma$, there exists $x \in \mathcal{V}^{m-1}$ such that $(x + v_{\psi(m)})(\boldsymbol{n}) \in W(\boldsymbol{n})$. Let $\tilde{\mathcal{V}}^{m-1} = \mathcal{V}^{m-2} + \sum_{\lambda \prec \psi} \mathcal{V}_{\lambda(m-1)}$, where $\mathcal{V}_{\lambda(m-1)}$ is the weight space of the sl_d-module $\mathcal{V}_{(m-1)}$ with weight λ. Let $h_i = E_{ii} - \frac{1}{d}I_d$ $(i = 1, 2, \ldots, d)$. Using (1.2) and (1.3), we have

$$\begin{aligned}
&(E_{ii} - E_{ii}^2) \cdot (x + v_{\psi(m)}) \\
&\quad = \left(\frac{b}{d} + \psi(h_i)\right)\left(1 - \frac{b}{d} - \psi(h_i)\right)(x + v_{\psi(m)}) + \frac{1}{d}\left(1 - 2\left(\frac{b}{d} + \psi(h_i)\right)\right)v_{\psi(m-1)} + u,
\end{aligned}$$

for some $u \in \tilde{\mathcal{V}}^{m-1}$. Since $\psi \neq 0$, there exist i, j such that $\psi(h_i - h_j) = \psi(E_{ii} - E_{jj}) \neq 0$. So we have $1 - 2(\frac{b}{d} + \psi(h_i)) \neq 0$ for some i. Using Lemma 2.3, we have

$$\left(\frac{1}{d}\left(1 - \frac{2b}{d} - 2\psi(h_i)\right)v_{\psi(m-1)} + u\right)(\boldsymbol{n}) \in (W \cap F^\alpha(\mathcal{V}^{m-1})) \backslash F^\alpha(\mathcal{V}^{m-2}).$$

By induction, we get $W \cap F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^{m-1})$. Hence $W = W + F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^m)$ as required. This completes the proof.

Theorem 2.5. *Suppose that* $\psi = 0$. *For any positive integer* m, $F^\alpha(\mathcal{V}^m)$ *is a uniserial* $\mathrm{Der}A$*-module. Moreover,*

(i) *if* $\alpha \notin \Gamma$ *or* $b \notin \{0, d\}$, *then the chain of its submodules is*

$$\{0\} \subset F^\alpha(\mathcal{V}^1) \subset F^\alpha(\mathcal{V}^2) \subset \cdots \subset F^\alpha(\mathcal{V}^m);$$

(ii) *if* $\alpha \in \Gamma$ *and* $b = 0$, *then the chain of its submodules is*

$$\begin{aligned}
\{0\} &\subset \mathcal{V}_{(1)}(-\alpha) \subset F^\alpha(\mathcal{V}^1) \subset F^\alpha(\mathcal{V}^1) + \mathcal{V}_{(2)}(-\alpha) \subset F^\alpha(\mathcal{V}^2) \\
&\subset \cdots \subset F^\alpha(\mathcal{V}^{m-1}) \subset F^\alpha(\mathcal{V}^{m-1}) + \mathcal{V}_{(m)}(-\alpha) \subset F^\alpha(\mathcal{V}^m);
\end{aligned}$$

(iii) *if* $\alpha \in \Gamma$ *and* $b = d$, *then the chain of its submodules is*

$$\{0\} \subset \sum_{\boldsymbol{n}\neq -\alpha} \mathcal{V}_{(1)}(\boldsymbol{n}) \subset F^\alpha(\mathcal{V}^1) \subset F^\alpha(\mathcal{V}^1) + \sum_{\boldsymbol{n}\neq -\alpha} \mathcal{V}_{(2)}(\boldsymbol{n}) \subset F^\alpha(\mathcal{V}^2)$$
$$\subset \cdots \subset F^\alpha(\mathcal{V}^{m-1}) \subset F^\alpha(\mathcal{V}^{m-1}) + \sum_{\boldsymbol{n}\neq -\alpha} \mathcal{V}_{(m)}(\boldsymbol{n}) \subset F^\alpha(\mathcal{V}^m).$$

Proof. Since $\psi = 0$, we have $\mathcal{V} = \mathbb{C}v_0$ and $\mathcal{V}_{(j)} = \mathbb{C}v_{0(j)}$ $(j = 1, 2, \ldots, m)$. We use induction on m. The case m=1 is true by Lemma 2.2. Assume that $m \geqslant 2$ and the theorem holds for $m-1$. Let W be a submodule of $F^\alpha(\mathcal{V}^m)$. By induction, we may assume that $W \nsubseteq F^\alpha(\mathcal{V}^{m-1})$.

For (i), we only need to show that $W = F^\alpha(\mathcal{V}^m)$. By Lemmas 1.6 and 2.2, the quotient module $F^\alpha(\mathcal{V}^m)/F^\alpha(\mathcal{V}^{m-1})$ is irreducible. So we have $W + F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^m)$. Thus, choose $\boldsymbol{n} = \sum n_i e_i \in \Gamma$ so that $n_1 + \alpha_1 \neq 0$, there exists $x \in \mathcal{V}^{m-1}$ such that $(x + v_{0(m)})(\boldsymbol{n}) \in W(\boldsymbol{n})$. Using (1.1)–(1.3), we have

$$w := D\left(e_2 - \frac{d(n_2+\alpha_2)+b}{d(n_1+\alpha_1)} e_1, e_2\right)(x + v_{0(m)})(\boldsymbol{n}) = \frac{1}{d} v_{0(m-1)}(\boldsymbol{n} + e_2) + u,$$

for some $u \in F^\alpha(\mathcal{V}^{m-2})$. So we have $w \in (W \cap F^\alpha(\mathcal{V}^{m-1}))\backslash F^\alpha(\mathcal{V}^{m-2})$. By induction, we get $W \cap F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^{m-1})$. Hence we have $W = F^\alpha(\mathcal{V}^m)$ as required.

For (ii), it is clear that $F^\alpha(\mathcal{V}^{m-1}) + \mathcal{V}_{(m)}(-\alpha)$ is a Der$A$-submodule. By Lemmas 1.6 and 2.2, we have that if U is a proper submodule of $F^\alpha(\mathcal{V}^m)$ and $F^\alpha(\mathcal{V}^{m-1}) \subset U$ then $U = F^\alpha(\mathcal{V}^{m-1}) + \mathcal{V}_{(m)}(-\alpha)$. So we only need to show that $F^\alpha(\mathcal{V}^{m-1}) \subset W$. Since $W \nsubseteq F^\alpha(\mathcal{V}^{m-1})$, we have

$$W + F^\alpha(\mathcal{V}^{m-1}) \supseteq F^\alpha(\mathcal{V}^{m-1}) + \mathcal{V}_{(m)}(-\alpha).$$

Hence, there exists $x \in \mathcal{V}^{m-1}$ such that $(x + v_{0(m)})(-\alpha) \in W(-\alpha) \subseteq W$. By a simple computation, we have

$$w := D(e_1, e_1)(x + v_{0(m)})(-\alpha) = \frac{1}{d} v_{0(m-1)}(-\alpha + e_1) + u,$$

for some $u \in F^\alpha(\mathcal{V}^{m-2})$. So we have $w \in (W \cap F^\alpha(\mathcal{V}^{m-1}))\backslash(F^\alpha(\mathcal{V}^{m-2}) + \mathcal{V}_{(m-1)}(-\alpha))$. By induction, we get $W \cap F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^{m-1})$, which yields $F^\alpha(\mathcal{V}^{m-1}) \subset W$ as required.

For (iii), it is clear that $F^\alpha(\mathcal{V}^{m-1}) + \sum_{\boldsymbol{n}\neq-\alpha} \mathcal{V}_{(m)}(\boldsymbol{n})$ is a Der$A$-submodule. By Lemmas 1.6 and 2.2, we have that if U is a proper submodule of $F^\alpha(\mathcal{V}^m)$ and $F^\alpha(\mathcal{V}^{m-1}) \subset U$ then $U = F^\alpha(\mathcal{V}^{m-1}) + \sum_{\boldsymbol{n}\neq-\alpha} \mathcal{V}_{(m)}(\boldsymbol{n})$. So we only need to show that $F^\alpha(\mathcal{V}^{m-1}) \subset W$. Since $W \nsubseteq F^\alpha(\mathcal{V}^{m-1})$, we have

$$W + F^\alpha(\mathcal{V}^{m-1}) \supseteq F^\alpha(\mathcal{V}^{m-1}) + \sum_{\boldsymbol{n}\neq -\alpha} \mathcal{V}_{(m)}(\boldsymbol{n}).$$

Hence, there exists $x \in \mathcal{V}^{m-1}$ such that $(x + v_{0(m)})(-\alpha + e_1) \in W(-\alpha + e_1) \subseteq W$. By a simple computation, we have

$$w := D(e_1, -e_1)(x + v_{0(m)})(-\alpha + e_1) = -\frac{1}{d} v_{0(m-1)}(-\alpha) + u,$$

for some $u \in F^\alpha(\mathcal{V}^{m-2})$. So we have $w \in (W \cap F^\alpha(\mathcal{V}^{m-1}))\backslash(F^\alpha(\mathcal{V}^{m-2}) + \sum_{\boldsymbol{n}\neq-\alpha} \mathcal{V}_{(m-1)}(\boldsymbol{n}))$. By induction, we get $W \cap F^\alpha(\mathcal{V}^{m-1}) = F^\alpha(\mathcal{V}^{m-1})$, which yields $F^\alpha(\mathcal{V}^{m-1}) \subset W$ as required. These complete the proof of the theorem.

By Theorems 2.4 and 2.5, we have the following corollary:

Corollary 2.6. *Suppose that* $(\psi, b) \neq (\delta_k, k)$, $1 \leqslant k \leqslant d-1$. *Then for any positive integer* m, $F^\alpha(\mathcal{V}^m(\psi, b))$ *is an indecomposable* Der$A$*-module.*

3 Structure of $\boldsymbol{F}^\alpha(\mathcal{V}^m(\delta_k, k))$

In this section, we deal with the case for $(\psi, b) = (\delta_k, k)(1 \leqslant k \leqslant d-1)$. We first recall some facts about $\mathcal{V}(\delta_k, k)$ and $F^\alpha(\mathcal{V}(\delta_k, k))$ (see [8, 11]). The vector space $\mathcal{U}$ can be regarded as a gl$_d$-module by defining

$E_{ij}e_l = \delta_{jl}e_i$. Then one can see that $\mathcal{U} \cong \mathcal{V}(\delta_1, 1)$ as a gl_d-module. Consider the exterior product: $E^k(\mathcal{U}) = \mathcal{U} \wedge \cdots \wedge \mathcal{U}$ (k times), which is a gl_d-module with the following action:

$$X(v_1 \wedge \cdots \wedge v_k) = \sum_{i=1}^{k} v_1 \wedge \cdots \wedge v_{i-1} \wedge Xv_i \wedge \cdots \wedge v_k, \quad \forall X \in \mathrm{gl}_d. \tag{3.1}$$

It is a standard fact that $E^k(\mathcal{U}) \cong \mathcal{V}(\delta_k, k)$ as a gl_d-module for $1 \leqslant k \leqslant d-1$. In what follows, we identify the gl_d-module $\mathcal{V}(\delta_k, k)$ with $E^k(\mathcal{U})$. Then one can check from (1.1) and (3.1) that the Der$A$-module $F^\alpha(\mathcal{V}(\delta_k, k))$ is given by

$$\begin{aligned} llD(\boldsymbol{u}, \boldsymbol{r})v_1 \wedge \cdots \wedge v_k(\boldsymbol{n}) &= (\boldsymbol{u}, \boldsymbol{n}+\alpha)v_1 \wedge \cdots \wedge v_k(\boldsymbol{n}+\boldsymbol{r}) \\ &\quad + \sum_{i=1}^{k}(\boldsymbol{u}, v_i)v_1 \wedge \cdots \wedge v_{i-1} \wedge \boldsymbol{r} \wedge \cdots \wedge v_k(\boldsymbol{n}+\boldsymbol{r}). \end{aligned} \tag{3.2}$$

For $\boldsymbol{n} = \sum n_i e_i \in \Gamma$, let $W^k(\boldsymbol{n}) = \mathrm{span}_{\mathbb{C}}\{(\boldsymbol{n}+\alpha) \wedge v_1 \wedge \cdots \wedge v_{k-1}(\boldsymbol{n}) | v_i \in \mathcal{U}\}$ and $W(\alpha, k) = \bigoplus_{\boldsymbol{n}\in\Gamma} W^k(\boldsymbol{n})$ which is a submodule of $F^\alpha(\mathcal{V}(\delta_k, k))$ (see [11]).

Suppose that $\alpha = \sum \alpha_i e_i \in \Gamma$, then for $v_1 \wedge \cdots \wedge v_k \in \mathcal{V}(\delta_k, k)$,

$$D(\boldsymbol{u}, \boldsymbol{r})v_1 \wedge \cdots \wedge v_k(-\alpha) = \sum_{i=1}^{k}(\boldsymbol{u}, v_i)v_1 \wedge \cdots \wedge v_{i-1} \wedge \boldsymbol{r} \wedge \cdots \wedge v_k(-\alpha+\boldsymbol{r}),$$

which belongs to $W^k(-\alpha+\boldsymbol{r})$. Therefore, $\tilde{W}(\alpha, k) := W(\alpha, k) + \mathcal{V}(\delta_k, k)(-\alpha)$ is a submodule of $F^\alpha(\mathcal{V}(\delta_k, k))$, and $\mathcal{V}(\delta_k, k)(-\alpha)$ is a trivial d-dimensional submodule of $F^\alpha(\mathcal{V}(\delta_k, k))/W(\alpha, k)$ (see [11]).

Lemma 3.1 [11]**.** *Suppose* $(\psi, b) = (\delta_k, k), 1 \leqslant k \leqslant d-1$.

(i) $W(\alpha, k)$ *is an irreducible submodule of* $F^\alpha(\mathcal{V}(\delta_k, k))$;

(ii) *If* $\alpha \notin \Gamma$ *then* $F^\alpha(\mathcal{V}(\delta_k, k))/W(\alpha, k)$ *is irreducible*;

(iii) *If* $\alpha \in \Gamma$ *then* $F^\alpha(\mathcal{V}(\delta_k, k))/\tilde{W}(\alpha, k)$ *is irreducible.*

Proposition 3.2. *Suppose* $(\psi, b) = (\delta_k, k), 1 \leqslant k \leqslant d-1$.

(i) *If* $\alpha \notin \Gamma$ *then* $F^\alpha(\mathcal{V}(\delta_k, k))$ *is a uniserial* Der$A$*-module with the chain of submodules given as*

$$\{0\} \subset W(\alpha, k) \subset F^\alpha(\mathcal{V}(\delta_k, k));$$

(ii) *If* $\alpha \in \Gamma$ *then* U *is a nonzero proper submodule of* $F^\alpha(\mathcal{V}(\delta_k, k))$ *if and only if* $U = W(\alpha, k) + S$, *where* S *is a subspace of* $\mathcal{V}(\delta_k, k)(-\alpha)$;

(iii) $F^\alpha(\mathcal{V}(\delta_k, k))$ *is an indecomposable* Der$A$*-module.*

Proof. For (i), by Lemma 3.1, it is sufficient to show that if U is a submodule of $F^\alpha(\mathcal{V}(\delta_k, k))$ and $U \not\subseteq W(\alpha, k)$ then $U = F^\alpha(\mathcal{V}(\delta_k, k))$. Now since $F^\alpha(\mathcal{V}(\delta_k, k))/W(\alpha, k)$ is irreducible, we have $U + W(\alpha, k) = F^\alpha(\mathcal{V}(\delta_k, k))$. Thus, choose $\boldsymbol{n} = \sum n_i e_i \in \Gamma$ so that $n_d + \alpha_d \neq 0$, then there exists $x = \sum a_{i_1 i_2 \cdots i_{k-1}}(\boldsymbol{n}+\alpha) \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}$ such that $(x + e_1 \wedge e_2 \wedge \cdots \wedge e_k)(\boldsymbol{n}) \in U$, where $a_{i_1 i_2 \cdots i_{k-1}} \in \mathbb{C}$. Using (3.1), we have

$$\begin{aligned} u &:= \sum_{j=1}^{d}(n_j + \alpha_j)((n_d + \alpha_d)E_{j1} - (n_1 + \alpha_1)E_{jd})(x + e_1 \wedge e_2 \wedge \cdots \wedge e_k)(\boldsymbol{n}) \\ &= (n_d + \alpha_d)(\boldsymbol{n}+\alpha) \wedge e_2 \wedge \cdots \wedge e_k(\boldsymbol{n}). \end{aligned}$$

By the choice of $\boldsymbol{n}$ and Lemma 2.3, we get $0 \neq u \in U \cap W(\alpha, k)$. Since $W(\alpha, k)$ is irreducible, $U \cap W(\alpha, k) = W(\alpha, k)$. Thus we get $U = F^\alpha(\mathcal{V}(\delta_k, k))$ as required.

For (ii), it is clear that $W(\alpha, k) + S$ is a nonzero proper submodule of $F^\alpha(\mathcal{V}(\delta_k, k))$ for any subspace S of $\mathcal{V}(\delta_k, k)(-\alpha)$. On the other hand, let U be a nonzero submodule of $F^\alpha(\mathcal{V})$, we have the following two claims.

Claim 3.3. If $U \not\subseteq \tilde{W}(\alpha, k)$ then $U = F^\alpha(\mathcal{V}(\delta_k, k))$.

In fact, since $U \nsubseteq \tilde{W}(\alpha,k)$ and the quotient module $F^\alpha(\mathcal{V}(\delta_k,k))/\tilde{W}(\alpha,k)$ is irreducible, we get $U+\tilde{W}(\alpha,k) = F^\alpha(\mathcal{V}(\delta_k,k))$. Thus, choose $\boldsymbol{n} = \sum n_i e_i \in \Gamma$ so that $n_d+\alpha_d \neq 0$, there exists $x = \sum a_{i_1 i_2\cdots i_{k-1}}(\boldsymbol{n}+\alpha)\wedge e_{i_1}\wedge\cdots\wedge e_{i_{k-1}}$ such that $(x+e_1\wedge e_2\wedge\cdots\wedge e_k)(\boldsymbol{n})\in U$, where $a_{i_1 i_2\cdots i_{k-1}}\in\mathbb{C}$. As we have done in (i), we get $W(\alpha,k)\subseteq U$. Since $U+\tilde{W}(\alpha,k) = F^\alpha(\mathcal{V}(\delta_k,k))$ and $\tilde{W}(\alpha,k) = W(\alpha,k)+\mathcal{V}(\alpha,k)(-\alpha)$, we have $F^\alpha(\mathcal{V}(\delta_k,k))(\boldsymbol{n})\subseteq U$ for all $\boldsymbol{n}\in\Gamma\backslash\{-\alpha\}$. So, for $1\leqslant i_1<\cdots<i_k\leqslant d$, there exists $j\in\{1,\dots,d\}\backslash\{i_1,\dots,i_k\}$ such that $e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}(-\alpha+e_j)\in U$. Since

$$D(e_j,-e_j)e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}(-\alpha+e_j) = e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}(-\alpha),$$

we get $e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}(-\alpha)\in U$, for $1\leqslant i_1<\cdots<i_k\leqslant d$. So $F^\alpha(\mathcal{V}(\delta_k,k))(-\alpha)\subseteq U$. Therefore, we have $U = F^\alpha(\mathcal{V}(\delta_k,k))$ as required.

Claim 3.4. If $U\subseteq\tilde{W}(\alpha,k)$ then $W(\alpha,k)\subseteq U$.

Suppose that $W(\alpha,k)\nsubseteq U$. Since $U\subseteq\tilde{W}(\alpha,k)$, there exists

$$x = \sum_{1\leqslant i_1<\cdots<i_k\leqslant d} a_{i_1\cdots i_k} e_{i_1}\wedge\cdots\wedge e_{i_k}.$$

so that $0\neq x(-\alpha)\in U$, where $a_{i_1\cdots i_k}\in\mathbb{C}$. Suppose that $a_{i'_1\cdots i'_k}\neq 0$, by simply computing, we have

$$\begin{aligned}D(e_{i'_1},e_{i'_1})x(-\alpha) &= E_{i'_1 i'_1}\sum_{1\leqslant i_1<\cdots<i_k\leqslant d} a_{i_1\cdots i_k}e_{i_1}\wedge\cdots\wedge e_{i_k}(-\alpha+e_{i'_1})\\ &= a_{i'_1\cdots i'_k}e_{i'_1}\wedge\cdots\wedge e_{i'_k}(-\alpha+e_{i'_1})\\ &\quad+\sum_{(i'_1,i_2,\dots,i_k)\neq(i'_1,i'_2,\dots,i'_k)} a_{i'_1 i_2\cdots i_k}e_{i'_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}(-\alpha+e_{i'_1}).\end{aligned}$$

So $0\neq D(e_{i'_1},e_{i'_1})x(-\alpha+e_{i'_1})\in U\cap W(\alpha,k)$. Since $W(\alpha,k)$ is irreducible, we get $U\cap W(\alpha,k) = W(\alpha,k)$, which implies $W(\alpha,k)\subseteq U$, a contradiction. Therefore $W(\alpha,k)\subseteq U$ as required.

Together with Claims 3.3 and 3.4, (ii) holds. (iii) follows from (i) and (ii). These complete the proof of the proposition.

Finally we consider $F^\alpha(\mathcal{V}^m(\delta_k,k))$ for any fixed positive integer m. Recall that $\mathcal{V}^m(\delta_k,k) := \bigoplus_{i=1}^m \mathcal{V}(\delta_k)_{(i)}$, where $\mathcal{V}(\delta_k)_{(i)} = \{v_{(i)}|v\in\mathcal{V}(\delta_k)\}$ is a linear copy of irreducible sl_d-module $\mathcal{V}(\delta_k)$. For $\boldsymbol{n}\in\Gamma$ and $i=1,\dots,m$, let

$$W^k_{(i)}(\boldsymbol{n}) = \mathrm{span}_{\mathbb{C}}\{(\boldsymbol{n}+\alpha)\wedge v_1\wedge\cdots\wedge v_{k-1(i)}(\boldsymbol{n})|v_j\in\mathcal{U}\}.$$

Let $W_{(i)}(\alpha,k) = \bigoplus_{\boldsymbol{n}\in\Gamma}W^k_{(i)}(\boldsymbol{n})$, then one can see that $W_{(i)}(\alpha,k)+F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))$ is a submodule of $F^\alpha(\mathcal{V}^i(\delta_k,k))$ for $i=1,\dots,m$.

Suppose $\alpha = \sum\alpha_i e_i\in\Gamma$. For $i=1,\dots,m$, let $\tilde{W}_{(i)}(\alpha,k) := W_{(i)}(\alpha,k)+\mathcal{V}(\delta_k)_{(i)}(-\alpha)$. By definition, one can see that $\tilde{W}_{(i)}(\alpha,k)+F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))$ is a submodule of $F^\alpha(\mathcal{V}^i(\delta_k,k))$, and $\mathcal{V}(\delta_k)_{(i)}(-\alpha)$ is a trivial d-dimensional submodule of $F^\alpha(\mathcal{V}^i(\delta_k,k))/(W_{(i)}(\alpha,k)+F^\alpha(\mathcal{V}^{i-1}(\delta_k,k)))$ for $i=1,\dots,m$.

Using Lemma 3.1 and Proposition 3.2 we have the following lemma. The proof of Lemma 3.5 is trivial, which is omitted.

Lemma 3.5. *For $1\leqslant i\leqslant m$, let U be a* Der$A$*-submodule so that*

$$F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))\subset U\subset F^\alpha(\mathcal{V}^i(\delta_k,k)).$$

(i) *$F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))+W_{(i)}(\alpha,k)$ is a submodule of $F^\alpha(\mathcal{V}^i(\delta_k,k))$ and the quotient module $(F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))+W_{(i)}(\alpha,k))/(F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))$ is isomorphic to $W(\alpha,k)$;*

(ii) *If $\alpha\notin\Gamma$, then $U = F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))+W_{(i)}(\alpha,k)$;*

(iii) *If $\alpha\in\Gamma$, then $U = F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))+W_{(i)}(\alpha,k)+S$, where S is a subspace of $\mathcal{V}(\delta_k)_{(i)}(-\alpha)$. Conversely, let S be a subspace of $\mathcal{V}(\delta_k)_{(i)}(-\alpha)$, then $F^\alpha(\mathcal{V}^{i-1}(\delta_k,k))+W_{(i)}(\alpha,k)+S$ is a* Der$A$*-submodule.*

Theorem 3.6. *Suppose $(\psi,b) = (\delta_k,k), 1\leqslant k\leqslant d-1$. For any positive integer m,*

(i) *if* $\alpha \notin \Gamma$ *then* $F^\alpha(\mathcal{V}^m(\delta_k, k))$ *is a uniserial* Der$A$*-module with the chain of submodules given as follow*:

$$\{0\} \subset W_{(1)}(\alpha, k) \subset F^\alpha(\mathcal{V}^1(\delta_k, k)) \subset F^\alpha(\mathcal{V}^1(\delta_k, k)) + W_{(2)}(\alpha, k) \subset F^\alpha(\mathcal{V}^2(\delta_k, k))$$
$$\subset \cdots \subset F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) \subset F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) + W_{(m)}(\alpha, k) \subset F^\alpha(\mathcal{V}^m(\delta_k, k));$$

(ii) *if* $\alpha \in \Gamma$ *and* $U \nsubseteq F^\alpha(\mathcal{V}^{m-1}(\delta_k, k))$, *then* U *is a proper submodule of* $F^\alpha(\mathcal{V}^m(\delta_k, k))$ *if and only if*

$$U = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) + W_{(m)}(\alpha, k) + S,$$

where S *is a subspace of* $\mathcal{V}(\delta_k)_{(m)}(-\alpha)$;

(iii) $F^\alpha(\mathcal{V}^m(\delta_k, k))$ *is an indecomposable* Der$A$*-module.*

Proof. We use induction on m for (i) and (ii). The case $m = 1$ is true by Proposition 3.2. Assume that $m \geqslant 2$ and (i), (ii) hold for $m - 1$.

For (i), let U be a submodule of $F^\alpha(\mathcal{V}^m(\delta_k, k))$. By induction we may assume that $U \nsubseteq F^\alpha(\mathcal{V}^{m-1}(\delta_k, k))$. By Lemma 3.5, we have

$$U + F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) + W_{(m)}(\alpha, k)$$
$$\text{or } \ U + F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) = F^\alpha(\mathcal{V}^m(\delta_k, k)).$$

So it is sufficient to show that

$$U \cap F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)).$$

Choose $\boldsymbol{n} = \sum n_i e_i \in \Gamma$ so that $n_1 + \alpha_1, n_d + \alpha_d \neq 0$, there exists $x \in \mathcal{V}^{m-1}(\delta_k, k)$ such that $u(\boldsymbol{n}) = (x + (\boldsymbol{n} + \alpha) \wedge e_2 \wedge \cdots \wedge e_{k(m)})(\boldsymbol{n}) \in U$. By simply computing, we have

$$(E_{11} - E_{11}^2)u(\boldsymbol{n}) = -\frac{2(n_1 + \alpha_1)}{d} e_1 \wedge e_2 \wedge \cdots \wedge e_{k(m-1)}(\boldsymbol{n}) + s,$$

for some $s \in F^\alpha(\mathcal{V}^{m-2}) + W_{(m-1)}(\alpha, k)$. By the choice of $\boldsymbol{n}$, we have $(E_{11} - E_{11}^2)u(\boldsymbol{n}) \in F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) \setminus (F^\alpha(\mathcal{V}^{m-2}(\delta_k, k)) + W_{(m-1)}(\alpha, k))$. By Lemma 2.3 and induction, we have $U \cap F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k))$ as required. Therefore (i) holds.

For (ii), it is clear that if $U = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) + W_{(m)}(\alpha, k) + S$, where S is a subspace of $\mathcal{V}(\delta_k)_{(m)}(-\alpha)$, then U is a proper submodule of $F^\alpha(\mathcal{V}^m(\delta_k, k))$. Now suppose that $U \nsubseteq F^\alpha(\mathcal{V}^{m-1}(\delta_k, k))$ is a proper submodule of $F^\alpha(\mathcal{V}^m(\delta_k, k))$. By Lemma 3.5, it is sufficient to show that

$$F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) \cap U = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)).$$

Since $U \nsubseteq F^\alpha(\mathcal{V}^{m-1}(\delta_k, k))$, using Lemma 3.5 we have

$$F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) + W_{(m)}(\alpha, k) \subseteq U + F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)).$$

Thus, choose $\boldsymbol{n} = \sum n_i e_i \in \Gamma$ so that $n_1 + \alpha_1, n_d + \alpha_d \neq 0$, there exists $x \in \mathcal{V}^{m-1}$ such that $u(\boldsymbol{n}) = (x + (\boldsymbol{n} + \alpha) \wedge e_2 \wedge \cdots \wedge e_{k(m)})(\boldsymbol{n}) \in U$. By simply computing, we have

$$(E_{11} - E_{11}^2)u(\boldsymbol{n}) = -\frac{2(n_1 + \alpha_1)}{d} e_1 \wedge e_2 \wedge \cdots \wedge e_{k(m-1)}(\boldsymbol{n}) + s$$

for some $s \in F^\alpha(\mathcal{V}^{m-2}(\delta_k, k)) + \tilde{W}_{(m-1)}(\alpha, k)$. By the choice of $\boldsymbol{n}$, we get $(E_{11} - E_{11}^2)u(\boldsymbol{n}) \in F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) \setminus (F^\alpha(\mathcal{V}^{m-2}(\delta_k, k)) + \tilde{W}_{(m-1)}(\alpha, k))$. By induction and Lemma 2.3, we have $U \cap F^\alpha(\mathcal{V}^{m-1}(\delta_k, k)) = F^\alpha(\mathcal{V}^{m-1}(\delta_k, k))$ as required. Therefore (ii) holds.

(iii) follows from (i) and (ii). Thus we have completed the proof.

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