Differential Galoisian approach to Jacobi integrability of general analytic dynamical systems and its application

Kaiyin Huang[1], Shaoyun Shi[2,3,*] & Shuangling Yang[2]

[1]*School of Mathematics, Sichuan University, Chengdu 610065, China;*
[2]*School of Mathematics, Jilin University, Changchun 130012, China;*
[3]*State Key Laboratory of Automotive Simulation and Control, Jilin University, Changchun 130012, China*

Email: huangky@scu.edu.cn, shisy@jlu.edu.cn, yangsl19@mails.jlu.edu.cn

Abstract The Morales-Ramis theory provides an effective and powerful non-integrability criterion for complex analytic Hamiltonian systems via the differential Galoisian obstruction. In this paper, we give a new Morales-Ramis type theorem on the meromorphic Jacobi non-integrability of general analytic dynamical systems. The key point is to show that the existence of Jacobian multipliers of a nonlinear system implies the existence of common Jacobian multipliers of Lie algebra associated with the identity component. In addition, we apply our results to the polynomial integrability of Karabut systems for stationary gravity waves in finite depth.

Keywords Morales-Ramis theory, Jacobi integrability, differential Galois groups, Karabut systems

MSC(2020) 32H04, 34C14, 34M03, 34M15, 34M45

1 Introduction

The fundamental problem in the field of dynamical systems is to detect whether a given system is integrable or not. Roughly speaking, a system is called integrable if it has a number of invariant tensors (for example, first integrals, symmetry fields or Jacobian multipliers) such that it can be solved by quadrature or in a closed form. The integrability structure of a system provides us its general solutions in an "explicit" way, and helps us obtain the global dynamics, the topological structure or the final evolution of phase curves for the considered system [15, 18, 28, 29, 49]. On the contrary, non-integrability of a system pushes us to expect that the system admits chaotic phenomena or complex dynamical behavior [9, 10, 47].

It should be pointed out that there is no unique definition of integrability for dynamical systems. In terms of Hamiltonian systems, the integrability is well defined in the Liouville sense, i.e., an n-degree-of-freedom Hamiltonian system is integrable if and only if it has n functionally independent first integrals

---

* Corresponding author

in involution. In terms of general non-Hamiltonian systems, there exist several different definitions of integrability as follows.

Let us recall some basic notions and facts concerning integrability of general non-Hamiltonian systems. Consider a general analytic system of differential equations

$$\dot{x} = F(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{C}^n \tag{1.1}$$

with smooth right-hand sides $F = (F_1, \ldots, F_n)$. Denote by $\mathcal{L}_F$ the Lie derivative associated with the system (1.1). A differential tensor field $T(x)$ is called an invariant tensor of the system (1.1) if $\mathcal{L}_F(T) = 0$. Three typical invariant tensors are first integrals, the symmetry field and the invariant n-form. More precisely, a scalar function $\Phi(x)$ is a first integral of (1.1) if and only if $\mathcal{L}_F(\Phi) := \langle \partial_x \Phi, F \rangle = 0$, where $\partial_x \Phi$ is the gradient of Φ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{C}^n$. An n-dimensional vector function $V(x)$ is a symmetry field of (1.1) if and only if $\mathcal{L}_F(V) := [V, F] = 0$. Clearly, the system (1.1) always has a *trivial* symmetry field, i.e., itself $V = F$. An n-form $\Omega = J(x)dx_1 \wedge \cdots \wedge dx_n$ is an invariant n-form of (1.1) if and only if $\mathcal{L}_F(\Omega) := \mathrm{div}(JF)dx_1 \wedge \cdots \wedge dx_n = 0$, where the scalar function $J(x)$ is called a Jacobian multiplier of (1.1).

System (1.1) is completely integrable if it admits $(n-1)$ functionally independent first integrals $\Phi_1, \ldots, \Phi_{n-1}$. Clearly, the orbits of a completely integrable system are contained in the curves

$$\mathcal{S}_{c_1 \cdots c_{n-1}} = \{x \mid \Phi_1(x) = c_1, \ldots, \Phi_{n-1} = c_{n-1}\}.$$

Let us mention that the system with $(n-1)$ C^r first integrals is orbitally equivalent to a linear differential system in a full Lebesgue subset [30].

Another definition of integrability is due to the classic work of Lie, who proved that the system admitting n linearly independent and commuting symmetries $V_1 = F, V_2, \ldots, V_n$ is integrable by quadrature [24]. Correspondingly, the system (1.1) is called integrable in the Lie sense if it has n linearly independent symmetries $V_1 = F, V_2, \ldots, V_n$ such that $[V_i, V_j] = 0$ for any $1 \leqslant i, j \leqslant n$.

Doing a careful analysis of the concept of the Liouville integrability of Hamiltonian systems, Bogoyavlenskij [8] proposed a new definition of integrability for non-Hamiltonian systems, which is regarded as a generalization of the Liouville integrability. More precisely, the system (1.1) is integrable in the Bogoyavlenskij sense if for some $k \in \{0, 1, \ldots, n-1\}$, it has k functionally independent first integrals $\Phi_1, \ldots, \Phi_k$ and $n - k$ linearly independent vector fields $V_1 = F, \ldots, V_{n-k}$ such that

$$[V_i, V_j] = 0 \quad \text{and} \quad \langle \partial_x \Phi_l, V_j \rangle = 0 \quad \text{for } 1 \leqslant l \leqslant k, \quad 1 \leqslant i, j \leqslant n - k.$$

Similar to the case of Hamiltonian systems, if the system (1.1) is integrable in the Bogoyavlenskij sense, then the invariant sets associated with first integrals are generically diffeomorphic to tori, cylinders or planes inside the phase space [8]. Obviously, complete integrability and Lie integrability are special cases of Bogoyavlenskij integrability, corresponding to $k = 0$ or $n - 1$, respectively.

The next definition of integrability is due to Jacobi [17], and is widely applied to nonholonomic mechanical systems [19,24]. System (1.1) is called integrable in the Jacobi sense if it has $n-2$ functionally independent first integrals $\Phi_1, \ldots, \Phi_{n-2}$ and a Jacobian multiplier $J(x)$. The integrability in terms of the existence of $n - 1$ Jacobian multipliers was studied in [16,48].

In 2013, Kozlov [25] combined the above definitions and proposed the Euler-Jacobi-Lie integrability: if the system (1.1) has k functionally independent first integrals $\Phi_1, \ldots, \Phi_k$, $(n - k - 1)$ independent symmetry fields $V_1 = F, V_2, \ldots, V_{n-k-1}$ generating a nilpotent Lie algebra of the vector fields, and an invariant volume n-form $\Omega = J(x)dx_1 \wedge \cdots \wedge dx_n$ such that

$$\mathcal{L}_{V_i}(\Phi_j) = 0, \quad \mathcal{L}_{V_i}(\Omega) = 0, \quad 1 \leqslant i \leqslant n - k - 1, \quad 1 \leqslant j \leqslant k,$$

then the system (1.1) can be integrated by quadrature. To summarize, all the above accepted definitions of the integrability follow the same philosophy: the existence of some invariant tensors, including first integrals, symmetry fields and Jacobian multipliers, whose total number is equal to the dimension n of the dynamical system (see Table 1).

Table 1   Summary of integrability for differential systems in terms of invariant tensors

Integrability	First integrals	Symmetry fields	Jacobian multipliers
Complete integrability	$n-1$	1	0
Lie integrability	0	n	0
Bogoyavlenskij integrability	k	$n-k$	0
Jacobi integrability	$n-2$	1	1
Integrability in [16, 48]	0	1	$n-1$
Euler-Jacobi-Lie integrability	k	$n-1-k$	1

There are few effective methods to decide whether a system is integrable or not. At the end of the 20th century, a significant development, called the Morales-Ramis theory, was made by Morales-Ruiz [37], Morales-Ruiz and Ramis [38], Morales-Ruiz and Simó [39], Baider et al. [6] and Churchill et al. [13], who used the properties of the differential Galois group of variational equations to give strong and effective necessary conditions for the Liouville integrability of Hamiltonian systems. This theory has been applied successfully to a number of nonlinear physical models such as the N-body problem [1, 44], Hill's problem [40], problems with homogeneous potentials [31, 32], geodesic motion [34] and other physical problems [2, 35, 43]. In the recent decades, inspired by the works of Morales-Ruiz and Ramis [37, 38], the differential Galoisian approach has been applied to studying the integrability of non-Hamiltonian systems. In 2010, Ayoul and Zung [5] used the cotangent lifting trick to naturally extend the Morales-Ramis theory into non-Hamiltonian systems, and gave necessary conditions for meromorphic integrability in the Bogoyavlenskij sense. The necessary conditions for the complete integrability are proposed in [27, 36], which are expressed in terms of finiteness of the differential Galois group. Based on the Malgrange pseudogroup and the Artin approximation, Casale [11] showed that if a rational vector field is rationally integrable in the Jacobi sense on an algebraic variety, then identity components of Galois groups of variational equations are solvable and their first derived Lie algebras are abelian (see [33, 41] for more details on Jacobi non-integrability).

Our main results of this work are summarized as follows.

(i) Instead of the tools of the Malgrange pseudogroup and the Artin approximation used in Casale's proof, we provide an elementary proof of necessary conditions for the Jacobi integrability in the category of meromorphic functions. Indeed, we show that if the system (1.1) has $k \in [0, n-2]$ meromorphic first integrals $\Phi_1, \ldots, \Phi_k$ and $n-1-k$ meromorphic Jacobian multipliers $J_1, \ldots, J_{n-k-1}$ such that

$$\Phi_1, \ldots, \Phi_k, \frac{J_2}{J_1}, \ldots, \frac{J_{n-1-k}}{J_1}$$

are functionally independent, then the identity component of the differential Galois group of the normal variational equations along a particular solution is abelian (see Theorem 3.7). Our proof strategy can be used to investigate necessary conditions for other integrability via differential Galoisian methods.

(ii) Based on Theorem 3.7, we study the polynomial integrability of Karabut systems for stationary gravity waves in finite depth. Witting [46] proposed a new formal series solution to the water waves problem when he studied the solitary wave in a fluid of finite depth. Then Karabut [20–22] showed the problem of exact summation of Witting's series can be reduced to solving or integrating some homogeneous ordinary differential equations, called Karabut systems. We show that the 3-dimensional Karabut system is integrable and admits infinitely many Hamilton-Poisson realizations and a Lax formulation (see Propositions 4.1–4.3), and also show that the 5-dimensional Karabut system has two and only two functionally independent polynomial first integrals (see Theorem 4.4), which answers the question by Karabut [22] and improves the result in [12] from the point of view of partial integrability.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary notions and results on the differential Galois theory and Jacobian multipliers. In Section 3, we formulate a general theorem which provides necessary conditions for the existence of k first integrals and $n-1-k$ Jacobian multipliers satisfying compatibility conditions. In Section 4, we apply our results to studying integrability

of 3-dimensional and 5-dimensional Karabut systems.

2 Preliminary results

2.1 Differential Galois theory

Differential Galois theory is a generalization of the classic Galois theory from polynomial equations to linear differential equations, which is also called the Picard-Vessiot theory. Here, we only review some parts of the differential Galois theory (see, for more details, [37, 45] and the references therein).

Consider the linear homogeneous differential equations on a differential field $(\mathbb{K}, \partial)$:

$$Y' = AY, \quad A \in \mathrm{Mat}(\mathbb{K}, n), \tag{2.1}$$

where $\mathrm{Mat}(\mathbb{K}, n)$ denotes the ring of $n \times n$ matrices with entries in $\mathbb{K}$. Recall that a differential field is a pair $(\mathbb{K}, \partial)$ consisting of a field $\mathbb{K}$ and a derivative ∂, where the derivative ∂ is an additive mapping $\partial : \mathbb{K} \to \mathbb{K}$ satisfying $\partial(ab) = \partial(a)b + a\partial(b)$, $a, b \in \mathbb{K}$. We also write a' instead of $\partial(a)$. The set of elements of $\mathbb{K}$ for which ∂ vanishes is called the field of constants of $\mathbb{K}$, denoted by $\mathrm{Const}(\mathbb{K}) := \{a \in \mathbb{K} \mid a' = 0\}$. In practical applications, $\mathbb{K}$ is the field of meromorphic functions on a Riemann surface endowed with a meromorphic vector field and the field of constants becomes the field of complex numbers $\mathbb{C}$.

A differential field extension, denoted by $\mathbb{L}/\mathbb{K}$, is a field extension such that $\mathbb{L}$ is also a differential field and the derivations on $\mathbb{L}$ and $\mathbb{K}$ coincide on $\mathbb{K}$, i.e., $\partial_{\mathbb{L}}|_{\mathbb{K}} = \partial_{\mathbb{K}}$, where $\partial_{\mathbb{L}}$ and $\partial_{\mathbb{K}}$ are the derivations of $\mathbb{L}$ and $\mathbb{K}$, respectively. Now we introduce two important differential field extensions. The first one is the Liouvillian extension.

Definition 2.1.   The differential field extension $\mathbb{L}/\mathbb{K}$ is called a Liouvillian extension if $\mathrm{Const}(\mathbb{L})$ $= \mathrm{Const}(\mathbb{K})$ and there exists a tower of extensions $\mathbb{K} = \mathbb{L}_0 \subset \mathbb{L}_1 \subset \cdots \subset \mathbb{L}_m = \mathbb{L}$ such that for $i = 1, \ldots, m$, $\mathbb{L}_i = \mathbb{L}_{i-1}(a_i)$, and one of the following cases holds:

(1) $a_i' \in \mathbb{L}_{i-1}$, and in this case, a_i is called an integral element of L_{i-1};

(2) $a_i \neq 0$ and $a_i'/a_i \in \mathbb{L}_{i-1}$, and in this case, a_i is called an exponential integral element of $\mathbb{L}_{i-1}$;

(3) a_i is algebraic over $\mathbb{L}_{i-1}$.

Remark 2.2.   Roughly speaking, the fact that $\mathbb{L}/\mathbb{K}$ is a Liouvillian extension means that each element of $\mathbb{L}$ can be built up from $\mathbb{K}$ by algebraic operations and taking exponentials or indefinite integrals.

The second one is the Picard-Vessiot (P-V) extension which is associated with a linear system of the differential equations (2.1).

Definition 2.3.   The differential field extension $\mathbb{L}/\mathbb{K}$ is a Picard-Vessiot (P-V) extension for the linear system (2.1) if and only if it satisfies the following three conditions:

(1) $\mathrm{Const}(\mathbb{L}) = \mathrm{Const}(\mathbb{K})$;

(2) there exists a fundamental matrix $\Phi \in GL(\mathbb{L}, n)$ for the linear homogeneous differential equations (2.1);

(3) $\mathbb{L}$ is generated over $\mathbb{K}$ as a differential field by the entries of the fundamental matrix Φ.

Remark 2.4.   Roughly speaking, the P-V extension is the smallest differential extension such that it contains n linearly independent solutions of (2.1) and no new constants are added. In addition, it is known that if the constant subfield of the differential field $(\mathbb{K}, \partial)$ is characteristic zero, for example $\mathrm{Const}(\mathbb{K}) = \mathbb{C}$, then (2.1) can admit a Picard-Vessiot extension which is unique up to isomorphism [45].

Fix a P-V extension $\mathbb{L}/\mathbb{K}$ and the fundamental matrix Φ, and all the differential $\mathbb{K}$-automorphisms $(\sigma : \mathbb{L} \to \mathbb{L}, \sigma(a') = (\sigma(a))', \forall a \in \mathbb{L}$ and $\sigma(a) = a, \forall a \in \mathbb{K})$ of $\mathbb{L}$ are called the differential Galois group of (2.1) and denoted by $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$. Let $\Phi(t)$ be a fundamental-solution matrix of (2.1). Note that for any $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{K})$, we see that $\sigma(\Phi)$ is also a fundamental matrix of (2.1). Therefore, $\sigma(\Phi) = \Phi M_\sigma$ with $M_\sigma \in GL(\mathbb{C}, n)$, which gives a faithful representation of the group of $\mathbb{K}$-automorphisms of $\mathbb{L}$ on the general linear group as

$$\rho : \mathrm{Gal}(\mathbb{L}/\mathbb{K}) \to GL(\mathbb{C}, n), \quad \sigma \mapsto M_\sigma.$$

Hence, the differential Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ can be regarded as a subgroup of $GL(\mathbb{C}, n)$. In what follows, we will be dealing with either the differential Galois group $G := \mathrm{Gal}(\mathbb{L}/\mathbb{K})$ or its matrix group $\rho(G)$. Moreover, G is a linear algebraic group [45], and is a union of a finite number of disjoint connected components. Then the differential Galois group G has a unique maximal connected subgroup G^0 containing the identity element of the group, called the identity component of G. We say that a group G is solvable if and only if there exists a chain of normal subgroups $e = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ such that the quotient G_{i+1}/G_i is abelian for $i = 0, \ldots, n-1$.

The following fundamental theorem concerns the deep relation between the solvability of the linear differential equations and that of the corresponding differential Galois group.

Theorem 2.5.   *Let $\mathbb{L}/\mathbb{K}$ be a P-V extension for (2.1). Then the linear system (2.1) is solvable by quadrature, i.e., $\mathbb{L}/\mathbb{K}$ is a Liouvillian extension if and only if the identity component $\mathrm{Gal}(\mathbb{L}/\mathbb{K})^0$ of the differential Galois group $\mathrm{Gal}(\mathbb{L}/\mathbb{K})$ is solvable. In particular, if the identity component $\mathrm{Gal}(\mathbb{L}/\mathbb{K})^0$ is abelian, (2.1) is solvable by quadrature.*

In general, for an n-dimensional linear differential system, it is difficult to compute or analyze the properties of the corresponding differential Galois group. When the (normal) variational equations can be reduced into a second-order linear differential equation with rational coefficients, the so-called Kovacic's algorithm [23] can help us calculate effectively the solvability of the differential Galois group.

Lemma 2.6 (See [23]).   *The differential Galois group G of*

$$\frac{d^2\chi}{dt^2} = r(t)\chi, \quad r(x) \in \mathbb{C}(t) \tag{2.2}$$

with $\mathbb{C}(t)$ being the rational function field on $\mathbb{C}$ can be classified into the following four cases:

Case 1.   *G is conjugate to a subgroup of the triangular group*

$$\mathbb{C}^* \ltimes \mathbb{C}^+ = \left\{ \begin{pmatrix} c & 0 \\ b & c^{-1} \end{pmatrix} \;\middle|\; b \in \mathbb{C},\, c \in \mathbb{C}^* \right\}. \tag{2.3}$$

Then (2.2) has a solution of the form $\mathrm{e}^{\int \omega dt}$ with $\omega \in \mathbb{C}(t)$.

Case 2.   *G is not of Case 1, but is conjugate to a subgroup of the infinite dihedral group*

$$D = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \;\middle|\; c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & c \\ -c^{-1} & 0 \end{pmatrix} \;\middle|\; c \in \mathbb{C}^* \right\}.$$

Then (2.2) has a solution of the form $\mathrm{e}^{\int \omega dt}$ with ω algebraic over $\mathbb{C}(t)$ of degree two.

Case 3.   *G is not of Cases 1 and 2, but is a finite group. Then all the solutions of (2.2) are algebraic over $\mathbb{C}(t)$.*

Case 4.   *$G = SL(2, \mathbb{C})$, where $SL(2, \mathbb{C})$ is the group of 2×2 matrices with elements in $\mathbb{C}$ and determinants one. Then (2.2) is not integrable in the Liouville sense.*

Proof.   See [23, Lemma 1.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 2.7.   Kovacic [23] presented a complete algorithm to analyze which cases the differential Galois group G of the system (2.2) falls into. Due to Lemma 2.6, there exist four cases in Kovacic's algorithm. We can obtain the Liouvillian solutions of (2.2) in Cases 1–3, but for Case 4 this system has no Liouvillian solutions. In addition, Kovacic's algorithm gives us only one solution ξ_1 in Cases 1–3, and the second independent solution ξ_2 can be obtained by

$$\xi_2 = \xi_1 \int \frac{dt}{\xi_1^2}.$$

Remark 2.8.   For a general second-order linear differential equation

$$\frac{d^2\xi}{dt^2} + a(t)\frac{d\xi}{dt} + b(t)\xi = 0, \quad a(t), b(t) \in \mathbb{C}(t), \tag{2.4}$$

we can make a well-known change of the variable

$$\xi = \chi \exp\left(-\frac{1}{2}\int_{t_0}^t a(s)ds\right),$$

and get the reduced form of (2.4), i.e.,

$$\frac{d^2\chi}{dt^2} = r(t)\chi, \quad r(t) = \frac{a^2}{4} + \frac{1}{2}\frac{da}{dt} - b. \tag{2.5}$$

It should be pointed out that the identity component of (2.4) is solvable if and only if that of (2.5) is solvable, since the above transformation does not affect the Liouvillian solvability of (2.4).

The next lemma is due to the work of Singer and Ulmer [42], which provides a more precise characterization of Case 1 of Lemma 2.6.

Lemma 2.9. *Assume that the differential Galois group G of (2.2) is conjugate to a subgroup of the triangular group. Then*

Subcase 1.1. *G is conjugate to a subgroup of the diagonal group*

$$\left\{\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \,\middle|\, c \in \mathbb{C}^* \right\}.$$

In this subcase, the system (2.2) has two independent solutions ξ_1 and ξ_2 such that $\xi_i'/\xi_i \in \mathbb{C}(t)$, $i = 1, 2$.

Subcase 1.2. *G is conjugate to the group*

$$\left\{\begin{pmatrix} c & 0 \\ b & c^{-1} \end{pmatrix} \,\middle|\, b \in \mathbb{C}, \ c^m = 1 \ with \ an \ m \in \mathbb{N} \right\},$$

which is a proper subgroup of the group $\mathbb{C}^* \ltimes \mathbb{C}^+$. In this subcase, the system (2.2) has only one solution ξ (up to constant multiples) such that $\xi'/\xi \in \mathbb{C}(t)$, and m is the smallest positive integer such that $\xi^m \in \mathbb{C}(t)$.

Subcase 1.3. *G is conjugate to the group $\mathbb{C}^* \ltimes \mathbb{C}^+$. In this subcase, the system (2.2) has only one solution ξ (up to constant multiples) such that $\xi'/\xi \in \mathbb{C}(t)$, and $\xi^m \notin \mathbb{C}(t)$ for any positive integer m.*

Proof. See [42, Proposition 4.2]. ☐

Remark 2.10. Based on the above results, let us point out that

(1) the identity component G^0 associated with (2.2) is not solvable if and only if G belongs to Case 4;

(2) the identity component G^0 associated with (2.2) is not abelian if and only if G belongs to either Subcase 1.3 or Case 4.

Acosta-Humánez and Blázquez-Sanz [3] gave a complete classification of the differential Galois group of a second-order differential equation with polynomial coefficients.

Lemma 2.11 (See [3]). *Let $Q(t) \in \mathbb{C}[t]/\mathbb{C}$ be a polynomial of degree $k > 0$. The differential Galois group G of*

$$\frac{d^2\chi}{dt^2} = Q(t)\chi, \quad Q(t) \in \mathbb{C}[t]/\mathbb{C}$$

is either $SL(2, \mathbb{C})$ or $\mathbb{C}^* \ltimes \mathbb{C}^+$, and in particular is non-abelian.

In applications, the next two results can help us reduce the dimension of linear differential equations.

Lemma 2.12 (See [14]). *Consider the differential Galois group G of the linear differential system*

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad A \in \mathrm{Mat}(\mathbb{K}, m), \quad C \in \mathrm{Mat}(\mathbb{K}, l). \tag{2.6}$$

The following statements hold:

(i) *If the identity component of G is solvable (abelian), then the identity component of the differential Galois group of the subsystem*

$$\frac{d}{dt}X_1 = AX_1 \tag{2.7}$$

is also solvable (abelian).

(ii) *Suppose $B = 0$. Then the system* (2.6) *is the direct sum of two subsystems* (2.7) *and*

$$\frac{d}{dt}X_2 = CX_2. \tag{2.8}$$

The identity component of G is solvable (abelian) if and only if the identity components of the differential Galois groups of both subsystems (2.7) *and* (2.8) *are solvable (abelian).*

The next result is a variant of the well-known Ziglin lemma [50] for which the meromorphic functions are replaced by C^2 functions, but it is very useful to reduce the dimension of linear differential equations.

Lemma 2.13. *Assume that $\psi(t)$ is a particular solution of the nonlinear system* (1.1)*. If a C^2 function $\Phi(x)$ is a first integral of the nonlinear system* (1.1)*, then the function $G(t, \xi) = \langle \nabla \Phi(\psi(t)), \xi \rangle$ is either a constant or a time-dependent linear first integral of the variational system*

$$\frac{d\xi}{dt} = A(t)\xi, \quad A(t) = \frac{\partial F}{\partial x}\bigg|_{x=\psi(t)}. \tag{2.9}$$

Proof. By definition, we have $d\Phi(x(t))/dt = 0$ for all the solutions $x(t)$ of (1.1), i.e.,

$$\langle \partial_x \Phi(x), F(x) \rangle = \sum_j F_j \frac{\partial \Phi}{\partial x_j} \equiv 0. \tag{2.10}$$

Taking the derivative of (2.10) with respect to x_i, we have

$$\sum_j \left(F_j \frac{\partial^2 \Phi}{\partial x_j \partial x_i} + \frac{\partial \Phi}{\partial x_j} \frac{\partial F_j}{\partial x_i} \right) \equiv 0, \quad i = 1, \dots, n.$$

Then

$$\begin{aligned}
\frac{dG}{dt} &= \frac{\partial G}{\partial t} + \sum_j \frac{\partial G}{\partial \xi_j} \frac{d\xi_j}{dt} \\
&= \sum_i \left(F_j \frac{\partial^2 \Phi}{\partial x_j \partial x_i} \right) \xi_i + \sum_j \left(\frac{\partial \Phi}{\partial x_j} \sum_i \left(\frac{\partial F_j}{\partial x_i} \xi_i \right) \right) \\
&= \sum_i \left(\sum_j \left(F_j \frac{\partial^2 \Phi}{\partial x_j \partial x_i} + \frac{\partial \Phi}{\partial x_j} \frac{\partial F_j}{\partial x_i} \right) \right) \xi_i \\
&= 0,
\end{aligned}$$

which completes the proof. $\qquad \square$

2.2 (Generalized) Jacobian multipliers and the characterization

The first result is due to the classic work of Poincaré, which gives an equivalent characterization of Jacobian multipliers. For a proof, please see [49, Proposition 2.2].

Lemma 2.14. *Let $J(x)$ be a non-zero continuously differentiable function. Then the following statements are equivalent*:

(i) *$J(x)$ is a Jacobian multiplier of the system* (1.1)*, i.e.,*

$$\mathrm{div}(JF) = J(x)\nabla_x \cdot F(x) + \langle \partial_x J, F \rangle \equiv 0.$$

(ii) *For any flow $\phi_t(t_0, x_0)$ of* (1.1) *satisfying $\phi_{t_0}(t_0, x_0) = x_0$, we have*

$$J(x_0) = J(\phi_t) \det \partial_{x_0}(\phi_t), \quad \forall t \geqslant t_0.$$

(iii) *For any bounded region D_0, the integral*

$$V(t) = \int_{D_t} J(y) dy$$

is independent of the time t, where $D_t = \{y \mid y = \phi_t(t_0, x_0), x_0 \in D_0\}$ is the evolution of D_0 under the flow ϕ_t.

Next, we discuss the relationship between Jacobian multipliers of equivalent differential systems.

Lemma 2.15.   *Assume that $x = G(y)$ is a continuously differentiable and invertible transformation. If $J(x)$ is a Jacobian multiplier of the system* (1.1)*, then $\hat{J}(y) := J(G(y)) \det \partial_y G(y)$ is a Jacobian multiplier of the system $\dot{y} = (\partial_y G)^{-1} F(G(y))$.*

Lemma 2.15 can be easily proved by using the definition of Jacobian multipliers or using the invariant of the integral $V(t)$; for a detailed proof, we refer to [49, Proposition 2.3].

Now we introduce the notion of *time-dependent* Jacobian multipliers for non-autonomous differential systems.

Definition 2.16.   A non-zero continuously differentiable function $J(t, x)$ is called a Jacobian multiplier of the non-autonomous differential system

$$\dot{x} = F(t, x), \quad x \in \mathbb{C}^n, \tag{2.11}$$

if

$$\partial_t J(t, x) + J(t, x) \nabla_x \cdot F(t, x) + \langle \partial_x J, F \rangle \equiv 0.$$

If we set $\omega = t$ and rewrite (2.11) as an autonomous differential system

$$\dot{x} = F(\omega, x), \quad \dot{\omega} = 1,$$

then the above definition coincides with the usual Jacobian multiplier of autonomous differential systems. Furthermore, as two corollaries of Lemmas 2.14–2.15, we immediately obtain the following results.

Lemma 2.17.   *Let $J(t, x)$ be a non-zero continuously differentiable function. Then $J(t, x)$ is a Jacobian multiplier of the system* (2.11) *if and only if for any flow $\phi_t(t_0, x_0)$ of* (2.11) *satisfying $\phi_{t_0}(t_0, x_0) = x_0$, we have*

$$J(t_0, x_0) = J(t, \phi_t) \det \partial_{x_0}(\phi_t), \quad \forall t \geqslant t_0.$$

Lemma 2.18.   *Assume that $x = G(t, y)$ is a continuously differentiable and invertible transformation for any fixed t. If $J(t, x)$ is a Jacobian multiplier of the system* (2.11)*, then*

$$\hat{J}(t, y) := J(t, G(t, y)) \det \partial_y G(t, y)$$

is a Jacobian multiplier of the system

$$\dot{y} = \hat{F}(t, y) = (\partial_y G)^{-1}(F(t, G(t, y)) - \partial_t G(t, y)).$$

3   Necessary conditions for Jacobi integrability of analytic differential systems

Let $\psi(t)$ be a non-equilibrium analytic solution of the system (1.1). Linearization of (1.1) around $\psi(t)$ yields the *variational equations* of the following form:

$$\frac{d\xi}{dt} = A(t)\xi, \quad A(t) = \frac{\partial F}{\partial x}\bigg|_{x = \psi(t)} \in \mathrm{Mat}(\mathbb{K}, n), \tag{3.1}$$

where elements of the field $\mathbb{K}$ are meromorphic functions on the phase curve $\Gamma = \{\psi(t)\}$. Note that $\dot{\psi}(t)$ is a nontrivial solution of the system (3.1) due to $\psi(t)$ being a non-equilibrium solution of the system (1.1). Using this fact, we can reduce the dimension of (3.1) by one.

Indeed, making a change of variables

$$\xi = P(t)\eta,$$

where $P(t) = (P^1, \ldots, P^{n-1}, \dot{\psi}(t))$ is a non-singular matrix with its components in $\mathbb{K}$, we see that (3.1) becomes the following equivalent form:

$$\frac{d\eta}{dt} = B(t, \eta) := P(t)^{-1}(A(t)P(t) - \dot{P}(t))\eta = \begin{pmatrix} C(t) & \theta \\ \alpha(t)^{\mathrm{T}} & 0 \end{pmatrix} \eta, \tag{3.2}$$

where θ denotes the $(n-1)$-dimensional zero vector and $\eta = (\zeta, \eta_1)^{\mathrm{T}}$. Therefore, we obtain a subsystem of (3.2), i.e.,

$$\frac{d\zeta}{dt} = C(t)\zeta, \quad C(t) \in \mathrm{Mat}(\mathbb{K}, n-1), \tag{3.3}$$

which is the so-called *normal variational equations* of (1.1) along Γ.

Equations (3.1)–(3.3) are linear differential equations. Then we can associate the differential Galois theory with them. Let us mention that the choice of $P(t)$ is not unique, and both (3.1) and (3.2) have the same differential Galois group of equations since they have the same P-V extension. Denote by G the differential Galois group of the normal variational equations (3.3). Recall that G is a linear algebraic group, thus in particular a Lie group, and one can consider its Lie algebra which reflects only the properties of the identity component G^0 of the group. We denote by $\mathcal{G} \subset gl(n, \mathbb{C})$ the Lie algebra of G. Then an arbitrary element $Y \in \mathcal{G}$ can be viewed as a linear vector field: $x \to Y(x) := Y \cdot x$ for $x \in \mathbb{C}^n$, and $\mathrm{e}^{Yt} \in G$ for all $t \in \mathbb{C}$.

The next result goes back to Ziglin [50] and plays a critical and fundamental role in the non-integrability approach (see [37, Chapter 4] for a proof).

Proposition 3.1.   *Assume that the system* (1.1) *has* k $(k \geqslant 1)$ *functionally independent meromorphic first integrals in a neighborhood of* Γ. *Then*

(i) *the normal variational equations in* (3.3) *have* k $(k \geqslant 1)$ *functionally independent first integrals which are rational functions in* ζ;

(ii) *each element of Lie algebra* $\mathcal{G}$, *as a linear vector field, has* k $(k \geqslant 1)$ *functionally independent rational first integrals.*

Let us give some remarks on Proposition 3.1. Observing that the dimension of (3.3) is $n-1$, some readers may doubt the correctness of Proposition 3.1 in the case of $k = n-1$. Indeed, for an n-dimensional general differential system $\dot{x} = f(x, t)$, the maximal number of *autonomous* functionally independent first integrals is $n - 1$, whereas that of *time-dependent* functionally independent first integrals is n (see [4, Chapters 10.5–10.7]). Equation (3.3) is an $(n-1)$-dimensional differential system and may have $(n-1)$ functionally independent first integrals. To further illustrate Proposition 3.1, we give two examples with $k = n - 1$.

Example 3.2 $(n = 2$ and $k = 1)$.   Consider a two-dimensional system

$$\dot{x} = xy, \quad \dot{y} = -y^2 - x + 1, \tag{3.4}$$

which has a first integral $\Phi(x, y) = -3x^2 + 2x^3 + 3x^2y^2$ and a non-equilibrium solution $\phi(t) = (0, \tanh t)$. The variational equations along $\phi(t)$ read

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} \tanh t & 0 \\ -1 & \tanh t \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{3.5}$$

and consequently, the normal variational equations read

$$\dot{\xi} = a(t)\xi = \tanh t\, \xi,$$

which has a first integral $\widetilde{\Phi}(t, \xi) = (\tanh t^2 - 1)\xi^2$.

Example 3.3 ($n = 3$ and $k = 2$).   Consider a three-dimensional system

$$\dot{x} = -x + yz, \quad \dot{y} = -y - xz, \quad \dot{z} = -z + xy, \tag{3.6}$$

which has two first integrals $\Phi_1 = (x^2 + y^2)/(x^2 - z^2)$ and $\Phi_2 = (y^2 + z^2)/(x^2 - z^2)$ and a non-equilibrium solution $\phi(t) = (0, 0, \mathrm{e}^{-t})$. The normal variational equations along $\phi(t)$ are given by

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} -1 & \mathrm{e}^{-t} \\ -\mathrm{e}^{-t} & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{3.7}$$

One can check that the system (3.7) has two functionally independent first integrals

$$\widetilde{\Phi}_1(\eta_1, \eta_2, t) = \mathrm{e}^t \sin(\mathrm{e}^{-t})\eta_1 - \mathrm{e}^t \cos(\mathrm{e}^{-t})\eta_2,$$
$$\widetilde{\Phi}_2(\eta_1, \eta_2, t) = \mathrm{e}^t \cos(\mathrm{e}^{-t})\eta_1 + \mathrm{e}^t \sin(\mathrm{e}^{-t})\eta_2.$$

In what follows, we aim to establish results analogous to Proposition 3.1 for which the choice of invariant tensors is changed from first integrals to n-forms (or Jacobian multipliers).

Lemma 3.4.   *Assume that the system* (1.1) *has a meromorphic Jacobian multiplier* $J(x)$ *in a neighborhood of* Γ. *Then the variational equations in* (3.1) *have a time-dependent Jacobian multiplier* $J_{VE}(t, \xi)$, *which is a rational function with respect to* ξ.

Proof.   Let $\xi(t) = x - \psi(t)$, which satisfies

$$\frac{d\xi}{dt} = \widehat{F}(t, \xi) := F(\xi + \psi(t)) - F(\psi(t)). \tag{3.8}$$

By Lemma 2.18, the system (3.8) has a Jacobian multiplier $\widehat{J}(t, \xi) = J(\xi + \psi(t))$, which is meromorphic with respect to ξ. By definition, we have

$$\partial_t \widehat{J} + \widehat{J} \nabla_\xi \cdot \widehat{F} + \langle \partial_\xi \widehat{J}, \widehat{F} \rangle \equiv 0. \tag{3.9}$$

Observing $\widehat{J}(t, \xi) = P(t, \xi)/Q(t, \xi)$ for certain functions P and Q, which is holomorphic with respect to ξ, we can rewrite (3.9) into

$$Q\partial_t P - P\partial_t Q + \langle Q\partial_\xi P - P\partial_\xi Q, \widehat{F} \rangle + PQ\nabla_\xi \cdot \widehat{F} \equiv 0. \tag{3.10}$$

We expand functions P and Q in the neighborhood of Γ:

$$P = P_m(t, \xi) + O(\|\xi\|^{m+1}), \quad Q = Q_k(t, \xi) + O(\|\xi\|^{k+1}), \quad P_m \neq 0, \quad Q_k \neq 0, \tag{3.11}$$

where P_m and Q_k are the leading terms of P and Q, i.e., the lowest-order nonvanishing terms of expansions, and P_m (or Q_k) are homogenous polynomials of degree m (or k) with respect to ξ. Similarly, we expand the vector field $\widehat{F}$ as

$$\widehat{F} = A(t)\xi + O(\|\xi\|^2) \tag{3.12}$$

in the neighborhood of Γ, where $A(t)$ is defined in (3.1). Substituting (3.11)–(3.12) into (3.10) and comparing the terms of the lowest order $m + k$, we get

$$Q_k \partial_t P_m - P_m \partial_t Q_k + \langle Q_k \partial_\xi P_m - P_m \partial_\xi Q_k, A(t)\xi \rangle + P_m Q_k \mathrm{tr}(A(t)) \equiv 0, \tag{3.13}$$

or equivalently,

$$\partial_t \left(\frac{P_m}{Q_k} \right) + \left\langle \partial_\xi \left(\frac{P_m}{Q_k} \right), A(t)\xi \right\rangle + \left(\frac{P_m}{Q_k} \right) \mathrm{tr}(A(t)) \equiv 0, \tag{3.14}$$

where tr is the trace of a matrix, i.e., the sum of the entries on the diagonal. It follows from (3.14) that the variational equations in (3.1) admit a Jacobian multiplier $J_{VE}(t, \xi) = P_m/Q_k$, which is a rational function with respect to ξ. $\hfill\square$

Lemma 3.5.   *Assume that the variational equations in* (3.1) *have a time-dependent Jacobian multiplier* $J_{VE}(t, \xi)$, *which is a rational function with respect to* ξ. *Then the normal variational equations in* (3.3) *have a time-dependent Jacobian multiplier* $J_{\mathrm{NVE}}(t, \zeta)$, *which is a rational function with respect to* ζ.

Proof.   Due to Lemma 2.18, the system (3.2) has a time-dependent Jacobian multiplier

$$\widehat{J}_{VE}(t, \eta) := J_{VE}(t, P(t)\eta)\det P(t),$$

which is a rational function with respect to $\eta = (\zeta, \eta_1)$. Hence, by definition we have

$$\partial_t \widehat{J}_{VE} + \partial_{\eta_1} \widehat{J}_{VE}\langle \alpha(t), \zeta \rangle + \langle \partial_\zeta \widehat{J}_{VE}, C(t)\zeta \rangle + \widehat{J}_{VE}\mathrm{tr}(C(t)) \equiv 0, \tag{3.15}$$

where we use $\mathrm{tr}(B(t)) = \mathrm{tr}(C(t))$. From the proof of Lemma 3.4, we see that $\widehat{J}_{VE}(t, \eta) = \widehat{P}_m(t, \eta)/\widehat{Q}_k(t, \eta)$ for certain functions $\widehat{P}_m$ and $\widehat{Q}_k$, which are polynomials of degrees m and k with respect to η, respectively. Hence, (3.15) becomes

$$\widehat{Q}_k \partial_t \widehat{P}_m - \widehat{P}_m \partial_t \widehat{Q}_k + (\widehat{Q}_k \partial_{\eta_1} \widehat{P}_m - \widehat{P}_m \partial_{\eta_1} \widehat{Q}_k)\langle \alpha(t), \zeta \rangle$$
$$+ \langle \widehat{Q}_k \partial_\zeta \widehat{P}_m - \widehat{P}_m \partial_\zeta \widehat{Q}_k, C(t)\zeta \rangle + \widehat{P}_k \widehat{Q}_m \mathrm{tr}(C(t)) \equiv 0. \tag{3.16}$$

We write

$$\widehat{P}_m = \sum_{j=0}^{m_1} P_{m,j}(t, \zeta)\eta_1^j, \quad m_1 \leqslant m, \quad P_{m,m_1} \neq 0 \tag{3.17}$$

with $P_{m,j}$ being polynomials of degree $m - j$ with respect to ζ. Similarly, we also write

$$\widehat{Q}_k = \sum_{j=0}^{k_1} Q_{k,j}(t, \zeta)\eta_1^j, \quad k_1 \leqslant k, \quad Q_{k,k_1} \neq 0 \tag{3.18}$$

with $Q_{k,j}$ being polynomials of degree $k - j$ with respect to ζ. Substituting (3.17)–(3.18) into (3.16) and comparing the terms of the highest order $m_1 + k_1$ with respect to η_1, we get

$$\partial_t \left(\frac{P_{m,m_1}}{Q_{k,k_1}}\right) + \left\langle \partial_\zeta \left(\frac{P_{m,m_1}}{Q_{k,k_1}}\right), C(t)\zeta \right\rangle + \left(\frac{P_{m,m_1}}{Q_{k,k_1}}\right)\mathrm{tr}(C(t)) \equiv 0, \tag{3.19}$$

i.e., the normal variational equations in (3.3) admit a Jacobian multiplier $J_{\mathrm{NVE}}(t, \zeta) = P_{m,m_1}/Q_{k,k_1}$, which is a rational function with respect to ζ. □

Lemma 3.6.   *Assume that the normal variational equations in* (3.3) *have a time-dependent Jacobian multiplier* $J_{\mathrm{NVE}}(t, \zeta)$, *which is a rational function with respect to* ζ. *Then each element of the Lie algebra* $\mathcal{G}$ *of the identity component* G^0 *of the differential Galois group* G *for* (3.3), *as linear vector fields, has a common rational Jacobian multiplier.*

Proof.   Fixing t_0, for any $x \in \mathbb{C}^{n-1}$ we consider the solution $\phi_t(t_0, x_0)$ of (3.3). Thanks to Lemma 2.17, we have

$$J_{\mathrm{NVE}}(t_0, x_0) = J_{\mathrm{NVE}}(t, \phi_t(t_0, x_0))\det \partial_{x_0}\phi_t(t_0, x_0), \quad \forall t \geqslant t_0. \tag{3.20}$$

Let $\Phi(t)$ be the fundamental-solution matrix of the normal variational equations (3.3) satisfying $\det \Phi(t_0) = \mathrm{Id}$. Then we have $\phi_t(t_0, x_0) = \Phi(t)x_0$ and (3.20) becomes

$$J_{\mathrm{NVE}}(t_0, x_0) = J_{\mathrm{NVE}}(t, \Phi(t)x_0)\det \Phi(t), \quad \forall t \geqslant t_0. \tag{3.21}$$

For any $\sigma \in G$, we have its representation M_σ through $\sigma(\Phi(t)) = \Phi(t)M_\sigma$. Furthermore, taking the group action σ on both sides of (3.21) yields

$$J_{\mathrm{NVE}}(t_0, x_0) = \sigma(J_{\mathrm{NVE}}(t, \Phi(t)x_0)\det \Phi(t))$$

$$= \sigma(J_{\mathrm{NVE}}(t, \Phi(t)x_0)) \cdot \sigma(\det(\Phi(t)))$$
$$= J_{\mathrm{NVE}}(t, \sigma(\Phi(t))x_0) \cdot \det(\sigma(\Phi(t)))$$
$$= J_{\mathrm{NVE}}(t, \Phi(t)M_\sigma x_0) \cdot \det(\Phi(t)M_\sigma)$$
$$= J_{\mathrm{NVE}}(t_0, M_\sigma x_0) \cdot \det(M_\sigma).$$

Define $J_{\mathrm{gal}}(x) = J_{\mathrm{NVE}}(t_0, x)$. Based on the above discussions, we see that the rational function $J_{\mathrm{gal}}(x)$ satisfies

$$J_{\mathrm{gal}}(x) = J_{\mathrm{gal}}(M_\sigma x) \det(M_\sigma), \quad \forall\, x \in \mathbb{C}^{n-1}, \quad \forall\, M_\sigma \in G. \tag{3.22}$$

On the other hand, recall that the arbitrary element $Y \in \mathcal{G}$ can be viewed as a linear vector field: $x \to Y(x) := Yx$ for $x \in \mathbb{C}^{n-1}$, and $\mathrm{e}^{Yt} \in G^0$ for all $t \in \mathbb{C}$. It follows from (3.22) that $J_{\mathrm{gal}}(x) = J_{\mathrm{gal}}(\mathrm{e}^{Yt}x) \det(\mathrm{e}^{Yt})$ for any $Y \in \mathcal{G}$, which means that $J_{\mathrm{gal}}(x)$ is a common Jacobian multiplier of a family of linear vector fields $\{\dot{x} = Yx, Y \in \mathcal{G}\}$. $\square$

Now we can state our main results.

Theorem 3.7. *Let $\psi(t)$ be a non-equilibrium analytic solution of an n-dimensional analytic system* (1.1) *of differential equations. If the system* (1.1) *has $k \in [0, n]$ meromorphic first integrals $\Phi_1(x), \ldots, \Phi_k(x)$ and $n-1-k$ meromorphic Jacobian multipliers $J_1(x), \ldots, J_{n-1-k}(x)$ such that*

$$\Phi_1, \ldots, \Phi_k, \frac{J_2}{J_1}, \ldots, \frac{J_{n-1-k}}{J_1}$$

are functionally independent of a neighborhood of Γ, then the following statements hold:

(i) *The identity component of the differential Galois group of the normal variational equations along $\psi(t)$ is abelian.*

(ii) *The identity component of the differential Galois group of the variational equations along $\psi(t)$ is solvable.*

Proof. We first show that J_i/J_1 $(i = 2, \ldots, n-1-k)$ are first integrals of the system (1.1). Indeed, by definition we have

$$J_i(x)\nabla \cdot F(x) + \langle \partial_x J_i, F \rangle \equiv 0, \tag{3.23}$$
$$J_1(x)\nabla \cdot F(x) + \langle \partial_x J_1, F \rangle \equiv 0. \tag{3.24}$$

Eliminating the divergence term $\nabla \cdot F(x)$ from (3.23)–(3.24) yields

$$J_1\langle \partial_x J_i, F \rangle - J_i\langle \partial_x J_1, F \rangle \equiv 0,$$

which can be rewritten as

$$\left\langle \partial_x\left(\frac{J_i}{J_1}\right), F \right\rangle \equiv 0.$$

Furthermore, J_i/J_1 cannot be constants, and otherwise $\Phi_1, \ldots, \Phi_k, J_2/J_1, \ldots, J_{n-1-k}/J_1$ will be functionally independent. Hence, J_i/J_1 $(i = 2, \ldots, n-1-k)$ are first integrals of the system (1.1).

Since (1.1) has $n-2$ meromorphic first integrals $\Phi_1, \ldots, \Phi_k, J_2/J_1, \ldots, J_{n-1-k}/J_1$, by Proposition 3.1 the differential Galois group G of the normal variational equations (3.3) has $n-2$ rational invariants, and then its Lie algebra $\mathcal{G}$ has $n-2$ rational first integrals, denoted by $I_1, \ldots, I_{n-2}$. Let $U \subset \mathbb{C}^{n-1}$ be a neighborhood of 0 such that $I_1, \ldots, I_{n-2}$ are functionally independent of it. Set the level surface

$$H_c = \{\eta \mid I_i(\eta) = c_i,\, \eta \in \mathbb{C}^{n-1},\, i = 1, \ldots, m\},$$

where $c_i \in \mathbb{C}$ are constants such that H_c is regular and $H_c \cap U \neq \emptyset$. Obviously, H_c is a one-dimensional manifold, and for any fixed point $x \in H_c$, the tangent space of H_c corresponding to x is a one-dimensional linear space containing $\mathcal{V}_x = \{Y(x) \mid Y \in \mathcal{G}\}$. Therefore, there exists a non-trivial element $Y_0 \in \mathcal{G}$

such that for any linear vector field $Y \in \mathcal{G}$, there exists a one-to-one rational function $a(x)$ such that $Y = a(x)Y_0$.

Similarly, since (1.1) has a meromorphic Jacobian multiplier J_1, by Lemma 3.6, the elements $Y(x)$ of the Lie algebra $\mathcal{G}$ have the common rational Jacobian multiplier $J_{\text{gal}}(x)$, i.e.,

$$J_{\text{gal}}\nabla \cdot Y + \langle \partial_x J_{\text{gal}}, Y \rangle = J_{\text{gal}}(\langle \partial_x a, Y_0 \rangle + a\nabla \cdot Y_0) + a\langle \partial_x J_{\text{gal}}, Y_0 \rangle \equiv 0. \tag{3.25}$$

In particular, for $a(x) = 1$, we also have

$$J_{\text{gal}}\nabla \cdot Y_0 + \langle \partial_x J_{\text{gal}}, Y_0 \rangle \equiv 0. \tag{3.26}$$

Combining (3.25) and (3.26) yields $Y_0(a) := \langle \partial_x a, Y_0 \rangle = 0$, i.e., $a(x)$ is a rational first integral of the linear vector field Y_0.

Consider any two elements $Y_1 = a(x)Y_0$ and $Y_2 = b(x)Y_0$ of $\mathcal{G}$ with $a(x), b(x) \in \mathbb{C}(x)$. We have $Y_0(a) = Y_0(b) = 0$. Then their Lie bracket

$$[Y_1, Y_2] = [aY_0, bY_0] = (aY_0(b) - bY_0(a))Y_0 - ab[Y_0, Y_0] = 0,$$

which means that the Lie algebra $\mathcal{G}$ is abelian. Hence, the Lie group G^0 is also abelian.

Finally, an abelian group is also a solvable group, and the identity component $\widehat{G}^0$ of the variational equations (3.2) is an extension of the identity component G^0 of the normal variational equations (3.3) by an algebraic group isomorphic to some additive group. This means that G^0 is solvable if and only if $\widehat{G}^0$ is solvable. We complete the proof. □

Corollary 3.8. *If the system* (1.1) *has* $n-1$ *functionally independent meromorphic Jacobian multipliers* $J_1(x), \ldots, J_{n-1}(x)$, *then the identity component of the differential Galois group of the normal variational equations along* $\psi(t)$ *is abelian, and that of variational equations along* $\psi(t)$ *is solvable.*

Proof.   It follows from Theorem 3.7 and the fact that the functional independence of J_i $(i = 1, \ldots, n-1)$ implies the functional independence of J_i/J_1 $(i = 2, \ldots, n-1)$. □

Corollary 3.9. *Assume that the system* (1.1) *is divergence-free, i.e.,* $\text{div}(F) = 0$. *If the system* (1.1) *has* $n - 2$ *functionally independent meromorphic first integrals* $\Phi_1(x), \ldots, \Phi_{n-2}(x)$, *then the identity component of the differential Galois group of the normal variational equations along* $\psi(t)$ *is abelian, and that of variational equations along* $\psi(t)$ *is solvable.*

Proof.   Recall that $\text{div}(F) = 0$ means that the system (1.1) has a Jacobian multiplier $J(x) = 1$. □

Remark 3.10.   In practical applications, as pointed out in [37, Subsection 4.2] or [38, Subsection 5.2], if the variational equations are non-Fuchsian, i.e., they have irregular singularities, one should extend Γ to a new bigger Riemann surface $\overline{\Gamma}$ by adding possible equilibrium points and points at infinity, and should treat the variational equations in a *small* field $\mathcal{M}\overline{\Gamma}$ such as $\mathbb{C}(t)$. Then meromorphic non-integrability near $\overline{\Gamma}$ gives rise to rational non-integrability near Γ.

At the end of this section, we provide two examples to further illustrate Theorem 3.7.

Example 3.11.   Consider the integrable stretch-twist-fold flow

$$\dot{x} = -8xy, \quad \dot{y} = 11x^2 + 3y^2 + z^2 + \beta xz - 3, \quad \dot{z} = 2yz - \beta xy, \tag{3.27}$$

which is proposed to model the stretch-twist-fold mechanism of the magnetic field generation and plays an important role in understanding the fast dynamo action for magnetohydrodynamics [7]. It admits a polynomial first integral $\Phi = x^3(x^2 + y^2 + z^2 - 1)^4$ and a particular solution $(x(t), y(t), z(t)) = (0, -\tanh(3t), 0)$. Then the variational equations of the system (3.27) along this solution are given by

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 8\tanh(3t) & 0 & 0 \\ 0 & -6\tanh(3t) & 0 \\ \beta\tanh(3t) & 0 & -2\tanh(3t) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{3.28}$$

According to Theorem 3.7, the identity component of the differential Galois group of the variational equations (3.28) along this solution is solvable, i.e., (3.28) can be integrable by quadrature. Indeed, one can easily check that

$$x(t) = c_1 \cosh^{8/3}(3t), \quad y(t) = \frac{c_2}{\cosh(6t) + 1}, \quad z(t) = \frac{c_3}{\cosh^{2/3}(3t)} + \frac{c_1 \beta}{10} \cosh^{8/3}(3t)$$

are the general solutions of (3.28).

Example 3.12. Theorem 3.7 can be applied to studying the non-existence of first integrals for some high-dimensional divergence-free systems. As a simple example, consider a dynamical system with the V_4 symmetry group [26], i.e.,

$$\dot{x} = x - yz, \quad \dot{y} = -2y + xz, \quad \dot{z} = z - xy, \tag{3.29}$$

which has a particular solution $(x(t), y(t), z(t)) = (0, 0, e^t)$. The variational equations read

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 1 & -e^t & 0 \\ e^t & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}. \tag{3.30}$$

The equations for (ξ, η) consist of a subsystem

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 1 & -e^t \\ e^t & -2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{3.31}$$

We make a time variable change $\tau = e^t$ to transform (3.31) into

$$\frac{d}{d\tau} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau} & -1 \\ 1 & -\frac{2}{\tau} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{3.32}$$

which is equivalent to the Bessel equation

$$\tau^2 \frac{d^2 \xi}{d\tau^2} + \tau \frac{d\xi}{d\tau} + (\tau^2 - n^2)\xi = 0 \tag{3.33}$$

with $n = 1$. It is well known that the Bessel equation (3.33) has Liouvillian solutions if and only if $n + 1/2$ is an integer (see [37, Chapter 2.8.2]). Hence, we see that the identity is not solvable. Suppose that the system (3.29) has a rational first integral. By Corollary 3.9, the identity component of the differential Galois group of (3.30) and (3.33) is solvable. This leads to a contradiction. Therefore, the system (3.29) has no rational first integrals.

4 An application to the stationary gravity wave problem in finite depth

Many studies have been devoted to the phenomenon of stationary gravity waves, which can lead to the formation of various patterns such as solitary waves, star-shaped waves and hexagon waves. In particular, Witting [46] proposed a new formal series solution of the water wave problem when he studied the solitary wave in a fluid of finite depth. Witting's method has the advantage of employing a systematic procedure and can give rise to higher approximations for the water waves. Karabut [20–22] showed that the problem of exact summation of this series can be reduced to solving or integrating some homogeneous ordinary differential equations, called Karabut systems. The aim of this section is to study Karabut systems from the integrability standpoint with the help of Theorem 3.7.

4.1   The mathematical formulation

To motivate Karabut systems to be studied, we give a brief account of the modeling. Consider an incompressible and irrotational stationary plane flow of a heavy fluid over a flat bottom. In the coordinate system (X, Y), the origin is located on the bottom, the X-axis is directed along the bottom from the left to the right, the Y-axis is directed vertically upward and the free-surface equation is given by $Y = \eta(X)$ (see Figure 1). Without loss of generality, we assume the fluid flows from the left to the right. Then we obtain the flow area

$$\mathcal{D}_1 = \{Z = X + \mathrm{i}Y : -\infty < X < +\infty, \, 0 \leqslant Y \leqslant \eta(X)\}.$$

Denote by $\Phi = \Phi(X, Y)$ the velocity potential and $\Psi = \Psi(X, Y)$ the streamline function. Then the flow in the domain $\mathcal{D}$ is to be potential with the complex potential $\Lambda = \Phi + \mathrm{i}\Psi$, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. In addition, we also denote by u_0 and h_0 the velocity and the depth at some point of the free surface, respectively. For a solitary wave problem, the point is located at infinity. For simplicity, we make a dimensionless rescaling $\chi = \phi + \mathrm{i}\psi = \theta(\Phi + \mathrm{i}\Psi)/h_0 u_0$ in the strip

$$\mathcal{D}_2 = \{\chi = \phi + \mathrm{i}\psi : -\infty \leqslant \phi \leqslant +\infty, \, 0 \leqslant \psi \leqslant \theta\}.$$

Here, the Stokes parameter θ is related to the Froude number $Fr := u_0/\sqrt{gh_0}$ via $Fr = \sqrt{\tan\theta/\theta}$.

  Now the water wave problem is reduced to finding an analytic function $Z = h_0 f(\chi)/\theta$, which maps the domain $\mathcal{D}_2$ conformally onto the domain $\mathcal{D}_1$. Set the function $f(\chi)$ in the form $f(\chi) = \chi + W(\chi)$. At the free surface $Y = \eta(X)$, by the Bernoulli integral, we have

$$\frac{1}{2}\left|\frac{d(\Phi + \mathrm{i}\Psi)}{dZ}\right|^2 + gY = \frac{1}{2}u_0^2 + gh_0,$$

which is equivalent to

$$\left|\frac{dW}{d\chi} + 1\right|^2 = \frac{1}{1 - 2\nu\mathrm{Im}W} \quad \text{for } \psi = \theta \tag{4.1}$$

with $\nu = \cot\theta$. At the flat bottom, we have

$$\mathrm{Im}W = 0 \quad \text{for } \psi = 0. \tag{4.2}$$

Moreover, for the solitary wave, the boundary condition at infinity

$$\lim_{\varphi \to -\infty} \mathrm{Im}W = \lim_{\varphi \to +\infty} \mathrm{Im}W = 0 \tag{4.3}$$

should be fulfilled. In a word, the mathematical statement of the bottom, sides and upper boundary condition is given in (4.1)–(4.3).
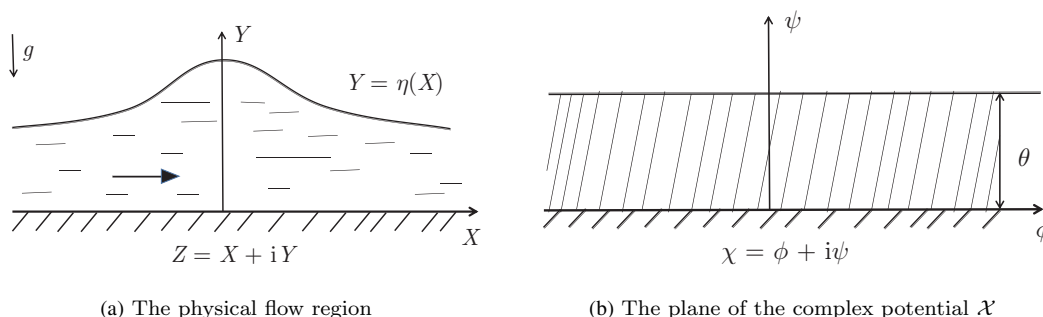

(a) The physical flow region   (b) The plane of the complex potential $\mathcal{X}$

Figure 1   Sketches of the flow and the complex potential

The shallow-water theory shows that it is reasonable to consider $W(\phi + \mathrm{i}\psi)$ as a periodic function of the variable ψ. For small-amplitude waves, one can consider the solution of (4.1)–(4.3) in the form

$$W = \sum_{j=1}^{\infty} \theta^{2j} W^{(j)}(\chi) \tag{4.4}$$

with $W^{(j)}(\chi)$ being polynomials of $\cosh(\chi/2)^{-2}$, which corresponds to the shallow-water expansion. Witting [46] proposed a new expansion parameter for solitary waves and constructed a solution of (4.1)–(4.2) in the form

$$W = \sum_{j=1}^{\infty} E_j(\theta)\zeta^j, \quad \mathrm{Im}E_j = 0, \quad \zeta = \mathrm{e}^\chi, \tag{4.5}$$

which permits calculation to extremely high-order terms. When $\theta = \pi/n$ with n being an odd integer, Karabut [20–22] introduced the new unknown functions

$$P_j(\chi) = W(\zeta \mathrm{e}^{\mathrm{i}(2j-2)\theta}), \quad j = 1, 2, \ldots, n$$

and proved that P_j $(j = 1, 2, \ldots, n)$ satisfy the following system of ordinary differential equations:

$$\left(\frac{dP_{j+1}}{d\chi} + 1\right)\left(\frac{dP_j}{d\chi} + 1\right) = \frac{1}{f_j}, \quad j = 1, 2, \ldots, n, \tag{4.6}$$

where $P_{n+1} = P_1$ and $f_j = 1 + \mathrm{i}\nu(P_{j+1} - P_j)$. Clearly, to solve the boundary-value problem (4.1)–(4.3) in the form of the Witting series, one only needs to integrate the system (4.6) and to take $W = P_1$.

Rewrite the equation (4.6) into the following equation with respect to the variables f_j:

$$\frac{df_j}{d\chi} = \mathrm{i}\nu \frac{\prod_{k=1}^{(n-1)/2} f_{[(2k+j-1) \bmod n]+1} - \prod_{k=1}^{(n-1)/2} f_{[(2k+j-2) \bmod n]+1}}{\sqrt{\prod_{k=1}^{n} f_k}}, \quad j = 1, 2, \ldots, n. \tag{4.7}$$

Making a time scale $dt = \mathrm{i}\nu/\sqrt{\prod_{k=1}^{n} f_k}\, d\chi$ and replacing f_j with x_j, we see that (4.7) becomes the following equivalent homogeneous equations:

$$\frac{dx_j}{dt} = \prod_{k=1}^{(n-1)/2} x_{[(2k+j-1) \bmod n]+1} - \prod_{k=1}^{(n-1)/2} x_{[(2k+j-2) \bmod n]+1}, \quad j = 1, 2, \ldots, n, \tag{4.8}$$

which are called a Karabut system.

4.2 Integrability analysis of the 3-dimensional Karabut systems

For $\theta = \pi/3$, we obtain the 3-dimensional Karabut system

$$\begin{cases} \dot{x}_1 = x_3 - x_2, \\ \dot{x}_2 = x_1 - x_3, \\ \dot{x}_3 = x_2 - x_1. \end{cases} \tag{4.9}$$

Obviously, the system (4.9) is a linear system of differential equations and one can get its general solutions, which implies that the Witting series for $\theta = \pi/3$ can be summed up exactly [20]. From the point of view of integrability, we have the following three results.

Proposition 4.1. *System* (4.9) *is completely integrable with two functionally independent first integrals*

$$I_1 = x_1 x_2 + x_2 x_3 + x_3 x_1, \quad I_2 = x_1 + x_2 + x_3.$$

Using first integrals I_1 and I_2, we can construct infinitely many Hamilton-Poisson realizations $(\mathbb{R}^3, \Pi_{a,b}, H_{c,d})$ of the system (4.9) parameterized by the group $SL(2, \mathbb{C})$, which implies that the system (4.9) is bi-Hamiltonian.

Proposition 4.2. *Let $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 1$. For the system (4.9), there exist infinitely many Hamilton-Poisson realizations $(\mathcal{P}, \Pi_{ab}, H_{cd})$ such that*

$$\dot{\boldsymbol{x}} = \Pi_{a,b} \cdot \nabla H_{c,d}, \quad \boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}},$$

where the Poisson structure $\Pi_{a,b}$ and the Hamiltonian $H_{c,d}$ are given by

$$\Pi_{a,b} = \begin{bmatrix} 0 & -b - ax_1 - ax_2 & b + ax_1 + ax_3 \\ b + ax_1 + ax_2 & 0 & -b - ax_2 - ax_3 \\ -b - ax_1 - ax_3 & b + ax_2 + ax_3 & 0 \end{bmatrix},$$

$$H_{c,d} = c(x_1 x_2 + x_2 x_3 + x_3 x_1) + d(x_1 + x_2 + x_3),$$

where a, b, c and d satisfy $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$.

The next result shows that the 3-dimensional Karabut system admits a Lax formulation.

Proposition 4.3. *The system (4.9) can be written in the Lax form $\dot{L} = [N, L]$, where the matrices L and N are, respectively, given by*

$$L := \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad N := \begin{pmatrix} 0 & -1 - x_3 & 1 + x_2 \\ 1 + x_3 & 0 & -1 - x_1 \\ -1 - x_2 & 1 + x_1 & 0 \end{pmatrix}.$$

4.3 Integrability analysis of the 5-dimensional Karabut systems

For $\theta = \pi/5$, we obtain the 5-dimensional Karabut system

$$\begin{cases} \dot{x}_1 = x_3 x_5 - x_2 x_4, \\ \dot{x}_2 = x_4 x_1 - x_3 x_5, \\ \dot{x}_3 = x_5 x_2 - x_4 x_1, \\ \dot{x}_4 = x_1 x_3 - x_5 x_2, \\ \dot{x}_5 = x_2 x_4 - x_1 x_3. \end{cases} \tag{4.10}$$

Karabut [22] found two functionally independent polynomial first integrals of (4.10), i.e.,

$$\Phi_1 = x_1 + x_2 + x_3 + x_4 + x_5, \quad \Phi_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2.$$

But as he said, *it is not known whether the system has other polynomial integrals*, so that he had to study the system (4.10) by numerical methods. The goal of this subsection is to deal with this problem. More specifically, we investigate its polynomial integrability, namely, *what is the maximal number of functionally independent polynomial first integrals that the system (4.10) can exhibit?*

A nice result is due to Christov [12], in which he used a Morales-Ramis type theory on Bogoyavlenskij integrability [5] and showed that the system (4.10) is not meromorphically integrable in the Bogoyavlenskij sense. Since we do not know if (4.10) admits other symmetry fields besides the trivial symmetry field, we can only conclude from Christov's result that the number of functionally independent polynomial first integrals of the system (4.10) is less than four. Otherwise, (4.10) has four independent polynomial first integrals, so it is integrable in the Bogoyavlenskij sense, which yields a contradiction. For this corollary, there is a simple proof: observing that the Karabut system is a quadratic homogeneous polynomial differential system, we can get its Kovalevskaya exponents $(-1, 1, 2, 3/2 + \sqrt{-7 - 8\mathrm{i}}/2, 3/2 - \sqrt{-7 - 8\mathrm{i}}/2)$ corresponding to a balance $c = (-\mathrm{i}, -1, 1, \mathrm{i}, 0)$, where $\mathrm{i} = \sqrt{-1}$. Then by [15, Theorem 5.5], the system (4.10) does not have four functionally independent polynomial first integrals. The non-rational Kovalevskaya exponents also imply that the system (4.10) does not enjoy the Painlevé/weak-Painlevé property. In addition, we mention that the analysis of differential Galois groups in Christov's work [12]

is obtained by computing local monodromy matrices. As our aim is to give a practical example to show the effectiveness of Theorem 3.7, we provide a systematic procedure to analyze the differential Galois group associated with Karabut systems with the help of Kovacic's algorithm, which can be applied to other high-dimensional systems.

Theorem 4.4.   *System* (4.10) *has two and only two functionally independent meromorphic first integrals.*

A direct consequence is that the system (4.10) has no additional polynomial first integrals which are functionally independent of Φ_1 and Φ_2.

Proof of Theorem 4.4.   The basic idea of our proof is to find an integrable invariant manifold $\mathcal{N}$ for the 5-dimensional Karabut system (4.10), to reduce the (normal) variational equations along a solution contained in $\mathcal{N}$ into a second-order equation with rational coefficients, and to prove that the identity component of the differential Galois group associated with this second-order equation is not solvable.

System (4.10) admits an *integrable invariant manifold* defined by

$$\mathcal{N} := \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 : x_1 + x_4 = 0, \ x_2 + x_3 = 0, \ x_5 = 0\}.$$

Indeed, the system (4.10) restricted to $\mathcal{N}$ is given by

$$\dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = -x_1^2, \tag{4.11}$$

which is completely integrable with a polynomial first integral $\Phi = x_1^2 + x_2^2$. Given the initial condition $(x_1(t_0), x_2(t_0)) = (x_{10}, x_{20}) \neq (0, 0)$, we get

$$x_2 = \pm\sqrt{x_{10}^2 + x_{20}^2 - x_1^2}, \quad \pm\frac{dx_1}{x_1\sqrt{x_{10}^2 + x_{20}^2 - x_1^2}} = dt,$$

i.e.,

$$x_1(t) = \frac{4\exp(\frac{t+C_2}{C_1})}{4 + C_1^2 \exp(\frac{2t+2C_2}{C_1})}, \quad x_2(t) = \frac{\dot{x}_1}{x_1},$$

or

$$x_1(t) = \frac{4\exp(\frac{t+C_2}{C_1})}{C_1^2 + 4\exp(\frac{2t+2C_2}{C_1})}, \quad x_2(t) = \frac{\dot{x}_1}{x_1},$$

where $C_1 = (x_{10}^2 + x_{20}^2)^{-1/2}$ and C_2 is an integration constant. For convenience, we set $C_1 = 1$ and $C_2 = 0$ and get a non-equilibrium solution of the system (4.10), i.e.,

$$\phi(t) = \left(\frac{4\exp(t)}{\exp(2t) + 4}, \frac{4 - \exp(2t)}{\exp(2t) + 4}, \frac{\exp(2t) - 4}{\exp(2t) + 4}, -\frac{4\exp(t)}{\exp(2t) + 4}, 0\right).$$

Let Γ be the phase curve corresponding to the solution $\phi(t)$. Then the variational equations along Γ read

$$\frac{d}{dt}\boldsymbol{\xi} = \boldsymbol{A}(t)\boldsymbol{\xi}, \tag{4.12}$$

where

$$\boldsymbol{A}(t) = \begin{pmatrix} 0 & \frac{4\exp(t)}{\exp(2t)+4} & 0 & \frac{\exp(2t)-4}{\exp(2t)+4} & \frac{\exp(2t)-4}{\exp(2t)+4} \\ -\frac{4\exp(t)}{\exp(2t)+4} & 0 & 0 & \frac{4\exp(t)}{\exp(2t)+4} & \frac{4-\exp(2t)}{\exp(2t)+4} \\ \frac{4\exp(t)}{\exp(2t)+4} & 0 & 0 & -\frac{4\exp(t)}{\exp(2t)+4} & \frac{4-\exp(2t)}{\exp(2t)+4} \\ \frac{\exp(2t)-4}{\exp(2t)+4} & 0 & \frac{4\exp(t)}{\exp(2t)+4} & 0 & \frac{\exp(2t)-4}{\exp(2t)+4} \\ \frac{4-\exp(2t)}{\exp(2t)+4} & -\frac{4\exp(t)}{\exp(2t)+4} & -\frac{4\exp(t)}{\exp(2t)+4} & \frac{4-\exp(2t)}{\exp(2t)+4} & 0 \end{pmatrix}.$$

Since the system (4.10) has two first integrals Φ_1 and Φ_2, by Lemma 2.13, the system (4.12) has two time-dependent first integrals

$$G_1(t, \xi) = \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5, \quad G_2(t, \xi) = \frac{8\exp(t)}{\exp(2t) + 4}(\xi_1 - \xi_4) + \frac{8 - 2\exp(2t)}{\exp(2t) + 4}(\xi_2 - \xi_3). \tag{4.13}$$

Then by means of the change of the independent variable

$$t \to \tau := \exp(t) - 4\exp(-t), \tag{4.14}$$

we transform (4.12) into the linear differential system with rational coefficients

$$\frac{d}{d\tau}\boldsymbol{\xi} = \boldsymbol{B}(\tau)\boldsymbol{\xi}, \tag{4.15}$$

where

$$\boldsymbol{B}(t) = \begin{pmatrix} 0 & \frac{4}{16+\tau^2} & 0 & \frac{\tau}{16+\tau^2} & \frac{\tau}{16+\tau^2} \\ -\frac{4}{16+\tau^2} & 0 & 0 & \frac{4}{16+\tau^2} & -\frac{\tau}{16+\tau^2} \\ \frac{4}{16+\tau^2} & 0 & 0 & -\frac{4}{16+\tau^2} & -\frac{\tau}{16+\tau^2} \\ \frac{\tau}{16+\tau^2} & 0 & \frac{4}{16+\tau^2} & 0 & \frac{\tau}{16+\tau^2} \\ -\frac{\tau}{16+\tau^2} & -\frac{4}{16+\tau^2} & -\frac{4}{16+\tau^2} & -\frac{\tau}{16+\tau^2} & 0 \end{pmatrix}.$$

It should be pointed out that the transformation (4.14) does not change the identity component of the differential Galois group (see [37, Theorem 2.5]). Make a linear transformation $\boldsymbol{\xi} = \boldsymbol{P}\boldsymbol{\eta}$ with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4.16}$$

and the system (4.15) becomes the following equivalent form:

$$\frac{d}{d\tau}\boldsymbol{\eta} = \boldsymbol{C}(t)\boldsymbol{\eta}, \tag{4.17}$$

where

$$\boldsymbol{C}(t) = \begin{pmatrix} \frac{\tau}{16+\tau^2} & \frac{4}{16+\tau^2} & \frac{2\tau}{16+\tau^2} & 0 & 0 \\ 0 & 0 & -\frac{2\tau}{16+\tau^2} & 0 & 0 \\ -\frac{\tau}{16+\tau^2} & -\frac{4}{16+\tau^2} & 0 & 0 & 0 \\ \frac{4}{16+\tau^2} & 0 & -\frac{\tau}{16+\tau^2} & 0 & -\frac{8}{16+\tau^2} \\ \frac{\tau}{16+\tau^2} & 0 & \frac{\tau}{16+\tau^2} & \frac{4}{16+\tau^2} & -\frac{\tau}{16+\tau^2} \end{pmatrix}.$$

Then we obtain a 3-dimensional subsystem

$$\frac{d}{d\tau}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \frac{\tau}{16+\tau^2} & \frac{4}{16+\tau^2} & \frac{2\tau}{16+\tau^2} \\ 0 & 0 & -\frac{2\tau}{16+\tau^2} \\ -\frac{\tau}{16+\tau^2} & -\frac{4}{16+\tau^2} & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}. \tag{4.18}$$

Due to (4.13) and (4.16), one can easily check that $H(\eta_1, \eta_2, \eta_3) = \eta_1(t) + \eta_2(t) + \eta_3(t)$ is a first integral of (4.18), i.e.,

$$\frac{d}{dt}(\eta_1(t) + \eta_2(t) + \eta_3(t)) = 0,$$

or $\eta_1(t) + \eta_2(t) + \eta_3(t) = \text{constant}$. Let this constant be zero. Then by (4.18), we obtain

$$\frac{d}{d\tau}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{\tau}{16+\tau^2} & \frac{4-2\tau}{16+\tau^2} \\ \frac{2\tau}{16+\tau^2} & \frac{2\tau}{16+\tau^2} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{4.19}$$

From the first equation of (4.19), we have

$$\eta_2 = \frac{16 + \tau^2}{4 - 2\tau} \frac{d\eta_1}{d\tau} + \frac{\tau\eta_1}{4 - 2\tau}. \tag{4.20}$$

Substituting (4.20) into the second equation of (4.19), we can eliminate the variable η_2 and get an equivalent second-order equation

$$\frac{d^2}{d\tau^2}\eta_1 + P(\tau)\frac{d}{d\tau}\eta_1 + Q(\tau)\eta_1 = 0, \tag{4.21}$$

where the coefficients are as follows:

$$P = -\frac{2\tau + 16}{(\tau - 2)(\tau^2 + 16)}, \quad Q = \frac{2\tau^3 - 14\tau^2 + 16\tau - 32}{(\tau + 16)^2(\tau - 2)}.$$

Next, under the change of the dependent variable

$$\eta_1 = \chi \exp\left[-\frac{1}{2}\int P d\tau\right],$$

(4.21) is converted to its reduced form

$$\frac{d^2}{d\tau^2}\chi = r(\tau)\chi \tag{4.22}$$

with

$$r = \frac{3}{4(\tau - 2)^2} - \frac{11 - 8i}{16(\tau + 4i)^2} - \frac{11 + 8i}{16(\tau - 4i)^2} + \frac{1}{20(\tau - 2)} - \frac{8 + 59i}{320(\tau + 4i)} - \frac{8 - 59i}{320(\tau - 4i)}.$$

We claim that the identity component of the differential Galois group of the system (4.22) is not solvable. To this end, we use the Kovacic's algorithm [23]. Obviously, (4.22) is Fuchsian (see [37]) with four regular singular points τ equaling 2, 4i, $-4i$ and ∞, and all of them are of order two. For Case 1 in Kovacic's algorithm, by simple computations, we get

$$\alpha_\infty^\pm = \frac{1}{2} \pm \frac{\sqrt{7}i}{2}, \quad \alpha_2^+ = \frac{3}{2}, \quad \alpha_2^- = \frac{1}{2},$$

$$\alpha_{4i}^\pm = \frac{1}{2} \pm \frac{\sqrt{-7 - 8i}}{4}, \quad \alpha_{-4i}^\pm = \frac{1}{2} \pm \frac{\sqrt{-7 + 8i}}{4},$$

and then $d := \alpha_\infty^\pm - \alpha_2^\pm - \alpha_{4i}^\pm - \alpha_{-4i}^\pm \in \mathbb{C}/\mathbb{R}$ is not a non-negative integer, which means that Case 1 of Lemma 2.6 is impossible. For Case 2, we can obtain the auxiliary sets E_i from Kovacic's algorithm

$$E_\infty = E_{4i} = E_{-4i} = \{2\}, \quad E_2 = \{-2, 2, 6\}.$$

Next, we should select elements $e_\infty \in E_\infty$, $e_{\pm 4i} \in E_{\pm 4i}$ and $e_2 \in E_2$ such that $d := (e_\infty - e_{4i} - e_{-4i} - e_2)/2$ is a non-negative integer. There is only one possible choice of $(e_\infty, e_{4i}, e_{-4i}, e_2) = (2, 2, 2, -2)$ with $d = 0$. We construct the rational function

$$\theta(\tau) = \frac{-1}{\tau - 2} + \frac{2\tau}{\tau^2 + 16},$$

and check if there exists a monic polynomial P of degree zero satisfying the equation

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4r)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r')P = 0. \tag{4.23}$$

However,

$$\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r' = \frac{8\tau^2 + 52\tau + 224}{(\tau - 2)^2(\tau^2 + 16)^2} \neq 0.$$

Hence, Case 2 of Lemma 2.6 is also excluded. For Case 3, by [23, Theorem 2.1], it is also impossible that the differential Galois group is finite since the necessary conditions cannot hold. In summary, the

differential Galois group of (4.22) falls into Case 4 of Lemma 2.6 and is $SL(2, \mathbb{C})$. In this case, the identity component is also $SL(2, \mathbb{C})$, which is not solvable.

Assume that the system (4.10) has three functionally independent meromorphic first integrals. Due to Corollary 3.9, the identity component of the differential Galois group of (4.12), (4.15) or its equivalent form (4.17) is solvable. Thanks to Lemma 2.12, the system (4.18) can be solved by quadrature. So the system (4.21) is also solvable. Finally, note that (4.21) and (4.22) have the same Liouvillian solvability, which implies that (4.22) is solvable, i.e., the identity component of (4.22) is solvable. This leads to a contradiction. □

Remark 4.5. Similar to $3D$ and $5D$ Karabut systems, Karabut systems with dimension $n \geqslant 7$ also have two first integrals $\Phi_1 = x_1 + \cdots + x_n$ and $\Phi_2 = x_1^2 + \cdots + x_n^2$, but their integrability analysis is still open. The main difficulty is to find an analytic particular solution lying in an integrable invariant manifold. The numerical simulations in [22] inspire us to believe that Karabut systems with dimension $n \geqslant 7$ are also not integrable either.

References

1   Acosta-Humánez P B, Álvarez-Ramírez M, Delgado J. Non-integrability of some few body problems in two degrees of freedom. Qual Theory Dyn Syst, 2009, 8: 209–239

2   Acosta-Humánez P B, Álvarez-Ramírez M, Stuchi T J. Nonintegrability of the Armbruster-Guckenheimer-Kim quartic Hamiltonian through Morales-Ramis theory. SIAM J Appl Dyn Syst, 2018, 17: 78–96

3   Acosta-Humánez P B, Blázquez-Sanz D. Non-integrability of some Hamiltonians with rational potentials. Discrete Contin Dyn Syst Ser B, 2008, 10: 265–293

4   Arnold V I. Ordinary Differential Equations. Berlin: Springer-Verlag, 1992

5   Ayoul M, Zung N T. Galoisian obstructions to non-Hamiltonian integrability. C R Math Acad Sci Paris, 2010, 348: 1323–1326

6   Baider A, Churchill R C, Rod D L, et al. On the infinitesimal geometry of integrable systems. In: Mechanics Day. Fields Institute Communications, vol. 7. Providence: Amer Math Soc, 1996, 7: 5–56

7   Bajer K, Moffatt H K. On a class of steady confined Stokes flows with chaotic streamlines. J Fluid Mech, 1990, 212: 337–363

8   Bogoyavlenskij O I. Extended integrability and bi-Hamiltonian systems. Comm Math Phys, 1998, 196: 19–51

9   Bolsinov A V, Taimanov I A. Integrable geodesic flows with positive topological entropy. Invent Math, 2000, 140: 639–650

10   Bountis T. Investigating non-integrability and chaos in complex time. Phys D, 1995, 86: 256–267

11   Casale G. Morales-Ramis theorems via Malgrange pseudogroup. Ann Inst Fourier (Grenoble), 2009, 59: 2593–2610

12   Christov O. Non-integrability of the Karabut system. Nonlinear Anal Real World Appl, 2016, 32: 91–97

13   Churchill R C, Rod D L, Singer M F. Group-theoretic obstructions to integrability. Ergodic Theory Dynam Systems, 1995, 15: 15–48

14   Duval G, Maciejewski A J. Jordan obstruction to the integrability of Hamiltonian systems with homogeneous potentials. Ann Inst Fourier (Grenoble), 2009, 59: 2839–2890

15   Goriely A. Integrability and Nonintegrability of Dynamical Systems. Singapore: World Scientific, 2001

16   Hu Y X. On the first integrals of n-th order autonomous systems. J Math Anal Appl, 2018, 459: 1062–1078

17   Jacobi C G J. C. G. J. Jacobi's Vorlesungen über Dynamik. Berlin: G. Reimer, 1884

18   Jiao J, Huang K Y, Liu W S. Stationary shear flows of nematic liquid crystals: A comprehensive study via Ericksen-Leslie model. J Dynam Differential Equations, 2022, 34: 239–269

19   Jovanović B. Geometry and integrability of Euler-Poincaré-Suslov equations. Nonlinearity, 2001, 14: 1555–1567

20   Karabut E A. Summation of the Witting series in the solitary wave problem. Sib Math J, 1995, 36: 287–304

21   Karabut E A. Asymptotic expansions in the problem of a solitary wave. J Fluid Mech, 1996, 319: 109–123

22   Karabut E A. Summation of the Witting series in the solitary-wave problem. J Appl Mech Tech Phys, 1999, 40: 36–45

23   Kovacic J J. An algorithm for solving second order linear homogeneous differential equations. J Symbolic Comput,

        1986, 2: 3–43

24   Kozlov V V. Symmetries, Topology and Resonances in Hamiltonian Mechanics. Berlin: Springer-Verlag, 1996

25   Kozlov V V. The Euler-Jacobi-Lie integrability theorem. Regul Chaotic Dyn, 2013, 18: 329–343

26   Letellier C, Gilmore R. Symmetry groups for 3D dynamical systems. J Phys A, 2007, 40: 5597–5620

27   Li W L, Shi S Y. Galoisian obstruction to the integrability of general dynamical systems. J Differential Equations, 2012, 252: 5518–5534

28   Llibre J, Messias M. Global dynamics of the Rikitake system. Phys D, 2009, 238: 241–252

29   Llibre J, Tian Y Z. Dynamics of the FitzHugh-Nagumo system having invariant algebraic surfaces. Z Angew Math Phys, 2021, 72: 15

30   Llibre J, Valls C, Zhang X. The completely integrable differential systems are essentially linear differential systems. J Nonlinear Sci, 2015, 25: 815–826

31   Maciejewski A J, Przybylska M. All meromorphically integrable 2D Hamiltonian systems with homogeneous potential of degree 3. Phys Lett A, 2004, 327: 461–473

32   Maciejewski A J, Przybylska M. Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential. J Math Phys, 2005, 46: 062901

33   Maciejewski A J, Przybylska M. Integrability analysis of the stretch-twist-fold flow. J Nonlinear Sci, 2020, 30: 1607–1649

34   Maciejewski A J, Przybylska M, Stachowiak T. Nonexistence of the final first integral in the Zipoy-Voorhees space-time. Phys Rev D, 2013, 88: 064003

35   Maciejewski A J, Przybylska M, Yaremko Y. Dynamics of a dipole in a stationary electromagnetic field. Proc R Soc A Math Phys Eng Sci, 2019, 475: 20190230

36   Maciejewski A J, Przybylska M, Yoshida H. Necessary conditions for super-integrability of Hamiltonian systems. Phys Lett A, 2008, 372: 5581–5587

37   Morales-Ruiz J J. Differential Galois Theory and Non-Integrability of Hamiltonian Systems. Basel: Birkhäuser, 1999

38   Morales-Ruiz J J, Ramis J P. Galoisian obstructions to integrability of Hamiltonian systems. Methods Appl Anal, 2001, 8: 33–95

39   Morales-Ruiz J J, Simó C. Picard-Vessiot theory and Ziglin's theorem. J Differential Equations, 1994, 107: 140–162

40   Morales-Ruiz J J, Simó C, Simon S. Algebraic proof of the non-integrability of Hill's problem. Ergodic Theory Dynam Systems, 2005, 25: 1237–1256

41   Przybylska M. Differential Galois obstructions for integrability of homogeneous Newton equations. J Math Phys, 2008, 49: 022701

42   Singer M F, Ulmer F. Galois groups of second and third order linear differential equations. J Symbolic Comput, 1993, 16: 9–36

43   Szumiński W, Woźniak D. Dynamics and integrability analysis of two pendulums coupled by a spring. Commun Nonlinear Sci Numer Simul, 2020, 83: 105099

44   Tsygvintsev A. The meromorphic non-integrability of the three-body problem. J Reine Angew Math, 2001, 537: 127–149

45   van der Put M, Singer M F. Galois Theory of Linear Differential Equations. Berlin-Heidelberg: Springer-Verlag, 2003

46   Witting J. On the highest and other solitary waves. J Appl Math, 1975, 28: 700–719

47   Yagasaki K. Galoisian obstructions to integrability and Melnikov criteria for chaos in two-degree-of-freedom Hamiltonian systems with saddle centres. Nonlinearity, 2003, 16: 2003–2012

48   Zhang X. Liouvillian integrability of polynomial differential systems. Trans Amer Math Soc, 2016, 368: 607–620

49   Zhang X. Integrability of Dynamical Systems: Algebra and Analysis. Singapore: Springer, 2017

50   Ziglin S L. Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics. I. Funct Anal Appl, 1983, 16: 181–189