Preliminary Research on the Mathematical Methods and Order of Problems in the “*Fangcheng*” Chapter of Yang Hui’s *Mathematical Methods* (1261 CE)

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Abstract: Yang Hui was one of the most important authors of mathematical works during the thirteenth century. *Mathematical Methods Explaining in Detail The Nine Chapters* (*Xiangjie jiuzhang suanfa* 詳解九章算法, 1261 CE) is the earliest extant work attributed to Yang Hui. From the thirteenth to the fifteenth century, this work played a crucial role in the circulation and popularization of *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu* 九章算術). However, the only surviving printed edition of *Mathematical Methods* is incomplete and contains many mistakes obstructing contemporary researchers’ understanding of this work. The “*Fangcheng*” chapter of *The Nine Chapters* deals with problems related to solving what today are known as simultaneous sets of linear equations. However, interpreting the text in this chapter of *Mathematical Methods* and recovering the mathematical practices relating to *fangcheng* are difficult. Through detailed textual and mathematical analyses, the author of this paper explains Yang Hui’s understanding and practice relating to “the *fangcheng* method” and “the method of the positive and the negative.” This paper includes an appendix that provides a detailed translation of the ambiguous text relating to “the method of the positive and the negative” and gives reasons supporting the interpretation provided here. Yang Hui’s understanding of the concepts of “positive” and “negative” and his practice relating to these two concepts may easily be confused with their apparent counterparts in modern mathematics. Also, careful analysis of the mathematical methods in this work reveal that the order of problems in Yang Hui’s *Reclassifications of Mathematical Methods Explaining in Detail The Nine Chapters* ([*Xiangjie jiuzhang suanfa zuanlei* 詳解九章算法纂類],

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namely, the last section of *Mathematical Methods*) were rearranged according to commentaries to specific methods that appear in *Mathematical Methods*. Some textual clues referring to the “previous question” (*qianwen* 前問) in certain commentaries of *Mathematical Methods* indeed reflect the order of problems in *Reclassifications*. Yang Hui made especially detailed commentaries on the problems that he arranged in a sequence that differs with respect to the original order of problems as they appear in the ancient classic work, *The Nine Chapters*. All these discoveries reveal and serve to prove a close relationship between Yang Hui’s *Mathematical Methods* and his *Reclassifications*.

Keywords: *Mathematical Methods Explaining in Detail The Nine Chapters*, *Reclassifications of Mathematical Methods Explaining in Detail The Nine Chapters*, *The Nine Chapters*, Yang Hui, the *fangcheng* method, the method of the positive and the negative, translations of ancient mathematical writing

摘 要: 杨辉是中国 13 世纪数学著作的重要作者之一,《详解九章算法》(1261 CE)是现存杨辉最早的一部著作。13~15 世纪,杨辉的这部著作在《九章算术》的流传与流行中起到了关键的作用。然而,现存的《详解》刻本残缺不全,并且包含大量舛误,这些阻碍了当代研究者对这部著作的理解。《九章算术》的方程章处理有关我们现在所视作线性方程组的问题,然而阐释《详解》中该章的文本并且恢复与方程有关的数学操作却有困难。通过细致的文本和算法分析,本文作者解释了杨辉对于“方程法”和“正负法”的理解和操作。《详解》中与“正负法”相关的一些文字含义模糊,本文的附录是对这些文字的细致的翻译,并且给出了支持这种阐释的理由。杨辉对于“正”和“负”概念的理解以及他的与这两个概念相关的操作,很容易与现代数学中的相应内容相混淆。对这部著作中算法的仔细分析揭示了杨辉《纂类》(即《详解》最后一部分)中问题的顺序是依据《详解》中算法的注释而安排的。《详解》注释中的一些文本线索提到“前问”,实际上反映的是《纂类》中的题目顺序。杨辉重新安排了一些题目的顺序,使之不同于古代经典《九章》中原来的顺序,并对这些改变顺序的题目做了详细的注释。所有这些发现揭示并证实了《详解》和《纂类》之间的密切关系。

关键词:《详解九章算法》,《详解九章算法纂类》,《九章算术》,杨辉,方程法,正负法,古代数学著作翻译

1 Introduction

In general twentieth-century scholarship on the history of mathematics, mathematics around the period from the tenth to the fourteenth century has often been regarded as “of high level” in the history of mathematics in China. For example, the fifth chapter of *Chinese Mathematics: A Concise History* (1987), written by Li Yan and Du Shiran,² is entitled “the zenith of the development of mathematics during the Song (960–1279 CE) and Yuan (1271–1368 CE) dynasties” (Li and Du 1987, 109). In such works that hold this positive assessment of Song-Yuan mathematics, historians usually take the thirteenth-

2 This book was translated from *A Concise History of Ancient Mathematics in China* (*Zhongguo gudai shuxue jianshi* 中国古代数学简史, 1963). Du Shiran 杜石然 (1929–) mentioned that the Chinese version of this book was written by himself (Du 2003, 684–91).

century practitioner of mathematics, Yang Hui (fl. 1261 CE), as one of the representatives of mathematicians of this period.

The exact dates of the birth and death of Yang Hui are still unknown. All his works were finished in the second half of the thirteenth century. Several mathematical works attributed to him have survived and are listed below.³

*Mathematical Methods Explaining in Detail The Nine Chapters (Xiangjie jiuzhang suanfa 詳解九章算法, 1261 CE, hereafter Mathematical Methods),*⁴

*Precious Reckoner of Variations of the Multiplication and Division (Chengchu tongbian suanbao 乘除通變算寶, 1274 CE),*⁵

*Selection of Unusual Mathematical Methods in the Wake of the Ancients (Xugu zhaiqi suanfa 續古摘奇算法, 1275 CE),*⁶

*Quick Methods for Multiplication and Division for the Surface of the Fields and Analogous Problems (Tianmu bilei chengchu jiefa 田畝比類乘除捷法, 1275 CE).*⁷

These books had a great influence on mathematical works over the following several centuries (Guo 2002). Some of his works were also transmitted to Korea and

3 For an introduction in English to the contents of Yang Hui's mathematical works, see Ho (1980, 538–46). For a critical study of some works of Yang Hui, see Lam (1977).

4 In Lam (1977), Li and Du (1987), and Shen, Crossley, and Lun (1999), the title of this book is translated as *A Detailed Analysis of the Mathematical Methods in the Nine Chapters*. Jean-Claude Martzloff (1997, 157) translates it as *Detailed Explanations of the Computational Methods in the Nine Chapters*. However, it is the words “mathematical methods” (*suanfa* 算法) that express the core of Yang Hui's work (also see Kong 1993), and the two expressions “mathematical methods” and “the nine chapters” in this title should not be understood as a combination that is the object of the activity of Yang Hui's “detailed analysis.” The main body of Yang Hui's work is the “mathematical methods” that were used to explain the contents of *The Nine Chapters*. This paper will demonstrate the core status of “mathematical methods” in this work in support of this interpretation. It is also the reason why, in this paper, *Mathematical Methods* is used as an abbreviation of *Mathematical Methods Explaining in Detail The Nine Chapters*, even though by convention the book's title has usually been abbreviated to “Xiangjie” 詳解 (lit. detailed explanation), to which refer the first two characters of this title. In addition, the abbreviation “*Mathematical Methods*” should not be confused with another of Yang Hui's works entitled *Yang Hui's Mathematical Methods (Yang Hui suanfa 楊輝算法)*. Since the latter is a collection of three treatises, when mentioning them the treatise's precise title will be used.

5 This title is translated as *Precious Reckoner for Mutually Varying Quantities* in Needham and Wang (1959, 696); is translated as *Precious Reckoner for Variations of Multiplication and Division* in Lam (1977, 3), and Li and Du (1987, 287); and is translated as *Computational Treasure of the Multiplicity of Variations of Multiplication and Division* in Martzloff (1997, 157).

6 This title is translated as *Continuation of Ancient Mathematical Methods for Elucidating the Strange (Properties of Numbers)* in Needham and Wang (1959, 703); Lam (1977, 137); and Li and Du (1987, 280). It was translated as *Continuation of the Tradition of Strange Computational Methods* in Martzloff (1997, 158).

7 This title is translated as *Practical Rules of Arithmetic for Surveying* in Lam (1977, 81), and Li and Du (1987, 287); and is translated as *Quick Methods of Multiplication and Division for the Computation of the Number of Mu of Fields, Analogies* in Martzloff (1997, 158).

Japan and studied by scholars there (Guo Shirong 2009, 163–87).

In the section *Reclassifications*, which is placed at the end of the printed edition of *Mathematical Methods*, Yang Hui reclassifies questions and answers from *The Nine Chapters* into his own categories and subcategories. The main questions which will be addressed in this paper are as follows. By comparing the extant complete text of *The Nine Chapters* with that in the printed *Mathematical Methods* and with that in *Reclassifications*, this research paper will address issues concerning the thirteenth-century Yang Hui's understanding and interpretation of the traditional mathematical procedures in *The Nine Chapters*. Moreover, such comparison also enables an in-depth understanding of Yang Hui's way of making categories and his ideas about the order of problems in the *Reclassifications*. Among all the texts, those in the chapter entitled “*Fangcheng*” 方程 in the related works can provide some clear evidence to answer these inquiries. Before turning to these pieces of evidence, an introduction to *Mathematical Methods* and its editions and emendations, and an introduction to the “*Fangcheng*” chapter will serve to further highlight several important issues.

1.1 Introduction to *Mathematical Methods Explaining in Detail The Nine Chapters*

The text that is the focus of this paper is extracted from Yang Hui's *Mathematical Methods*, and an introduction to this work must begin with an introduction to *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu* 九章算術, abbreviated as *The Nine Chapters*). *The Nine Chapters* is one of the most important mathematical works from ancient China. The completion date of *The Nine Chapters* in the form handed down is still in debate, but historians agree to place it somewhere between the first century BCE and the first century CE. In all the editions of *The Nine Chapters* that were handed down, there are two layers of commentary: one completed by Liu Hui 劉徽 (fl. the third century CE) in 263 CE, and the other completed under Li Chunfeng's 李淳風 (602–670 CE) supervision in 656 CE. In modern scholars' research, these commentaries are regarded as a part of the Canon of *The Nine Chapters*. In this paper, the expression “texts of *The Nine Chapters*” refers to problems, answers, procedures (*shu* 術) and commentaries written before the eighth century.

In the Song dynasty (960–1279 CE), scholars resumed writing commentaries on *The Nine Chapters*. This is the case of Jia Xian 賈憲 (fl. the eleventh century) who composed “detailed procedures of calculation” (*xicao* 細草) to this book, and Jia Xian's book was in turn entitled *The Yellow Emperor's Mathematical Canon in The Nine Chapters with Detailed Procedures of Calculation* (*Huangdi jiuzhang suanjing xicao* 黃帝九章算經細草). In 1261 Yang Hui relied on *The Nine Chapters* containing the commentaries of Liu Hui, Li

Chunfeng, and Jia Xian to compose his own commentaries on all these older layers of text. This work, entitled *Mathematical Methods Explaining in Detail The Nine Chapters*, was concluded by a final section *Reclassifications of Mathematical Methods Explaining in Detail The Nine Chapters* (Xiangjie jiuzhang suanfa zuanlei 詳解九章算法纂類, hereafter, *Reclassifications*). Each chapter of *The Nine Chapters* was composed of questions, answers, and procedures. In *Reclassifications*, Yang reclassifies the questions and answers from *The Nine Chapters* into his categories and subcategories (Sha 1993). *Mathematical Methods* is also the earliest among his extant work.

The main body of *Mathematical Methods* came from *The Nine Chapters* and its commentaries made by the two earlier commentators. Other excerpts of this work attributed to Yang Hui or Jia Xian⁸ usually begin with the expressions: “explanation of the problem” (*jieti* 解題), “the method says” (*fayue* 法曰), “the procedure of calculation says” (*caoyue* 草曰), and “analogy” (*bilei* 比類).

There are only a few sentences that follow the expression “explanation of the problem.” These passages give the key points of Yang Hui’s explanation of the problem from *The Nine Chapters* or a problem he has added. The passages after the expression “the method says” are more complicated. An important issue to be analyzed below is that the mathematical methods to which these passages refer are usually different from the methods that are placed after the expression “the procedure says” (*shu* 術曰) in *The Nine Chapters*.

The passages placed after the expression “the procedure of calculation says” are as a rule closely related to the texts placed after the expressions “the method says” in *Mathematical Methods* or “the procedure says” of *The Nine Chapters*. In these passages, Yang Hui often repeats the big-sized characters in “the method” or “the procedure” in the printed edition of *Mathematical Methods*, and he also provides numerical values mentioned in the question and used in the calculation, as well as values that were yielded by the process of calculation described in “the method” or “the procedure.” These numerical values are usually written in small-sized characters.

Under the expression “analogy,” Yang Hui usually gives one or two problems that share a certain similarity with the problem from *The Nine Chapters*. These problems could be called “analogical problems.” Depending on the case dealt with, the solution given to analogical problems follows what is introduced by the “method” or by the “procedure.”⁹

8 For the issue of the authorship of different texts in *Mathematical Methods*, see Guo (1988). I shall return to this topic and make a short remark on the former research later.

9 For research on the topic of “analogy” in the history of mathematics in China, see Volkov (1992) and Bréard (2019, 107–42). The latter translates the word “bilei” as “comparable categories.”

1.2 The edition and emendations of *Mathematical Methods*

Guo Shuchun 郭书春 has made a statistical analysis of the texts under the expressions “the method (procedure) says,” “the procedure of calculation says,” “explanation of the problem,” and “analogy” placed after each problem in the extant printed editions of *Mathematical Methods*. He suggests that his results (the extant incomplete edition of *Mathematical Methods* includes ninety-two problems containing text that does not belong to *The Nine Chapters*) contradict Yang Hui’s declaration in the preface: “[I (Yang Hui)] selected eighty problems as examples (*jinshi* 矜式)” (择八十题以為矜式) (Guo 1988, 330). To offer a solution to the contradiction that he suggested,¹⁰ Guo claims that, in this book, a considerable amount of the content (the texts now in *Mathematical Methods* but not belonging to *The Nine Chapters*) has to be attributed to Jia Xian, rather than to Yang Hui (Guo 1988, 333).¹¹ This conclusion was very inspiring, drawing the attention of modern scholars to the authorship of different layers of the text in this work. This issue does, however, require further discussion. In fact, in *Mathematical Methods* there are two examples that refer to two different layers of texts. Clues in them clearly demonstrate that one layer of text was written by Yang Hui, while the other layer, to which Yang refers, was composed by a person writing prior to him, but after Liu Hui and Li Chunfeng. Only in these places can one assert, therefore, that the author of the text of the second layer is Jia Xian.¹² This paper will not address further the issue of authorship, and will tentatively still hold to the conventional view of Yang Hui as the author of *Mathematical Methods*, though this does not exclude the possibility that parts of the text may be attributed to Jia Xian. In some parts of this paper, “Yang Hui” could be replaced by an expression “Yang/Jia,” representing the uncertainty over the actual authorship.

In order to gain a better understanding of the thesis of this paper, namely the mathematical methods and the order of problems in *Mathematical Methods*, it is necessary to be clear about how the work has been handed down. There is only one

10 *Stricto sensu*, there is a contradiction only if one takes for granted that the problems containing text that do not belong to *The Nine Chapters* are exactly these “examples” (*jinshi* 矜式). However, the context from which the sentence “[I (Yang Hui)] selected eighty questions to be examples” comes only expresses the fact that Yang Hui added text to eighty problems. These problems (in Yang Hui’s word, *jinshi*), are likely to be only certain problems among the ninety-two problems that Guo Shuchun picked up.

11 Specifically, Guo suggests that in this book the following textual parts were all written by Jia Xian: 1. text in big-sized characters except for citations from *The Nine Chapters* (excluding, however, the texts placed after the expressions “explanation of the problem,” “analogy,” and “detailed explanation”); 2. All the texts under the expression “the method (procedure) says”; 3. The diagrams attached to the “methods (procedures)”; 4. The majority of commentaries (small-sized characters) in the “methods (procedures)” (Guo 1988, 333).

12 For details, see Kong (1993, 36), Zhou (2018, 114–22), and Guo (2019, 55–59).

extant edition of *Mathematical Methods Explaining in Detail The Nine Chapters*. In 1843 CE, a Shanghai merchant, Yu Songnian 郁松年 (1822–1886 CE), published a printed version of *Mathematical Methods* with other ancient books. The name of this series of books was *Collection of Books in the Yijia Studio* (*Yijiatang congshu* 宜稼堂叢書; here *yijia* literally means “appropriate for planting and multiplying [precious ancient books]”). According to Yu Songnian’s preface to a *Corrigenda* (*Zhaji* 札記) of *Mathematical Methods* (Guo 1993b, 1:1026), this printed version was based on a version on each page of which were five characters “manuscript of the *Shiyan Studio* (石研齋抄本).”¹³ There is no further information about the origins of this manuscript or its whereabouts subsequent to the Yu’s publication of *Mathematical Methods*.¹⁴

The *Corrigenda* placed at the end of the printed edition of *Mathematical Methods* was written by Song Jingchang 宋景昌 (1790–1861 CE), a student (*menren* 門人) of a famous nineteenth-century practitioner of mathematics, Shen Qinpei 沈欽裴 (ca. 1790–ca. 1870 CE) (Feng 2012, 534). At Yu

Figure 1: The title page of the *Yijia Studio* edition (1843 CE) of *Mathematical Methods Explaining in Detail The Nine Chapters* (1261 CE). The seal reads “Collected by the *Shuangxiao Studio*” (雙嘯室珍藏). The owner of this studio was Qiu Chongman 裘冲曼 (1890–1974). This edition is preserved in the library of the Institute for the History of Natural Sciences, Chinese Academy of Sciences.

13 The *Shiyan Studio* (lit. Inkstone Studio) was owned by a Qing scholar-official, book collector Qin Enfu 秦恩复 (1760–1843 CE) (Guo 1993a, 947; Fan 2013, 194).

14 Guo Shuchun points out that a lot of mistakes in the manuscript are the same as those in the edition of *The Nine Chapters* printed by Bao Huanzhi 鮑澣之 in 1213 CE, and also the same as those contained in the extant sections of the *Yongle Encyclopaedia* (*Yongle dadian* 永樂大典). He suggests that this manuscript was copied from the *Yongle Encyclopaedia* (Guo 1994). Guo’s main argument is that the chapter “Discussing Works (*Shanggong* 商功)” in the manuscript lacks problems concerning the volumes of “Millet in Piles (*Weisu* 委粟),” while these problems were accurately classified into one fascicle (*juan* 卷) in the *Yongle Encyclopaedia* (Guo 1993a, 947). In fact, this fascicle in the *Yongle Encyclopaedia* has not survived, thus its contents are still unknown, and the only extant evidence to support this is that there was a fascicle bearing this title in the *Yongle Encyclopaedia*. Guo’s assumption is reasonable, though more evidence is needed to prove that the manuscript was copied from the *Yongle Encyclopaedia*.

Songnian's invitation, Song read the *Shiyan* Studio manuscript along with two other versions of *The Nine Chapters* published in the Qing dynasty (Guo 1993b, 1:1026). One was *The Nine Chapters* in the *Weibo xie* (微波榭) edition published by Kong Jihan 孔繼涵 (1739–1783 CE), and the other was *Detailed Procedures of Calculation and Diagrammatic Explanations of The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu xicao tushuo* 九章算術細草圖說) by Li Huang 李潢 (?–1812 CE). Both contained the text of *The Nine Chapters*. However, these editions of *The Nine Chapters* had undergone changes made by Dai Zhen 戴震 (1724–1777 CE), Kong Jihan, or Li Huang.¹⁵ With the help of these two editions, Song was able to understand the problematic passages of the manuscript. He corrected them and wrote the *Corrigenda*, which he attached to the edition.¹⁶ Both were printed in 1843.

Song recorded in his *Corrigenda* details of every correction he made. Since he did not make any mention of the deletion of a mathematical problem or that of a chapter, it is reasonable to believe that he made no such deletions from the manuscript in the process of preparation of the printed edition of *Mathematical Methods*. Moreover, he did not record any changes in the order of the chapters or in that of the problems in a chapter. Therefore, the printed edition of *Mathematical Methods* is deemed to faithfully reflect the original order of chapters and the original order of problems in the manuscript.

A quick examination of the contents of the printed edition of *Mathematical Methods* shows that the manuscript was not a complete version of the book. The *Yijia* Studio edition of *Mathematical Methods* contains a preface written by Yang Hui, which provides some information on the original, complete *Mathematical Methods*.

[I] compiled procedures of multiplication and division to facilitate learning for beginners, [and I] reclassified methods and questions into orderly categories, which is to be seen after [all] the chapters. This makes twelve fascicles (*juan* 卷) in total. (Yang [1261] 1993, 951)¹⁷

The *Yijia* Studio edition of *Mathematical Methods* does not contain a textual marker

15 For an introduction to the differences between these editions, see Chemla and Guo (2004, 74–78), and Guo (2014, 571–603).

16 Note that *The Nine Chapters* in the *Weibo xie* edition and the book written by Li Huang do not include any layer of text by Yang Hui or Jia Xian. These two editions contain only the text of *The Nine Chapters* (including the commentaries written by Liu Hui and Li Chunfeng). Song Jingchang compared these texts with those in the manuscript, helping him to understand parts of the manuscript and correct them. Song also emended parts of the text written by Yang Hui and Jia Xian. He made these changes on the basis of his own understanding of the text and his calculations.

17 “編乘除諸術以便入門，纂法問類次，見之章末。總十有二卷。”

designating a division between different chapters.¹⁸ By comparing the contents of *Mathematical Methods* with those of *The Nine Chapters* in the form in which it has been handed down it is straight-forward to establish that the text of the *Yijia Studio* version belongs to five different chapters of *The Nine Chapters*. Following their order in the printed edition of *Mathematical Methods*, the parts belong respectively to the following chapters in *The Nine Chapters*: “Excess and Deficit” (Yingbuzu 盈不足), “Fangcheng” (方程), “Base and Height [of the Right-Angled Triangle]” (Gougu 勾股), “Discussing Works” (Shanggong 商功), and “Equally [Paying Tax and Corvée Labor According to the] Transport” (Junshu 均輸).

It is highly probable that the text belonging to each of the chapters mentioned above formed one fascicle (*juan*) among those mentioned in Yang Hui’s preface. Together with *Reclassifications*, which is placed at the end of the *Yijia Studio* edition, these make only six fascicles. This is at odds with the “twelve fascicles” mentioned in Yang Hui’s preface.¹⁹ In addition, the order of chapters mentioned above is different from that in the extant editions of *The Nine Chapters*.²⁰ No explanation is provided concerning the rearrangement of chapters in the *Yijia Studio* edition. One likely assumption is that the printed edition followed the (dis)order of the pages in the problematic manuscript, but the reason for the missing chapters and the rearrangement of chapters is still unknown.

In the printed *Mathematical Methods*, there are only ninety-eight problems coming from *The Nine Chapters*.²¹ There are no problems from the chapters “Rectangular Fields” (Fangtian 方田), “Unhusked and Husked Grains” (Sumi 粟米), “Sharing

18 In the *Corrigenda*, Song Jingchang suggests that at the beginning of each chapter there should have been a title marking the separation between different chapters, though he did not add such titles directly into the printed edition.

19 In the extant printed edition of *Mathematical Methods*, Yang Hui makes no mention of the titles of the twelve fascicles. On the basis of information collected from other mathematical books written by Yang Hui, Yan Dunjie put forward an assumption with respect to the titles and provided a brief introduction to each fascicle (Yan 1966). However, Yan’s assumption is based on his hypothesis that the book entitled *Xiangjie suanfa* 詳解算法 (lit. *Mathematical Methods Explaining in Detail*) is precisely *Mathematical Methods Explaining in Detail The Nine Chapters*. Since *Xiangjie suanfa* is not extant, this view cannot be confirmed yet. Lü Bianting 吕变庭 (2014) makes use of *Reclassifications* and gives his reconstruction of *Mathematical Methods* according to his understanding of *Reclassifications*, which are slightly different from what Yan reconstructed.

20 In the extant editions of *The Nine Chapters*, these chapters are arranged in the following order: “Discussing Works,” “Equally [Paying Tax and Corvée Labor According to the] Transport,” “Excess and Deficit,” “Fangcheng,” and “Base and Height [of the Right-angled Triangle].”

21 A count of the number of problems from *The Nine Chapters* in each chapter of the *Yijia Studio* edition of *Mathematical Methods* is as follows: “Excess and Deficit” has twenty problems (that is, the same number as the original *The Nine Chapters*), “Fangcheng” has fourteen problems (whereas the original *The Nine Chapters* had eighteen problems), “Base and Height” has twenty-four problems (that is, the same number), “Discussing Works” has thirteen problems, and “Equally [Paying Tax and Corvée Labor According to the] Transport” has twenty-seven problems (whereas the original *The Nine Chapters* has twenty-eight problems in “Discussing Works” and “Equally [Paying Tax and Corvée Labor According to the] Transport” respectively).

According to a Given Proportion” (Cuifen 衰分), and “Reducing the Width” (Shaoguang 少廣) of the original *The Nine Chapters*. Moreover, the chapters “Fangcheng,” “Discussing Works,” and “Equally [Paying Tax and Corvée Labor According to the] Transport” in *Mathematical Methods* all lack problems from their corresponding chapters in *The Nine Chapters*. There is another phenomenon deserving attention. “Fangcheng” lacks four problems (nos. 9–12) that follow one after another in *The Nine Chapters*; “Discussing Works” lacks fifteen problems, the first seven of which are placed one after another in *The Nine Chapters*, the remaining eight also; and “Equally [Paying Tax and Corvée Labor According to the] Transport” lacks one problem. In each group of lost problems, the problems do not share any obvious similarity, which suggests that they were dropped intentionally by scribes or that their omission is the result of a systematic error. A very likely assumption is that the missing texts were caused by accidental reasons which took place on the *Shiyan* Studio manuscript. Another important phenomenon is that in *Mathematical Methods* the extant problems in each of the five chapters are placed in the same order as those from *The Nine Chapters* in the form handed down. Since Song Jingchang did not mention any emendations to the order of problems, it is believed the *Shiyan* Studio manuscript recorded these problems in the same order.²²

22 Apparently, it seems that these remarks do not support the view that the manuscript was copied from the *Yongle Encyclopaedia*. According to the extant catalog of *Yongle Encyclopaedia*, *Yongle Encyclopaedia* reclassified problems from *The Nine Chapters* into its own categories. See *Yongle Encyclopaedia* ([1408] 2012, 10:508–9). If the manuscript was copied by picking up problems from the *Yongle Encyclopaedia*, the order of the problems would not be the same as that of *The Nine Chapters* seen today. However, the question does not seem to be that simple. The issue of the order of problems in *The Nine Chapters* is complex. In brief, after the eighteenth century, the order of problems in *The Nine Chapters*, especially that in the last four chapters, was influenced by Dai Zhen’s 戴震 work. Dai Zhen also picked up the problems from the *Yongle Encyclopaedia*. If the *Shiyan* Studio manuscript scribe also picked up problems from the *Yongle Encyclopaedia*, it is not surprising that the orders of problems in the four extant chapters in the manuscript would be the same as those in the editions of *The Nine Chapters* influenced by Dai Zhen. The extant Southern Song printed edition of *The Nine Chapters* contains only the first five chapters. Fortunately, the chapter “Discussing Works” in this edition also appears in *Mathematical Methods*. It seems that one can rely on it, rather than on the other chapters, which were influenced by Dai Zhen’s compilation, to compare the order of problems in it with that in the manuscript of the *Shiyan* Studio. Except for the lost fifteen problems, the order of the rest of the problems is the same as that in the Southern Song printed edition. It seems that this coincidence implies that the manuscript had a source that was different from the fragmented and disordered contents of *The Nine Chapters* in the *Yongle Encyclopaedia*. However, this is not enough yet to disprove the assumption that the manuscript was copied from the *Yongle Encyclopaedia*. The reason is that the *Yongle Encyclopaedia* also included two *juan* (not extant) that were titled “Discussing Works.” It is not clear whether the edition included in the *Yongle Encyclopaedia* had changed the order of the problems in this chapter. If it did not change the order, it is also not surprising to find that the manuscript copied from the two *juan* has the same order of problems as that in the Southern Song printed edition.

1.3 Introduction to the “*Fangcheng*” chapter

Modern scholars all agree that, from the perspective of modern mathematics, the main contents of this chapter are related to systems of linear equations. However, translations of this title and the ways used to interpret its contents vary from scholar to scholar. For example, Wang Ling and Needham translate the title as “The Way of Calculating by Tabulation” (1959, 26), and describe its contents as: “The counting-rods, in Chinese style, were placed in different squares in a table so as to represent coefficients of the different unknowns, and the constant terms. The *Chiu Chang*²³ (chapter 8) has numerous problems requiring the solution of equations of the type $ax+by=c$, $a'x+b'y=c'$ ” (1959, 115). Shen Kangshen 沈康身 et al. (1999, 386) translate it as “Rectangular Arrays,” and explain the procedures as “the coefficients are laid out in an array and then the coefficient matrix of the array is reduced to triangular form using elementary matrix operations.” Xu Yibao 徐义保 and Joseph Dauben (2013, 801) also adopt this translation. Lam Lay Yong and Ang Tian Se (2004, 145) translate the title as “Rectangular Tabulation (lit. square procedure),” and suggest that this method solves a set of simultaneous linear equations. Roger Hart (2011, 67–84) thinks there is no definitive explanation of the meaning of this title at the time of the compilation of *The Nine Chapter* and leaves it untranslated. He thought that the “*fangcheng* procedure” is in many ways similar to modern Gaussian elimination except for the system for writing numbers and the orientation of the matrix. Karine Chemla leaves it untranslated, but in the glossary interprets this title as “mesures en carré” [measures (*cheng* 程) in square (*fang* 方)] (2004, 599, 922).²⁴ She explores the relationship between the mathematical practice of *fangcheng* and that of *chu* 除 (lit. division/elimination), and claims that “the operation appears to be a generalization of the division *chu*. Instead of having a single divisor and dividend, ‘measures in square’ opposes a square of divisors to a row of dividends, and the operation is executed using the key process of division first vertically, and then horizontally.” The vertical division refers to an operation “which [relying on the physical properties of numbers represented with rods] brings into play two columns to eliminate the upper position in one of them, [and] is prescribed as ‘an upright division’ (*zhichu* 直除).” The horizontal division refers to “a plain division” (*chu* 除) (Chemla 2019, 106–7).²⁵

In this paper, Romanized Pinyin *fangcheng* is used to refer to this title. Instead of

23 In the (slightly modified) Wade-Giles romanization system, the characters *Jiu Zhang* 九章 (*Nine Chapters*) are romanized as *Chiu Chang*.

24 This interpretation also appeared in her earlier publications. See Chemla (1994) and Chemla (2000).

25 For other research written in English on the *fangcheng* procedure in *The Nine Chapters*, see Lam and Ang (1987); Martzloff (1997, 249–58). For research written in Chinese on this topic, see Qian (1964, 51–56); Mei (1984); Li (1982); Guo (1985); Zhu (2010); Wei and Guo (2016); Horng (2018).

using modern mathematical terms such as “simultaneous set of linear equations,” “coefficients,” “constant terms,” or “matrix,” this paper shows how Yang Hui expressed, understood, and demonstrated the mathematical methods from his perspective in the “*Fangcheng*” chapter. They comprise the first goal of this paper.

By comparing the complete text of *The Nine Chapters* with that in the printed *Mathematical Methods*, it is easy to find that the “*Fangcheng*” chapter in the latter contains only fourteen problems from *The Nine Chapters*. Problems 1 to 8 and 9 to 14 in “*Fangcheng*” in *Mathematical Methods* are, respectively, the same as problems 1 to 8 and 13 to 18 in “*Fangcheng*” of *The Nine Chapters*. Furthermore, they are arranged in the same order. Four problems, from 9 to 12, in *The Nine Chapters* do not appear in the printed version of *Mathematical Methods*. According to the analysis in section 1.2, these problems may have been lost in the manuscript by accident. *Reclassifications* contains all the problems in “*Fangcheng*” of *The Nine Chapters*, and it also includes two problems that belong to chapters six and seven of *The Nine Chapters* respectively. In *Mathematical Methods*, under these two problems Yang Hui gives an alternative solution using the *fangcheng* method, which could support classifying them into “*Fangcheng*” in *Reclassifications*. In *Reclassifications*, the problems in “*Fangcheng*” are placed in an order that is quite different from the order of problems in *The Nine Chapters* and that in *Mathematical Methods*. The results of a comparison of each problem in the three works are shown in Figure 2.²⁶

Except for the questions asked above concerning Yang Hui’s mathematical methods, this paper also inquiries into the reason why Yang Hui reclassified the problems into his subcategories in this way. Sha Na 沙娜 (1993, 38–39) has pointed out the relationship between *Reclassifications* and *Mathematical Methods*. The paper below gives a reasonable explanation for this order; moreover, it may also prove that Yang’s reclassification of problems in “*Fangcheng*” has a close relationship with his understanding of these mathematical methods attested to the text in the work *Mathematical Methods*.

2 The *fangcheng* procedure and the “flexible methods” (*huofa* 活法)

The first problem of the “*Fangcheng*” chapter in *The Nine Chapters* is also the first

²⁶ Zhang Jiuchun was first to pay attention to the order of problems and to give a table of comparison between problems from the chapter “Base and Height” in *The Nine Chapters*, *Mathematical Methods*, and *Reclassifications* respectively (Zhang 2003, 56). However, in that paper, he focuses on the source of the fifteenth-century mathematical work *Great Compendium of The Nine Chapters on Mathematical Methods with Analogies* (*Jiuzhang suanfa bilei daquan* 九章算法比類大全, 1450 CE), and does not give an explanation of the reason why the problems in *Reclassifications* were arranged in that order.

Figure 2: An illustration showing the relationship between problems in the "Fangcheng" chapter of *The Nine Chapters*, *Mathematical Methods*, and *Reclassifications*.

problem in *Mathematical Methods* and in *Reclassifications*.²⁷ This problem was set in an assumed context of knowing the following conditions: the production of three *bing* 秉 (lit. bundle) of high-quality grain, two *bing* of middle-quality grain, and one *bing* of low-quality grain is thirty-nine *dou*; the production of two *bing* of high-quality grain, three *bing* of middle-quality grain, and one *bing* of low-quality grain is thirty-four *dou* 斗 (a unit of capacity); the production of one *bing* of high-quality grain, two *bing* of middle-quality grain, and three *bing* of low-quality grain is twenty-six *dou*. The question is how much is produced by one *bing* of high-, middle-, and low-quality grain respectively.²⁸

After the text of *The Nine Chapters, Mathematical Methods* includes a detailed “procedure” (*shu* 術) showing how the *fangcheng* procedure is carried out. Under the expression “the procedure says,” Yang Hui’s commentary on this problem begins with a sentence written in small-sized characters:

At the very beginning [of introducing this method], [we] do not have (i.e. give) the “flexible methods” (*huofa* 活法, lit. vital methods). [We] state the meaning [of the (original) method] here. (Yang [1261] 1993, 963)²⁹

The term “flexible methods” (*huofa* 活法) in this sentence does not appear in other places in the printed version of *Mathematical Methods*. In this paper, the translation and further interpretation of this term rely on the following: Firstly, the literal meaning of the word *huo* (活) implies that the methods designated by it are probably flexible, and not a fixed routine of a mathematical procedure. Secondly, the term “*huofa*” also appears in other mathematical works written by Yang Hui. At present, it is still difficult to find a suitable explanation of this term in *Quick Methods for Multiplication and Division for the Surface of the Fields and Analogous Problems*.³⁰ However, in Yang’s *Selection of Unusual Mathematical Methods in the Wake of the Ancients*, the word is combined with the word “transformation” (*bianhuan* 變換) (Guo 1996, 354) to form a subheading that can be interpreted as “flexible methods of (all kinds of) transformation” (*bianhuan huofa* 變換活法). The meanings of “transformation” and “flexible” match each other. Last but not least,

27 As far as the collation of the procedure of the first problem in *The Nine Chapters* is concerned, Qian Baocong 錢寶琮 (1955) points out the mistakes made by Dai Zhen, and then uses other editions, including the printed version of *Mathematical Methods*, to correct the text of the procedure.

28 For English translations of this problem, see Shen, Crossley, and Lun (1999, 399); Dauben, Xu, and Guo (2013, 909–11).

29 “初無活法，今述此意。”

30 In this book, the term “*huofa*” appears in the sentence “the ‘*huo*’ method was recorded in detail in [the chapter] ‘Reducing the Width’ of *The Nine Chapters*” (活法詳載《九章·少廣》) (Guo 1996, 230). Guo Xihan 郭熙漢 (1996, 234) interprets it as “concrete method.” It seems that no evidence can support this interpretation.

the following part of this paper shows that Yang Hui demonstrates “flexible methods” through his mathematical methods or “procedures of calculation” (*cao* 草). They will constitute the direct proof supporting the translation of “*huofa*” as “flexible methods.”

Since the quotation above says that “At the very beginning [of introducing this method], [we] do not have (i.e. give) the ‘flexible methods,’” what Yang wants to introduce after this sentence should be a method unlike the “flexible methods.” The method that he introduces after this sentence is designated as “this meaning” (*ciyi* 此意). It refers to a “standard” method for solving such problems, one rooted in the procedure of the first problem of *The Nine Chapters*. Yang Hui introduces this method first perhaps because he considered it to be the basis on which other methods of this chapter rely. The analysis below about the *fangcheng* method will begin with this “standard” method.

2.1 The procedure introduced in the first problem of *Mathematical Methods*

Following the sentence quoted above, some text and some non-discursive configurations composed of numbers and text demonstrate how Yang Hui understood the *fangcheng* method and how he carried out the procedure.

Below is the translation of each sentence in the procedure placed before the configurations. Letters are added as labels at the beginning of these sentences for later reference.

A: [One] disposes [on the counting board] in order the numbers of the items asked about [in the problem].³¹

Small-sized characters:³² [The number of] things [of one kind] and [the number] of things [of another kind] [as well as] the amount of money they are worth together form one column; [the number of] things [of one kind] and [the number] of things [of another kind] [as well as] the amount of money they are worth together form [another] column.³³

B: [One] assigns the greater [number of] the thing at the first position as the “main [number].”³⁴

Small-sized characters: There is seven and here is five. The [number] seven is greater [and thus it is taken as the main number].³⁵

C: [One] makes the number in the neighboring column augmented, [and] finds the

31 A: “排列逐項問數。” (Sentences A to J are quoted from Yang [1261] 1993, 963.)

32 The expression “small-sized characters” is added in the translation in order to distinguish the small-sized characters from the big-sized characters in the original Chinese text.

33 “某物某物共直錢為一行，某物某物共直錢為一行。”

34 B: “命首位物多者為主。”

35 “彼七此五，以七為多。”

equal [number] (*deng* 等, i.e. the greatest common divisor).³⁶

Small-sized characters: The equal [number] can be subtracted [from the augmented number].³⁷

D: The rest of [the number of] things and [their] price³⁸

Small-sized characters: [The price] is the total numbers [corresponding to the things].³⁹

E: are also multiplied by it (the same times) one by one.⁴⁰

Small-sized characters: Once [the number of] one thing has been augmented, the rest of [the number of] the things and their price are raised to the same level (i.e. be augmented by the same times).⁴¹

F: [One] uses [the column] which contains the original greater [number] of the thing.⁴²

Small-sized characters: [each] number in the column⁴³

G: to subtract [it from the column which contains the smaller number between positions that] correspond [to each other].⁴⁴

Small-sized characters: That is to say [one] subtracts [the number of] thing from [the number of the same] thing, [one] subtracts money from money. The aim is to eliminate (*qing* 輕) [the number] in one position.⁴⁵

H: The other [columns] are augmented and subtracted one by one.⁴⁶

Small-sized characters: [One augments] the smaller number and makes it equal (*ting* 停⁴⁷) to [several integral times of] the greater number. [One] wants to make [the

36 C: “以鄰行數增乘求等。”

For the interpretation of the word “*deng*” 等 in this sentence, see Zhu (2010, 84). This interpretation is supported by evidence in Chapter One of *The Nine Chapters* (Li 1990, 235–37; Chemla and Guo 2004, 157, 762, 916). Zhu suggests that if there are two numbers *A* and *B* ($A > B$, *A* is the main [number]), then *A* is used to multiply *B* and one gets *AB*. The “equal [number]” of *AB* and *A* is *A*, and then *A* is subtracted from *AB*.

37 “等數可以減損。”

38 D: “餘物與價”

39 “即總數也。”

40 E: “亦例乘之。”

41 “一物既增，餘物與價亦各陞為一體。”

42 F: “以原多物”

43 “行內數目”

44 G: “對減。”

45 “謂物減物，錢減錢，求輕一位。”

46 H: “其餘次第增減。”

47 A common literal meaning of *ting* (停) is “stop,” but it also means “equal” in ancient Chinese (Luo 1986–1993, 1:1556). The meaning in this sentence is “equal.”

number] in the corresponding positions [be able to be] eliminated, [and the ultimate] aim is to make the position simple.⁴⁸

I: [When] the price could be [regarded as] the dividend (*shi* 實) and [the number] of the thing could be [regarded as] the divisor (*fa* 法), [one] stops.⁴⁹

Small-sized characters: The divisor (*fa*) and the dividend (*shi*) [all only appear in] one position respectively.⁵⁰

J: [One] divides it (the dividend) by the divisor.⁵¹

Small-sized characters: [One uses the method of] “division of consulting (estimation) (*shangchu* 商除)” [to do this calculation].⁵²

The above text written in big-sized characters gives a general procedure for finding the dividend and divisor concerning one thing. Through division of the two numbers, the procedure yields the price of one thing. The small-sized characters among these big-sized characters have several functions. Those following the sentences A, B, D, G⁵³, F, I, and J give explanations or examples for certain operations to serve as a guide to how the operations are carried out. The small-sized characters following sentence C and E give reasons why the practitioner should do the operations described by the big-sized characters. Those following the sentences G⁵⁴ and H give the aims of the operations described by the big-sized characters placed before them. However, even though the paragraph above combines with the descriptions, examples, and explanations of reasons and aims of each step of the procedure, Yang Hui may still have considered that they did not provide enough help for his readers to understand how the operations are carried out. He thus then uses the detailed procedure designed for the solution of problem 1 to demonstrate how this procedure is carried out. A solution for this problem could be written in a modern mathematical way as follows, in which, x , y , and z represent the production by one *bing* of high-, middle-, and low-quality grain respectively.

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 26 \\ x + 2y + 3z = 26 \end{cases}$$

Yang Hui gives four configurations of numbers representing the results after some

48 “增少數與多數為停。如求對除，以求位簡。” The first character *ru* 如 in the latter sentence cannot be interpreted at present, so the translation above temporarily omits it.

49 I: “價可為實，物可為法而止。”

50 “法實皆一位也。”

51 J: “以法除之。”

52 “商除”

53 The first two clauses under sentence G.

54 The last clause under sentence G.

operations on the counting board (Figure 3).⁵⁵ Each configuration corresponds to one or several steps indicated by the text placed before the configuration.

High 1	High 2	High 3	High 3	High 6	High 3	High	High	High 3	High	High	High 3
Middle 2	Middle 3	Middle 2	Middle 6	Middle 9	Middle 2	Middle 4	Middle 5	Middle 2	Middle	Middle 5	Middle 2
Low 3	Low 1	Low 1	Low 9	Low 3	Low 1	Low 8	Low 1	Low 1	Low 6	Low 1	Low 1
26 <i>dou</i>	34 <i>dou</i>	39 <i>dou</i>	78 <i>dou</i>	102 <i>dou</i>	39 <i>dou</i>	39 <i>dou</i>	24 <i>dou</i>	39 <i>dou</i>	99 <i>dou</i>	24 <i>dou</i>	39 <i>dou</i>
The first configuration			The second configuration			The third configuration			The fourth configuration		

Figure 3: Four configurations of numbers in problem 1 of *Mathematical Methods*.

The first configuration in Figure 3 corresponds to sentence A, which reads “[One] disposes [on the counting board] in order the numbers of the items asked about [in the problem]” (排列逐項問數). The extant copies of *The Nine Chapters* and the commentaries written by Liu Hui and Li Chunfeng do not contain any configuration of numbers, even though these configurations would have appeared in the operation of solving the problem and they could be reconstructed. However, thirteen of the fourteen problems in this chapter of *Mathematical Methods* contain such a configuration as a part of readable elements of the work.⁵⁶ In *Mathematical Methods*, this operation is the first step in solving problems concerning *fangcheng*, and the first configuration is also the most fundamental configuration. In problems 12, 13, and 14 of *Mathematical Methods*, after the text from *The Nine Chapters*, there is only a configuration containing numbers from the conditions of the problem, and Yang Hui does not add anything after this configuration. From his perspective, the different forms of this first configuration are

⁵⁵ In these configurations, “High” (*shang* 上), “Middle” (*zhong* 中), and “Low” (*xia* 下) refer to grain of three kinds as mentioned in the problem, that of “high-quality grain,” “middle-quality grain,” and “low-quality grain” respectively. They should not be interpreted as referring to the respective positions (“top,” “middle,” or “bottom”) in the configuration. My thanks to the anonymous reviewer for pointing this out.

⁵⁶ Problem 2 in *Mathematical Methods* does not contain any configuration of numbers.

also related to different methods (procedures) for solving the problem. I shall return to this point in section 3.4 of this paper.

Before the second configuration in Figure 3, there is a sentence expressing a similar meaning to sentences B, C, D, and E translated above. This sentence is also composed of big-sized characters and small-sized characters. Comparing it with sentences B to E, it is easy to see that, even though the big-sized characters are very similar, the small-sized characters in sentences B to E have been replaced by the concrete conditions and numbers from this problem.⁵⁷ This phenomenon also happens in the text before the third configuration, where the sentence placed before it (see Figure 3) is similar to sentences F and G translated above.

The first sentence placed before the last configuration in Figure 3 corresponds to sentence H. However, this sentence indicates a procedure in a loop structure. It tells the practitioners not to stop until “the price could be [regarded as] the dividend (*shi* 實) and [the number] of the thing could be [regarded as] the divisor (*fa* 法) (The divisor [*fa*] and the dividend [*shi*] all only appear in one position.)” (see “sentence I” translated above in the “procedure”). So, after this sentence, there are a series of varied versions of the sentences placed before the second and third configurations. They also correspond to sentences B to G.

After the last configuration, big-sized characters repeat a part of sentence I. All the text placed after this sentence is written in small-sized characters. These small-sized characters describe in detail the process of finding the answers for the high-, the middle-, and the low-quality grain respectively. The main body of the procedure Yang Hui wanted to demonstrate ends with finding the dividend and divisor of the low-quality grain, which serves as the first pair of numbers to be calculated.

Following the translation above, the procedure that the four configurations depict, step-by-step, leading to a solution of the first *fangcheng* problem is not difficult for contemporary readers to understand. The text composed of big-sized characters and small-sized characters, together with the four configurations of numbers, demonstrates the “*fangcheng* method,” which was stated in opposition to the “flexible method.” The next section of this paper will examine what the “flexible methods” were, how Yang Hui demonstrated them, and for what reasons he used the “flexible methods” in his commentaries on *The Nine Chapters*.

⁵⁷ This phenomenon leads to an assumption that before the first configuration there was perhaps a subheading “The Procedure of Calculation Says” (*Caoyue* 草曰). In other problems in *Mathematical Methods*, the content of “procedure of calculations” (*cao* 草) usually repeats the big-sized characters under the expressions “the method says” or “the procedure says,” and replaces the explanatory small-sized characters by concrete information from the problem. For more about this point, see section 4.4 of Zhou (2018).

2.2 The flexible methods

2.2.1 The method of “cross-multiplication”

After the second problem in *Mathematical Methods*, Yang Hui mentions an operation designated by the term “cross-multiply” (*hucheng* 互乘, lit. “multiply with each other”) in the text of “the procedure of calculation” (*cao*).

The two [numbers in the positions of] the high-quality grain cross-multiply [the numbers in the] two columns. (Yang [1261] 1993, 964)⁵⁸

Yang does not give any direct explanation of the term “cross-multiply” in problem 2.⁵⁹ However, on the bases of the literal meaning of this term and the analysis below, it is not difficult to see how the “cross-multiplication” was carried out. To make the discussion more concise and to focus on the key points, an interpretation is provided below in modern mathematical terms.⁶⁰

$$\begin{cases} 7x + 2y = 11 \\ 2x + 8y = 9 \end{cases}$$

One uses number two, the first coefficient in the second equation, to multiply all the coefficients in the first equation. And then one uses the number seven, the first coefficient in the first equation, to multiply all the coefficients in the second equation. After these operations, the two equations above are transformed into those below.

$$\begin{cases} 2 \times 7x + 2 \times 2y = 2 \times 11 \\ 7 \times 2x + 7 \times 8y = 7 \times 9 \end{cases}$$

Through “cross-multiplication,” the two lines of coefficients become a form in which one can eliminate the number in the first position through subtraction between the numbers in the two lines. This method for eliminating the number of a certain position is different from that which Yang Hui introduces in the first problem. There one assigns the greater number in the first position as the “main number.” One then makes the numbers in the neighboring columns augmented and equal to several integral times of the “main number.” Through

58 “上禾二位互乘两行。”

59 The reason why Yang Hui did not give any explanation about how the “cross-multiplication” is carried out is related to the fact that this problem was arranged as the thirteenth problem in *Reclassifications*, so the procedure had already been introduced or exemplified in earlier problems. For a detailed discussion on the order of the problems in *Mathematical Methods* and *Reclassifications*, see section 4 of this paper.

60 *Stricto sensu*, when “cross-multiplication” is carried out by means of counting rods on a counting board, an extra column of numbers is probably disposed which do not correspond to the systems of linear equations above. For a detailed interpretation of “cross-multiplication,” see Zhu (2010, 75–83).

“several times of subtraction” one eliminates the number of a certain position.

In fact, the method of “cross-multiplication” appears in the commentary written by the third-century commentator Liu Hui to problem 7 in *The Nine Chapters*.⁶¹ The problem is set in an assumed concrete context where two conditions were known, each of which gives the total value of several oxen and several sheep. The problem asks for the price of each ox and each sheep. Modern mathematical notation is used to represent the problem thus.

$$\begin{cases} 5x + 2y = 10 \\ 2x + 5y = 8 \end{cases}$$

In the commentary, Liu Hui explains the reasoning behind the “cross-multiplication.”

If one supposes that one makes a “homogenization-equalization” (*tongqi* 同齊), the head positions consisting of the ox, [one] should multiply with each other. The right column is determined (*ding* 定, i.e. does not change). [One] disposes once again (*gengzhi* 更置) [beside] ten oxen, four sheep, and the gold they are worth, twenty *liang* 兩⁶²; in the left column, ten oxen, twenty-five sheep, and the gold they are worth, forty *liang*. The numbers of oxen are the same, the surplus twenty *liang* of gold, the difference of sheep, twenty-one, makes it [twenty *liang*]. [One] subtracts the smaller column (in which the number of oxen is smaller) [before multiplication] from the greater column (in which the number of oxen is greater). The number of oxen is exhausted [*jin* 盡, i.e. eliminated], and only the numbers of sheep and the gold [they] are worth appear. [One] can get to know [the result]. [One] deduces the greater [situation] (the number of columns, the number of unknowns) from the smaller [situation]: even if [one has] four or five columns, it is no different.⁶³

An interesting phenomenon is that, in the text of the “procedure of calculation” placed after the quotation above, Yang Hui does not use configurations to demonstrate

61 Qian Baocong has pointed out that the text of the “*Fangcheng*” chapter which was influenced by Dai Zhen’s collations are problematic. Dai Zhen’s collation used the method of “cross-multiplication” improperly and distorted Liu Hui’s commentaries. Qian suggests that it was not until problem 7 of *The Nine Chapters* that Liu Hui’s commentary implies the idea of “cross-multiplication” (Qian 1955). Textual criticism of *The Nine Chapters* testifies to this view, and subsequent scholarship has adopted this suggestion (Guo Shuchun 2009, 323–30, 341–43; Li 1990, 625, 630, 648–49).

62 *Liang* (兩) is a unit of money.

63 “假令爲同齊，頭位爲牛，當相乘。右行定，更置牛十、羊四直金二十兩，左行牛十、羊二十五直金四十兩。牛數等同，金多二十兩者，羊差二十一，使之然也。以少行減多行，則牛數盡，惟羊與直金之數見，可得而知也。以小推大，雖四五行不異也。” The English translation of the quotation above is modified from Chemla and Guo’s French translation (Chemla and Guo 2004, 633) since their interpretation of the “*Fangcheng*” chapter in *The Nine Chapters* conforms to the analysis of *Mathematical Methods* in this paper. In addition, the translation of the term “*gengzhi*” (更置) in this quotation follows Zhu Yiwen’s 朱一文 (2010, 76) interpretation.

Ox 2	Ox 5		Ox 5
Sheep 5	Sheep 2	Sheep 21	Sheep 2
Gold 8 <i>liang</i>	Gold 10 <i>liang</i>	Gold 20 <i>liang</i>	Gold 10 <i>liang</i>

Figure 4: Two configurations of numbers in problem 7 of *Mathematical Methods*.

the method Liu Hui has explained. He disposed the numbers in the two conditions as the configuration below (on the left of Figure 4) and transforms it into the configuration on the right.⁶⁴

In the configuration on the right, the first position of the left column is empty. Between the two configurations, Yang describes a method of subtraction which is different from what Liu Hui explained.

[One] uses the greater number, five oxen, to multiply throughout all [the numbers] in the left column respectively. After that, [one] uses the right column twice to subtract [it] correspondingly from [the left column]. (Yang [1261] 1993, 966)⁶⁵

The method of “subtracting [between positions that] correspond [to each other]” (*duijian* 對減) several times is precisely the one Yang demonstrated in the first problem above. However, this method is different from the one in Liu Hui’s commentary on this problem, where, after the operation of “cross-multiplication,” the practitioner only needs one subtraction to make the number of oxen exhausted (*jin* 盡, i.e. eliminated).⁶⁶

Why did Yang Hui make the method in the “procedure of calculation” different from that in Liu Hui’s commentary?⁶⁷ One assumption, which will be proved to be erroneous below, is that he “preferred” the method of after “multiplying throughout”

64 The two configurations correspond to the following two systems of linear equations in modern mathematics:
$$\begin{cases} 5x + 2y = 10 \\ 2x + 5y = 8 \end{cases} \quad \begin{cases} 5x + 2y = 10 \\ 21y = 20 \end{cases}$$

65 “以多數五牛遍乘左行，訖，以右行兩度對減。”

66 Another difference between the two methods lies in which line is subtracted. In the method from *The Nine Chapters* and Liu Hui’s commentary, the “smaller line” (少行) was subtracted from the “greater line” (多行), while according to Yang’s text, the “greater line which one had originally multiplied” (原乘多行) is subtracted from the other one. This difference also appears in other problems and the statements about the two methods in *Mathematical Methods*. See the quotations below.

67 Zhu Yiwen (2010, 86–87) pointed out this phenomenon. He suggests that Jia Xian’s (Yang Hui’s) consistent method in this chapter is still *zhichu* 直除 (or in Jia/Yang’s words, subtracting [between positions that] correspond [to each other] [*duijian* 對減]), even though the numbers written by characters in the configuration of numbers enable Jia/Yang to choose freely between the two methods. The author of this paper agrees with Zhu’s conclusion that written numbers changed the way practitioners dealt with the *fangcheng* method; however, the following part of this paper suggests that Yang Hui laid more emphasis on the method of “cross-multiplication.”

(*biancheng* 徧乘) using “subtracting [between positions that] correspond [to each other]” several times rather than the method of “cross-multiplication.” For the convenience of reference in this paper, expression “subtracting [between positions that] correspond [to each other]” (*duijian*, in Yang’s words) are used to refer to the method of after “multiplying throughout,” “subtracting [between positions that] correspond [to each other]” several times.⁶⁸ This is different from the method in which after “cross-multiplication” only one subtraction was carried out. This latter method was referred to as the method of “cross-multiplication,” also for convenience.

The assumption about Yang’s “preference” apparently can explain the fact that he intentionally uses the method demonstrated in the first problem to solve the seventh problem in which Liu Hui’s commentary indicated the other method. However, after examining all of his commentaries on the extant fourteen problems in *Mathematical Methods*, it is surprising that five of them use the method of “cross-multiplication” in the process of eliminating the number of a certain position (problems 2, 4, 5, 6, and 8), and four problems use the method of “subtracting [between positions that] correspond [to each other]” (*duijian* 對減) (problems 1, 3, 7, and 8), in which “subtracting correspondingly several times” (*jidu duijian* 幾度對減) or similar expressions, like “two times of subtractions with each other [for numbers with] ‘different *ming*’ 名 (lit. name, i.e. sign)” (*liangdu yiming xiangjian* 兩度異名相減) (Yang [1261] 1993, 968) appear. Problems 9 and 11 both present a situation where one of the two corresponding numbers that should be eliminated is the number “one,” so the difference between the application of the method of “cross-multiplication” and that of the method of “subtracting [between positions that] correspond [to each other]” is not obvious. The remaining four problems (problems 10, 12, 13, and 14) do not mention which method was used to eliminate the number in a position. The two methods coexisted in different steps of the demonstration of the process for solving problem 8. This is related to Yang Hui’s demonstration of the use of “the method of the positive and the negative,” which will be discussed below. This statistic does not support the assumption that Yang “preferred” the method of “subtracting [between positions that] correspond [to each other].”

Section 4 of this paper will explain further the order of the problems in *Reclassifications* and its relationship with the order of the problems in *Mathematical Methods*. A brief conclusion is that Yang Hui’s commentaries in *Mathematical Methods* are related to the order and the classification of the problems in *Reclassifications*. If one

68 This method corresponds to the method that is usually referred as “*zhichu* 直除” (lit. directly eliminate, in Liu Hui’s words) by modern scholars (Li 1990, 630; Guo Shuchun 2009, 325, 342). Zhu Yiwen (2010) has studied the history of *zhichu* and that of “cross-multiplication” in the “*Fangcheng*” chapter, and he interprets the change of the two methods on the basis of his analysis of the way of using calculation tools.

takes the position of the problem (problem 7 from *Mathematical Methods*) in *Reclassifications* into consideration, there is another interpretation for the aim of the method used in Yang's "procedure of calculation" for problem 7. Problem 7 of *Mathematical Methods* quoted above is advanced as the second problem under the subheading "*Fangcheng Method*" (方程法) in *Reclassifications*. This move implies that Yang intended to demonstrate to his readers (learners of mathematics) that the method of "cross-multiplication" and the method of "subtracting [between positions that] correspond [to each other]" in his commentaries on the first problem lead to the same correct answer. The two methods could, thus, be regarded as equivalent. After the demonstration of the equivalence between the two methods, he could use both ways to eliminate the number of a certain position in any other problem.

In fact, Yang Hui laid greater emphasis on the method of "cross-multiplication" than the method of "subtracting [between positions that] correspond [to each other]." The former method appears in the "overall statement" (*zongshuo* 總說) at the end of this chapter in *Mathematical Methods*.

The overall statement: . . . For example, in columns A (*jia* 甲) and B (*yi* 乙), [one] disposes all sorts of things and their prices, the procedure uses [the number in] the first position of column A (*jia*) to multiply [each number in] column B (*yi*). And then, [the procedure] uses [the number in] the first position of column B (*yi*) to multiply [each number in] column A (*jia*). The aim [of this operation] is to make [the numbers in the first position of the two columns] the same. Then [one] subtracts the smaller column [i.e. the column in which the original number at the first position is smaller] from the greater column [i.e. the column in which the original number at the first position is greater]. This is to remove the thing [in the first position] and to reduce the total money. [When] only one divisor (*fa* 法) and dividend (*shi* 實) appear, [the procedure] uses [the method of] division (*shangchu* 商除) [to solve] it. (Yang [1261] 1993, 973)⁶⁹

The method of "cross-multiplication" also appears in the text under the subheading "*Fangcheng Method*" in *Reclassifications*.

[The method uses] the "*lü* (率) of the thing we are looking for" to cross-multiply [all the numbers] in the neighboring columns. [One] subtracts the smaller [column in which the original number at the first position is smaller] from the greater [column in which the original number at the first position is greater]. And then [one] aims at eliminating (*sun* 損) [the positions] through subtraction, [until the remaining] money becomes the dividend and [the remaining number of] thing becomes the divisor. One

69 “總說：……如甲乙行列諸物與價，術以甲行首位徧乘其乙，復以乙行首位徧乘其甲。求其有等。以少行減多行，是去其物，減其錢。見一法一實，如商除之。”

divides the dividend by the divisor. (Yang [1261] 1993, 1020)⁷⁰

The fact that Yang lays more emphasis on the method of “cross-multiplication” is also related to the interpretation of his way of demonstrating “the method of the positive and the negative.” This will be returned to in great detail in section 4 of this paper.

2.2.2 A “flexible method” is used to avoid using the “procedure of the positive and the negative”

In *Mathematical Methods*, the “flexible method” does not only refer to the method of “cross-multiplication.” In “procedure of calculation” of some problems, Yang Hui uses other kinds of “flexible method” to show how a problem is solved without a certain mathematical method.

Under the third problem of *Mathematical Methods*, Yang Hui first disposed the numbers in the problem’s conditions in columns according to the instructions given by the sentence A. The result is the configuration of numbers on the left of Figure 5.

If one uses the method introduced in the first problem of the “*Fangcheng*” chapter, negative numbers will appear in the second position in the process for eliminating the number of high-quality grain in the left column. *The Nine Chapters* and Liu Hui’s commentary on this problem introduced and explained the “procedure of the positive and the negative” (*zhengfu shu* 正負術), solving the question of the emergence of negative numbers in the process of elimination.

However, in *Reclassifications* Yang Hui presents this problem as the third problem under the subheading “*Fangcheng* Method,” rather than under the subheading “Method of the Positive and the Negative.” The reason for such a classification is that he had his own view on the conditions for using the “method of the positive and the negative,” and his own view on the difference between this method and the “*fangcheng* method” (see section 3.4 of this paper). The section here first shows that, in *Mathematical Methods*, he successfully uses a kind of “flexible method” to avoid the appearance of a negative number in the process of elimination of the number in a certain position. The key operation lies in the transformation from the first configuration to the second (see Figure 5).⁷¹

70 “所求率互乘鄰行，以少減多，再求減損，錢為實，物為法，實如法而一。”

71 The two configurations correspond to the following two systems of linear equations in

modern mathematics:
$$\begin{cases} 2x + y + z = 1 \\ 3y + z = 1 \\ 1x + 4z = 1 \end{cases} \quad \begin{cases} 2x + y + z = 1 \\ 3y + z = 1 \\ 2y + 9z = 2 \end{cases}$$

High 1		High 2			High 2
	Middle 3	Middle 1	Middle 2	Middle 3	Middle 1
Low 4	Low 1		Low 9	Low 1	
<i>Shi</i> (divisor)	<i>Shi</i> (divisor)	<i>Shi</i> (divisor)			
1 <i>dou</i>	1 <i>dou</i>	1 <i>dou</i>	2 <i>dou</i>	1 <i>dou</i>	1 <i>dou</i>

Figure 5: The first configuration of numbers (on the left) and the second configuration of numbers (on the right) in problem 3 of *Mathematical Methods*.

in the middle column [to the left column correspondingly]” is absolutely essential to this mathematical procedure. By adding the numbers of the middle column to the numbers in the corresponding positions in the left column, the following operation of subtraction between the left column and the right column successfully avoids the emergence of a negative number (see the configuration on the right of Figure 5).⁷²

Why did Yang Hui demonstrate this method including addition between columns, rather than use a configuration of numbers to demonstrate the “procedure of the positive and the negative” presented by the text of *The Nine Chapters* and earlier commentaries? Once more, it is necessary to consider the order in which these problems are presented. In *Reclassifications*, following problem 7 from *The Nine Chapters* quoted above, the third

Between the two configurations, Yang Hui says:

[One chooses] those [columns in which] the high-quality grain exists. [One] should use [the number] “two” of grains on the upper right [corner] to multiply [all the numbers in] the left column and add the numbers in the middle column [to the left column correspondingly]. [One] subtracts the right column from it (the left column).⁷²

In the first configuration of Figure 5, because the middle position of the left column is empty, there is no number from which the practitioner can subtract the number “one” in the middle position of the left column. The sentence “add the numbers

72 “存上禾者。當以右上二禾乘左行，加中行數，以右行減之。”

73 If this kind of special operation of “adding” numbers of a column to another one is not included, as Zou Dahai 邹大海 (2010, 497–98) has pointed out, a negative number must be involved into the following procedure of *fangcheng*, no matter which number in the first position is eliminated. In fact, this operation of “adding” is unique in the tradition of the *fangcheng* method. Karine Chemla points out that the operation between columns in the configuration established in the process of solving problems by the *fangcheng* method is always “elimination.” The application of the “procedure of the positive and the negative” also always makes two columns to be eliminated to each other, rather than adding the columns to each other when the numbers are of different *ming* 名 (lit. name i.e. sign) (Chemla 1994, 2000). This paper will prove that Yang Hui’s way of using the “method of the positive and the negative” conforms to this conclusion. Considering these facts, the operation of addition between columns in the “procedure of calculation” (*cao*) of the third problem is very special and deserves more attention and further research.

problem in *Mathematical Methods* is also the third problem under the subheading “*Fangcheng* Method.” The second problem in the order of *Reclassifications* contains a “procedure of calculation” that is interpreted as a kind of “flexible method.” This “flexible method” is different from that of Liu Hui’s commentary on this problem. In the different layers of commentaries on the third problem of *Mathematical Methods*, the contrast between Yang Hui’s “procedure of calculation” (*cao*) and the “procedure” (*shu*) (including Liu Hui’s commentary) of *The Nine Chapters* again becomes apparent. Yang held his own view on the “method of the positive and the negative,” which will be discussed in detail in the next section. This led him not to include this problem in the subcategory “method of the positive and the negative” in his classification. Nevertheless, he includes this problem in the subcategory “*fangcheng* method” in *Reclassifications*. In order to support his classification, in *Mathematical Methods*, he constructs a “flexible method” that does not involve any negative numbers to solve this problem and demonstrates the “flexible method” using configurations.

3 “The method of the positive and the negative” in *Mathematical Methods* and *Reclassifications*

Many modern scholars have suggested that the “procedure of the positive and the negative” in *The Nine Chapters* corresponds to what today is known as a “sign rule” of addition and subtraction between positive and negative numbers (Guo Shuchun 2009, 333–36; Shen, Crossley, and Lun 1999, 404–6).⁷⁴ Mei (1984, 69) has pointed out that this interpretation is problematic. Karine Chemla (2000, 205–8) (Chemla and Guo 2004, 924 [“negative”], 964 [“name”], and 1031 [“positive”]) and Zhu Yiwen 朱一文 (2010, 92–98) have pointed out that the number calculated in the procedure is only the number of counting rods. This view suggests that the procedure is only used to deal with numbers in columns of the configuration of *fangcheng* problem, instead of as a general sign rule of addition and subtraction between positive and negative numbers.⁷⁵ Section 3 and the appendix of this paper analyze the “method of the positive and the negative” in *Mathematical Methods* and support such an interpretation with detailed evidence.

In *Mathematical Methods*, the “flexible method” in the “procedure of calculation” can avoid the use of the “method of the positive and the negative,” but it can also demonstrate this same method. Such a kind of “flexible method” can be found in problems 8 and 6 of *Mathematical Methods*.

74 For research on the transmission of the “procedure of the positive and the negative” to Korea and Japan, see Horng (2018).

75 In a personal discussion with Zhu Yiwen (May 12, 2020), Zhu said that Chemla and he agree on the view that in *The Nine Chapters* “the positive number” and “the negative number” only exist in the context of the “*Fangcheng*” chapter, and with the exception of the cases in “*Fangcheng*,” numbers are not differentiated as “positive” and “negative.”

In problem 8, Yang Hui has a sentence to explain his understanding and operation of the “method of the positive and the negative.” This problem is the first problem placed after the subheading “Method of the Positive and the Negative” in *Reclassifications*. Why did Yang advance this problem and make so much comment on it? The sentence in the “explanation of the problem” (*jieti* 解題) gives some reasons:

[The number of things] sold is positive, [the number of things] bought is negative. This problem relies on [the meaning] of selling and buying as positive and negative and adds conditions [which include] “deficit,” “excess,” and “rightly sufficient” [amount] to make the question. Is the meaning [of this problem] not profound as well? (Yang [1261] 1993, 967)⁷⁶

The statement above reveals that Yang thought this problem contains two sorts of conditions: the nature of numbers (positive and negative), and the results of equaling under some additional conditions by adding or subtracting. It is the combination of these complicated conditions that made him think that this problem was meaningful and deserved his detailed comments.

3.1 The definition of “positive” and “negative”

After the “explanation of the problem,” Yang Hui then gives the contents of the “method of the positive and the negative” by using big-sized characters. Among these, small-sized characters are inserted to explain the words or the sentences placed before them.

The first two words written in big-sized characters Yang Hui explains are “*zheng* 正” (lit. positive) and “*fu* 負” (lit. negative). He gives a definition of the two characters separately as:

What [is designated as] “*zheng* 正” (positive) is the positive number. What [is designated as] “*fu* 負” (negative) is the owed number. (Yang [1261] 1993, 967)⁷⁷

Firstly, Yang uses the explanation above to tell his readers what the two characters (*zheng* and *fu*) stand for when one uses the method. Since he will introduce a very concise rule for using this method, the characters, like a single “positive” (*zheng*) and “negative” (*fu*) that appear in his subsequent concise sentences, need to be explained first. Secondly, once the two opposite natures that are designated as characters “positive” and “negative” are converted to “positive numbers” and “owed numbers,” the operation on numbers with different *ming* 名 (lit. name i.e. sign) could be carried

76 “賣爲正數，買爲負數。題中借賣買爲正負，又加少、剩、適足爲問。此意不亦遠乎？”

77 “正者，正數也。負者，欠數也。”

out by practitioners of mathematics at his time. In fact, Yang refers to a “foolish” (*yu* 愚) method for the calculation of numbers with different *ming*.

[One] can put them (numbers with different *ming*) [in the places] beside the positions [of the numbers in the configuration] and [use the method of] “deduction” (*zhechu* 折除) of a positive number and a negative number. The ancient people spoke negatively of this method, so they established the ready-made procedure . . . to replace the foolishness of [the method of] “deduction” after putting [numbers] beside the positions. (Yang [1261] 1993, 967)⁷⁸

The quotation above shows that he thought that before the “method of the positive and the negative” there was a method of “deduction” (*zhechu*), which was replaced by the ancient people. In his eyes, this method was only regarded as foolish, but not regarded as wrong. Meanwhile, the method for dealing with calculations of numbers from the perspective of “positive and owed numbers” did exist and was understandable by the audience of *Mathematical Methods*. This is the reason why he first converts the characters “positive and negative” into numbers that could be calculated.

Thirdly, this explanation about “*zheng*” (positive) and “*fu*” (negative) works with his commentary on conditions in some problems of this chapter. In these problems, there are commentaries in which the first sentence converts the conditions in an assumed concrete context into the nature of “positive” and “negative.” For instance:

[One] regards [the situation of] adding the product (*ji* 積) to itself as *zheng* 正 (positive), [and the situation that after supplements, the product produced by some grains] should (*dang* 當) [equal to but, actually] not (*wei* 為) [equal to those produced by the certain amount of other grains] as *fu* 負 (negative). (Yang [1261] 1993, 966)⁷⁹

[The situation of] selling is *zheng* (positive); [the situation of] buying is *fu* (negative). [The situation of being] just sufficient [means that] the numbers are *ting* 停 (equal). Excess is *zheng* (positive), deficit is *fu* (negative). (Yang [1261] 1993, 967)⁸⁰

[One] defines [the item after the term] “more than” as *fu* (negative), the original weight as *zheng* (positive). (Yang [1261] 1993, 970)⁸¹

The explanation of the “*zheng*” (positive) and “*fu*” (negative) in each sentence quoted above converts opposite situations, like “selling and buying,” “excess and

78 “可副置位傍，正負折除。古人謂非其法，故立成術。……以代副置折除之愚也。”

79 “以本身并添積為正，當未為負。”

80 “賣為正，買為負，適足數停，多為正，少為負。”

81 “以多為負，本重為正。”

deficiency” and so forth into two natures: “*zheng* (positive) and *fu* (negative).” Since Yang also states that “What [is designated as] ‘*zheng*’ (positive) is the positive number. What [is designated as] ‘*fu*’ (negative) is the owed number” at the beginning of “method of the positive and the negative,” the nature of “positive” and “negative” that appear after these explanations in those problems will then be represented by “positive numbers” and “owed numbers.” These numbers can be disposed in the configurations and calculated by the method introduced below.

3.2 The aim of the operation of the “method of the positive and the negative”

Before introducing the use of the “method of the positive and the negative,” Yang Hui gives the way and the aim of his treatment for cases in which positive numbers and owed numbers appear in the configuration:

[One] uses [numbers in] the neighboring columns to multiply each other to make the “equality” [between the numbers in certain corresponding positions of different columns]. In the corresponding positions, [one] makes the operation of “elimination” (*chu* 除) to simplify the position, and to find the original (*yuan* 源) [quantity of the remaining things]. (Yang [1261] 1993, 967)⁸²

One interpretation of the word “*deng*” 等 in this paragraph is “the equal number (i.e. the greatest common divisor)” (see sentence C of section 2.1 of this paper). The “equal number” is the number in a certain position of the configuration, and it is subtracted from the number in its corresponding position that has been multiplied by this number. The number in the corresponding position is thereby eliminated. Another translation that is also suitable for this context is as above, and it can be supported by evidence in *Mathematical Methods*. According to the translation above, through multiplication between numbers in certain positions of the neighboring columns, the situation of “equality” is achieved. However, this “equality” is different from the modern understanding of this word. In the procedure for solving a problem, the numbers in the first configuration could be marked by the character “positive” or “negative” after the number. The “equality” after multiplication refers to the equality of numbers before the character “positive” or “negative” (i.e. the equality of the absolute values of the numbers). No matter which interpretation is taken, these operations aim at “simplifying the position” (簡其位), namely, removing the numbers in the configuration. After simplifying all the other positions, the relationship between the dividend and divisor of the last single thing appears in the configuration. In fact, this way and its aim are the same as those described by the small-sized characters in the *fangcheng* procedure quoted in section 2.1 of

82 “以鄰行相乘求等。對位爲除，而簡其位，求源。”

this paper: “[One] wants to make [the number] in the corresponding positions [be able to be] eliminated, [and the ultimate] aim is to make the position simple.”

3.3 “Pieces of procedure of calculation” (草段) following the “method of the positive and the negative”

The text of the “method of the positive and the negative” is very complex and is translated in the appendix to this paper. The translation and interpretation in the appendix are crucial for understanding this paper. In the following texts, sentences 1-A, 1-B, 1-C, 2-A, 2-B, and 2-C refer to the sentences translated in the appendix. Yang Hui thought that his readers needed “pieces of procedure of calculation” (*caoduan* 草段) for understanding how the method was used. Under the text of the method, he writes “[we] use pieces of procedure of calculation in the problem as explanation, [the readers] can understand the method of practice” (以問中草段為解，就明作法也) and “[we] extract several pieces of procedure of calculation again as explanation [of the method] (更摘草段為解)” (Yang [1261] 1993, 967). Horng Wann-Sheng 洪萬生 (2018, 4–7) quotes and analyzes Yang Hui’s procedure of calculation of problem 8 of *Mathematical Methods*. Using an interpretation of the “method of the positive and the negative” that is different from Horng’s, section 3.3 of this paper gives an analysis that conforms to more of the textual details of Yang’s text.

Yang uses the “flexible method” once again, this time in order to demonstrate the use of the “method of the positive and the negative,” in problem 8 of *Mathematical Methods* (see Figure 6 on the next page).⁸³

According to the procedure introduced in the first problem, the first position to be eliminated is usually the top position in the configuration (see the first configuration of Figure 6). More precisely, the top position in the middle column is eliminated first, and then the top position in the left column is eliminated. However, the method given before the configuration is “the first method of the positive and the negative” (see Appendix), a condition of which is that the numbers involved should have different *ming* 名 (i.e. sign). In order to demonstrate how to use this method, Yang Hui would need to find positions in which the two numbers have different *ming*. Several pairs of numbers meet the requirement. Yang chooses the second positions of the left column and middle column.⁸⁴

⁸³ The problem could be solved by the following system of linear equations:

$$\begin{cases} 2x + 5y - 13z = 1000 \\ x - 3y + z = 0 \\ -5x + 6y + 8z = -600 \end{cases} \quad (x, y, \text{ and } z \text{ represent the price of each ox, sheep, and pig, respectively}).$$

⁸⁴ In the first configuration of Figure 6, the first position in the left column and the right column meet this requirement too. There are two hypotheses explaining why Yang did not choose them. Either he wanted to include a column containing an empty position in his demonstration of the “first method,” or he wanted to make his first demonstration to be relatively straightforward, with fewer multiplications involved.

			Positive and negative is deducted [to acquire] this number 3	Minuend	Subtrahend		
5 Negative	1 Positive	2 Positive	Negative [sentence 1-A]	5 Negative	2 Positive	2 Positive	2 Positive
6 Positive	3 Negative	5 Positive	Empty Positive [sentence 1-A]	6 Positive	6 Negative	6 Negative	5 Positive
8 Positive	1 Positive	13 Negative	10 Positive [sentence 1-B]	8 Positive	2 Positive	2 Positive	13 Negative
600 Negative	Empty	1000 Positive	600 Negative There is no [number] to be added, do not move	600 Negative	Empty	Empty	1000 Positive

The first configuration

The second configuration

		Remove the number through deduction	Minuend	Subtrahend	
the ready-made methods and do not repeat the explanation (<i>shuo</i>)	3 Negative	0x Empty [sentence 2-A]	2 Positive	2 Positive	2 Positive
	Sheep	11	6	5	5
	Empty	Negative [sentence 2-B]	Negative	Positive	Positive
	10	15	2	13	13
	Positive	Positive [sentence 2-B]	Positive	Negative	Negative
	600	1000	No [Number] to enter	1000	1000
	Negative Thereafter, [we] will always use	Negative Positive has [no number] to enter, to be negative		Positive	Positive

The fourth configuration

Figure 6: Some configurations of numbers in problem 8 of *Mathematical Methods*.

The grey shading has been added by the author of this paper.⁸⁵

⁸⁵ In Figure 6, the fourth configuration of problem 8 contains a misprint. Instead of “*liubai fu* 六百負” (600 negative), it contains “*liubai guan* 六百貫” (600,000) (my thanks to the anonymous reviewer for pointing this out). The transcription above includes a correction. The Chinese texts in the second configuration written above or below the numbers are: “正負折除此數,” “異名相

To demonstrate how the method is used, after the multiplication of the middle column, he uses a specially inserted configuration⁸⁶ in his procedure of calculation.

In the second configuration in Figure 6, the three left-hand columns are grouped to form an inserted configuration. These have been shaded in grey in this transcription. This configuration was intentionally inserted by Yang in order to present the way the method of the positive and the negative is used. There are two pieces of evidence supporting this view. Firstly, this kind of inserted configuration appears twice, the second of which serves to demonstrate the use of “the second method.” Secondly, after using the inserted configuration for the second time, Yang declares that “thereafter, [we] will always use the ready-made methods and will not repeat the explanation (*shuo* 說) [of the use of the method]” (以後竝用成法, 更不重說) (Yang [1261] 1993, 968).

In this inserted configuration, the first column from the left contains the results of the “subtraction” between the two columns to its right. Under each result, there is a clause quoting from “the first method” or Yang’s explanation for the “subtraction” in this position. The subtraction aims at “simplifying a position” by eliminating the number in that position. In this case, the first place where Yang applies the method is the middle position, where the two numbers are “six of positive” and “six of negative” respectively. Once the number of the minuend has been eliminated, the position is overwritten with the character “*kong* 空” (empty). Clauses derived from the same method are used at other positions of the inserted configuration.

In the case in Figure 7, under the result “three of negative,” there is sentence 1-A from the “first method.” In order to show how the method was applied, in this case let us put the concrete number into the text⁸⁷ written in small-sized characters placed after sentence 1-A. “One uses ‘two of positive’ to subtract [it] from the ‘five of negative,’ this is not a real subtraction. The number (two) marked by ‘*zheng* 正’ (positive) was deducted from the number (five) marked by ‘*fu* 負’ (negative).” Only “two” was deducted, so the number “three” is left, the *ming*

減,” “異名相減,” “同名相加,” and “無加不動” (in the first column from the left); “原數” and “減數” (in the second and the third column from the left). The Chinese texts in the last configuration written above or below the numbers are: “以後竝用成法, 更不重說” (in the first and the second column from the left); “折半此數,” “同名相減,” “異名相加,” “異名相加,” and “正無入負” (in the third column from the left); “原數” and “減數” (in the fourth and the fifth column from the left respectively). In the third column from the left, the character “*ban* 半” (half) should be mistaken. It has been emended by the character “*qu* 去” (remove). For the original layout of the configurations of numbers, see Yang ([1261] 1993, 967–968).

86 According to Horng Wann-Sheng 洪萬生 (2018, 6), the right column of the inserted configuration corresponds to Yang Hui’s words “[One] can put [it (the column of “subtrahend” [*jianshu* 減數))] [in the places] beside the positions” (副置位旁). It can be also related to Zhu’s interpretation of the word *gengzhi* (更置, i.e. dispose once again) in Liu Hui’s words (Zhu 2010, 76) (see section 2.2 of this paper).

87 “以正減負者, 非減也, 是正折其去負矣。”

Positive and negative is deducted [to acquire] this number 3	Minuend	Subtrahend
Negative	5	2
[sentence 1-A]	Negative	Positive

Figure 7: Parts of configuration extracted from the second configuration of numbers in problem 8 of *Mathematical Methods*.

to demonstrate the use of the method. In this inserted configuration (shaded in grey), the number in the position of the oxen is to be eliminated. The two numbers “two” in these positions have the same *ming*; thus the practitioner chooses “the second method.” Here let us focus on two positions on which the methods are applied to determine the *ming* of the results.

In the bottom position of the inserted configuration (see the configuration in the upper part of Figure 8), the subtrahend is marked by “*zheng* 正” (positive) and the position of minuend is empty.⁸⁸ Thus, the practitioner uses “If the positive has no number to enter, one makes it to be negative” (a part of sentence 2-C) and gets “one thousand of negative” as the result.

In the second position from the top of the inserted configuration (see the configuration in the lower part of Figure 8), the minuend is “six of negative” and the subtrahend is “five of positive.” Thus, the practitioner uses sentence 2-B, which is written below the result. The sentence reads “[the numbers with] different *ming* are added to each other.” It guides the practitioner to add the number five (without sign) to the number six (without sign). The small-sized characters placed after sentence 2-B in the appendix tell how to determine the *ming* of the result. These small-sized characters explain that once the subtrahend is positive, the nature of positive should make up (*bu* 補) the nature of negative by returning (*huan* 還). Thus, the final result of the “addition,” the number eleven, should be marked by “*fu* 負” (negative).

名 (lit. name i.e. sign) of which is still negative. In a similar way, in the third position from the top of this inserted configuration, it is easy to understand how the sentence “[the numbers with] the same *ming* are added to each other” (1-B) was applied on numbers there.

Under the texts of the second method of the positive and the negative (see Appendix), a configuration (see the last configuration of Figure 6) is given

⁸⁸ In the inserted configuration, the characters in this position read “no [number] to enter” (無入), which precisely corresponds to the words used in sentence 2-C.

1000 Negative Positive has [no number] to enter, to be negative	No [Number] to enter	1000 Positive
11 Negative [sentence 2-B]	6 Negative	5 Positive

Figure 8: Parts of configurations extracted from the fourth configuration of numbers in problem 8 of *Mathematical Methods*.

The text above is an analysis of Yang Hui's way of using the "method of the positive and the negative" demonstrated by the procedure of calculation and the configurations.⁸⁹ This analysis may help to answer the following question: What were the reasons why Yang separated the "method of the positive and the negative" from the *fangcheng* method in his *Reclassifications*?

3.4 The reasons why Yang Hui separated the "method of the positive and the negative" from the *fangcheng* method

In *Reclassifications*, five problems are classified after the subheading "Method of the Positive and the Negative": problems 8, 6, 11, 5, and 4 of *Mathematical Methods*. All of the first configurations of the procedure in these problems are listed below.

⁸⁹ Due to the limits of the focus in this section, the analysis of problem 8 in *Mathematical Methods* is not completed. However, the remaining procedure of calculation and the configurations in this problem need to be studied further. It is believed that some parts of the text of the *Yijia* Studio edition contain errors. One of them has been pointed out in a footnote in the section on "determining the *ming* (i.e. sign) of result" of the appendix of this paper. Other emendations will be discussed in another paper.

5 Negative	1 Positive	2 Positive	High 2 Negative	High 3 Positive	1 Negative	Jia Empty	2 Positive	High 5 bing Negative	High 6 bing Positive	7 Positive	5 Positive
6 Positive	3 Negative	5 Positive	Low 5 Positive	Low 10 Negative	Yi Empty	3 Positive	1 Negative	15 bing Positive	Low 10 bing Negative	5 Negative	7 Negative
8 Positive	1 Positive	13 Negative	Added 1 dou Positive	Added 6 dou Positive	4 Positive	1 Negative	Bing Empty	Shi 5 sheng Positive	Shi (xia) 1 dou 8 sheng Positive	2 dou 5 sheng Positive	1 dou 1 sheng Positive
600 Negative	Empty	1000 Positive			1 shi Positive	1 shi Positive	1 shi Positive				

Figure 9: The first configuration of numbers in problems 8, 6, 11, 5, and 4 of *Mathematical Methods* respectively.

All the configurations above contain numbers marked by “zheng 正” (positive) and numbers marked by “fu 負” (negative).⁹⁰ In fact, from Yang Hui’s perspective, all the conditions in these problems contain words implying opposite meanings that could be converted to two natures, “positive and negative.” He makes these conversions at the beginning of the procedures dealing with these problems. As has been mentioned in section 3.1 of this paper, commentaries on problems 6, 8, and 11 in *Mathematical Methods* convert the conditions in the different concrete problems into the nature of “positive” and “negative.” The numbers marked by “positive” and “negative” are written directly in the configuration.

A phenomenon worthy of attention here is that, apparently, the conditions of the problem containing the verbal formula “diminish (*sun* 損) a certain amount of product” might imply a negative number from the perspective of modern mathematics. Moreover, Yang did classify problems 4 and 5 containing this character into the subcategory “method of the positive and the negative.” However, the condition containing the opposite words “diminish” (*sun* 損) and “augment” (*yi* 益) is not a sufficient condition for classifying the problem into the subcategory “the method of the

⁹⁰ The five configurations correspond to the following systems of linear equations in modern

mathematics:
$$\begin{cases} 2x + 5y - 13z = 1000 \\ 1x - 13y + z = 0 \\ -5x + 6y + 8z = -600 \end{cases} \begin{cases} 3x - 10y = 6 \\ -2x + 5y = 1 \end{cases} \begin{cases} 2x - y = 1 \\ 13y - z = 1 \\ -x + 4x = 1 \end{cases} \begin{cases} 6x - 10y = 18 \\ -5x + 15y = 5 \end{cases} \begin{cases} 5x - 7y = 11 \\ 7x - 5y = 25 \end{cases}$$

positive and the negative.” For example, problem 2 in *Mathematical Methods* also contains the conditions of “diminish” and “augment,” but in Yang’s text of this problem, its first configuration in *Mathematical Methods* does not contain any negative number; thus it was classified into another subcategory, namely, “the method of diminishing and augmenting” in *Reclassifications*.⁹¹

In *Mathematical Methods*, the problems that were referred to in *Reclassifications* after the subheading “*Fangcheng* Method” are problems 1, 7, 3, 10, 12, 13, and 14. Two of these problems are not in *Mathematical Methods*. Among all the extant first configuration under these problems, none contains a negative number. More interestingly, in problems 10, 12, 13, and 14, procedures from *The Nine Chapters* all include the sentence “[one] introduces the procedure of the positive and the negative into it” (以正負術入之). According to the commentaries written by Liu Hui to these problems, it is during the process of solving them that negative numbers appear.

However, the emergence of a negative number in the process of solving the problem is not a criterion for classifying problems in *Reclassifications*. In Yang Hui’s eyes, if the first configuration constructed for the problem in *Mathematical Methods* did not contain a negative number, the problem should be put in the subcategory “*fangcheng* method” in *Reclassifications*.⁹² Only the problems in which the first configuration contains a negative number are classified under the subcategory “the method of the positive and the negative.”

4 How did Yang Hui reclassify the problems from the “*Fangcheng*” chapter?

The discussion above has shown that *Reclassifications* is crucial for understanding Yang Hui’s views on the several methods concerning *fangcheng*. These views arguably provide some reasons for his classification of the problems found in *Reclassifications*.

This section of this paper focuses on the issue of Yang’s way of reclassifying problems from *The Nine Chapters* in each subcategory. As shown above, in *Reclassifications*, there are only questions and answers to the problems. Thus, the issue in question is why Yang reclassified the problems in a certain order. Were these problems randomly arranged after each subheading? In the “*Fangcheng*” chapter of *Mathematical Methods*, are there some clues to explain the order in which the problems were arranged in *Reclassifications*?

91 For a detailed discussion of “the method of diminishing and augmenting,” see Zhou (2018, 208–15).

92 Four problems are exceptional, and they were classified into two other subcategories. These exceptional problems are concerned with fractions or conditions that contain “diminishing and augmenting” (*sunyi* 損益). In *Reclassifications*, there are special methods for dealing with these problems.

4.1 The order of the problems in the subcategory “the method of the positive and the negative” in *Reclassifications* and an interpretation of this order

The first sentence in the “procedure of calculation” (*cao* 草) of problem 6 is the first place deserving a great attention:

The [commentary/commentaries on the] previous problem(s) is/are not enough to explain [the method of] the positive and the negative. (Yang [1261] 1993, 965)⁹³

Which problem(s) does the expression “the previous problem(s)” in this sentence refer to? Let us first check all the problems placed before problem 6 in *Mathematical Methods*.

4.1.1 To which problem(s) does the “previous problem(s)” (前問) not refer?

In Yang Hui’s commentaries on problems 5 and 4, the method of the positive and the negative is used to solve them. However, his commentaries only appear in the “procedure of calculation,” each of which is composed of only one configuration and a very brief description of procedure of calculation. If the expression “the previous problem(s)” refers to them, some contradictions arise. On the one hand, these two problems do not contain any explanation of the method of the positive and the negative. This phenomenon to some extent contradicts the sentence quoted above that the commentary/commentaries on the “previous problem(s)” is/are not enough to explain the method. The fact that “it does not explain [at all]” and the statement that it/they “is/are not enough to explain” are contradictory.

On the other hand, in the “procedures of calculation” of problems 4 and 5, there is the sentence “[the numbers with] the same *ming* are subtracted from each other” (同名相減) or there are the abbreviated expressions “[the numbers with] different *ming* are subtracted” (異名減) and “[the numbers with] the same *ming* are added” (同名加). According to the order of the problems in the *Yijia* Studio printed version, in the main text of *Mathematical Methods* these sentences or expressions are not explained by Yang Hui before problem 4 and problem 5.⁹⁴ The method of the positive and the negative is explained in detail by Yang Hui in problem 8. Moreover, the commentaries on these

93 “前問未足以發明正負。”

94 In problem 3 of *Mathematical Methods*, the text of *The Nine Chapters* introduces the “procedure of the positive and the negative.” However, the word “subtract” (*jian* 減) in Yang’s commentary in problems 4 and 5 is different from the word “eliminate” (*chu* 除) in the text of *The Nine Chapters*. When the “procedure of calculation” of problems 4 and 5 mentions sentences or expressions concerning the method of the positive and the negative, it should refer to the method explained by Yang Hui.

two problems do not refer its readers to problem 8 for understanding these sentences or expressions. It is somewhat awkward that problems 4 and 5 are explained by the commentary in a problem that appears later, and without any reference to this fact. This phenomenon will be explained later. When the commentary on problem 6 uses the expression “the previous problem,” could it refer to these two problems? The answer is probably no.

In the other problems placed before problem 6, only problem 3 is concerned with the method of the positive and the negative. Is it possible that the “previous problem” is problem 3 of *Mathematical Methods*? Section 3.2 of this paper has analyzed problem 3. In Yang’s commentary on this problem, he first demonstrates a method that he thought “was the same as that of the first problem” (與第一問同), after which, nevertheless, he uses a flexible method of the “standard” *fangcheng* method to intentionally and successfully avoid the occurrence of a negative number. His commentary on this problem does not contain anything pertaining to the method of the positive and the negative. Thus, “the previous problem” cannot be problem 3 either.

4.1.2 The “previous problem” probably refers to problem 8 in *Mathematical Methods*

In *Reclassifications*, problem 6 ranks second under the subheading “Method of the Positive and the Negative.” Is it possible that the expression “the previous problem” refers to the first problem in this subcategory? This is problem 8 in *Mathematical Methods*, which has been quoted and analyzed above. Yang Hui’s commentary and “procedure of calculation” for this problem do indeed explain and demonstrate the method of the positive and the negative. If this assumption is correct, why did Yang think that his commentary on problem 8 was not enough to explain the method of the positive and the negative?

In the “procedure of calculation” of problem 8, there are four cases of eliminations of unknowns. In the first and the third case, only one multiplication is carried out, after which one “subtraction [between positions that] correspond [to each other]” (對減) or “two times of subtraction of different *ming*” (兩度異名相減) makes the position empty. In the second case of elimination, the numbers of oxen in the middle and right columns are the same. They are both “two positive” (二正); thus there is no need to do any multiplication before using the method of the positive and the negative. However, the last elimination requires one to eliminate the number of sheep in the middle column.

Before this configuration (see Figure 10), there is a sentence indicating an operation of “cross-multiplication” between the numbers of sheep in the middle and right columns (以右中羊互乘). The numbers and their *ming* in the position of sheep of the

3 Negative	Ox Empty	Ox Empty
Sheep Empty	11 Negative	15 Positive
10 Positive	15 Positive	19 Negative
600 Positive	1000 Negative	1800 Positive

Figure 10: The fifth configuration of numbers in problem 8 of *Mathematical Methods*.

two columns are “eleven of negative” (十一負) and “fifteen of positive” (十五正). In terms of this configuration, in order to eliminate the number of sheep in the middle column, Yang did not use the method that made one multiplication and several times of subtraction.

Section 2.2 of this paper has shown that Yang may have thought that the method of “cross-multiplication” was a kind of “flexible method” (活法). Moreover, he introduces it intentionally by putting forward problem 7 in *Mathematical Methods* as the second problem in *Reclassifications* and demonstrating the equivalence between the “standard method” and the “flexible method” in *Mathematical Methods*.

In problem 8 of *Mathematical Methods*, there is a case in which the operands of the “cross-multiplication” have opposite *ming*. However, there is no further explanation for such a “cross-multiplication” in the “procedure of calculation” (*cao*) of problem 8. In the method of the positive and the negative, did Yang Hui need to introduce the way of “cross-multiplication” in a particular fashion? From his perspective, the answer to this question is probably yes. For the *fangcheng* method, in which all the numbers operated on were positive, the practitioner did not need to consider the *ming* (i.e. sign) of the result of cross-multiplication. If one of the operands of “cross-multiplication” was negative, the *ming* of the result of each multiplication would directly affect the following process of elimination using the method of the positive and the negative. How could the practitioner determine the result of such cross-multiplication?

Yang Hui does not leave this question to his readers. Neither does he explain this point in problem 8. However, in the next problem, problem 6 of *Mathematical Methods*, he applies the method of “cross-multiplication” for a positive number and a negative number twice. When he applies the method for the first time, he also provides further explanation written in small-sized characters on how to handle such “cross-multiplication.” His commentary on the cross-multiplication of a positive number and negative number has been analyzed in the appendix of this paper⁹⁵ and will not be repeated here.

Furthermore, his commentary on this method is crucial for his uniformly

⁹⁵ For details, see the footnote about the issue that “it is the number without any sign that is operated and calculated” in the appendix to this paper.

solving the problems by the method of cross-multiplication. Section 3.2 of this paper has shown that, in the “overall statement” (總說) at the end of the “*Fangcheng*” chapter, Yang summarizes the *fangcheng* method, in which the method of “cross-multiplication” played an important role. Without knowing how to “cross-multiply” between negative and positive numbers, Yang could not summarize it in that way.

All the evidence above suggests that the point, Yang Hui presumably thought, that was not explained clearly in the commentary to the “previous problem (problem 8 of *Mathematical Methods*),” was precisely the point he explains specifically in the procedure of calculation of problem 6. This point was the method of cross-multiplication between numbers marked by “positive” and “negative.”

4.1.3 The order of the problems in the subcategory “the method of the positive and the negative” in *Reclassifications*

Once the assumption above is accepted, the order of the other problems under this subcategory in *Reclassifications* becomes understandable. In the commentaries to problems 4 and 5 in *Mathematical Methods*, there are expressions which had not been explained in the text prior to them. Moreover, the commentaries on these two problems are very brief, and problem 5 also uses the “cross-multiplication” of positive and negative numbers.

In *Reclassifications*, under the subheading “The Method of the Positive and the Negative,” problems 8, 6, 11, 5, and 4 of the “*Fangcheng*” chapter in *Mathematical Methods* are arranged according to this order. This order matches very well with some textual phenomena in the commentaries on these problems in *Mathematical Methods*. The textual phenomena can be summarized as follows. Yang Hui provides a lot commentary to problem 8 and gives his explanations and “procedure of calculation” for demonstrating the use of “the method of the positive and the negative.” The commentary to problem 6 explains further how to deal with the case of “cross-multiplication” between positive and negative numbers. Then, he still provides a detailed “procedure of calculation” to problem 11. However, compared with the commentaries on the first two problems, the “procedure of calculation” of problem 11 is much simpler. As for the last two problems, only one configuration and a brief procedure are given, perhaps because the method used in the two problems had been explained in great detail earlier. Such cases are normal in this work. In Yang’s activity of commenting on the problems in *Mathematical Methods*, for the contents which have been stated previously, he would write that “thereafter, [we] will always use the ready-made

methods and do not repeat the explanation (*shuo*) [of the use of the method]" (以後並用成法，更不重說) (Yang [1261] 1993, 968).

4.2 The order of the other problems in the “*Fangcheng*” chapter of *Reclassifications* and an interpretation of this order

Is the analyses of the order of the problems under the subcategory “method of the positive and the negative” in *Reclassifications* suitable for other subcategories? Let us analyze the order of the problems under the subheading “*Fangcheng* Method” in *Reclassifications*.

4.2.1 The order of the problems in the subcategory “*fangcheng* method” in *Reclassifications*

In *Mathematical Methods*, the commentary on the first problem is very detailed and is composed of the text of the “procedure” written in big-sized characters, of explanations (written in small-sized characters) of the “procedure,” and of the “procedure of calculation” of configurations displaying a standard procedure of elimination. However, Yang admits that “at the very beginning (*chu* 初) [of introducing this method],” the “standard procedure” of elimination was not a “flexible method.” He consequently uses the next two problems and the commentaries on them to show several kinds of “flexible method.” He does not put forward the flexible methods by himself, however, using problem 7 of *The Nine Chapters* as the second one in *Reclassifications*. Liu Hui’s commentary on this problem had already explained the reason for “cross-multiplication.” By means of Liu’s commentary, which was already regarded as a part of Canon, Yang presents the method of “cross-multiplication.” This was a method he stressed and it was included in his conclusive statement (“overall statement”) of the *fangcheng* method. This matches well with the fact that after he put forward problem 7, his “procedure of calculation” for this problem demonstrates the standard *fangcheng* method. According to the analysis in section 2.2, his aim was to prove that this method of cross-multiplication was equivalent to the standard *fangcheng* method. For the third problem in *Reclassifications*, problem 3 in *The Nine Chapters*, because his understanding of the condition of using the “method of the positive and the negative” was different from that described in *The Nine Chapters*, as discussed in section 3.4 of this paper, he intentionally uses a “flexible method” when he eliminates the numbers and successfully avoids the emergence of negative number (see section 2.2 of this paper). All the first three problems in *Reclassifications* in the subcategory “*fangcheng* method” seem to be concerned with his introduction and

demonstration of “flexible methods” and the “standard method” of the *fangcheng* method.

Unfortunately, problems 9 and 12 in the “*Fangcheng*” chapter in *The Nine Chapters*, though present after the above three problems in *Reclassifications*, are not in the extant version of *Mathematical Methods*. Relying on the problems in Wu Jing’s 吳敬 (fl. 1450 CE) *Great Compendium of The Nine Chapters on Mathematical Methods with Analogies* (*Jiuzhang suanfa bilei daquan* 九章算法比類大全 1450 CE, hereafter, *Great Compendium*)⁹⁶, some commentaries which should have appeared in *Mathematical Methods* can be partly recovered.⁹⁷ There is no evidence that Yang Hui used other “flexible methods” for these two missing problems. However, based on a reconstruction of the text, attention can be drawn to two expressions in the two problems’ commentaries that perhaps belonged to *Mathematical Methods*:

The remaining method is the same as [that] of the previous method.⁹⁸ (Guo 1993b, 2:249; commentary on problem 4 of the “*Fangcheng*” chapter in *Great Compendium*)

This problem relies on the horse [to express the context of the problem], [it] is also the same as the meaning of [the problem] which relies on the grain.⁹⁹ (Guo 1993b, 2:249; commentary on the problem 5 of the “*Fangcheng*” chapter in *Great Compendium*)

Yang’s commentaries on these two problems all refer to the “previous” (前) problem(s). It seemed that he thought that he had nothing further or nothing special to say on them. Thus, he just refers his readers to the previous problems, which he had already explained and commented on.

In fact, such referencing expressions can be found in the commentaries on problems 10, 12, 13, and 14 of *Mathematical Methods*. The expression in the last four problems are the same: “[It] is the same as the previous body (problem) to look for [the answer]” (同

⁹⁶ This title is translated as *Comparative Study of the “Nine Chapters”* in Needham and Wang (1959, 79), as *Fully Comprehensive [Collection of] Computational Methods in Nine Chapters with [New Problems and Rules] Devised by Analogy with [Ancient Problems and Rules]* in Martzloff (1997, 20), and as *Great Compendium of Analogical Categories to The Nine Chapters* in Bréard (2014, 177).

⁹⁷ Through a detailed comparison between the texts of Yang Hui’s and Wu Jing’s work, some ways in which Wu took text from *Mathematical Methods* to compile his *Great Compendium* can be found. On the basis of these rules gained through analyzing Wu Jing’s ways of exacting texts, some text lost in the printed edition of *Mathematical Methods* can be partly recovered. For details, see Zhou (2018, 210–11; 224–39).

⁹⁸ “餘法并同前問。”

⁹⁹ “此問借馬，亦同前借禾之意也。”

前體求)¹⁰⁰. Moreover, under the subheading “Fangcheng Method” in *Reclassifications*, these four problems were placed exactly after problem 12 of *The Nine Chapters*, which is the second missing problem in *Mathematical Methods*. In conclusion, the “previous body (problem)” in these expressions refers to the problems preceding them according to the order of the problems in *Reclassifications*, rather than to the problems preceding them according to the order in *Mathematical Methods*.

4.2.2 A further interpretation of the order of the problems in the subcategory “fangcheng method” in *Reclassifications*

The commentaries on the problems in *Mathematical Methods* are precisely related to the order of the problems placed after the subheading “Fangcheng Method” in *Reclassifications*. Now let us re-examine the order of these problems in *Reclassifications* from another angle. The problems from *The Nine Chapters* are listed below according to the order in which they are listed in the subcategory “fangcheng method” of *Reclassifications*:

Problems 1, 7, 3, 9, 12, 14, 16, 17, and 18 of *The Nine Chapters*.

Problems 9 and 12 of *The Nine Chapters* are lost from the printed version of *Mathematical Methods*. From this order, it can be found that, in fact, with the exception of problem 7 which was indeed put before problem 3, all the other problems are still arranged according to their order in *The Nine Chapters*. That is to say, Yang Hui only selected those problems that he thought belonged in his new subcategory “fangcheng method,” and did not change the original order of the problems from *The Nine Chapters* if he had no special intention to provide commentaries on them. He respected the original order of the problems in *The Nine Chapters*. For the last six (or four) problems in his subcategory “fangcheng method” in *Mathematical Methods*, he does not comment on them any further, merely referring his readers to “the previous problems,” and leaving them in the original order in *The Nine Chapters*.

If the above assumption is accepted, the following fact is very important. The reorganization and reordering of problems in each subcategory made by Yang were intentional and significant. He made finely detailed commentaries on those problems that disrupted the original order of the problems in *The Nine Chapters*. This paper’s analyses of problems 8, 6, and 15, which were put before problems 5 and 4 in the subcategory “the method of the positive and the negative” of *Reclassifications*, can also

100 The character “ti 體” (lit. body) is, perhaps, an error, and should be the character “ti 題” (problem) in the manuscript, for the two characters have a similar pronunciation. The manuscript contains many such errors due to similarities in pronunciation (Zhou 2018, 108–10). The latter character is more suitable in this expression and matches other expressions in which Yang Hui used the word “problem.”

support this view.¹⁰¹

4.2.3 The order of the rest of the problems in *Reclassifications*

The conclusions concerning the order in *Reclassifications* could be applied to two other subcategories: “the method of diminishing and augmenting,” and “the procedure of the numerator and the denominator” (分母子術). It is a pity that only two problems were put under each of these subcategories, and that one problem has been lost from each. However, analysis of Wu Jing’s *Great Compendium* can still allow one to make some reliable assumptions.

In the subcategory “the method of diminishing and augmenting,” problem 11 was put ahead of problem 2 of *Mathematical Methods*. Under the first problem, Yang Hui should have provided a relatively detailed commentary and “procedure of calculation” to explain his understanding of the method “diminishing and augmenting.” *Great Compendium* can be used once again to recover some crucial pieces of text belonging to this problem in *Mathematical Methods* (Zhou 2018, 209–13). For the second problem in the subcategory, the “procedure of calculation” also mentions the method of “cross-multiplication” (Yang [1261] 1993, 964).¹⁰² This manner of especially introducing “cross-multiplication” also appears in the subcategories “the *fangcheng* method” and “the method of the positive and the negative,” where “cross-multiplication” is also introduced in the second problem.

Similarly, the problems under the subcategory “the method of the numerator and the denominator” probably follow the same pattern, but the first problem of these problems is lost, and *Great Compendium* does not preserve enough information about the commentary originally present in *Mathematical Methods*.¹⁰³

101 The reason why Yang put problem 5 of *The Nine Chapters* before problem 4 is still unclear.

102 “Cross-multiplication” was applied to the number of high-quality grain in both columns of this problem.

103 Another point about the problems in the subcategory “the procedure of the numerator and the denominator” remains unclear and obstructs an examination of the order of the problems in this subcategory. In *Mathematical Methods*, the commentary on the second problem of this subcategory (problem 13 of *Mathematical Methods*) uses the “method of the positive and the negative.” This method is not introduced until the next subcategory, namely, the subcategory “the method of the positive and the negative.” The order of the problems in *Reclassifications* was elaborately designed by Yang, and it is abnormal to find a problem solved by a method without any introduction to it. In *Great Compendium*, this problem was put at the end of this chapter, after the five problems of the subcategory “the method of the positive and the negative.” More information concerning the editions and emendations of *Mathematical Methods* is needed to facilitate further research on the “procedure of the numerator and the denominator.”

5 Conclusion

This paper has addressed the mathematical methods and the order of the problems in the “*Fangcheng*” chapter of Yang Hui’s work. It also has demonstrated that the issues of giving an in-depth interpretation of these methods and explaining the order of the problems are interrelated.

Mathematical Methods contains a detailed “procedure” showing how the fundamental method for *fangcheng* is carried out. After introducing the *fangcheng* procedure using the example of the first problem, this paper focused on the expression “flexible methods” that Yang Hui used in the commentary on this problem. One such “flexible method” refers to the method of “cross-multiplication” in the process of eliminating a number in a certain position of the configuration (that is, from a modern mathematical perspective, a coefficient in one of the linear equations). Yang lays greater emphasis on the method of “cross-multiplication” than the method of “subtracting [between positions that] correspond [to each other]” (*duijian* 對減), which was introduced as “vertical elimination” (*zhichu* 直除) in *The Nine Chapters*. The method of “cross-multiplication” had already appeared in the commentary written by the third-century commentator Liu Hui to a problem in *The Nine Chapters*. However, in *Reclassifications*, the position of this problem was changed, and it became the second problem under the subheading “*Fangcheng* Method.” Yang Hui’s commentary on this problem uses the method of “subtracting [between positions that] correspond [to each other]” (*duijian*). He made use of this treatment to show that the method of “cross-multiplication” is equivalent to the method of “subtracting [between positions that] correspond [to each other].” Subsequently, he uses both procedures to carry out elimination in other problems.

The term “flexible method” does not only refer to a certain kind of method for multiplication. Another kind of “flexible method” is also employed by him to avoid using the “method of the positive and the negative.” If one uses the method introduced in the first problem to solve the third problem of the “*Fangcheng*” chapter in *The Nine Chapters*, negative numbers must be introduced in the process for eliminating the number. By contrast with this procedure, Yang’s alternative method of eliminating numbers managed to avoid the appearance of negative numbers. Moreover, in *Reclassifications*, he puts this as the third problem under the subheading “*Fangcheng* Method,” rather than under the subheading “The Method of the Positive and the Negative.” The reason why he carried out such a change seems to be related to his understanding of the difference between the two methods, and of the different circumstances in which one should opt for one or the other of them.

The appendix to this paper provides a detailed interpretation of “the method of the positive and the negative” in *Mathematical Methods* and *Reclassifications*. Yang Hui

discusses this method in great detail in his commentary on problem 8 of *Mathematical Methods*. The operation performed upon the columns of numbers, in one of which a number in a certain position should be eliminated (or, to use Yang Hui's term, will be "simplified"), is always "subtraction" (*jian* 減). However, this "subtraction" (*jian*) is different from "subtraction" in modern mathematics. What "positive" and "negative" refer to are not identical to what are called "positive numbers" or "negative numbers" in modern mathematics. *Stricto sensu*, in *Mathematical Methods* "positive" and "negative" only refer to a number that is marked by a character written as "zheng" 正 or as "fu" 負 respectively.

On the basis of the analysis of the above-mentioned methods in the "*Fangcheng*" chapter, Yang Hui's ordering of the problems in the "*Fangcheng*" chapter in *Reclassifications* can be understood.

Firstly, the expression "the previous problem" in the commentary to problem 6 in *Mathematical Methods* does not refer to any of the problems placed before problem 6 according to the order of the problems in *Mathematical Methods*, but, rather, refers to problem 8 of *Mathematical Methods*. What is more, problem 8 was placed before problem 6 specifically according to the order in *Reclassifications*. With problem 8, Yang Hui explains the use of the "method of the positive and the negative," and with problem 6, the commentary introduces how the method of "cross-multiplication" is used for a positive number and a negative number. These commentaries explain the reason why these two problems were placed together and why Yang says that "the [commentary to the] previous problem is not enough to explain [the method of] the positive and the negative" (前問未足以發明正負).

Secondly, the analysis of the order of the other problems in *Reclassifications* also supports the conclusion that the commentaries on the problems in *Mathematical Methods* are directly related to their positions under the subheading "*Fangcheng* Method" in *Reclassifications*. The order of some problems in *The Nine Chapters* was regarded as problematic by Yang Hui. The reorganization and especially the problems which were put forward in each category were intentionally designed by him. Yang provided especially detailed commentaries on those problems the position of which had changed from the original order in the sequence of problems.

With respect to the methodology of doing research on ancient mathematical works, this paper provides a detailed textual analysis and a mathematical analysis of the mathematical methods recorded in *Mathematical Methods*, especially on "the method of the positive and the negative." Using these research methods, rather than merely relying on modern mathematical terms and formulas, contemporary researchers can gain a more subtle understanding of the mathematical ideas of ancient practitioners, and, in particular, a better understanding of the ancient and medieval author's ways in which they wrote or compiled their mathematical works.

Appendix: A translation and interpretation of the “method of the positive and the negative”

The “method of the positive and the negative” Yang Hui introduces in *Mathematical Methods* contains two parts, each of which is composed of big-sized characters giving the text of the method, and small-sized characters that explain the big-sized characters that precede them. In *Reclassifications*, under the subheading “Method of the Positive and the Negative,” Yang repeats the parts of the method printed in big-sized characters in *Mathematical Methods* (Yang [1261] 1993, 1022). On the basis of the text from *Reclassifications* and the meaning the small-sized characters convey, it can be confirmed that one sentence in the second part of this method contains a mistake.¹⁰⁴ After modification, the big- and small-sized characters in *Mathematical Methods* are interpreted as below. For the sake of convenience for reference, numbers and letters have been added before each sentence written in big-sized characters. Since the text printed in big-sized characters is very concise, and the text in small-sized characters also allows of many different interpretations, in some cases, footnotes have been added as explanation for the interpretation adopted. The analysis after the translation and the “pieces of procedure of calculation” (*caoduan* 草段) translated in section 3.3 of this paper will also explain the interpretation here.

1 Translation of the “method of the positive and the negative”

1-A: The first method¹⁰⁵: [If the numbers in the position which is to be eliminated

104 The first character (“zheng 正” [positive]) in the fourth column from the left of folio thirty-three of the *Yijia* Studio edition is an error (Yang [1261] 1993, 967). It should be “fu 負” (negative). The sixth character (“fu 負”) in the same column should be “zheng 正.” The *Corrigenda* written by Song Jingchang did not mention these two mistakes.

105 The “first method” refers to the “first method of the positive and the negative.” In the *Yijia* Studio printed edition, between the big-sized characters “zhengfu 正負” (positive and negative) and “yifa 一法” (the first method), there are several sentences, parts of which have been translated above (Yang [1261] 1993, 967). These were carved in a smaller size than the characters “zhengfu 正負” and “yifa 一法,” so they are clearly inserted commentaries. Looking at the big-sized characters only, “zhengfu 正負” and “yifa 一法” are connected. The characters “erfa 二法” (the second method) below written in big-sized characters are also connected to the word “zhengfu 正負,” even though between them there are some text and configurations of numbers. In *Reclassifications*, Yang Hui designates the method as “the method of the positive and the negative” (zhengfu fa 正負法), after which the two parts of text were started by expressions “the first” (其一) and “the second” (其二) respectively.

have] different *ming* 名 (lit. name i.e. sign),¹⁰⁶ [the numbers with different *ming*] are subtracted from each other,¹⁰⁷

Small-sized characters: [Cases in which] the positive meets the negative are considered as [the type having] different *ming*. The case in which using the positive to subtract [it] from the negative is not subtraction. This means that the positive is deducted from the

106 Whether in “the first method” or “the second method,” the first clause of the method always implies a conditional letting the readers know in which cases they should choose one method or the other. This is the reason why a translation like “different *ming* are eliminated from one another” or “Opposite signs subtract” (Shen, Crossley, and Lun 1999, 404; Dauben 2007, 280) is not adopted. The text of *The Nine Chapters* includes the “procedure of the positive and the negative” (正負術). The text of the procedure is very similar to that of “*zhengfu fa* 正負法” (the method of the positive and the negative) in *Mathematical Methods* (Chemla and Guo 2004, 624–26). Karine Chemla translates the sentence in the procedure, “*Yiming xiangchu, tongming xiangyi*” (異名相除, 同名相益), that corresponds to the first sentence of the method in *Mathematical Methods*, as follows: “If the different names are eliminated from one another, the same names are increasing each other” (Chemla 1994, 10); “Si des nombres de noms différents sont éliminés l’un de l’autre, les nombres de même nom s’augmentent l’un l’autre” (Chemla and Guo 2004, 627). Some changes are made on the basis of an understanding of Yang Hui’s texts. Firstly, the small-sized characters under the two big-sized characters “*zhengfu* 正負” (the positive and the negative) explain that, prior to the application of “the method of the positive and the negative,” through multiplication the practitioners had made the numbers (without signs of positive or negative) in a certain position of two columns “equal” (see section 3.2 of this paper, or according to the other interpretation there, the number in the certain position has been augmented and the “equal number [i.e. the greatest common divisor]” has been found). Moreover, the aim is to eliminate the number in a certain position (相乘求等, 對位為除, 以簡其位) (Yang [1261] 1993, 967). When Yang first mentions “the two methods [of the positive and the negative],” he says that the two methods were “chosen and used according to the problems (*ti* 題)” (參題取用) (Yang [1261] 1993, 967). A question now arises concerning the precise meaning of this sentence: based on what aspect of the problem (*ti*) is a certain method (the first one/the second one) chosen? The text of this method should include conditionals informing its readers of the basis on which a certain method is chosen. Since the aim of using the methods has been determined (elimination of the number in a certain position), and the numbers without signs had been made equal (or the number had been augmented), the thing that could become the variable condition is the mark placed after the numbers, which is called “*ming*” 名 (lit. name, i.e. sign) in the methods. Thus, the first clause (1-A, 2-A) in each method should include a conditional, and “different *ming*” and “the same *ming*” are conditions by which the readers choose between the two methods. Also see Mei (1984, 70). Secondly, the first clause (1-A, 2-A) in each method serve as more than a conditional. In the second configuration of the problem 8 (Yang [1261] 1993, 967), the clause “[the numbers with] different *ming* are subtracted from each other” appears twice. This configuration will be analyzed below. On one occasion, the clause tells its readers which method they should choose, and in the place where this clause is repeated, since the method had already been chosen, this clause serves as an instruction that could be regarded as parallel to the second clause (1-B, 2-B) in each method. Even though in Chemla’s translation the first clause is only a conditional, in the footnote to the translation, she points out that “La première de ces deux clauses permet également de pratiquer ‘l’élimination’ (*chu*) entre dividendes et dividendes disposés, dans la seconde partie de l’algorithme” (Chemla and Guo 2004, 868). (In the second part of the algorithm, the first of the two clauses also permits [one] to practice “elimination” between dividendes and arranged dividendes.)

107 1-A: “一法: 異名相減, ”

negative.¹⁰⁸ [Cases in which] the negative meets the positive are also considered as [the type having] different *ming*. The case in which using the negative to subtract [it] from the positive is real subtraction. [In the case in which the number of] the positive is larger than [that of] the negative, one deducts [the negative from the positive].¹⁰⁹

1-B: [the numbers with] the same *ming* are added to each other.¹¹⁰

Small-sized characters: [Cases in which] the positive meets the positive or the negative meets the negative are all considered as [the type having] the same *ming*. The text above is that [the type of having] different *ming* uses subtraction; [the text] below¹¹¹ should be [the type of having] the same *ming* “makes up” (*buhuan* 補還) [with each other].¹¹²

1-C: [If] the positive has no [number] to enter (*wuru* 無入, i.e. the other operand is equal to zero), [one makes it] to be positive; [If] the negative has no [number] to enter, [one makes it] to be negative.¹¹³

Small-sized characters: [This method] originally is “[the numbers with] the same *ming* are added to each other.” Because the neighboring positions have no counting rods (*suan* 算) to enter, [one] says that “[in] the case in which the positive has no [number] to enter, [one] still [makes it] to be positive. [In] the case in which the negative has no [number] to enter, [one] still [makes it] to be negative.” The ancient edition mistakenly carved [the characters] as “there is no people,”¹¹⁴ [this is] wrong.¹¹⁵

108 This paper will interpret the word *zhe* 折 as “deduct” in the two sentences (“是正折其去負矣” and “正多負而折去矣”) of this paragraph. However, a proper translation of the two sentences cannot be achieved if no emendation is made here. One possible emendation of the former sentence is placing the character “*qu* 去 (lit. remove)” after “*zhe* (deduct)” and forming a word “*zhequ*,” which does not change the meaning of “*zhe* (deduct).” Another possible emendation is simply to remove “*qu*” in the former sentence.

109 “正見負為異名，以正減負者，非減也，是正折其去負矣。負見正亦異名，以負減正者，誠減也，正多負而折去矣。”

110 1-B: “同名相加。”

111 An interpretation of the words “the text above” (*shangwen* 上文) and “below” (*xia* 下) is as follows: This sentence of small-sized characters tells the reader that once they have used the first clause above ([the type of having] different *ming* uses subtraction 異名為減), the text below that will be used should be “[the type of having] the same *ming* make up with each other” (同名補還). The character “*ji* 即” (lit. precisely) reveals the close relationship between the two clauses and that they are not methods independent of each other. These two words also appear in the small-sized characters following sentence 2-B. There they also imply that once the first clause has been used, the second clause used should be what the small-sized characters describe placed after the character “*xia* 下” (below).

112 “正見正或負見負，皆為同名。上文異名為減，下即同名補還。”

113 1-C: “正無入正之，負無入負之。”

114 Yang Hui thought that a character in the woodblock edition he saw included an error, and that the character “*ren* 人” (people) should be replaced by the character “*ru* 入” (enter). The two characters have a very similar shape. Qing scholars Wang Lai 汪萊 and Li Huang 李潢 did not agree on this view (Guo Shuchun 2009, 334). Guo insists on the fact that the character “*ren* 人” is not wrong and interprets it as “partner” (偶, 伴侶) (Guo Shuchun 2009, 334). In order to interpret the text in *Mathematical Methods*, the translation follows Yang Hui’s suggestion.

2-A: The second method:¹¹⁶ [If the numbers in the position which is to be eliminated are] the same *ming*, [the numbers with the same *ming*] are subtracted from each other,¹¹⁷

Small-sized characters: [Cases in which] the positive meet the positive, or the negative meets the negative are considered as [the type having] the same *ming*. [One] subtracts [them] from each other.¹¹⁸

2-B: [the numbers with] different *ming* are added to each other.¹¹⁹

Small-sized characters: [If the text which is used] above [deals with the case of] using the positive to subtract [it] from the positive, [the text which will be used] below is using the negative to return to the positive or using the positive to return to the negative. [If the text which is used] above [deals with the case of] using the negative to subtract [it] from the negative, [the text which will be used] below is using the negative to return to the positive or using the positive to return to the negative. [This] is like the meaning of “making up” in the previous¹²⁰ method.¹²¹

2-C: [If] the positive has no [number] to enter, [one makes it] to be negative; [if] the negative has no [number] to enter, [one makes it] to be positive.¹²²

Small-sized characters: [This method] is also “[the numbers with] different *ming* are added to each other,” after which the reason is “making up.” Originally, the neighboring position has no counting rods (*suan* 算) to enter; therefore [one] says this is opposite to the previous procedure.¹²³

2 Two kinds of “subtraction” in the text of the method

In the quotations above, pay attention to the fact the Chinese expression of “using (*yi* 以) *X* to subtract [it] from *Y*” does not correspond to the expression “*Y-X*” in modern mathematical symbols, in which *X* and *Y* could be a positive number, negative number or zero. There are two meanings of “subtract” (*jian* 減) in the quotations. The

115 “本是同名相加，因鄰位無算可入，故云‘正無入者仍爲正，負無入者仍爲負’。古本誤刻‘無人’者，非。”

116 The words “first method” and the “second method” are not contiguous in the text. Between them are some text and configurations of calculation. The two methods are put together and are translated in such a way for the convenience of comparing and interpreting them.

117 2-A: “二法：同名相減，”

118 “正見正，負見負，謂之同名，相減。”

119 2-B: “異名相加。”

120 The last character of 2-C is supposed to be “*fa* 法” (method), rather than “*qu* 去.” Concerning the interpretation of this sentence, see the text after the translation and the footnote about “make up” (補還) and “return” (還) there.

121 “上以正減正，下以負還正，或以正還負。上以負減負，下以負還正，或以正還負。猶前去相補之意。”

122 2-C: “正無入負之，負無入正之。”

123 “亦是異名相加，補還之理。原其鄰位無算可入，故云是反前術。”

commentary after the sentence 1-A, “[the case in which] using the positive to subtract [it] from the negative is not subtraction” (以正減負者非減也), reveals the different meanings of the word “*jian*” (lit. subtract). A key point for understanding the text lies in the fact that Yang Hui was to explain the methods (written in big-sized characters) to his thirteenth-century readers by means of these small-sized characters. It is still unclear how his readers understood and used “subtraction” before they learned from this book.¹²⁴ Since Yang was explaining “[the case in which] using the positive to ‘subtract’ [it] from the negative,” the meaning of the second “subtraction” should be a common one. “Common” here means that it was known and understandable by the *Mathematical Methods*’ readers of the thirteenth century, because Yang would not use a term that was not familiar to his readers to explain the method he was introducing. Let us name this “subtraction” as “subtraction-common,” and then try to inquire into what Yang and his audience could do and how they did it within framework of the concept of “subtraction-common.” In contrast, the first “subtraction” in this sentence refers to one of the two cases of the “subtraction” written in the big-sized characters. Let us name it “subtraction-explained.” It means that this “subtraction” was to be explained by Yang Hui.

In order to make the analysis below clear and precise, two operands of subtraction in *Mathematical Methods* need to be introduced. They are “subtrahend” (*jianshu* 減數) and “minuend” (*yuanshu* 原數, lit. original number), which appear in two configurations of this problem (Yang [1261] 1993, 967). Even though the two English words that are used to translate the two Chinese terms are also used in modern mathematics, and the meanings of the Chinese terms and those of the corresponding English words share some similarities, they are not identical. According to the configurations where these terms appear, the two words can be interpreted as follows. The “minuend” (*yuanshu*) refers to each number in the column from which the practitioner subtracted another one. The “subtrahend” (*jianshu*) refers to each number in the column which was to be subtracted. As introduced above, the columns borne on “simplifying position” (求位簡) are always subtracted from each other.¹²⁵ Thus, the numbers in the two columns are named as operands of subtraction. However, when practitioners were using “the method of the positive and the negative,” the way they dealt with the “subtrahend” and the “minuend” is the main point to be analyzed below.

124 One should not take for granted that the thirteenth-century readers had already mastered the mathematical knowledge and practices that had “progressed” up to Yang Hui’s time. Many clues in this work show that the target readers of *Mathematical Methods* were learners of mathematics. Another assumption is that this work might have been written for mathematics teachers to teach beginners, see Zhou (2018, 131–32).

125 See translations of the sentences G and H in section 2.1 of this paper.

Another sentence of small-sized characters in “the first method” shares a similar structure to “[the case in which] using the positive to subtract [it] from the negative is not subtraction.” The sentence is “[the cases in which] using the negative to subtract [it] from the positive is real subtraction” (以負減正者誠減也). The similarity of the structure of the two sentences shows the latter “subtraction” in this sentence is also “subtraction-common.” That is to say, in the case in which the subtrahend of “subtraction-explained” is “negative” and the minuend is “positive,” the “subtraction-explained” is precisely equivalent to the “subtraction-common.”

The small-sized characters following the sentence 2-A are “[cases in which] the positive meet the positive, or the negative meets the negative are considered as [the type having] the same *ming*. [One] subtracts [them] from each other” (正見正，負見負，謂之同名，相減). In this sentence, Yang Hui does not explain the sentence 2-A by differentiating two “subtractions” as above, but writes “subtract” directly. It seems to imply that, in the cases in which the subtrahend and the minuend are simultaneously positive or simultaneously negative, the “subtraction-explained” applied on the operands was also the same as the “subtraction-common” the thirteenth-century readers could understand.

Another issue that deserves great attention is that, in the text of *Mathematical Methods*, what “positive” and “negative” refer to are not equivalent to what we call “positive numbers” or “negative numbers” in modern mathematics.¹²⁶ This paper will demonstrate below that the text of the method, and the “procedure of calculation” placed after the method, especially the text in the configurations, all support the following view: in *Mathematical Methods*, “positive” refers to the number that was marked by a character written as “*zheng* 正” (positive), and “negative” refers to the number that was marked by a character written as “*fu* 負” (negative).¹²⁷ For the three cases in which “subtraction-explained” is the same as the “subtraction-common,” even though the operands could be marked by “positive” and “negative,” it was the numbers without any marked sign, rather than a “positive number” or a “negative

126 For research on the concepts of “positive number” and “negative number” in early Chinese history, see Zou (2010), where, on the basis of the earliest Chinese mathematical texts of the first millennium BCE written on bamboo or wooden slips, the subtle meaning of these concepts and the process of their emergence are analyzed.

127 This understanding of “positive number” and “negative number” could be supported by the use of counting rods of different color or of different shape as calculation tools. The opposite color or opposite shapes are marks of the nature of “positive” and “negative.” Only the number of counting rods was calculated. For a detailed discussion of this topic, see Zhu (2010, 92–98). In *Mathematical Methods*, the character “positive” or “negative” is written after the number. In cases in which a number is combined with its unit, the character “positive” or “negative” is written after the unit.

number” as treated in modern mathematics, that were operated and calculated.¹²⁸ Karine Chemla and Zhu Yiwen have pointed out that in *The Nine Chapters*, “positive” and “negative” should be understood in the same way as introduced above.¹²⁹ Yang Hui must have inherited the concept of “positive” and “negative” from *The Nine Chapters*.

Let us return to the first case in which “subtraction-explained” and “subtraction-common” are not the same. In the case in which the minuend is positive and the subtrahend is negative, readers of *Mathematical Methods* might have treated “subtraction-common” in the following way. Since the number marked by the character “negative” was understood as an “owed number” (欠數) stated in the small-sized character commentary (see section 3.1 of this paper), from a simple idea about things or money borrowed from other places, common readers at that time could have thought that when they used a number (positive) to subtract [it] from an “owed” number, the result will “owe” more. This thought would lead the readers to deal with the number without sign (*ming*) in the way of addition. However, for the “subtraction-explained,” supported by the analysis below, when the subtrahend of “subtraction-explained” is marked by “positive” and the minuend of “subtraction-

128 From this point of view, an issue concerning an emendation of the character “positive” in the sixth problem of *Mathematical Methods* can be clarified (Yang [1261] 1993, 966). Neither Yang Hui nor Jia Xian corrected this character in the text of the procedure of *The Nine Chapters*; however, modern critical editions of *The Nine Chapters* usually do. Karine Chemla was the first to point out this issue and explain the different conceptions about “positive” and “negative” in *The Nine Chapters* and *Mathematical Methods* from their modern counterparts (Chemla 2000, 213–14). Zhu Yiwen discusses this issue in great detail, and also suggests that the original text of *The Nine Chapter* is correct and that modern emendations are unnecessary (Zhu 2010, 96–98). In problem 6, *Mathematical Methods* finally gives a configuration in which the divisor is negative, and the dividend is positive. This case is not acceptable from the perspective of modern mathematics, because according to the context of the problem, the result of the division between the two operands should be positive. However, this case was perhaps acceptable for Yang Hui, Jian Xian, and their audiences, and it should not be arbitrarily regarded as a mistake that must be corrected. The reason why Yang accepted this case is that it was the numbers without any sign that were manipulated and calculated with. There are other pieces of evidence to support this view. After the first configuration of the sixth problem in *Mathematical Methods*, small-sized characters disclose the way of “cross-multiplication” between a “number of positive” and a “number of negative” (Yang [1261] 1993, 966). Even though the number in the left top position is “two of negative,” according to the small-sized characters, it was only “two” that was multiplied by the number in the right top of the configuration. Section 4.1 of this paper is also concerned with this evidence. There is also evidence from Wu Jing’s fifteenth-century book. In the “*Fangcheng*” chapter of *Great Compendium*, lots of cases show that when a “number of negative” was used to multiply another number, the result of which keeps the *ming* (i.e. sign) of the number being multiplied. This contradicts the rule of sign for multiplication in modern mathematics, but it proves that in Wu Jing’s mind, only the number without sign participates in the calculation. Wu Jing might have inherited the concepts of positive and negative from *Mathematical Methods*. Section 3.6.3 of Zhou Xiaohan’s dissertation gives an such example in *Great Compendium* (Zhou 2018).

129 See Chemla (2000, 205–8); Chemla and Guo (2004, 924 [“negative”], 964 [“name”], 1031 [“positive”]); Zhu (2010, 92–98).

explained" is marked by "negative," only the numbers without sign (*ming*) are subtracted.¹³⁰ This is different from the operation based on the audience's conception of "subtraction-common" in this case. This is the reason why Yang Hui commented "it is not 'subtraction.'"

3 Interpretation of the word "deduct" (*zhe* 折)

The sentences after the expressions "is not a 'subtraction'" and "is a real 'subtraction'" read: "This means that the positive is deducted from the negative" (是正折其[去]負矣) and "[In the case in which the number of] the positive is larger than [that of] the negative, one deducts [the negative from the positive]" (正多負而折去矣) respectively. These two sentences are crucial, because they are Yang Hui's explanations of how the "subtraction-explained" (in the case of "is not a 'subtraction'") and the "subtraction-explained" or "subtraction-common" (in the case of "is a real 'subtraction'") were carried out. Moreover, since in the commentary on sentence 2-A there is only "subtraction-common," and Yang does not explain how to carry out this "subtraction-common," he may have thought that what is stated about the "subtraction" in the commentary on sentence 1-A is also suitable for the commentary on sentence 2-A. That is to say, if we can understand the sentences after the expressions "is not a subtraction" and "is a real subtraction," we can also get to know how the operation instructed by sentence 2-A was carried out.

Unfortunately, it is difficult to give an interpretation that takes into account every character in the two sentences above without emendation. Whatever the case, a reliable interpretation can be given for the word "deduct" (*zhe* 折) appearing in the two sentences. This word also relates to the word "deduction" (*zhechu* 折除) mentioned in section 3.1 of this paper. Yang's commentary on the two characters "*zheng* 正" (positive) and "*fu* 負" (negative) mentioned that, before the "method of the positive and the negative," there was a method of "deduction of the positive and the negative" (正負折除) (Yang [1261] 1993, 967). This word is very crucial to understanding the operation of "subtraction" in the two sentences above. There is an interpretation of this word that fits the analysis of the whole method and the "procedure of calculation" placed below it. The "deduction" means deducting the smaller number from the larger number without involving the mark of *ming* (i.e. sign) (positive or negative) placed after the number. In one of the configurations quoted below, there is an example of using the "first method" in which the subtrahend is "two-positive" (i.e. +2) and

130 The following sections of this appendix will address how the practitioner did the subtraction and determine the *ming* (i.e. sign, positive or negative) placed after the result of the subtraction.

the minuend is “five-negative” (i.e. -5). The result is “three-negative” (i.e. -3) and more importantly, above the result, the text reads “positive and negative are deducted [to acquire] this number” (正負折除此數). However, having acquired the number after “deduction,” the *ming* (i.e. sign) of the result is not indicated directly by these two sentences.

4 Which sentences determine the *ming* (i.e. sign) of the result and how did Yang explain the sentences through “addition”?

Which parts of the text deals with the *ming* (i.e. sign) of the result? In the big-sized characters, it is sentences 1-C and 2-C that play this role. The configurations in section 3.3 of this paper will demonstrate how to use sentences 1-C and 2-C to determine the *ming* (i.e. sign) of the result in the cases in which a certain position of the configuration is empty. The section here reveals the function of sentences 1-C and 2-C in the process of determining the *ming* (i.e. sign) of the result after the application of sentences 1-A and 2-A. The method of “deduction” only deals with numbers without a sign. The remaining number from the larger number after the deduction still keeps its *ming*, and the position that the smaller number occupied previously becomes empty. This forms the case in which one number with *ming* “has no [number] to enter” (無入), then the *ming* of the result will be determined according to sentence 1-C or 2-C.¹³¹

What is the reasoning behind sentences 1-C and 2-C? The small-sized characters following them give the explanations. Let us examine how Yang Hui explains the reason and why he explains it in that way. In each case described by each of the four clauses in sentence 1-C and in sentence 2-C there is always “subtraction” between two numbers. This “subtraction” is determined by the subtraction between two columns.¹³² Moreover, the first character in each clause refers to the subtrahend (*jianshu* 減數). The number in the place where it will be “entered” (入) by the

131 A piece of evidence supports this view on determining the signs after “subtraction-explained.” In the last configuration of problem 8 in *Mathematical Methods*, after the sentence “[the numbers with] the same *ming* are subtracted from each other” (同名相減), it reads “no number to enter, one makes it to be positive” (無入, 正之) (Yang [1261] 1993, 968). According to the context and the interpretations adopted in this paper, the “same *ming*” in the first sentence should be replaced by “different *ming*.” The emergence of sentences 1-A and 1-C simultaneously conforms to the analysis above on how to determine the sign of the result. In this case, for “19800 of positive” (subtrahend) and “15000 of negative” (minuend), the practitioner uses sentence 1-A and deducts 15000 from 19800. The number 4800 is left in the position of the subtrahend and its sign is still positive. Then the first part of sentence 1-C is used, “the number of the positive has no number to enter, one makes it to be positive.” Thus “4800 of positive” is the final result written in the configuration.

132 See section 3.2 of this paper.

subtrahend is the minuend (*yuanshu* 原數). In each case described by each of the four clauses, there is no minuend. In the configuration, the position of the minuend is marked by the character “kong” 空 (empty) or characters “wuru” 無入 (cannot enter). The interpretations above about the subtrahend and minuend in the four clauses can be supported by the procedure of calculation in section 3.3 of this paper, and so are not elaborated on here. The following fact is emphasized here: The small-sized characters placed after sentence 1-C reflect that Yang Hui regarded the case in which the subtrahend is positive (or negative) and the minuend is empty as the case of “addition for the numbers with the same *ming*.”

Originally, [this is the case of] “[the numbers with] the same *ming* are added to each other. (Yang [1261] 1993, 967)¹³³

The empty minuend is regarded as having the same *ming* as that of the subtrahend. The word “originally is” (本是) reflects that Yang Hui regarded this way as natural and certain. However, was “regarding the subtrahend and minuend as having the same *ming*” the real perspective from which Yang looked at sentence 1-C, and after that he could apply the rule of “[the numbers with] the same *ming* are added to each other”?

The answer is no. Let us examine the small-sized characters placed after sentence 2-C.

[This is] also the [case of] “[the numbers with] different *ming* are added to each other.” (Yang [1261] 1993, 967)¹³⁴

Why did Yang Hui use “also is” (亦是) in this sentence? The only thing in common between this sentence and the sentence above is that they both contain the word “addition.” This sentence shows that he regarded the empty position as one with a different *ming* from that of the subtrahend. The “different *ming*” is exactly opposite to the “same *ming*” in the small-sized characters following sentence 1-C. The two sentences quoted above show that the real perspective from which Yang looked at sentences 1-C and 2-C was that of looking at the “subtraction between a subtrahend and an ‘empty’ minuend” as an “addition.” Namely, according to the conditions that allowed practitioners to apply an “addition” of the methods, Yang Hui regarded the empty operand in sentences 1-C or 2-C as having the same *ming* as the subtrahend or as having a different *ming*

133 “本是同名相加。”

134 “亦是異名相加。”

from the subtrahend.¹³⁵

The next question is why Yang wanted to use the result of “addition” to explain the reasoning of sentences 1-C and 2-C. An explanation is that in the cases in which the method of “addition” is applied, he could know the *ming* of the result, which is always the same as the *ming* of the minuend.¹³⁶ The phenomenon that the *ming* of the result is the same as the *ming* of minuend is indicated by him by using the following words: “make up” (*buhuan* 補還) or “return” (*huan* 還).¹³⁷ For example, when he writes “[the type of having] the same *ming* ‘makes up’ [with each other]” (同名補還) subsequent to sentence 1-B, he means that the numbers are added and the *ming* is kept. When he says “using negative to return to the positive” (以負還正) following sentence 2-B, he means that the number (without the *ming* of negative) was added to the number which was marked as positive, and the *ming* remained the same as the positive.

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135 There is further evidence supporting the view that it was through looking at “subtraction” as “addition” that Yang determined the *ming* (i.e. sign) of the result. In the second configuration of this problem, the minuend is “six hundred, negative” and the subtrahend is “empty,” thus one cannot use 1-C or 2-C in this case. The text below the result is “there is no [number] to be added; [thus the original number] did not move” (無加不動), rather than “there is no [number] to be subtracted; [thus the original number] did not move.” The treatment of this case as an “addition” conforms to the analysis on the perspective from which Yang determined the *ming* of the result.

136 As has been discussed in section 3.2 of this paper, columns are always subtracted from one another. Even though when the practitioner is using sentences 1-B and 2-B to carry out addition of the numbers (without signs), the two operands are still the subtrahend and the minuend.

137 The last sentence written in small-sized characters placed after sentence 2-B is “[This] is like the meaning of ‘making up’ in the previous method” (猶前去[法]相補之意). The “meaning of ‘making up’ in the previous method” in this sentence refers precisely to the meaning conveyed by the small-sized characters placed after sentence 1-B. In both places, the word “*bu* 補” (make up) appears.

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