MATHEMATICS

Differentiable structure of ω-subsets of symplectic groups

LONG Yiming (龙以明)[1,2] & ZHU Chaofeng (朱朝锋)[1]

1. Nankai Institute of Mathematics, Nankai University, Tianjin 300071, China;
2. Key Laboratory of Pure Mathematics and Combinatorics, Ministry of Education, Tianjin 300071, China
Correspondence should be addressed to Long Yiming

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Abstract In this paper, we study the differentiable structure of the ω-subset of $\mathrm{Sp}(2n)$, which is formed by all matrices in $\mathrm{Sp}(2n)$ possessing ω as an eigenvalue, for ω on the unit circle in the complex plane. Based on this result the ω-index theory parametrized by all ω on the unit circle for arbitrary symplectic paths is defined.

Keywords: symplectic group, ω-subset, stratification, differentiable structure, transversal structure.

1 Main results

Denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of natural numbers, real numbers and complex numbers, respectively. For $n \in \mathbb{N}$, as usual we define

$$\mathrm{Sp}(2n,\mathbb{C}) = \{M \in \mathrm{GL}(2n,\mathbb{C}) \mid M^* JM = J\},$$
$$\mathrm{Sp}(2n) = \{M \in \mathrm{GL}(2n,\mathbb{R}) \mid M^{\mathrm{T}} JM = J\},$$

where $J = J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix on $\mathbb{R}^n$. When there is no confusion we shall omit the subindex n. In ref. [1], Long established the ω-index theory based on his study of the topological structure of ω-subsets of $\mathrm{Sp}(2n)$. The ω-index theory plays an important role in the iteration theory of the Maslov-type index (cf. refs. [1—3]). Let $\mathbb{U}$ be the unit circle of $\mathbb{C}$ and $\omega \in \mathbb{U}$. Following Long's work[4], we define the ω-subsets of $\mathrm{Sp}(2n)$ as follows.

For $0 \leqslant k \leqslant 2n$ and $\omega \in \mathbb{U}$, define

$$\mathrm{Sp}^k(2n,\mathbb{C}) = \{M \in \mathrm{Sp}(2n,\mathbb{C}) \mid \dim_{\mathbb{C}}\ker_{\mathbb{C}}(M - I) = k\}, \tag{1}$$
$$\mathrm{Sp}_\omega^k(2n) = \{M \in \mathrm{Sp}(2n) \mid \dim_{\mathbb{C}}\ker_{\mathbb{C}}(M - \omega I) = k\}. \tag{2}$$

It is clear that there are stratifications for every $\omega \in \mathbb{U}$:

$$\mathrm{Sp}(2n,\mathbb{C}) = \bigcup_{0 \leqslant k \leqslant 2n} \mathrm{Sp}^k(2n,\mathbb{C}), \tag{3}$$
$$\mathrm{Sp}(2n) = \bigcup_{0 \leqslant k \leqslant 2n} \mathrm{Sp}_\omega^k(2n). \tag{4}$$

Our main results in this paper are the following

Theorem 1.1. (i) For every $0 \leqslant k \leqslant 2n$, $\mathrm{Sp}^k(2n, \mathbb{C})$ is a codimension k^2 smooth submanifold of $\mathrm{Sp}(2n,\mathbb{C})$, and $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(M\mathrm{e}^{Jt})$, with $M \in \mathrm{Sp}^1(2n, \mathbb{C})$, forms a transverse structure of $\mathrm{Sp}^1(2n, \mathbb{C})$ in $\mathrm{Sp}(2n, \mathbb{C})$. Moreover, we have

$$\overline{\mathrm{Sp}^k(2n,\mathbb{C})} = \bigcup_{l\geqslant k}\mathrm{Sp}^l(2n,\mathbb{C}).$$

(ii) For every $0\leqslant k\leqslant 2n$, $\mathrm{Sp}^k_{\pm 1}(2n)$ is a codimension $\frac{1}{2}k(k+1)$ smooth submanifold of $\mathrm{Sp}(2n)$, and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(M\mathrm{e}^{Jt})$, with $M\in\mathrm{Sp}^1_{\pm 1}(2n)$, forms a transverse structure of $\mathrm{Sp}^1_{\pm 1}(2n,\mathbb{C})$ in $\mathrm{Sp}(2n,\mathbb{C})$. Moreover, we have

$$\overline{\mathrm{Sp}^k_{\pm 1}(2n)} = \bigcup_{l\geqslant k}\mathrm{Sp}^l_{\pm 1}(2n).$$

Theorem 1.2. Let $\omega\in\mathbb{U}\setminus\mathbb{R}$. Then $\mathrm{Sp}^1_\omega(2n)$ is a codimension 1 smooth submanifold of $\mathrm{Sp}(2n)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(M\mathrm{e}^{Jt})$, with $M\in\mathrm{Sp}^1_\omega(2n)$, forms a transverse structure of $\mathrm{Sp}^1_\omega(2n)$ in $\mathrm{Sp}(2n)$. Moreover, $\bigcup_{l\geqslant k}\mathrm{Sp}^l_\omega(2n)$ is a codimention k^2 real algebraic subvariety of $\mathrm{Sp}(2n)$ for every $0\leqslant k\leqslant n$, and is an empty set for $n<k\leqslant 2n$.

Based on these theorems, for every path $\gamma:[0,1]\to\mathrm{Sp}(2n)$ and $\omega\in\mathbb{U}$, one can define the ω-index $i_{1,\omega}(\gamma)$ to be the intersection number of γ and $\bigcup_{k\geqslant 1}\mathrm{Sp}^k_\omega(2n)$. This ω-index theory has been used in the study of the iteration theory of the Maslov-type index in refs. [1,5].

2 Some topological facts

In this section we list some well-known facts which will be used later in our study of fiber maps.

Proposition 2.1. Let X, Y be topological spaces and $p: Y\to X$ be a continuous map. If $\mathscr{G}$ is a topological group and $\mathscr{G}$ acts on X, Y continuously so that

(a) $\mathscr{G}$ acts on X transitively.

(b) $p(gy)=gp(y)$ for all $g\in\mathscr{G}$ and $y\in Y$.

(c) There is an $x_0\in X$, a neighborhood U of x_0 in X, and a continuous map $s: U\to\mathscr{G}$ such that $s(x)x_0 = x$ for $x\in U$.

Then p is a fiber map.

Proof. The proof is similar to that of Theorem 4.13 in ref. [6] and therefore is omitted.

Proposition 2.2. Let $\mathscr{G}$ be a Lie group and $\mathscr{H}$ be a closed subgroup of $\mathscr{G}$. Then $\mathscr{H}$ is a Lie subgroup of $\mathscr{G}$ and the natural projection $\pi:\mathscr{G}\to\mathscr{G}/\mathscr{H}$ is a fiber map.

Proof. Cf. Theorem 3.58 in ref. [7].

Lemma 2.1. Let $\mathscr{G}$ be a Lie group, $\mathscr{H}$ be its closed subgroup, and $\mathscr{K}$ be a Lie subgroup of $\mathscr{G}$. Then the natural map

$$\varphi:\mathscr{K}/\mathscr{H}\cap\mathscr{K}\to\mathscr{G}/\mathscr{H}$$

is an embedding, provided that im φ is locally compact.

Proof. The proof is similar to that of Theorem 3.62 in ref. [7] and is thus omitted. We only point out that since $\mathscr{K}$ satisfies the second countable axiom and imφ is a locally compact Hausdorff space, we can apply the theorem in § 2.13 of ref. [8] to prove that φ is a homeomorphism.

Remark 2.1. Note that an open or closed subspace of a locally compact Hausdorff space is also locally compact Hausdorff.

3 Proof of Theorem 1.1

Let $V=\mathbb{C}^{2n}\oplus\mathbb{C}^{2n}$, and $(\cdot,\cdot)$ be the standard Hermitian inner product of V. We define

$$\{v,w\} = (\mathscr{J}v,w), \ \forall\, v,w \in V, \text{ where } \mathscr{J} = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

A complex subspace X of V is called Lagrangian if and only if

(a) X is isotropic, i.e. $\{v,w\}=0$, $\forall\, v,w\in X$, and

(b) $\dim_{\mathbb{C}} X = 2n$.

We denote by Lag(V) the set of Lagrangian subspaces of V and topologize it as a subspace of $G_{2n}(V)$, where $G_k(V)$ is the Grassmannian of all k-dimensional complex subspaces of V.

Let g: $\mathrm{Sp}(2n,\mathbb{C})\to \mathrm{Lag}(V)$ be the embedding

$$g(M) = \mathrm{Gr}(M) \equiv \left\{\begin{pmatrix} x \\ Mx \end{pmatrix} \middle|\, x \in \mathbb{C}^{2n}, \ \forall\, M \in \mathrm{Sp}(2n,\mathbb{C}).\right. \tag{5}$$

Proof of Theorem 1.1. (i) Note that the argument in ref. [9] can be generalized into the complex case, the stratification structure follows from the stratification structure of Lag(V) and the fact that g is an open embedded submanifold.

By Proposition 3.2 of ref. [10], the transversal structure follows.

(ii) The case $\omega=1$ follows from the same method of the complex case, and the case $\omega=-1$ follows from the case $\omega=1$ by considering the diffeomorphism $\mathrm{Sp}(2n)\to\mathrm{Sp}(2n): M\mapsto -M$.

4 Proof of Theorem 1.2

Our aim in this section is to prove Theorem 1.2. We carry out the proof in three steps.

4.1 The structure of $\mathrm{im}\,\varphi_{k,\omega}$ for $\omega\in\mathbb{U}\setminus\mathbb{R}$

For $k\in\{0,1,\cdots,2n\}$ and $\omega\in\mathbb{U}$ we define $\varphi_{k,\omega}:\mathrm{Sp}_\omega^k(2n)\to G_k(\mathbb{C}^{2n})$ by

$$\varphi_{k,\omega}(M) = \ker_{\mathbb{C}}(M-\omega I), \qquad \forall\, M\in \mathrm{Sp}_\omega^k(2n). \tag{6}$$

It is clear that locally $\varphi_{k,\omega}$ is rational hence of class C^ω. We will use it to study the structure of the ω-subsets of $\mathrm{Sp}(2n)$. The first step is to study im $\varphi_{k,\omega}$. Suggested by ref. [11], we have

Definition 4.1. Let $V\in G_k(\mathbb{C}^{2n})$. We call V admissible if V satisfies

(a) For any $\xi_1+\mathrm{i}\eta_1\in V$, $\xi_2+\mathrm{i}\eta_2\in V$, with $\xi_1,\xi_2,\eta_1,\eta_2\in\mathbb{R}^{2n}$, we have

$$\xi_1^{\mathrm T}J\xi_2 = \eta_1^{\mathrm T}J\eta_2, \qquad \xi_1^{\mathrm T}J\eta_2 = -\,\eta_1^{\mathrm T}J\xi_2. \tag{7}$$

(b) For any complex linearly independent elements $\xi_1+\mathrm{i}\eta_1,\cdots,\xi_l+\mathrm{i}\eta_l$ of V where $\xi_j,\eta_j\in\mathbb{R}^{2n}, j=1,\cdots,l, 1\leqslant l\leqslant k$, $\xi_1,\eta_1,\cdots,\xi_l,\eta_l$ are real linearly independent.

Lemma 4.1. Let $k\geqslant 1$, $\omega\in\mathbb{U}\setminus\mathbb{R}$ and $V\in\mathrm{im}\,\varphi_{k,\omega}$. Then V is admissible and $k\leqslant n$.

Proof. Property (a) of Definition 4.1. Let v, $w\in V$ and $M\in\mathrm{Sp}_\omega^k(2n)$ such that $\varphi_{k,\omega}(M)=V$. Then we have $Mv=\omega v$ and $Mw=\omega w$, so $v^{\mathrm T}M^{\mathrm T}JMw=(\omega v)^{\mathrm T}J(\omega w)=\omega^2 v^{\mathrm T}Jw$. Since $\omega^2\neq 1$, we have $v^{\mathrm T}Jw=0$. Therefore (a) is verified.

Property (b) of Definition 4.1. Let $M\in\mathrm{Sp}_\omega^k(2n)$ such that $\varphi_{k,\omega}(M)=V$. Let $a_j,b_j\in\mathbb{R}$, $j=1,\cdots,l$ such that $\sum_{j=1}^l(a_j\xi_j+b_j\eta_j)=0$. Let $v=\sum_{j=1}^l(b_j+\mathrm{i}a_j)(\xi_j+\mathrm{i}\eta_j)$. Then $v=\sum_{j=1}^l(b_j\xi_j-a_j\eta_j)\in\mathbb{R}^{2n}\cap V=\{0\}$. So $v=0$ and therefore $a_j=b_j=0$, $j=1,\cdots,l$. Hence (b) is verified.

That $k\leqslant n$ follows from property (b). Our lemma is proved.

Recall that $G=\mathrm{i}J$ is the Krein form on $\mathbb{C}^{2n}$.

Lemma 4.2. Let V be admissible. Consider the Hermitian form $h_V=\frac{1}{2}G\,|_V$, and we have

(a) $h_V(\xi_1+\mathrm{i}\eta_1,\xi_2+\mathrm{i}\eta_2) = -\,\xi_1^{\mathrm T}J\eta_2-\mathrm{i}\xi_1^{\mathrm T}J\xi_2=\eta_1^{\mathrm T}J\xi_2-\mathrm{i}\eta_1^{\mathrm T}J\eta_2$, where $\xi_1+\mathrm{i}\eta_1$, ξ_2+

$i\eta_2 \in V$ and $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^{2n}$.

(b) V has a unique orthogonal decomposition with respect to h_V

$$V = V_+ \oplus V_- \oplus V_0, \tag{8}$$

such that h_V is positive definite, negative definite, and null on V_+, V_-, and V_0 respectively.

(c) There is a standard basis of V

$$\begin{aligned} &v_1, \cdots, v_s \in V_+, \\ &v_{s+1}, \cdots, v_{s+t} \in V_-, \quad s, t \geqslant 0, s + t \leqslant k, \\ &v_{s+t+1}, \cdots, v_k \in V_0, \end{aligned} \tag{9}$$

such that $h_V(v_p, v_q) = \lambda_p \delta_{pq}, 1 \leqslant p, q \leqslant k$, where

$$\lambda_p = \begin{cases} 1, & \text{if } 1 \leqslant p \leqslant s, \\ -1, & \text{if } s+1 \leqslant p \leqslant s+t, \\ 0, & \text{if } s+t+1 \leqslant p \leqslant k. \end{cases}$$

Proof. By direct computation.

By virtue of Lemma 4.2 we can introduce

Notation 4.1. For $0 \leqslant k \leqslant 2n$, $0 \leqslant s, t \leqslant s+t \leqslant k$, we denote by

$A^k = \{V \in G_k(\mathbb{C}^{2n}) \mid V \text{ is admissible}\}$,

$h_V = \frac{1}{2} G|_V$, $\forall\, V \in A^k$, where G is the Krein form,

$A^{k,l} = \{V \in A^k \mid \dim_{\mathbb{C}} V_0 = l$ where V_0 is the null space of $h_V\}$,

$A^{k,s,t} = \{V \in A^k \mid \dim_{\mathbb{C}} V_+ = s, \dim_{\mathbb{C}} V_- = t\}$, where V_+, V_- are defined by (8) according to V.

Remark 4.1. We have

$$A^k = \bigcup_{0 \leqslant l \leqslant k} A^{k,l}, \tag{10}$$

$$A^{k,l} = \bigcup_{s+t=k-l} A^{k,s,t}, \tag{11}$$

$$A^k = \emptyset \text{ for } k > n, \text{ and} \tag{12}$$

$$A^{k,l} = \emptyset \text{ for } k + l > n. \tag{13}$$

Moreover, $A^{k,s,t}$ is closed in $A^{k,l}$, $\bigcup_{p \geqslant l} A^{k,p}$ is closed in A^k, and A^k is an open subset of a closed subset of $G_k(\mathbb{C}^{2n})$. Therefore by Remark 2.1 all the subsets of A^k that is defined by Notation 4.1 are locally compact Hausdorff spaces.

Proof. (10) and (11) are clear, (12) follows from Lemma 4.1, and we will prove (13) in Lemma 4.3 below. The second part of the remark follows from Lemma 4.2 and standard arguments. QED

For any topological transformation group $(\mathscr{G}, X)$ and $x \in X$, we denote

$$\mathrm{Stab}\,x = \{g \in \mathscr{G} \mid gx = x\}.$$

Then $\mathrm{Stab}\,x$ is a closed subgroup of G. Denote by e_k the k-th unit vector of $\mathbb{C}^n$. It is convenient to give the following notations (cf. refs. [4,11]).

Notation 4.2. We define $\diamond$-product $\diamond: \mathrm{Sp}(2k, \mathbb{C}) \times \mathrm{Sp}(2l, \mathbb{C}) \to \mathrm{Sp}(2k+2l, \mathbb{C})$ by

$$M_1 \diamond M_2 = \begin{pmatrix} A_{11} & 0 & A_{12} & 0 \\ 0 & B_{11} & 0 & B_{12} \\ A_{21} & 0 & A_{22} & 0 \\ 0 & B_{21} & 0 & B_{22} \end{pmatrix},$$

where $M_1 \in \mathrm{Sp}(2k, \mathbb{C})$ and $M_2 \in \mathrm{Sp}(2l, \mathbb{C})$ are in square block forms

$$M_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad M_2 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Moreover, we define k-fold $\diamond$-product of $M \in \mathrm{Sp}(2n, \mathbb{C})$ by $M^{\diamond k} = M \diamond \cdots \diamond M$.

Lemma 4.3. Let $k, s, t \geqslant 0$ and $l = k - s - t$ such that $k \leqslant n$.

(a) $A^{k,l} = \emptyset$ if and only if $k + l > n$.

(b) Assume that $k + l \leqslant n$. Let

$$X_{k,s,t} = \mathrm{span}_{\mathbb{C}}\{x_p, y_q \mid 1 \leqslant p \leqslant s + t, 1 \leqslant q \leqslant l\},$$

where $x_p = e_p + \mathrm{i}\lambda_p J e_p$, $y_q = e_{s+t+q} + \mathrm{i} e_{k+q}$, $\lambda_p = \mathrm{sign}\left(s - p + \frac{1}{2}\right)$, $1 \leqslant p \leqslant s + t$, $1 \leqslant q \leqslant l$.
Then we have $X_{k,s,t} \in A^{k,s,t}$ and

$$A^{k,s,t} = \mathrm{Sp}(2n)/\mathrm{Stab}X_{k,s,t}, \tag{14}$$

where $\mathrm{Stab}X_{k,s,t}$ is defined by the group action (15).

Proof. Since $\mathrm{Sp}(2n)$ preserves symplectic structure, there is a group action $f: \mathrm{Sp}(2n) \times A^k \to A^k$ defined by

$$f(M, X) = MX, \quad \forall X \in A^k. \tag{15}$$

Moreover, $A^{k,s,t}$ is invariant under this action. We shall prove that $A^{k,s,t} \neq \emptyset \Leftrightarrow k + l \leqslant n$, and if $k + l \leqslant n$, $A^{k,s,t}$ is an orbit of the action. Therefore our lemma will follow from Proposition 2.2 and Lemma 2.1.

In fact, it is clear that $X_{k,s,t} \in A^{k,s,t}$ if $k + l \leqslant n$. Let $V \in A^{k,s,t}$ and $v_p (1 \leqslant p \leqslant k)$ be a standard basis of V satisfying the conditions in (c) of Lemma 4.2. Let $\xi_p, \eta_p \in \mathbb{R}^{2n}$, $1 \leqslant p \leqslant k$, such that $v_p = \xi_p + \mathrm{i}\eta_p$. Let $W_1 = \mathrm{span}\{\xi_1, \cdots, \xi_{s+t}, \eta_1, \cdots, \eta_{s+t}\}$, W_2 be the symplectic complement of W_1, and $W_3 = \mathrm{span}\{\xi_{s+t+1}, \cdots, \xi_k, \eta_{s+t+1}, \cdots, \eta_k\}$. Then W_1, W_2 are symplectic subspace of $\mathbb{R}^{2n}$, and W_3 is an isotropic subspace of W_2. So we have

$$\dim W_3 = 2l \leqslant \frac{1}{2}\dim W_2 = \frac{1}{2}(2n - 2(s + t)) = n - s - t.$$

That is, $s + t + 2l \leqslant n$. Since $s + t + l = k$, $k + l \leqslant n$ and (a) is proved.

Now we prove that $A^{k,s,t}$ is an orbit of the action (15) by constructing an $M \in \mathrm{Sp}(2n)$ such that $MV = X_{k,s,t}$. By Lemma 4.2, it is not difficult to show that there is an $M_1 \in \mathrm{Sp}(2n)$ such that $M_1\xi_p = e_p$, $M_1\eta_p = \lambda_p J e_p$ for $p = 1, \cdots, s + t$. Since $\mathrm{Sp}(2n)$ acts transitively on the Lagrangian Grassmannian $\sum(n)$ and $M_1 W_2 = \mathrm{span}\{e_{s+t+1}, \cdots, e_n, Je_{s+t+1}, \cdots, Je_n\}$, there is an $M_2 \in \mathrm{Sp}(2n - 2s - 2t)$ such that $N_2 M_1 W_3 \subset \mathrm{span}\{e_{s+t+1}, \cdots, e_n\}$, where $N_2 = I_{2s+2t} \diamond M_2$. Let $M_3 \in \mathrm{GL}(n - s - t, \mathbb{R})$ and $N_3 = \mathrm{diag}(I_{s+t}, M_3)$ such that

$$\mathrm{diag}(N_3, I_n) N_2 M_1 \xi_p = e_p, \qquad \mathrm{diag}(N_3, I_n) N_2 M_1 \eta_p = e_{p+l}$$

for $p = s + t + 1, \cdots, k$. Let $M = \mathrm{diag}(N_3, (N_3^T)^{-1}) N_2 M_1$. Then $M \in \mathrm{Sp}(2n)$ and $MV = X_k, s, t$. Therefore $A^{k,s,t}$ is an orbit of the group action.

Lemma 4.4. Let n, k, s, t be non-negative integers and $l = k - s - t$ such that $0 \leqslant l \leqslant k \leqslant k + l \leqslant n$. Then we have

$$\dim\mathrm{Stab}X_{k,s,t} = 2n^2 + n - 4nk + 3k^2 - k + l^2.$$

Proof. As in linear algebra, a matrix $M \in \mathrm{gl}(n, \mathbb{C})$ is viewed as a representation of a linear transformation A on $\mathbb{C}^n$ by choosing the basis $e_1, \cdots, e_n$. Let N be a representation of A by choosing a permutation g on $e_1, \cdots, e_n$. We call g a block decomposition, and under g, M is in block form N. With this convention, we give our proof as follows.

By a suitable block decomposition, $J \equiv J_n$ is in block form

$$\operatorname{diag}(J_{s+t}, J_{2l}, J_{n-k-l}).$$

Consider the Lie algebra of $\operatorname{Stab} X_{k,s,t}$, $g = \{M \in \operatorname{sp}(2n) \mid MX_{k,s,t} \subset X_{k,s,t}\}$. Let $\lambda_p = \operatorname{sign}\left(s - p + \frac{1}{2}\right)$, $1 \leqslant p \leqslant s+t$ and $M \in \operatorname{gl}(2n, \mathbb{R})$. By the definition of $X_{k,s,t}$, $M \in g$ if and only if $M \in \operatorname{sp}(2n)$ and there are $a_{pq}, b_{pq} \in \mathbb{R}$ $(1 \leqslant p, q \leqslant k)$ such that

$$M(e_q + \mathrm{i}\lambda_q J e_q) = \sum_{1 \leqslant p \leqslant s+t} (a_{pq} + \mathrm{i} b_{pq})(e_p + \mathrm{i}\lambda_p J e_p) + \sum_{s+t+1 \leqslant p \leqslant k} (a_{pq} + \mathrm{i} b_{pq})(e_p + \mathrm{i} e_{p+l}), \tag{16}$$

for $1 \leqslant q \leqslant s+t$, and

$$M(e_q + \mathrm{i} e_{q+l}) = \sum_{1 \leqslant p \leqslant s+t} (a_{pq} + \mathrm{i} b_{pq})(e_p + \mathrm{i}\lambda_p J e_p) + \sum_{s+t+1 \leqslant p \leqslant k} (a_{pq} + \mathrm{i} b_{pq})(e_p + \mathrm{i} e_{p+l}), \tag{17}$$

for $s+t+1 \leqslant q \leqslant k$. Let $A = (a_{pq})_{1 \leqslant p,q \leqslant k} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, $B = (b_{pq})_{1 \leqslant p,q \leqslant k} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_{s+t})$, then (16) and (17) are equivalent to that M is in block form

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \tag{18}$$

where M_1 is in block form

$$\begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ 0 & 0 & N_{33} \end{pmatrix} \tag{19}$$

and

$$\begin{aligned} N_{11} &= \begin{pmatrix} A_1 & B_1 \Lambda \\ -\Lambda B_1 & \Lambda A_1 \Lambda \end{pmatrix}, \quad N_{12} = \begin{pmatrix} A_2 & B_2 \\ -\Lambda B_2 & \Lambda A_2 \end{pmatrix}, \\ N_{21} &= \begin{pmatrix} A_3 & B_3 \\ -B_3 & A_3 \end{pmatrix}, \quad N_{22} = \begin{pmatrix} A_4 & B_4 \\ -B_4 & A_4 \end{pmatrix}, \end{aligned} \tag{20}$$

M_3 is in block form

$$(0 \quad N_3). \tag{21}$$

By (18) we have $M \in \operatorname{sp}(2n)$ if and only if

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} J_{k+l} & 0 \\ 0 & J_{n-k-l} \end{pmatrix} + \begin{pmatrix} J_{k+l} & 0 \\ 0 & J_{n-k-l} \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = 0,$$

if and only if

$$M_1 \in \operatorname{sp}(2k + 2l), \tag{22}$$

$$M_4 \in \operatorname{sp}(2n - 2k - 2l), \tag{23}$$

$$M_3^{\mathrm{T}} J_{n-k-l} + J_{k+l} M_2 = 0. \tag{24}$$

By (19) we have $M_1 \in \operatorname{sp}(2k + 2l)$ if and only if

$$\begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ 0 & 0 & N_{33} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} J_{s+t} & 0 & 0 \\ 0 & 0 & -I_{2l} \\ 0 & I_{2l} & 0 \end{pmatrix} + \begin{pmatrix} J_{s+t} & 0 & 0 \\ 0 & 0 & -I_{2l} \\ 0 & I_{2l} & 0 \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ 0 & 0 & N_{33} \end{pmatrix} = 0,$$

if and only if

$$N_{11} \in \mathrm{sp}(2s + 2t), N_{12} = 0, \; N_{13} = - JN_{21}^{\mathrm{T}}, \; N_{33} = - N_{22}^{\mathrm{T}}, \; N_{23} = N_{23}^{\mathrm{T}}. \tag{25}$$

By (20) we have $N_{11} \in \mathrm{sp}(2s + 2t)$ if and only if

$$\Lambda A_1 = -(\Lambda A_1)^{\mathrm{T}}, \qquad \Lambda B_1 = (\Lambda B_1)^{\mathrm{T}}. \tag{26}$$

From (18) to (26) we have

$$\begin{aligned} \dim \mathrm{Stab} X_{k,s,t} &= \dim g \\ &= \frac{1}{2}(k - l)(k - l - 1) + \frac{1}{2}(k - l)(k - l + 1) \\ &\quad + 2l(k - l) + 2l^2 + l(2l + 1) + 2l(2n - 2k - 2l) \\ &\quad + 2(n - k - l)^2 + n - k - l \\ &= 2n^2 + n - 4nk + 3k^2 - k + l^2. \end{aligned}$$

GED

Corollary 4.1. If $1 \leqslant k \leqslant k + l \leqslant n$ and $s + t = k - l$, we have $\dim A^{k,s,t} = 4nk - 3k^2 + k - l^2$.

Lemma 4.5. If $\omega \in \mathbb{U} \setminus \mathbb{R}$ and $1 \leqslant k \leqslant 2n$, there holds $\mathrm{im}\varphi_{k,\omega} = A^k$.

Proof. Let $V \in A^k$ and

$$Y_{k,s,t} = \mathrm{span}_{\mathbb{C}}\{x_p, z_q \mid 1 \leqslant p \leqslant s + t, 1 \leqslant q \leqslant l\},$$

where $x_p = e_p + \mathrm{i}\lambda_p J e_p$, $z_q = e_{s+t+2q-1} + \mathrm{i}e_{s+t+2q}$, $\lambda_p = \mathrm{sign}\left(s - p + \frac{1}{2}\right)$, $1 \leqslant p \leqslant s + t$, $1 \leqslant q \leqslant l$. By Lemmas 4.2 and 4.3, there is a $P \in \mathrm{Sp}(2n)$ such that $PY_{k,s,t} = V$. Let $\omega = \mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in (-\pi, 0) \cup (0, \pi)$. Let $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathrm{gl}(2, \mathbb{R})$ such that $(b_2 - b_3)\cos\theta = (b_1 + b_4)\sin\theta$ and $b_2 \neq b_3$. Let $R(\theta) = I_2\cos\theta + J_1\sin\theta$ and $N = \begin{pmatrix} R(-\theta) & B \\ 0 & R(-\theta) \end{pmatrix}$. By Lemma 3.3 in ref. [4] there holds $N \in \mathrm{Sp}(4)$ and $\ker_{\mathbb{C}}(N - \omega I) = \mathbb{C}(e_1 + \mathrm{i}e_2)$. Let

$$M_{k,s,t} = R(-\theta)^{\diamond s} \diamond R(\theta)^{\diamond t} \diamond N^{\diamond l} \diamond I_{2n-2k-2l}, \tag{27}$$

then $M_{k,s,t} \in \mathrm{Sp}(2n)$ and $\varphi_{k,\omega}(M_{k,s,t}) = Y_{k,s,t}$. Let $M = PM_{k,s,t}P^{-1} \in \mathrm{Sp}(2n)$. Then $\varphi_{k,\omega}(M) = V$ and $V \in \mathrm{im}\varphi_{k,\omega}$. Hence $A^k \subset \mathrm{im}\varphi_{k,\omega}$. By Lemma 4.1 our Lemma follows.

QED.

4.2 The structure of ω-subsets of $\mathrm{Sp}(2n)$, case $\omega \in \mathbb{U} \setminus \mathbb{R}$

Lemma 4.6. Define $\tilde{\varphi}_k : \mathrm{Sp}_1^k(2n) \to G_k(\mathbb{R}^{2n})$ by

$$\tilde{\varphi}_k(M) = \ker(M - I), \qquad \forall\, M \in \mathrm{Sp}_1^k(2n), \tag{28}$$

for $k = 1, \cdots, 2n$, then $\tilde{\varphi}_k$ is a surjective map.

Proof. Let $W \in G_k(\mathbb{R}^{2n})$. There is a $P \in O(2n)$, such that $W = P\mathbb{R}^k$. Let $P^{-1}JP = \begin{pmatrix} * & Q \\ * & R \end{pmatrix}$ where Q and R are $k \times (2n - k)$ and $(2n - k) \times (2n - k)$ matrices respectively.

Let $A = \mathrm{diag}(I_k, 0)$, $B = \mathrm{diag}(0, I_{2n-k})$ and $C = P^{-1}JPB = \begin{pmatrix} 0 & Q \\ 0 & R \end{pmatrix}$ be $2n \times 2n$ matrices. Pick a symmetric $S \in \mathrm{gl}(2n - k, \mathbb{R})$ such that $S + R$ is invertible. Let X be the subspace of $\mathbb{R}^{4n}$

defined by

$$X = \operatorname{diag}(P, P)\begin{pmatrix} A & D \\ 0 & B \end{pmatrix}\mathbb{R}^{4n},$$

where $D = \operatorname{diag}(0, S) \in \mathrm{gl}(2n, \mathbb{R})$. Since D is symmetric, X is a Lagrangian subspace of $(\mathbb{R}^{4n}, J_2)$, where

$$J_2 = \begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix}.$$

Direct computation shows $X \cap \mathbb{R}^{2n} = W$. Moreover, we have

$$X \cap \operatorname{Gr}(J) = \{(-JPBy, PBy) \in \mathbb{R}^{4n} \mid Ax + (D + C)y = 0\}.$$

Since $S + R$ is invertible, there holds

$$\operatorname{rank}(D + C) = \operatorname{rank} B(D + C) = 2n - k.$$

So for any $(x, y) \in \mathbb{R}^{4n}$ such that $Ax + (D + C)y = 0$, we have

$$B(D + C)y = BAx + B(D + C)y = 0.$$

Therefore $(D + C)y = 0$, and then $By = 0$. From this we have $X \cap \operatorname{Gr}(J) = \{0\}$. Since X is Lagrangian, there is an $M \in \operatorname{Sp}(2n)$ such that

$$X = \{((I + M)x, J(I - M)x) \mid x \in \mathbb{R}^{2n}\}.$$

Then $\tilde{\varphi}_k(M) = \ker(M - I) = X \cap \mathbb{R}^{2n} = W$ and $\tilde{\varphi}_k$ is surjective. QED

Lemma 4.7. For any $V \in \operatorname{im}\varphi_{k,\omega}$, the inverse image $\varphi_{k,\omega}^{-1}(V)$ is a smooth manifold of dimension $2n^2 + n - 4nk + 2k^2 - k$.

Proof. Let $M \in \varphi_{k,\omega}^{-1}(V)$, $\xi_j + \mathrm{i}\eta_j (\xi_j, \eta_j \in \mathbb{R}^{2n})$, $j = 1, \cdots, k$ be a basis of V, and $W = \operatorname{span}\{\xi_1, \cdots, \xi_k, \eta_1, \cdots, \eta_k\}$. Let $N \in \operatorname{Sp}(2n)$. Then

$$\ker_{\mathbb{C}}(NM - \omega I) \supset V \Leftrightarrow NM\mid_V = \omega I\mid_V \Leftrightarrow N\mid_W = I\mid_W. \tag{29}$$

Let $\mathscr{G}_W = \{N \in \operatorname{Sp}(2n) \mid N\mid_W = I\mid_W\}$. Since $\mathscr{G}_W$ is a closed subgroup of $\operatorname{Sp}(2n)$, it is a Lie subgroup of $\operatorname{Sp}(2n)$. By the definition of $\mathscr{G}_W$ and (29), we have $\ker_{\mathbb{C}}(M - \omega I) \supset V$ for any $N \in \mathscr{G}_W$. Since $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}}\ker_{\mathbb{C}}(M - \omega I) = k$, there is a neighborhood U_1 of I in $\mathscr{G}_W$ such that $\dim_{\mathbb{C}}\ker_{\mathbb{C}}(NM - \omega I) \leqslant k$ and hence $\ker_{\mathbb{C}}(M - \omega I) = V$, for $N \in U_1$. So we get $U_1 M \subset \varphi_{k,\omega}^{-1}(V)$. By (29) there holds $\varphi_{k,\omega}^{-1}(V) \subset \mathscr{G}_W M$. Therefore there exists a neighborhood U_2 of M in $\operatorname{Sp}(2n)$ such that

$$U_1 M = \varphi_{k,\omega}^{-1}(V) \cap U_2. \tag{30}$$

So $\varphi_{k,\omega}^{-1}(V)$ is a manifold of dimension $\dim \mathscr{G}_W$. From (30) we obtain that $\varphi_{k,\omega}^{-1}(V)$ is smooth by the definition of smoothness and the Lie group structure of $\mathscr{G}_W$.

Now we calculate the dimension $\dim\mathscr{G}_W$. By Lemma 4.6 $\tilde{\varphi}_{2k}$ is surjective. By (ii) of Theorem 1.1, each fiber of the map $\tilde{\varphi}_{2k}$ is a smooth manifold of dimension

$$\begin{aligned} \dim \operatorname{Sp}_1^{2k}(2n) - \dim G_{2k}(\mathbb{R}^{2n}) &= 2n^2 + n - k(2k + 1) - 2k(2n - 2k) \\ &= 2n^2 + n - 4nk + 2k^2 - k. \end{aligned}$$

Since $W \in G_{2k}(\mathbb{R}^{2n})$, the argument of the above paragraph shows that $\tilde{\varphi}_{2k}^{-1}(W)$ is a manifold of dimension $\dim \mathscr{G}_W$. So we have

$$\dim\varphi_{k,\omega}^{-1}(V) = \dim\mathscr{G}_W = \dim\tilde{\varphi}_{2k}^{-1}(W) = 2n^2 + n - 4nk + 2k^2 - k. \qquad \text{QED}$$

Proposition 4.1. (i) There are fiber maps

$$\varphi_{k,\omega}^{-1}(A^{k,s,t}) \to A^{k,s,t}, \tag{31}$$

where k, s, t, l are non-negative integers and $l = k - s - t, k + l \leqslant n$.

(ii) Let k, s, t, l be as (i). Then $\varphi_{k,\omega}^{-1}(A^{k,l})$ is a smooth manifold of dimension $2n^2 + n - k^2 - l^2$.

(iii) $\mathrm{Sp}_\omega^k(2n)$ is an analytic variety of dimension $2n^2 + n - k^2$.

Proof. (i) Let $\mathrm{Sp}(2n)$ act on $\mathrm{Sp}(2n)$ by

$$P(M) = PMP^{-1}. \tag{32}$$

For any $M \in \mathrm{Sp}_\omega^k(2n)$, we have

$$\varphi_{k,\omega}(P(M)) = P\varphi_{k,\omega}(M). \tag{33}$$

Since $\mathrm{Sp}(2n)$ acts on $A^{k,s,t}$ transitively, our results follow from Proposition 2.1 and Lemmas 4.3, 4.5;

(ii) follows from Corollary 4.1, Lemma 4.7 and (i);

(iii) follows from (ii).

4.3 The top stratum of the ω-singular subset of Sp $(2n)$

Lemma 4.8. Let $\omega \in \mathbb{U}$. Then the submanifolds $\omega\mathrm{Sp}(2n), \mathrm{Sp}^1(2n, \mathbb{C})$ of $\mathrm{Sp}(2n, \mathbb{C})$ are transversal to each other.

Proof. Let $M \in \omega\mathrm{Sp}(2n) \cap \mathrm{Sp}^1(2n, \mathbb{C})$. By (i) of Theorem 1.1, the curve $\gamma(t) = Me^{Jt}, t \in \mathbb{R}$ is transversal to $\mathrm{Sp}^1(2n, \mathbb{C})$ at M. Since $\gamma(t) \in \omega\mathrm{Sp}(2n)$ and $\mathrm{Sp}^1(2n, \mathbb{C})$ is a codimension 1 submanifold of $\mathrm{Sp}(2n, \mathbb{C})$,

$$T_M(\omega\mathrm{Sp}(2n)) + T_M(\mathrm{Sp}^1(2n, \mathbb{C})) \supset T_M\gamma \oplus T_M(\mathrm{Sp}^1(2n, \mathbb{C})) = T_M(\mathrm{Sp}(2n, \mathbb{C})).$$

So $\omega\mathrm{Sp}(2n)$ is transversal to $\mathrm{Sp}^1(2n, \mathbb{C})$ at M.

Proposition 4.2. $\mathrm{Sp}_\omega^1(2n)$ is a smooth submanifold of codimension 1 of $\mathrm{Sp}(2n)$ and $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Me^{Jt}, M \in \mathrm{Sp}_\omega^1(2n)$ gives a transverse structure of $\mathrm{Sp}_\omega^1(2n)$ in $\mathrm{Sp}(2n)$ at M.

Proof. Let $f: \omega\mathrm{Sp}(2n) \to \mathrm{Sp}(2n, \mathbb{C})$ be the obvious embedding. By Lemma 4.8, f intersects $\mathrm{Sp}^1(2n, \mathbb{C})$ transversely. By (i) of Theorem 1.1, $\mathrm{Sp}^1(2n, \mathbb{C})$ is a smooth codimension 1 submanifold of $\mathrm{Sp}(2n, \mathbb{C})$. Moreover, we have

$$f^{-1}(\mathrm{Sp}^1(2n, \mathbb{C})) = \omega\mathrm{Sp}_\omega^1(2n).$$

By Theorem II. 4.4 of ref. [12], $\omega\mathrm{Sp}_\omega^1(2n)$ is a smooth codimension 1 submanifold of $\omega\mathrm{Sp}(2n)$. Hence the first part of the proposition follows. The second part follows from (i) of Theorem 1.1. QED

Proof of Theorem 1.2. For $0 \leqslant k \leqslant n$, note that $\bigcup_{l \geqslant k} \mathrm{Sp}_\omega^l(2n)$ consists of all matrices M in $\mathrm{Sp}(2n)$ such that any $(2n - k + 1) \times (2n - k + 1)$ submatrix N of $M - \omega I$ satisfies $\det N = 0$. Thus it is a real subvariety of $\mathrm{Sp}(2n)$. Since $M \in \mathrm{Sp}(2n)$ is real, ω and $\bar{\omega} \in \sigma(M)$ possess the same geometric multiplicity. Therefore $\bigcup_{l \geqslant k} \mathrm{Sp}_\omega^l(2n)$ is empty when $n < k \leqslant 2n$. Combining with Propositions 4.1 and 4.2, we obtain Theorem 1.2.

5 The definition of the ω-index theory

Let $\gamma: [0, \tau] \to \mathrm{Sp}(2n)$ be a continuous symplectic path ($\tau > 0$) and $\omega \in \mathbb{C}$ be on the unit circle. By (ii) of Theorem 1.1 and Theorem 1.2, there exists an $\varepsilon > 0$ such that $\gamma(t)e^{Js} \in \mathrm{Sp}_{\omega}^{0}(2n)$ for $t = 0$ or τ and $s \in (-\varepsilon, 0) \cup (0, \varepsilon)$. By (i) of Theorem 1.1 and Theorem 1.2 we can make the following definition.

Definition 5.1. The ω-index $i_{\tau,\omega}(\gamma)$ for the symplectic path γ is defined to be the intersection number of path $\gamma(t)e^{-Js}$, $t \in [0, \tau]$ and the codimension 1 algebraic subvariety $\bigcup_{k \geq 1} \mathrm{Sp}_{\omega}^{k}(2n)$ of $\mathrm{Sp}(2n)$ for $s \in (0, \varepsilon)$.

Remark 5.1. It is proved in ref. [5] that this definition coincides with that of ref. [1].

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