Life Span of Solution to Cauchy Problem for a Semilinear Heat Equation*

WANG Ming-Xin (王明新)

(*Institute of Applied Mathematics, Academia Sinica, Beijing 100080, PRC; Department of Mathematics, Henan University, Kaifeng 475001, PRC*)

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It is known that for $1<p<+\infty$, $n=2$ or $1<p<(n+2)/(n-2)$, $n\geqslant 3$, the elliptic problem

$$\begin{cases} -\Delta u=u^p-u, & x\in R^n, \\ u(x)>0,\ u(x)\to 0 \text{ as } |x|\to+\infty \end{cases}$$

has a unique solution $\bar u(x)$ which is radially symmetric and satisfies $\bar u'(0)=0$, $\bar u'(r)<0$, $\bar u(x)\sim Me^{-\alpha|x|^2}$ as $|x|\to+\infty$. Let $u(x, t)$ be the solution (positive) of

$$\begin{cases} u_t-\Delta u=u^p-u, \ x\in R^n,\ 0<t<T, \\ u(x,\ 0)=\lambda\bar u(x),\ x\in R^n,\ \lambda>0. \end{cases} \tag{1}$$

In Ref. [1] it has been proved that if $\lambda>1$, then $u(x, t)$ blows up in finite time. We denote the life span of $u(x, t)$ by T_λ. In this note, we discuss the estimate of T_λ, and obtain the following

Theorem. *Let* $\lambda>1$.

(i) *If* $1<p<+\infty$, $n=2$ *or* $1<p<(n+2)/(n-2)$, $n\geqslant 3$, *then*

$$\lambda^{1-p}\bar u^{1-p}(0)/(p-1)\leqslant T_\lambda\leqslant C/(\lambda^{p-1}-1),$$

$$(1/(p-1))^{1/(p-1)}e^{-T_\lambda}\leqslant \lim_{t\to T_\lambda^-}(T_\lambda-t)^{1/(p-1)}\cdot u(0,\ t)\leqslant C.$$

(ii) *If in addition* $p\geqslant 2$, *then as* $\lambda\to 1^+$, $T_\lambda=O(-\ln(\lambda^{p-1}-1))$.

Proof. (i) Choose $\alpha=(p-1)/(p+1)$, and set

$$I(t)=\int_{R^n}u^{p+1}(x,\ t)\,dx,\ J(t)=I^{-\alpha}(t).$$

According to the proof of Theorem 3 in Ref. [1] we know that there exists $0<T_1\leqslant T_0<-J(0)/J'(0)$, such that $\lim_{t\to T_0^-}\|u(\cdot,t)\|_{p+1}=+\infty$, $\lim_{t\to T_1^-}\|u(\cdot,t)\|_\infty=+\infty$. Direct computation gives

$$-\frac{J(0)}{J'(0)}=\frac{\int_{R^n}\bar u^{p+1}(x)dx}{(p-1)\int_{R^n}\bar u^{2p}(x)dx}\times\frac{1}{\lambda^{p-1}-1}=T^*. \tag{2}$$

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Therefore $T_\lambda = T_1 \leqslant T_0 < T^* = C/(\lambda^{p-1} - 1)$.

Let $e^t u = v$. Then $\lim\limits_{t \to T_\lambda^-} u(x, t) = +\infty$ iff $\lim\limits_{t \to T_\lambda^-} v(x, t) = +\infty$.

$$\begin{cases} v_t - \Delta v = e^{(1-p)t} v^p, & x \in R^n, \ 0 < t < T_\lambda, \\ v(x, 0) = \lambda \bar{u}(x), & x \in R^n. \end{cases} \tag{3}$$

For any $A > 1$, define

$$X = \{v(t) : [0, t_1] \to L^\infty(R^n) \text{ is continuous}, \ v(t) \geqslant 0 \text{ and } \|v(t)\|_\infty \leqslant A\lambda \bar{u}(0)\},$$

$$F(v(t)) = e^{t\Delta}(\lambda \bar{u}) + \int_0^t e^{-(p-1)s} e^{(t-s)\Delta} v^p(s)\, ds, \quad v \in X, \ 0 \leqslant t \leqslant t_1.$$

Then we have

$$0 \leqslant F(v(t)) \leqslant \lambda \bar{u}(0) + A^p \lambda^p \bar{u}^p(0) \int_0^t e^{-(p-1)s} ds \leqslant \lambda \bar{u}(0) + A^p \lambda^p \bar{u}^p(0) t.$$

Choose $0 < t_1 = (A-1)/(A^p \lambda^{p-1} \bar{u}^{p-1}(0))$. Then $F(v(t)) \leqslant A\lambda \bar{u}(0)$ for any $0 \leqslant t \leqslant t_1$, i.e.

$$F(v) \in X.$$

Define sequence

$$v_0(t) = e^{t\Delta}(\lambda \bar{u}), \quad v_m(t) = F(v_{m-1}(t)), \quad m = 1, 2, \cdots.$$

Then we have $v_m(t) \geqslant v_{m-1}(t)$ for all $0 \leqslant t \leqslant t_1$, $m = 1, 2, \cdots$. And hence

$$\lim_{m \to +\infty} v_m(t) = v(t) \in X \text{ and } v(t) = F(v(t)).$$

This shows that $v(t)$ is the solution of the following integral equation

$$v(t) = e^{t\Delta}(\lambda \bar{u}) + \int_0^t e^{-(p-1)s} e^{(t-s)\Delta} v^p(s)\, ds, \quad 0 \leqslant t \leqslant t_1.$$

That is, $v(t)$ is $L^\infty(R^n)$ solution of (3) defined on $[0, t_1]$. From the smooth property of $\bar{u}(x)$, it is easily proved that $v(x, t) = v(t)$ is classical solution of (3). $v(t) \in X$ implies that $0 \leqslant v(x, t) \leqslant A\lambda \bar{u}(0)$ on $R^n \times [0, t_1]$.

Considering $v(x, t_1)$ as the initial data of (3) and using $v(x, t_1) \leqslant A\lambda \bar{u}(0)$, similar to the above arguments we prove that if $t_2 - t_1 = (A-1)/(A^p (A\lambda)^{p-1} \bar{u}^{p-1}(0))$, then (3) has classical solution $v(x, t)$ on $[t_1, t_2]$. Continuing the above procedure, we deduce that if

$$t_k - t_{k-1} = (A-1)/(A^p (A^{k-1}\lambda)^{p-1} \bar{u}^{p-1}(0)),$$

then (3) has classical solution $v(x, t)$ on $[t_{k-1}, t_k]$, $k = 1, 2, \cdots$. Therefore

$$T_\lambda \geqslant t_\infty = \sum_{k=1}^\infty (t_k - t_{k-1}) = (A-1) A^{-p} \lambda^{1-p} \bar{u}^{1-p}(0) \sum_{k=1}^\infty A^{(1-p)(k-1)}$$

$$= (A-1) A^{-1} (A^{p-1} - 1)^{-1} \lambda^{1-p} \bar{u}^{1-p}(0).$$

Let $A \to 1^+$. It is deduced that

$$T_\lambda \geqslant \lambda^{1-p} \bar{u}^{1-p}(0)/(p-1).$$

The first result of (i) holds.

If we consider $v(x, t)$ as the initial data of (3), similar to the above proof we have

$$T_\lambda - t \geqslant \|v(\cdot, t)\|_\infty^{1-p}/(p-1) = e^{(1-p)t}\|u(\cdot, t)\|_\infty^{1-p}/(p-1).$$

Because $\bar{u}(x) = \bar{u}(r)$ and $\bar{u}'(r) < 0$, it follows that $u(x, t) = u(r, t)$ is radial function of x and $u_r(r, t) < 0$ (for $r > 0$), so that $\|u(\cdot, t)\|_\infty = u(0, t)$ and

$$\lim_{t \to T_\lambda} (T_\lambda - t)^{1/(p-1)} u(0, t) \geqslant e^{-T_\lambda} \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

Using the method of Ref. [2] it is easy to prove that there exists $C > 0$ such that

$$\lim_{t \to T_\lambda^-} (T_\lambda - t)^{1/(p-1)} u(0, t) \leqslant C.$$

The second result of (i) holds.

(ii) In Ref. [1] we have proved that the solution $u(x, t) = u(r, t)$ of (1) satisfies $u_t(x, t) \geqslant 0$, $\not\equiv 0$ and for any $0 < T < T_\lambda$, there exist $M_0 > 0$ and $\alpha_0 > 0$ such that $u(x, t) \leqslant M_0 \exp\{-\alpha_0 |x|^2\}$ on $R^n \times [0, T]$. Moreover, $|\nabla u|$ and $|\nabla u_t|$ are uniformly bounded on $R^n \times [0, T]$. Therefore, the following integration by parts are reasonable.

If $p \geqslant 2$, set $\tau = e^t$, $v(x, \tau) = u(x, \ln\tau)$. Then $t \geqslant 0$ iff $\tau \geqslant 1$.

$$\begin{cases} \tau v_\tau - \Delta = v^p - v, \ x \in R^n, \ 1 < \tau < \tau_0, \\ v(x, 1) = \lambda \bar{u}(x), \ x \in R^n. \end{cases}$$

Let $I(\tau) = \int_{R^n} h(v(x, \tau))\, dx$, $h(v) = v^{p+1}$. Write $f(v) = v^p - v$. Then

$$I'(\tau) = \int_{R^n} h'(v) v_\tau dx,$$

$$\tau I'(\tau) = \int_{R^n} h'(v)(\tau v_\tau) dx = \int_{R^n} h'(v)(\Delta v + f(v)) dx$$

$$= -\int_{R^n} h''(v)|\nabla v|^2 dx + \int_{R^n} h'(v) f(v) dx.$$

$$(\tau I'(\tau))' = -\int_{R^n} h'''|\nabla v|^2 v_\tau dx + \int_{R^n} (h'' f + h' f') v_\tau dx$$

$$-\int_{R^n} h'' \frac{d}{d\tau} |\nabla v|^2. \tag{4}$$

On the other hand,

$$(\tau I'(\tau))' = \int_{R^n} [h'(v)(\tau v_\tau)]_\tau dx$$

$$= \int_{R^n} \tau h''(v) v_\tau^2 dx + \int_{R^n} h'(v)(\Delta v + f(v))_\tau dx$$

$$= \int_{R^n} \tau h''(v) v_\tau^2 dx + \int_{R^n} h'(v) f'(v) v_\tau dx - \frac{1}{2} \int_{R^n} h''(v) \frac{d}{d\tau} |\nabla v|^2 dx. \tag{5}$$

$(5) \times 2 - (4)$ gives

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$$(\tau I'(\tau))' = 2\tau\int_{R^n} h''(v)v_\tau^2dx + \int_{R^n} h'''(v)|\nabla v|^2 v_\tau dx + \int_{R^n} (h'f' - h''f)v_\tau dx.$$

Using $I'(\tau) = \int_{R^n} h'(v)v_\tau dx$, from the above equation we get

$$\tau I''(\tau) = 2\tau\int_{R^n} h''(v)v_\tau^2dx + \int_{R^n} h'''(v)|\nabla v|^2 v_\tau dx + \int_{R^n} (f'h' - h''f - h')v_\tau dx.$$

$u_t \geqslant 0$ implies $v_\tau \geqslant 0$. Direct computations show that

$$h'''(v) > 0,\ h'(v)f'(v) - h''(v)f(v) - h'(v) \geqslant 0.$$

Hence

$$I''(\tau) \geqslant 2\int_{R^n} h''(v)v_\tau^2dx.$$

Choose $\alpha = (p-1)/(p+1)$, $J(\tau) = I^{-\alpha}(\tau)$. Similar to the proof of Theorem 3 in Ref. [1] it can be proved that there exists $1 < \tau_0 \leqslant 1 - J(1)/J'(1)$, such that $J(\tau_0) = 0$, thus

$$\lim_{\tau\to\tau_0^-} I(\tau) = +\infty, \qquad \text{i.e.}\ \lim_{\tau\to\tau_0^-} \|v(\cdot,\ \tau)\|_{p+1} = +\infty.$$

Therefore, there exists $T_0 \leqslant \ln\tau_0$ such that $\lim_{t\to T_0^-} \|u(\cdot,\ t)\|_{p+1} = +\infty$. From the construction of $J(\tau)$ it is easy to know that $-J(1)/J'(1) = T^*$, where T^* is defined by (2), so that

$$\begin{aligned} T_\lambda &\leqslant T_0 \leqslant \ln\tau_0 \leqslant \ln(1+T^*) = \ln(1 + C/(\lambda^{p-1}-1)) \\ &= \ln(C + \lambda^{p-1} - 1) - \ln(\lambda^{p-1}-1) = O(-\ln(\lambda^{p-1}-1)) \end{aligned} \tag{6}$$

as $\lambda > 1$ and close to 1.

In the sequel we will prove that if $1 < p < +\infty$, $n = 2$ or $1 < p < (n+2)/(n-2)$, $n \geqslant 3$, then

$$T_\lambda \geqslant \bar{u}^{1-p}(0)\lambda^{p-1}\int_0^{\lambda^{1-p}} \frac{s^p}{1-s^{p-1}}\, ds.$$

In fact, choose $A = \bar{u}^{p-1}(0) > 1$. It is easy to prove that $\bar{u}(x) - e^{-t}e^{t\Delta}\bar{u}(x) \leqslant At\bar{u}(x)$ for all $x \in R^n$ and $t \geqslant 0$. Write (1) as the integral equation

$$u(t) = e^{-t}e^{t\Delta}(\lambda\bar{u}) + e^{-t}\int_0^t e^s e^{(t-s)\Delta}\bar{u}^p(s)\, ds. \tag{7}$$

$\bar{u}(x)$ satisfies $\bar{u} = e^{-t}e^{t\Delta}\bar{u} + e^{-t}\int_0^t e^s e^{(t-s)\Delta}\bar{u}^p ds$. Choose $B > 1$ and define

$$X = \{u(t):[0,\ t_1] \to L^\infty(R^n) \text{ is continuous, } 0 \leqslant u(t) \leqslant B\lambda\bar{u}(x)\},$$

$$F(u) = e^{-t}e^{t\Delta}(\lambda\bar{u}) + e^{-t}\int_0^t e^s e^{(t-s)\Delta}u^p(s)\, ds,\ u \in X.$$

For $0 \leqslant t \leqslant t_1$,

$$0\leqslant F(u(t))\leqslant e^{-t}e^{t\Delta}(\lambda\bar{u})+B^p\lambda^pe^{-t}\int_0^t e^se^{(t-s)\Delta}\bar{u}^pds$$
$$=\lambda\bar{u}+\lambda(B^p\lambda^{p-1}-1)(\bar{u}-e^{-t}e^{t\Delta}\bar{u})$$
$$\leqslant\lambda\bar{u}+\lambda(B^p\lambda^{p-1}-1)A\bar{u}t.$$

Choose $t_1:(B^p\lambda^{p-1}-1)At_1=B-1$. Then for any $0\leqslant t\leqslant t_1$, $0\leqslant F(u(t))\leqslant B\lambda\bar{u}$ holds, i.e. $F(u(t))\in X$. Define sequence

$$u_0(t)=e^{-t}e^{t\Delta}(\lambda\bar{u}),\ u_m(t)=F(u_{m-1}(t)),\ m=1,\ 2,\ \cdots.$$

It is easy to verify $u_m(t)\geqslant u_{m-1}(t)$, for all $m\in\mathscr{N}$ and $0\leqslant t\leqslant t_1$. Thus

$$\lim_{m\to+\infty}u_m(t)=u(t)\in X \text{ and } u(t)=F(u(t)).$$

This shows that $u(t)$ is the solution of (7) on $[0,\ t_1]$ and $u(x,\ t_1)\leqslant B\lambda\bar{u}$.

Considering $u(x,\ t_1)$ as the initial data and using $u(x,\ t_1)\leqslant B\lambda\bar{u}$, similar to the above arguments we can prove that if t_2-t_1 satisfies $[B^p(B\lambda)^{p-1}-1]A(t_2-t_1)=B-1$, then (7) has solution $u(x,\ t)$ on $[t_1,\ t_2]$. So, (7) has solution $u(x,\ t)$ on $[0,\ t_2]$ and $u(x,\ t)\leqslant B^2\lambda\bar{u}$ for all $x\in R^n$ and $0\leqslant t\leqslant t_2$.

By induction we see that if t_k-t_{k-1} satisfies

$$t_k-t_{k-1}=(B-1)/A[B^p(B^{k-1}\lambda)^{p-1}-1], \tag{8}$$

then (7) has solution $u(x,\ t)$ on $[0,\ t_k]$. Therefore,

$$T_\lambda\geqslant t_\infty=\sum_{k=1}^{\infty}(t_k-t_{k-1}),\ t_0=0.$$

Choose $B=\lambda^{p-1}>1$. From (8) we obtain

$$T_\lambda\geqslant(B-1)A^{-1}\sum_{k=1}^{\infty}(B^{p+(k-1)(p-1)}\lambda^{p-1}-1)^{-1}$$
$$=(B-1)A^{-1}\sum_{k=1}^{\infty}(B^{p+1+(k-1)(p-1)}-1)^{-1}.$$

In view of the inequality $x^q-1\leqslant q(x-1)x^{q-1}$ $(x>1,\ q>1)$, we obtain

$$T_\lambda\geqslant\frac{1}{A}\sum_{k=1}^{\infty}\frac{1}{[2+k(p-1)]B^{1+k(p-1)}}\ .$$

For $y<1$, set $F(y)=\sum_{k=1}^{\infty}\frac{1}{2+k(p-1)}y^{1+k(p-1)}$. Then

$$yF(y)=\sum_{k=1}^{\infty}\frac{1}{2+k(p-1)}y^{2+k(p-1)}=\int_0^y\sum_{k=1}^{\infty}y^{1+k(p-1)}dy=\int_0^y\frac{s^p}{1-s^{p-1}}ds.$$

Consequently,

$$T_\lambda\geqslant\frac{B}{A}\int_0^{B^{-1}}\frac{s^p}{1-s^{p-1}}ds=\bar{u}^{1-p}(0)\lambda^{p-1}\int_0^{\lambda^{1-p}}\frac{s^p}{1-s^{p-1}}ds.$$

Direct computation shows that

$$\lim_{\lambda \to 1^+} \left\{ -\ln(\lambda^{p-1}-1) \Big/ \int_0^{\lambda^{1-p}} \frac{s^p}{1-s^{p-1}}\, ds \right\} = p-1.$$

And hence,

$$T_\lambda \geqslant O(-\ln(\lambda^{p-1}-1)) \qquad \text{as } \lambda \to 1^+.$$

This inequality and (6) show that the result of (ii) holds.

The proof of our theorem is complete.

References

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