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Tensor absolute value equations

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Abstract This paper is concerned with solving some structured multi-linear systems, which are called tensor absolute value equations. This kind of absolute value equations is closely related to tensor complementarity problems and is a generalization of the well-known absolute value equations in the matrix case. We prove that tensor absolute value equations are equivalent to some special structured tensor complementary problems. Some sufficient conditions are given to guarantee the existence of solutions for tensor absolute value equations. We also propose a Levenberg-Marquardt-type algorithm for solving some given tensor absolute value equations and preliminary numerical results are reported to indicate the efficiency of the proposed algorithm.

Keywords M-tensors, absolute value equations, Levenberg-Marquardt method, tensor complementarity problem

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1 Introduction

The systems of multi-linear equations can be expressed by tensor-vector products, just as we rewrite linear systems by matrix-vector products. Let $\mathcal{A}$ be an m-th-order tensor in $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $\boldsymbol{b}$ be a vector in $\mathbb{R}^n$. Then a multi-linear equation can be expressed as

$$\mathcal{A}\boldsymbol{x}^{m-1} = \boldsymbol{b}, \tag{1.1}$$

where $\mathcal{A}\boldsymbol{x}^{m-1}$ is a vector in $\mathbb{R}^n$ (see [24, 25]) with

$$(\mathcal{A}\boldsymbol{x}^{m-1})_i = \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, \ldots, n.$$

Solving multi-linear systems is always an important problem in engineering and scientific computing (see [10, 15, 36, 37]). In this paper, we consider the system of multi-linear absolute value equations, which can be expressed as

$$\mathcal{A}\boldsymbol{x}^{m-1} - |\boldsymbol{x}|^{[m-1]} = \boldsymbol{b}, \tag{1.2}$$

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where $|\boldsymbol{x}|^{[m-1]}$ is a vector in $\mathbb{R}^n$ with

$$|\boldsymbol{x}|^{[m-1]} = (|x_1|^{m-1}, \ldots, |x_n|^{m-1})^{\mathrm{T}}.$$

It is easy to see that the system of multi-linear absolute value equations (1.2) is a generalization of the well-known absolute value equations

$$A\boldsymbol{x} - |\boldsymbol{x}| = \boldsymbol{b}$$

with a matrix $A \in \mathbb{R}^{n\times n}$. The absolute value equations (AVEs) has wide applications in applied science and technology such as optimization physical and economic equilibrium problems (see [20–22]). As was shown in [22], the general NP-hard linear complementarity problem (see [8]), which subsumes many mathematical programming problems, can be formulated as an AVE. This implies that the AVE is NP-hard in its general form. Analogous to AVE, we call (1.2) tensor absolute value equations (TAVEs). Obviously, the TAVE is also NP-hard. Thus, investigating the existence of solutions to the TAVE is a significant problem.

Recently, Song and Qi [29] introduced a class of complementarity problems, called tensor complementarity problems, where the involved function is defined by some homogenous polynomial of degree n with $n > 2$. It is known that the tensor complementarity problem is a generalization of the linear complementarity problem (see [8]), and a subclass of nonlinear complementarity problems (see [12]). The tensor complementarity problem has many applications in n-person non-cooperative games (see [14]), nonlinear compressed sensing (see [19]), and so on. Recently, there has been growing interest in the study of the theory of tensor complementarity problems such as the structure of the solution set (see [32]) and the solvability of the problem (see [2]). More results on tensor complementarity problems can be found in [5, 30, 35] and the references therein. Another kind of complementarity problems related to tensors, named tensor eigenvalue complementarity problem, is considered in [6,17,18,31,38]. In [22], it was shown that the AVE is equivalent to a generalized linear complementarity problem. Can we show that the TAVE is equivalent to a generalized tensor complementarity problem? Although some computational methods have been presented for the AVE, it is very difficult to extend these algorithms to solve the TAVE because the TAVE (1.2) is a nonlinear equation. The Levenberg-Marquardt method is one of the important algorithms for solving nonlinear equations (see [11]). Can we propose an efficient algorithm such as the Levenberg-Marquardt method for solving the TAVE (1.2)? To our best knowledge, there is no general answer to these questions. Therefore, we shall focus on some special tensor absolute value equations.

Let $\mathcal{I}$ be an m-th-order n-dimensional unit tensor, whose entries are 1 if and only if $i_1 = \cdots = i_m$ and otherwise zero. A tensor $\mathcal{A}$ is called a non-negative tensor if all its entries are non-negative, denoted by $\mathcal{A} \geqslant 0$. A tensor is called a Z-tensor, if all its diagonal entries are non-negative and off-diagonal entries are nonpositive. M-tensor is a special class of Z-tensor, which was first introduced and studied in [9,39]. To define the M-tensors, we need to introduce the tensor eigenvalues first. Let $\mathcal{A}$ be an m-th-order n-dimensional tensor. If a scalar $\lambda \in \mathbb{R}$ and a nonzero vector $\boldsymbol{x} \in \mathbb{R}^n$ satisfy

$$\mathcal{A}\boldsymbol{x}^{m-1} = \lambda \boldsymbol{x}^{[m-1]},$$

where $\boldsymbol{x}^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^{\mathrm{T}}$. Then we call λ an eigenvalue of $\mathcal{A}$ and x a corresponding eigenvector. Qi [24] and Lim [16] first defined the eigenvalues of tensors independently. The spectral radius of a tensor $\mathcal{A}$ is defined by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

A tensor $\mathcal{A}$ is called an M-tensor, if it can be written as $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ with $\mathcal{B} \geqslant 0$ and $s \geqslant \rho(\mathcal{B})$; furthermore, it is called a strong M-tensor if $s > \rho(\mathcal{B})$. One can refer to a survey [4] for the spectral theory of non-negative tensors. In this paper, we first investigate the existence of solutions for the TAVE (1.2). We show that the TAVE (1.2) with the positive right-hand side $\boldsymbol{b}$ always has a unique solution when $\mathcal{A} - \mathcal{I}$ is strong M-tensor. Another sufficient condition for the existence of solution is also given. Can we compute the solution? We propose an inexact Levenberg-Marquardt method for solving the TAVE (1.2).

The rest of this paper is organized as follows. In Section 2, we introduce the tensor absolute value equations which are a generalization of absolute value equations with the matrix case. In Section 3, some sufficient conditions for the existence of solutions to the TAVE are given. In Section 4, we first reformulate the TAVE as a special tensor complementarity problem and then we propose an inexact Levenberg-Marquardt-type algorithm for solving the TAVE. Some numerical results are reported in Section 5. Finally, some conclusions are given in Section 6.

Throughout this paper, we assume that $m > 2$. We use $x, y, \ldots$ for scalars, $\boldsymbol{x}, \boldsymbol{y}, \ldots$ for vectors, $A, B, \ldots$ for matrixes, $\mathcal{A}, \mathcal{B}, \ldots$ for tensors, and $\mathcal{D}$ for the diagonal tensor whose diagonal elements are 1 or -1. All the tensors discussed in this paper are real. $T(m,n)$ denotes the set of all m-th-order n-dimensional tensors. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in T(m,n)$. Then $\mathcal{A}$ is called a symmetric tensor if its entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of their indices. $S(m,n)$ denotes the set of all symmetric tensors. We denote the transpose of A by A^{T}. The identity matrix of arbitrary dimensions will be denoted by I.

2 Tensor absolute value equations

In this section, we present some basic definitions and properties in absolute value equations, nonlinear complementarity problems, and nonsmooth analysis, which will be used in the sequel.

We recall the absolute value equations of the type

$$A\boldsymbol{x} - |\boldsymbol{x}| = \boldsymbol{b}, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^n$ and $|\boldsymbol{x}|$ denotes the vector with absolute values of each component of $\boldsymbol{x}$. The AVE (2.1) has been widely investigated in many literatures such as [20–22]. In [22], some results about the AVE are given, which we list as follows:

(i) The AVE (2.1) is equivalent to the bilinear program

$$0 = \min\{((A+I)\boldsymbol{x} - \boldsymbol{b})^{\mathrm{T}}((A-I)\boldsymbol{x} - \boldsymbol{b}) \,|\, (A+I)\boldsymbol{x} - \boldsymbol{b} \geqslant \boldsymbol{0}, (A-I)\boldsymbol{x} - \boldsymbol{b} \geqslant \boldsymbol{0}\},$$

and the generalized linear complementarity problem

$$(A+I)\boldsymbol{x} - \boldsymbol{b} \geqslant \boldsymbol{0}, \quad (A-I)\boldsymbol{x} - \boldsymbol{b} \geqslant \boldsymbol{0}, \quad ((A+I)\boldsymbol{x} - \boldsymbol{b})^{\mathrm{T}}((A-I)\boldsymbol{x} - \boldsymbol{b}) = 0.$$

(ii) Let $C \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$. Then

$$(C - I)\boldsymbol{z} = \boldsymbol{b},\ \boldsymbol{z} \geqslant \boldsymbol{0} \text{ has a solution } \boldsymbol{z} \in \mathbb{R}^n,$$

which implies that

$$A\boldsymbol{x} - |\boldsymbol{x}| = \boldsymbol{b} \text{ has a solution} \quad \text{for any } A = CD \quad \text{with } D = \mathrm{diag}(\pm 1).$$

Clearly, the tensor absolute value equation (1.2) is a generalization of the AVE (2.1) from the matrix case to the tensor case. Take an equation with the coefficient tensor $\mathcal{A} \in \mathbb{R}^{2\times 2\times 2}$ as an example. The tensor absolute equation

$$\mathcal{A}\boldsymbol{x}^2 - |\boldsymbol{x}|^{[2]} = \boldsymbol{b}$$

is a condense form of

$$\begin{cases} a_{111}x_1^2 + (a_{112} + a_{121})x_1x_2 + a_{122}x_2^2 - |x_1|^2 = b_1, \\ a_{211}x_1^2 + (a_{212} + a_{221})x_1x_2 + a_{222}x_2^2 - |x_2|^2 = b_2. \end{cases}$$

We want to find x_1 and x_2 that satisfy the above two equations.

The following example shows a specific tensor absolute value equation.

Example 2.1. Let a tensor $\mathcal{A} \in T(4,2)$ be defined by $a_{1111} = a_{2222} = 1$, $a_{2111} = -2$, $a_{1222} = -1$, and zero otherwise. Let $\boldsymbol{b} = (1,2)^{\mathrm{T}}$. Then the corresponding tensor absolute value equation is

$$\begin{cases} x_1^3 - x_2^3 - |x_1|^3 = 1, \\ -2x_1^3 + x_2^3 - |x_2|^3 = 2. \end{cases} \tag{2.2}$$

By a simple computation, we see that the TAVE (2.2) in Example 2.1 has no solution. In the next section we discuss the existence of solutions to the TAVE (1.2). We can extend the result (ii) to the TAVE and obtain a similar condition for the existence of solutions to (1.2).

Below, we introduce the classical nonlinear complementarity problem. The tensor complementarity problem recently studied in [2,5,19,29,30,32,35] is a special kind of nonlinear complementarity problems. It will be shown in Section 4 that the TAVE (1.2) can be reformulated as a special kind of generalized tensor complementarity problems.

Definition 2.2. Given a mapping $F:\mathbb{R}^n\to\mathbb{R}^n$, the nonlinear complementarity problem, denoted by NCP(F), is to find a vector $\boldsymbol{x}\in\mathbb{R}^n$ satisfying

$$\boldsymbol{x}\geqslant\boldsymbol{0},\quad F(\boldsymbol{x})\geqslant\boldsymbol{0},\quad \boldsymbol{x}^{\mathrm{T}}F(\boldsymbol{x})=0.$$

Many solution methods developed for NCP(F) or related problems are based on reformulating them as a system of equations using so-called NCP-functions (see [12]). Here, a function $\phi:\mathbb{R}^2\to\mathbb{R}$ is called an NCP-function if

$$\phi(a,b)=0\Leftrightarrow a\geqslant 0,\quad b\geqslant 0,\quad ab=0.$$

Given an NCP-function ϕ, let us define

$$\Phi(\boldsymbol{x})=\left(\phi(x_1,F_1(\boldsymbol{x})),\ldots,\phi(x_n,F_n(\boldsymbol{x}))\right)^{\mathrm{T}}.$$

It is obvious that $\boldsymbol{x}\in\mathbb{R}^n$ is a solution of NCP(F) if and only if it solves the system of nonsmooth equations

$$\Phi(\boldsymbol{x})=\boldsymbol{0}.$$

For the solution to $\Phi(\boldsymbol{x})=\boldsymbol{0}$, we recall some definitions in nonsmooth analysis. Suppose that $\Theta:U\subseteq\mathbb{R}^{n_1}\to\mathbb{R}^{n_2}$ is a locally Lipschitz function, where U is nonempty and open. By Rademacher's theorem, Θ is differentiable almost everywhere. Let $D_\Theta\subseteq\mathbb{R}^{n_1}$ denote the set of points at which Θ is differentiable. For any $\boldsymbol{x}\in D_\Theta$, we write $J\Theta(\boldsymbol{x})$ for the usual $n_2\times n_1$ Jacobian matrix of partial derivatives. The *B-subdifferential* of Θ at $\boldsymbol{x}\in U$ is the set defined by

$$\partial_B\Theta(\boldsymbol{x})=\{V\in\mathbb{R}^{n_2\times n_1}\mid\exists\{\boldsymbol{x}^k\}\subseteq D_\Theta\text{ with }\boldsymbol{x}^k\to\boldsymbol{x},\ J\Theta(\boldsymbol{x}^k)\to V\}.$$

The Clarke's *generalized Jacobian* of Θ at $\boldsymbol{x}$ is the set defined by

$$\partial\Theta(\boldsymbol{x})=\mathrm{co}(\partial_B\Theta(\boldsymbol{x})),$$

where "co" denotes the convex hull. Then, $\partial\Theta(\boldsymbol{x})$ is a nonempty convex compact subset of $\mathbb{R}^{n_2\times n_1}$ (see [7]). The function Θ is *semismooth* (see [27]) at $\boldsymbol{x}\in\mathbb{R}^{n_1}$ if

$$\lim_{\substack{V\in\partial\Theta(\boldsymbol{x}+t\tilde{\boldsymbol{d}})\\ \tilde{\boldsymbol{d}}\to\boldsymbol{d},\,t\downarrow 0}} V\tilde{d}$$

exists for all $\boldsymbol{d}\in\mathbb{R}^{n_1}$. If Θ is semismooth at all $\boldsymbol{x}\in U$, we call Θ semismooth on U. The function Θ is called *strongly semismooth* (see [28]) if it is semismooth and for any $\boldsymbol{x}\in U$ and $V\in\partial\Theta(\boldsymbol{x}+t\boldsymbol{d})$,

$$Vd-\Theta'(\boldsymbol{x};\boldsymbol{d})=O(\|\boldsymbol{d}\|^2),\quad \boldsymbol{d}\to\boldsymbol{0},$$

where $\Theta'(\boldsymbol{x};\boldsymbol{d})$ denotes the directional derivative (see [3]) of Θ at $\boldsymbol{x}$ in direction $\boldsymbol{d}$, i.e.,

$$\Theta'(\boldsymbol{x};\boldsymbol{d})=\lim_{t\downarrow 0}\frac{\Theta(\boldsymbol{x}+t\boldsymbol{d})-\Theta(\boldsymbol{x})}{t}.$$

Note that if the function Θ is semismooth at $\boldsymbol{x}$, the directional derivative $\Theta'(\boldsymbol{x};\boldsymbol{d})$ exists for all $\boldsymbol{d}\in\mathbb{R}^{n_1}$ and

$$\Theta'(\boldsymbol{x};\boldsymbol{d})=\lim_{\substack{V\in\partial\Theta(\boldsymbol{x}+t\tilde{\boldsymbol{d}})\\ \tilde{\boldsymbol{d}}\to\boldsymbol{d},\,t\downarrow 0}} V\tilde{d}.$$

We now present some NCP-functions which are widely used in nonlinear complementarity problems (see [33]). For more details about NCP-functions and their smoothing approximations, one can refer to [26, 40] and the references therein.

Here, we give some well-known NCP-functions as follows:

- The minimum function:

$$\phi(a,b)=\min\{a,b\}.$$

- The Fischer-Burmeister function:

$$\phi_{FB}(a,b)=a+b-\sqrt{a^2+b^2}.$$

It has been shown that all these NCP-functions are globally Lipschitz continuous, directionally differentiable, and strongly semismooth (see [13, 34]). For example, the generalized gradient $\partial\phi_{FB}(a,b)$ of $\phi_{FB}(a,b)$ is equal to the set of all (v_a,v_b) such that

$$(v_a,v_b)=\begin{cases}\left(1-\dfrac{a}{\sqrt{a^2+b^2}},1-\dfrac{b}{\sqrt{a^2+b^2}}\right), & \text{if } (a,b)\neq(0,0),\\ (1-\xi,1-\varsigma), & \text{if } (a,b)=(0,0),\end{cases}$$

where (ξ,ς) is any vector satisfying $\xi^2+\varsigma^2\leqslant 1$.

In Section 4, we use the Fischer-Burmeister function to reformulate the TAVE (1.2) as a system of equations and then we propose an algorithm to solve the system of equations.

We now introduce the tensor complementarity problem which is first defined by Song and Qi [29].

Definition 2.3. Given any tensor $\mathcal{A}\in T(m,n)$ and vector $\boldsymbol{q}\in\mathbb{R}^n$, the tensor complementarity problem, denoted by TCP$(\mathcal{A},\boldsymbol{q})$, is to find a vector $\boldsymbol{x}\in\mathbb{R}^n$ satisfying

$$\boldsymbol{x}\geqslant\mathbf{0},\quad \mathcal{A}\boldsymbol{x}^{m-1}+\boldsymbol{q}\geqslant\mathbf{0},\quad \boldsymbol{x}^{\mathrm{T}}(\mathcal{A}\boldsymbol{x}^{m-1}+\boldsymbol{q})=0.$$

Note that when $n=2$, the tensor $\mathcal{A}$ reduces to a matrix, denoted by A, and the TCP$(\mathcal{A},\boldsymbol{q})$ becomes: find a vector $\boldsymbol{x}\in\mathbb{R}^n$ such that

$$\boldsymbol{x}\geqslant\mathbf{0},\quad A\boldsymbol{x}+\boldsymbol{q}\geqslant\mathbf{0},\quad \boldsymbol{x}^{\mathrm{T}}(A\boldsymbol{x}+\boldsymbol{q})=0,$$

which is just the linear complementarity problem (see [8]). Very recently, a class of n-person noncooperative games were given in [14] and the authors reformulated this problem as a tensor complementarity problem to handle. This is regarded as an application of tensor complementarity problems. Some semismooth Newton-type methods are recently proposed for solving the tensor complementarity problems (see [6]). More results on TCP can be found in [2, 5, 17–19, 29–32, 35] and the references therein. In Section 4, we extend the result (i) to the TAVE (1.2) and show that the TAVE (1.2) is equivalent to a bi-multilinear program and a generalized tensor complementarity problem.

3 Existence of solutions

In this section, we give some sufficient conditions for the existence of solutions to the TAVE (1.2). Specially, we extend the result (ii) about the AVE (2.1) to the TAVE (1.2).

We need the following lemmas which are recently established in [10, Theorems 3.2–3.4].

Lemma 3.1. *Let $\mathcal{A}\in T(m,n)$. If $\mathcal{A}$ is a strong M-tensor, then for every positive vector $\boldsymbol{b}$ the multilinear system of equations $\mathcal{A}\boldsymbol{x}^{m-1}=\boldsymbol{b}$ has a unique positive solution.*

Lemma 3.2. *Let $\mathcal{A}\in T(m,n)$ be a Z-tensor. Then it is a strong M-tensor if and only if the multilinear system of equations $\mathcal{A}\boldsymbol{x}^{m-1}=\boldsymbol{b}$ has a unique positive solution for every positive vector $\boldsymbol{b}$.*

Lemma 3.3. *Let $\mathcal{A}\in T(m,n)$ be an M-tensor and $\boldsymbol{b}\geqslant\mathbf{0}$. If there exists $\boldsymbol{v}\geqslant\mathbf{0}$ such that $\mathcal{A}\boldsymbol{v}^{m-1}\geqslant\boldsymbol{b}$, then the multilinear system of equations $\mathcal{A}\boldsymbol{x}^{m-1}=\boldsymbol{b}$ has a non-negative solution.*

By the above lemmas, we have the following theorems.

Theorem 3.4. *Let $\mathcal{A} \in T(m,n)$. If $\mathcal{A}$ can be written as $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ with $\mathcal{B} \geqslant 0$ and $c > \rho(\mathcal{B}) + 1$, then for every positive vector $\boldsymbol{b}$ the TAVE* (1.2) *has a unique positive solution.*

Proof. Let $s = c - 1$. Then $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ yields

$$\mathcal{A} - \mathcal{I} = s\mathcal{I} - \mathcal{B}, \quad \mathcal{B} \geqslant 0, \quad s > \rho(\mathcal{B}),$$

which implies that $\mathcal{A} - \mathcal{I}$ is a strong M-tensor. By Lemma 3.1, the multilinear system of equations

$$(\mathcal{A} - \mathcal{I})\boldsymbol{x}^{m-1} = \boldsymbol{b}$$

has a unique positive solution for every positive vector $\boldsymbol{b}$. Hence, for every positive vector $\boldsymbol{b}$, the TAVE (1.2) has a unique positive solution. □

Combining [9, Theorem 3] and Lemma 3.2, we can rewrite the above theorem into an equivalent condition for $\mathcal{A} - \mathcal{I}$ being a strong M-tensor.

Theorem 3.5. *Let $\mathcal{A} \in T(m,n)$ be a Z-tensor. Then $\mathcal{A}$ can be written as the form of*

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}, \quad \mathcal{B} \geqslant 0, \quad c > \rho(\mathcal{B}) + 1 \tag{3.1}$$

if and only if for every positive vector $\boldsymbol{b}$ the TAVE (1.2) *has a unique positive solution.*

Proof. On one hand, by Theorem 3.4, we have the existence and uniqueness of the positive solution of the TAVE (1.2) for every positive vector $\boldsymbol{b}$. On the other hand, if for every positive vector $\boldsymbol{b}$ the TAVE (1.2) has a unique positive solution, then there exists a vector $\boldsymbol{x} > \boldsymbol{0}$ such that

$$(\mathcal{A} - \mathcal{I})\boldsymbol{x}^{m-1} = \boldsymbol{b} > \boldsymbol{0}.$$

Since $\mathcal{A}$ is a Z-tensor, $\mathcal{A} - \mathcal{I}$ is also a Z-tensor. Thus, by [9, Theorem 3], $\mathcal{A} - \mathcal{I}$ is a strong M-tensor and then the form of (3.1) holds. □

Remark 3.6. The sufficient condition in Theorem 3.5 can be weakened as follows: if the TAVE (1.2) has a non-negative solution for every positive vector $\boldsymbol{b}$, then we also have the form (3.1). In fact, let $\boldsymbol{x} \geqslant \boldsymbol{0}$ be a solution of the TAVE (1.2). Then there exists $\boldsymbol{x} \geqslant \boldsymbol{0}$ such that $(\mathcal{A} - \mathcal{I})\boldsymbol{x}^{m-1} > \boldsymbol{0}$. By [9, Theorem 3], we can obtain the conclusion.

Theorem 3.7. *Let $\boldsymbol{b} \geqslant \boldsymbol{0}$ and $\mathcal{A} \in T(m,n)$ be in the form of $\mathcal{A} = c\mathcal{I} - \mathcal{B}$ with $\mathcal{B} \geqslant 0$ and $c = \rho(\mathcal{B}) + 1$. If there exists a vector $\boldsymbol{v} \geqslant \boldsymbol{0}$ such that $(\mathcal{A} - \mathcal{I})\boldsymbol{v}^{m-1} \geqslant \boldsymbol{b}$, then the TAVE* (1.2) *has a non-negative solution.*

Proof. It follows from

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}, \quad \mathcal{B} \geqslant 0, \quad c = \rho(\mathcal{B}) + 1$$

that $\mathcal{A} - \mathcal{I}$ is an M-tensor. By Lemma 3.3, there is $\boldsymbol{x}^* \geqslant \boldsymbol{0}$ such that

$$(\mathcal{A} - \mathcal{I})(\boldsymbol{x}^*)^{m-1} = \boldsymbol{b}.$$

Thus, we have

$$\mathcal{A}(\boldsymbol{x}^*)^{m-1} - |\boldsymbol{x}^*|^{[m-1]} = \boldsymbol{b}.$$

This completes the proof. □

Note that the conditions given in the above theorems only can guarantee the existence of non-negative solutions to a TAVE. We next extend the result (ii) about the AVE (2.1) to the TAVE (1.2). This condition can guarantee the existence of solutions to a TAVE. Here, we assume that m is even and $\mathcal{D} \in T(m,n)$ is a diagonal tensor with diagonal elements being 1 or -1. We first introduce the product of a tensor and a diagonal tensor.

Definition 3.8. Let $\mathcal{C} = (c_{i_1 i_2 \cdots i_m}) \in T(m,n)$ and $\mathcal{B} \in T(m,n)$ be a diagonal tensor with diagonal elements $b_{i\cdots i}$. We denote by $\mathcal{CB} = (a_{i_1 i_2 \cdots i_m})$ their product, whose elements are defined as

$$a_{i_1 i_2 \cdots i_m} = c_{i_1 i_2 \cdots i_m} (b_{i_2 \cdots i_2})^{\frac{1}{m-1}} \cdots (b_{i_m \cdots i_m})^{\frac{1}{m-1}}, \quad 1 \leqslant i_1, \ldots, i_m \leqslant n.$$

Obviously, Definition 3.8 is well-defined due to the assumption that m is even.

By a simplicity computation, we have the following proposition.

Proposition 3.9. *Let $\mathcal{C} = (c_{i_1 i_2 \cdots i_m}) \in T(m,n)$ and $\boldsymbol{x} \in \mathbb{R}^n$. Define a vector $\boldsymbol{u} \in \mathbb{R}^n$ as*

$$\boldsymbol{u} = (\mathcal{D}\boldsymbol{x}^{m-1})^{[\frac{1}{m-1}]}.$$

Then, we have

$$(\mathcal{CD})\boldsymbol{x}^{m-1} = \mathcal{C}\boldsymbol{u}^{m-1}.$$

Proof. By some definitions introduced in Section 1, the i-th component of the vector $\mathcal{C}\boldsymbol{u}^{m-1}$ can be written as

$$\begin{aligned}
(\mathcal{C}\boldsymbol{u}^{m-1})_i &= \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} c_{i i_2 \cdots i_m} u_{i_2} \cdots u_{i_m} \\
&= \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} c_{i i_2 \cdots i_m} (d_{i_2 \cdots i_2} x_{i_2}^{m-1})^{\frac{1}{m-1}} \cdots (d_{i_m \cdots i_m} x_{i_m}^{m-1})^{\frac{1}{m-1}} \\
&= \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} c_{i i_2 \cdots i_m} d_{i_2 \cdots i_2}^{\frac{1}{m-1}} \cdots d_{i_m \cdots i_m}^{\frac{1}{m-1}} x_{i_2} \cdots x_{i_m}.
\end{aligned} \tag{3.2}$$

Let $\mathcal{A} = \mathcal{CD}$. Then by Definition 3.8, the i-th component of the vector $(\mathcal{CD})\boldsymbol{x}^{m-1}$ can be written as

$$\begin{aligned}
((\mathcal{CD})\boldsymbol{x}^{m-1})_i &= \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \\
&= \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} c_{i i_2 \cdots i_m} d_{i_2 \cdots i_2}^{\frac{1}{m-1}} \cdots d_{i_m \cdots i_m}^{\frac{1}{m-1}} x_{i_2} \cdots x_{i_m}.
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$(\mathcal{CD})\boldsymbol{x}^{m-1} = \mathcal{C}\boldsymbol{u}^{m-1}.$$

This completes the proof. □

It is easy to see that

$$|\boldsymbol{x}|^{[m-1]} = \mathcal{D}\boldsymbol{x}^{m-1} \tag{3.4}$$

holds for any vector $\boldsymbol{x} \in \mathbb{R}^n$, because the i-th component of the vectors $|\boldsymbol{x}|^{[m-1]}$ and $\mathcal{D}\boldsymbol{x}^{m-1}$ are in the form of

$$(|\boldsymbol{x}|^{[m-1]})_i = |x_i|^{m-1}, \quad (\mathcal{D}\boldsymbol{x}^{m-1})_i = d_{i\cdots i} x_i^{m-1}.$$

Here, the sign of x_i corresponds to the diagonal element 1 or -1 of $\mathcal{D}$.

The following theorem is a generalization of the result (ii) from AVE to TAVE.

Theorem 3.10. *Let $\mathcal{C} \in T(m,n)$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\mathcal{A} = \mathcal{CD}$. If the multilinear system of equations*

$$(\mathcal{C} - \mathcal{I})\boldsymbol{z}^{m-1} = \boldsymbol{b}, \quad \boldsymbol{z} \geqslant \boldsymbol{0} \tag{3.5}$$

has a solution, then the tensor absolute value equation

$$\mathcal{A}\boldsymbol{x}^{m-1} - |\boldsymbol{x}|^{[m-1]} = \boldsymbol{b}$$

also has a solution.

Proof. Let $\boldsymbol{z}^*$ be the solution of the multilinear system of (3.5). Then we have

$$(\mathcal{C}-\mathcal{I})(\boldsymbol{z}^*)^{m-1}=\boldsymbol{b},\quad \boldsymbol{z}^*\geqslant \mathbf{0}. \tag{3.6}$$

Set

$$\mathcal{D}(\boldsymbol{x}^*)^{m-1}=(\boldsymbol{z}^*)^{[m-1]},\quad \boldsymbol{u}=(\mathcal{D}(\boldsymbol{x}^*)^{m-1})^{[\frac{1}{m-1}]}.$$

Then (3.6) can be rewritten as

$$\mathcal{C}\boldsymbol{u}^{m-1}-\mathcal{D}(\boldsymbol{x}^*)^{m-1}=\boldsymbol{b},$$

which, together with Proposition 3.9 and (3.4), implies that $\boldsymbol{x}^*$ is a solution of the tensor absolute value equation

$$\mathcal{A}\boldsymbol{x}^{m-1}-|\boldsymbol{x}|^{[m-1]}=\boldsymbol{b}.$$

Thus, we complete the proof. □

We give an example to verify the above theorem.

Example 3.11. Let $\mathcal{C}\in T(4,2)$ with $c_{1111}=c_{1222}=c_{2111}=c_{2222}=1$ and zero otherwise, and $\boldsymbol{b}=(8,8)^{\mathrm{T}}$. Consider the multilinear system of equations

$$(\mathcal{C}-\mathcal{I})\boldsymbol{z}^{m-1}=\boldsymbol{b}.$$

It is rewritten as

$$z_1^3=8,\quad z_2^3=8.$$

This implies that $\boldsymbol{z}^*=(2,2)^{\mathrm{T}}$ is a solution of

$$(\mathcal{C}-\mathcal{I})\boldsymbol{z}^{m-1}=\boldsymbol{b},\quad \boldsymbol{z}\geqslant\mathbf{0}.$$

Let $\mathcal{D}\in T(4,2)$ be a diagonal tensor with $d_{1111}=1$ and $d_{2222}=-1$. Then we have $\mathcal{A}\in T(4,2)$ with $a_{1111}=a_{2111}=1$, $a_{1222}=a_{2222}=-1$, and zero otherwise, i.e., $\mathcal{A}=\mathcal{C}\mathcal{D}$. By Theorem 3.10, $\boldsymbol{x}^*=(2,-2)^{\mathrm{T}}$ is just a solution of the tensor absolute value equation

$$\mathcal{A}\boldsymbol{x}^{m-1}-|\boldsymbol{x}|^{[m-1]}=\boldsymbol{b}. \tag{3.7}$$

We now verify the conclusion. We rewrite (3.7) as

$$\begin{cases} x_1^3-x_2^3-|x_1|^3=8,\\ -x_2^3+x_1^3-|x_2|^3=8. \end{cases}$$

By a simple computation, the above equation has a solution $x_1=2, x_2=-2$.

4 Reformulation and algorithm

In this section, we extend the result (i) from AVE to TAVE. We show that the TAVE (1.2) is equivalent to a bi-multiliear program and a generalized tensor complementarity problem. We first introduce the following definition.

Definition 4.1. Let $\mathcal{A}\in T(m,n)$ and $\boldsymbol{x},\boldsymbol{b}\in\mathbb{R}^n$. Define

$$F(\boldsymbol{x})=(\mathcal{A}+\mathcal{I})\boldsymbol{x}^{m-1}-\boldsymbol{b},\quad G(\boldsymbol{x})=(\mathcal{A}-\mathcal{I})\boldsymbol{x}^{m-1}-\boldsymbol{b}.$$

The generalized tensor complementarity problem is to find $\boldsymbol{x}\in\mathbb{R}^n$ satisfying

$$F(\boldsymbol{x})\geqslant\mathbf{0},\quad G(\boldsymbol{x})\geqslant\mathbf{0},\quad F(\boldsymbol{x})^{\mathrm{T}}G(\boldsymbol{x})=0. \tag{4.1}$$

We call the following nonlinear program as a bi-multiliear program:

$$0=\min\{F(\boldsymbol{x})^{\mathrm{T}}G(\boldsymbol{x})\,|\,F(\boldsymbol{x})\geqslant\mathbf{0},G(\boldsymbol{x})\geqslant\mathbf{0}\}. \tag{4.2}$$

Theorem 4.2. *Let $\mathcal{A} \in T(m,n)$ and $\boldsymbol{b} \in \mathbb{R}^n$. Then the TAVE (1.2) is equivalent to the generalized tensor complementarity problem (4.1) and the bi-multilinear program (4.2).*

Proof. Clearly, the generalized tensor complementarity problem (4.1) is equivalent to the bi-multilinear program (4.2), i.e., (4.1) $\Leftrightarrow$ (4.2).

We only need to prove (1.2) $\Leftrightarrow$ (4.2). In fact, $|\boldsymbol{x}|^{[m-1]} = |\boldsymbol{x}^{[m-1]}|$. Hence, we have

$$|\boldsymbol{x}|^{[m-1]} \leqslant \mathcal{A}\boldsymbol{x}^{m-1} - \boldsymbol{b} \Leftrightarrow (\mathcal{A}+\mathcal{I})\boldsymbol{x}^{m-1} - \boldsymbol{b} \geqslant \boldsymbol{0}, \quad (\mathcal{A}-\mathcal{I})\boldsymbol{x}^{m-1} - \boldsymbol{b} \geqslant \boldsymbol{0}.$$

This implies that $\boldsymbol{x}$ is a feasible solution of (4.2). Since

$$|\boldsymbol{x}|^{[m-1]} = \mathcal{A}\boldsymbol{x}^{m-1} - \boldsymbol{b} \Leftrightarrow ((\mathcal{A}+\mathcal{I})\boldsymbol{x}^{m-1} - \boldsymbol{b})^{\mathrm{T}}((\mathcal{A}-\mathcal{I})\boldsymbol{x}^{m-1} - \boldsymbol{b}) = 0,$$

we have

$$|\boldsymbol{x}|^{[m-1]} = \mathcal{A}\boldsymbol{x}^{m-1} - \boldsymbol{b} \Leftrightarrow 0 = \min\left\{F(\boldsymbol{x})^{\mathrm{T}}G(\boldsymbol{x}) \mid F(\boldsymbol{x}) \geqslant \boldsymbol{0}, G(\boldsymbol{x}) \geqslant \boldsymbol{0}\right\}.$$

This completes the proof. □

By the above theorem, in order to solve the TAVE (1.2), we propose an algorithm for solving the generalized tensor complementarity problem (4.1). Using the Fischer-Burmeister function ϕ_{FB}, we can reformulate (4.1) as the following equation:

$$H(\boldsymbol{x}) = \begin{pmatrix} \phi_{FB}(F_1(\boldsymbol{x}), G_1(\boldsymbol{x})) \\ \vdots \\ \phi_{FB}(F_n(\boldsymbol{x}), G_n(\boldsymbol{x})) \end{pmatrix} = \boldsymbol{0}.$$

Hence, $\boldsymbol{x}$ is a solution of (1.2) if and only if $H(\boldsymbol{x}) = \boldsymbol{0}$. Moreover, $H(\boldsymbol{x})$ is strongly semismooth since the composition of strongly semismooth function is again strongly semismooth (see [23]), and according to the Jacobian chain rule, we have the following result.

Theorem 4.3. *Let $\mathcal{A} \in S(m,n)$. Then the function $H(\boldsymbol{x})$ is strongly semismooth. Moreover, for any $\boldsymbol{x} \in \mathbb{R}^n$, we have*

$$\partial H(\boldsymbol{x}) \subseteq D_a(\boldsymbol{x})JF(\boldsymbol{x}) + D_b(\boldsymbol{x})JG(\boldsymbol{x}),$$

where $D_a(\boldsymbol{x}) = \mathrm{diag}(a_i(\boldsymbol{x}))$ and $D_b(\boldsymbol{x}) = \mathrm{diag}(b_i(\boldsymbol{x}))$ are diagonal matrices in $\mathbb{R}^{n\times n}$ with entries

$$(a_i(\boldsymbol{x}), b_i(\boldsymbol{x})) \in \partial\phi_{FB}(F_i(\boldsymbol{x}), G_i(\boldsymbol{x})),$$

where $\partial\phi_{FB}(F_i(\boldsymbol{x}), G_i(\boldsymbol{x}))$ denotes the set $\partial\phi_{FB}(a,b)$ with (a,b) being replaced by $(F_i(\boldsymbol{x}), G_i(\boldsymbol{x}))$, and $JF(\boldsymbol{x})$ and $JG(\boldsymbol{x})$ are given by

$$JF(\boldsymbol{x}) = (\mathcal{A}+\mathcal{I})\boldsymbol{x}^{m-2}, \quad JG(\boldsymbol{x}) = (\mathcal{A}-\mathcal{I})\boldsymbol{x}^{m-2}.$$

Here, for a tensor $\mathcal{T} = (t_{i_1\cdots i_m}) \in T(m,n)$ and a vector $\boldsymbol{x} \in \mathbb{R}^n$, let $\mathcal{T}\boldsymbol{x}^{m-2}$ be a matrix in $\mathbb{R}^{n\times n}$ whose (i,j)-th component is defined by

$$(\mathcal{T}\boldsymbol{x}^{m-2})_{ij} = \sum_{i_3}^{n}\cdots\sum_{i_m}^{n} t_{iji_3\cdots i_m} x_{i_3}\cdots x_{i_m}.$$

In order to propose an algorithm for the solution of $H(\boldsymbol{x}) = \boldsymbol{0}$, we define a merit function as

$$\Psi(\boldsymbol{x}) = \frac{1}{2}\|H(\boldsymbol{x})\|^2.$$

We present some properties of the merit function, which can be obtained by [7, Theorems 2.2.4 and 2.6.6].

Theorem 4.4. *Let $\mathcal{A} \in S(m,n)$. Then the merit function $\Psi(\boldsymbol{x})$ is continuously differentiable with*

$$\nabla\Psi(\boldsymbol{x}) = Q^{\mathrm{T}}H(\boldsymbol{x})$$

for any $Q \in \partial H(\boldsymbol{x})$.

We are now in the position to propose a Levenberg-Marquardt-type algorithm to solve the semismooth system of equations $H(\boldsymbol{x}) = \boldsymbol{0}$, which is an extension of the nonsmooth inexact Levenberg-Marquardt-type method in [11]. To ensure global convergence, a line search is performed to minimize the smooth merit function Ψ. Because the problem with data in a structure of tensor is large scale, and the inexact version is more suitable to the large-scale case [11], we have the following algorithm.

Algorithm 1 Inexact Levenberg-Marquardt-type method

Step 0. Given a starting vector $\boldsymbol{x}^0 \in \mathbb{R}^n$ and some scales $p > 2$, $0 < \beta < 1/2$, $\rho > 0$, $\epsilon \geqslant 0$. Set $k := 0$.

Step 1. If $\|H(\boldsymbol{x}^k)\| \leqslant \epsilon$, stop. Otherwise, compute $Q^k \in \partial H(\boldsymbol{x}^k)$.

Step 2. Find a solution $\boldsymbol{d}^k$ satisfying

$$((Q^k)^{\mathrm{T}} Q^k + \mu_k I)\boldsymbol{d} = -(Q^k)^{\mathrm{T}} H(\boldsymbol{x}^k) + \boldsymbol{r}^k, \tag{4.3}$$

where $\mu_k \geqslant 0$ is the Levenberg-Marquardt parameter. If the condition

$$\nabla\Psi(\boldsymbol{x}^k)^{\mathrm{T}}\boldsymbol{d}^k \leqslant -\rho\|\boldsymbol{d}^k\|^p$$

is not satisfied, set

$$\boldsymbol{d}^k = -\nabla\Psi(\boldsymbol{x}^k).$$

Step 3. Find the smallest integer $i^k \in \{0, 1, 2, \dots\}$ such that $t_k = 2^{-i^k}$ and

$$\Psi(\boldsymbol{x}^k + t_k\boldsymbol{d}^k) \leqslant \Psi(\boldsymbol{x}^k) + \beta t_k \nabla\Psi(\boldsymbol{x}^k)^{\mathrm{T}}\boldsymbol{d}^k.$$

Step 4. Set $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + t_k\boldsymbol{d}^k$, $k := k + 1$, and go to Step 1.

In what follows, we analyze the global convergence of Algorithm 1. We shall assume that Algorithm 1 produces an infinite sequence $\{\boldsymbol{x}^k\}$. By [11, Theorems 15 and 16], we immediately obtain the following theorems.

Theorem 4.5. *Assume that the sequence* $\{\mu_k\}$ *is bounded and that the sequence* $\{\boldsymbol{r}^k\}$ *satisfies*

$$\|\boldsymbol{r}^k\| \leqslant \alpha_k\|\nabla\Psi(\boldsymbol{x}^k)\|,$$

where $\{\alpha_k\}$ *is a sequence of numbers with* $0 < \alpha_k < 1$ *and* $\alpha_k \to 0$ *as* $k \to \infty$*. Then each accumulation point of* $\{\boldsymbol{x}^k\}$ *is a stationary point of* Ψ*.*

Theorem 4.6. *Let the assumptions of Theorem* 4.5 *hold. If one of the accumulation points of* $\{\boldsymbol{x}^k\}$*, denoted by* $\boldsymbol{x}^*$*, is an isolated solution of the TAVE* (1.2)*, then*

$$\lim_{k\to\infty} \boldsymbol{x}^k = \boldsymbol{x}^*.$$

In the implementation of Algorithm 1, the most intensive part is to compute the approximation solution of (4.3) with $\boldsymbol{r}^k = \boldsymbol{0}$ at the k-th iteration. We note that the system is always solvable. In fact, if $\mu_k > 0$, the matrix $(Q^k)^{\mathrm{T}}Q^k + \mu_k I$ is symmetric positive definite and hence (4.3) is surely solvable. If $\mu_k = 0$, the matrix $(Q^k)^{\mathrm{T}}Q^k + \mu_k I$ reduces to $(Q^k)^{\mathrm{T}}Q^k$, which is guaranteed to be only positive semidefinite. However, in this case, (4.3) reduces to the normal gradient equation $Q^k\boldsymbol{d} = -H(\boldsymbol{x}^k)$, which is therefore solvable. We now have to specify which element $Q^k \in \partial H(\boldsymbol{x}^k)$ we select at the k-th iteration. By Theorem 4.3, we have that an element of $\partial H(\boldsymbol{x}^k)$ can be obtained in the following way. Let

$$\Lambda = \{i : F_i(\boldsymbol{x}^k) = 0 = G_i(\boldsymbol{x}^k)\}$$

be the set of "degenerate indices" and define $\boldsymbol{z} \in \mathbb{R}^n$ to be a vector whose components z_i are 1 if $i \in \Lambda$ and 0 otherwise. Then, the matrix Q^k, defined by

$$Q^k = A(\boldsymbol{x}^k)JF(\boldsymbol{x}^k) + B(\boldsymbol{x}^k)JG(\boldsymbol{x}^k),$$

where A and B are $n \times n$ diagonal matrices whose i-th diagonal elements are given, respectively, by

$$A_{ii}(\boldsymbol{x}^k) = \begin{cases} 1 - \dfrac{F_i(\boldsymbol{x}^k)}{\sqrt{F_i^2(\boldsymbol{x}^k) + G_i^2(\boldsymbol{x}^k)}}, & \text{if } i \notin \Lambda, \\ 1 - \dfrac{\nabla F_i(\boldsymbol{x}^k)^{\mathrm{T}} z}{\sqrt{(\nabla F_i(\boldsymbol{x}^k)^{\mathrm{T}} z)^2 + (\nabla G_i(\boldsymbol{x}^k)^{\mathrm{T}} z)^2}}, & \text{if } i \in \Lambda, \end{cases}$$

and by

$$B_{ii}(\boldsymbol{x}^k) = \begin{cases} 1 - \dfrac{G_i(\boldsymbol{x}^k)}{\sqrt{F_i^2(\boldsymbol{x}^k) + G_i^2(\boldsymbol{x}^k)}}, & \text{if } i \notin \Lambda, \\ 1 - \dfrac{\nabla G_i(\boldsymbol{x}^k)^{\mathrm{T}} z}{\sqrt{(\nabla F_i(\boldsymbol{x}^k)^{\mathrm{T}} z)^2 + (\nabla G_i(\boldsymbol{x}^k)^{\mathrm{T}} z)^2}}, & \text{if } i \in \Lambda, \end{cases}$$

belongs to $\partial H(\boldsymbol{x}^k)$. In the next section, we compute Q^k as the formulation.

5 Numerical results

In this section, we present the numerical performance of Algorithm 1 for the TAVE (1.2). All codes were written using Matlab Version R2015b and Tensor Toolbox Version 2.6 (see [1]). The numerical experiments were done on a laptop with an Intel Core i7-4720HQ CPU (2.6GHz) and RAM of 7.89GB.

In the implementation of Algorithm 1, we set $\varepsilon = 10^{-6}$, $\rho = 10^{-10}$, $p = 2.1$, $\beta = 10^{-4}$ and the Levenberg-Marquardt parameter $\mu_k = 0.3$ $(k \in \mathbb{N})$. We also set a maximum iteration step for the algorithm, i.e., $N_{\max} = 300$.

The first numerical experiment focuses on the behavior of algorithm's iterations. We generate a random symmetric non-negative tensor $\mathcal{A} \in S_{6,8}$ and a random vector $\boldsymbol{x}^* \in \mathbb{R}^8$. All entries of $\mathcal{A}$ and $\boldsymbol{x}^*$ are uniform random numbers in the interval $[0,1]$. We calculate $\boldsymbol{b} = \mathcal{A}\boldsymbol{x}^{*m-1} - |\boldsymbol{x}^*|^{[m-1]}$ in order to make TAVE have at least one solution. Then we use Algorithm 1 to solve TAVE: $\mathcal{A}\boldsymbol{x}^{m-1} - |\boldsymbol{x}|^{[m-1]} = \boldsymbol{b}$, with a random initial point chosen randomly from $[0,1]^8$ which is shown as $\boldsymbol{x}^0$ in Table 1. After 13 iterations, Algorithm 1 can find the solution. The iteration of Algorithm 1 is shown in Table 1. From the table, we see that $\|H(\boldsymbol{x}^k)\|$ tends to 0 as the number of iterations k increases. In addition, $\|\nabla\Psi(\boldsymbol{x}^k)\|$ also tends to 0 except that it increases from $k = 2$ to $k = 4$. This shows that $\|\nabla\Psi(\boldsymbol{x}^k)\|$ does converge to 0 but not monotonically when the algorithm converges.

The second numerical experiment aims to verify Theorem 3.10. We first generate a random symmetric non-negative tensor $\mathcal{C} \in S(4,10)$ and a random vector

$$\boldsymbol{z}^* = (0.1040, 0.7455, 0.7363, 0.5619, 0.1842, 0.5972, 0.2999, 0.1341, 0.2126, 0.8949)^{\mathrm{T}} \in \mathbb{R}^{10}.$$

All entries of $\mathcal{C}$ are uniform random numbers in the interval $[0,1]$. Let

$$\boldsymbol{b} = (\mathcal{C} - \mathcal{I})\boldsymbol{z}^{*m-1}.$$

Table 1 Iterations of Algorithm 1 for a random tensor $\mathcal{A} \in S_{6,8}$ and the corresponding $\boldsymbol{b}$

k	$\boldsymbol{x}^k$	$\|H(\boldsymbol{x}^k)\|$	$\|\nabla\Psi(\boldsymbol{x}^k)\|$
0	$(0.8143, 0.2435, 0.9293, 0.3500, 0.1966, 0.2511, 0.6160, 0.4733)^{\mathrm{T}}$	562.2589	1500602.8826
1	$(0.7407, 0.2435, 0.6880, 0.3545, 0.1072, 0.3670, 0.4555, 0.3271)^{\mathrm{T}}$	148.1702	203193.6101
2	$(0.4542, 0.4477, 0.4349, 0.4348, -0.2944, 0.7819, 0.4209, 0.3007)^{\mathrm{T}}$	25.5263	23486.6536
3	$(1.0757, 0.3147, 0.2655, 0.4343, -0.2368, 0.4908, 0.1690, 0.3781)^{\mathrm{T}}$	20.2932	48494.4079
4	$(1.2158, 0.3379, 0.4481, 0.5825, -0.2075, 0.1750, 0.3290, 0.0197)^{\mathrm{T}}$	18.6526	49990.1630
5	$(0.8865, 0.3812, 0.3176, 0.5179, -0.3075, 0.3684, 0.4486, 0.2840)^{\mathrm{T}}$	10.0354	20912.3905
6	$(0.8742, 0.2928, 0.3744, 0.5308, -0.3895, 0.5269, 0.2838, 0.3997)^{\mathrm{T}}$	2.9292	3867.9206
7	$(0.8798, 0.2888, 0.3406, 0.6301, -0.3722, 0.4799, 0.3198, 0.3293)^{\mathrm{T}}$	1.3213	1522.0099
8	$(0.8664, 0.2829, 0.3003, 0.6746, -0.3890, 0.4936, 0.3325, 0.3355)^{\mathrm{T}}$	0.7455	1084.3075
9	$(0.8684, 0.2850, 0.2737, 0.6914, -0.3985, 0.4960, 0.3394, 0.3411)^{\mathrm{T}}$	0.1766	482.0095
10	$(0.8690, 0.2852, 0.2752, 0.6895, -0.3976, 0.4957, 0.3383, 0.3411)^{\mathrm{T}}$	0.0144	21.2907
11	$(0.8692, 0.2853, 0.2753, 0.6894, -0.3975, 0.4956, 0.3383, 0.3410)^{\mathrm{T}}$	0.0029	2.4370
12	$(0.8692, 0.2853, 0.2754, 0.6893, -0.3975, 0.4956, 0.3383, 0.3410)^{\mathrm{T}}$	0.0002	0.1396
13	$(0.8692, 0.2853, 0.2754, 0.6892, -0.3975, 0.4956, 0.3383, 0.3410)^{\mathrm{T}}$	0.0001	0.0008
14	$(0.8692, 0.2853, 0.2754, 0.6892, -0.3975, 0.4956, 0.3383, 0.3410)^{\mathrm{T}}$	0.0000	0.0000

Table 2 Diagonal elements of $\mathcal{D}_t$

t	diag of $\mathcal{D}_t$
1	$(-1,-1,-1,-1,-1,-1,-1,-1,-1,-1)^{\mathrm{T}}$
2	$(-1,1,-1,1,-1,1,-1,-1,-1,1)^{\mathrm{T}}$
3	$(1,1,-1,-1,1,1,-1,-1,-1,-1)^{\mathrm{T}}$
4	$(-1,1,-1,1,-1,1,-1,-1,1,1)^{\mathrm{T}}$
5	$(1,-1,1,1,1,1,-1,1,-1,1)^{\mathrm{T}}$

Table 3 Numerical results for tensors $\mathcal{A}_t$ with type-I initial points

t	$\boldsymbol{x}_t^*$	$\lVert H(\boldsymbol{x}_t^*)\rVert$	**Iter.**	**Time**
1	$(-0.3485,-0.0971,-0.7753,-1.2447,-0.7739,$ $-0.5628,-0.4868,0.4480,0.2925,-0.9003)^{\mathrm{T}}$	0.00000012	15	0.2135
2	$(0.4184,-0.0423,-0.2989,1.0357,-1.0340,$ $0.3109,-0.3686,-0.2755,-0.6852,0.9528)^{\mathrm{T}}$	0.00000022	15	0.2060
3	$(0.7454,0.5055,-0.6641,0.3093,-0.1769,$ $1.1273,-0.4514,-1.1430,-0.0619,-0.2421)^{\mathrm{T}}$	0.00000003	18	0.2673
4	$(-0.9570,0.5494,-2.1429,-0.1959,-1.8247,$ $-0.3996,0.8803,-0.3457,0.0458,0.1694)^{\mathrm{T}}$	0.00000000	11	0.1355
5	$(0.3385,-1.1498,1.0413,0.3533,0.7606,$ $-0.1214,-0.3290,-0.0458,-0.2049,0.4027)^{\mathrm{T}}$	0.00000006	10	0.1265

Table 4 Numerical results for tensors $\mathcal{A}_t$ with type-II initial points

t	$\boldsymbol{x}_t^*$	$\lVert H(\boldsymbol{x}_t^*)\rVert$	**Iter.**	**Time**
1	$(-0.1040,-0.7455,-0.7363,-0.5619,-0.1842,$ $-0.5972,-0.2999,-0.1341,-0.2126,-0.8949)^{\mathrm{T}}$	0.00000072	20	0.2523
2	$(-0.1040,0.7455,-0.7363,0.5619,-0.1842,$ $0.5972,-0.2999,-0.1341,-0.2126,0.8949)^{\mathrm{T}}$	0.00000090	17	0.2050
3	$(0.1040,0.7455,-0.7363,-0.5619,0.1842,$ $0.5972,-0.2999,-0.1341,-0.2126,-0.8949)^{\mathrm{T}}$	0.00000091	24	0.2838
4	$(-0.1040,0.7455,-0.7363,0.5619,-0.1842,$ $0.5972,-0.2999,-0.1341,0.2126,0.8949)^{\mathrm{T}}$	0.00000064	16	0.1896
5	$(0.1040,-0.7455,0.7363,0.5619,0.1842,$ $0.5972,-0.2999,0.1341,-0.2126,0.8949)^{\mathrm{T}}$	0.00000075	14	0.1638

Since $\mathcal{D}\in S(4,10)$ is a diagonal tensor whose diagonal elements are 1 or -1, there are at most $2^{10}=1{,}024$ different $\mathcal{D}$. The first attempt is to generate all these 1,024 tensors. For each tensor $\mathcal{D}_t$, set $\mathcal{A}_t=\mathcal{C}\mathcal{D}_t$ (see Definition 3.8) and $\boldsymbol{x}_t^*=(\mathcal{D}_t\boldsymbol{z}^{*m-1})^{[\frac{1}{m-1}]}$. We check whether $\mathcal{A}_t(\boldsymbol{x}_t^*)^{m-1}-|\boldsymbol{x}_t^*|^{[m-1]}$ is equal to $\boldsymbol{b}$ for all $t\in\{1,2,\ldots,1024\}$. The result shows that each $\boldsymbol{x}_t^*$ is just one of the solutions to the corresponding TAVE problem $\mathcal{A}_t\boldsymbol{x}^{m-1}-|\boldsymbol{x}|^{[m-1]}=\boldsymbol{b}$.

The second attempt of the second numerical experiment is to generate five $\mathcal{D}_t$ of all 1,024 tensors randomly and use Algorithm 1 to solve the corresponding TAVE. The diagonal elements of the five $\mathcal{D}_t$ is shown in Table 2.

We first select the initial points for Algorithm 1 by using normal distribution, i.e., entries are from standardized normal distribution $N(0,1)$ independently. Here, we call these initial points *type-I* initial points. The results of corresponding TAVE with *type-I* initial points are summarized in Table 3. We can easily find out that none of the five $\boldsymbol{x}_t^*$ is in the form of $(\mathcal{D}_t\boldsymbol{z}^{*m-1})^{[\frac{1}{m-1}]}$. Because Algorithm 1 is based on the thoughts of Newton method, thus its convergence relies heavily on the initial point. In order to detect the solution which is mentioned in Theorem 3.10 by Algorithm 1, we should choose the initial points

Table 5 A random symmetric non-negative tensor $\mathcal{B} = (b_{i_1 i_2 i_3 i_4}) \in S(4,4)$

$b_{1111} = 0.8147$	$b_{1112} = 0.9058$	$b_{1113} = 0.1270$	$b_{1114} = 0.9134$	$b_{1122} = 0.6324$
$b_{1123} = 0.0975$	$b_{1124} = 0.2785$	$b_{1133} = 0.5469$	$b_{1134} = 0.9575$	$b_{1144} = 0.9649$
$b_{1222} = 0.1576$	$b_{1223} = 0.9706$	$b_{1224} = 0.9572$	$b_{1233} = 0.4854$	$b_{1234} = 0.8003$
$b_{1244} = 0.1419$	$b_{1333} = 0.4218$	$b_{1334} = 0.9157$	$b_{1344} = 0.7922$	$b_{1444} = 0.9595$
$b_{2222} = 0.6557$	$b_{2223} = 0.0357$	$b_{2224} = 0.8491$	$b_{2233} = 0.9340$	$b_{2234} = 0.6787$
$b_{2244} = 0.7577$	$b_{2333} = 0.7431$	$b_{2334} = 0.3922$	$b_{2344} = 0.6555$	$b_{2444} = 0.1712$
$b_{3333} = 0.7060$	$b_{3334} = 0.0318$	$b_{3344} = 0.2769$	$b_{3444} = 0.0462$	$b_{4444} = 0.0971$

Table 6 The symmetric tensor $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in S(4,4)$ based on $\mathcal{B}$

$a_{1111} = 40.8037$	$a_{1112} = -0.9058$	$a_{1113} = -0.1270$	$a_{1114} = -0.9134$	$a_{1122} = -0.6324$
$a_{1123} = -0.0975$	$a_{1124} = -0.2785$	$a_{1133} = -0.5469$	$a_{1134} = -0.9575$	$a_{1144} = -0.9649$
$a_{1222} = -0.1576$	$a_{1223} = -0.9706$	$a_{1224} = -0.9572$	$a_{1233} = -0.4854$	$a_{1234} = -0.8003$
$a_{1244} = -0.1419$	$a_{1333} = -0.4218$	$a_{1334} = -0.9157$	$a_{1344} = -0.7922$	$a_{1444} = -0.9595$
$a_{2222} = 40.9627$	$a_{2223} = -0.0357$	$a_{2224} = -0.8491$	$a_{2233} = -0.9340$	$a_{2234} = -0.6787$
$a_{2244} = -0.7577$	$a_{2333} = -0.7431$	$a_{2334} = -0.3922$	$a_{2344} = -0.6555$	$a_{2444} = -0.1712$
$a_{3333} = 40.9124$	$a_{3334} = -0.0318$	$a_{3344} = -0.2769$	$a_{3444} = -0.0462$	$a_{4444} = 41.5213$

Table 7 Numerical results for the third experiment

$\boldsymbol{x}$	$\boldsymbol{b}$	**Iter.**	**Time**	$\max \Vert H(\boldsymbol{x})\Vert$	**Attempts**
$(0.8100, 0.7881, 0.7786, 0.8003)^{\mathrm{T}}$	$(1.4193, 0.2916, 0.1978, 1.5877)^{\mathrm{T}}$	31.00	0.6783	0.00000098	20/100
$(0.7285, 0.7212, 0.7156, 0.7098)^{\mathrm{T}}$	$(0.8045, 0.6966, 0.8351, 0.2437)^{\mathrm{T}}$	19.40	0.3109	0.00000099	20/157
$(0.7219, 0.7313, 0.7230, 0.7098)^{\mathrm{T}}$	$(0.2157, 1.1658, 1.1480, 0.1049)^{\mathrm{T}}$	19.55	0.3456	0.00000099	20/205
$(0.8453, 0.8603, 0.8294, 0.8276)^{\mathrm{T}}$	$(0.7223, 2.5855, 0.6669, 0.1873)^{\mathrm{T}}$	13.65	0.1907	0.00000082	20/244
$(0.8445, 0.8584, 0.8321, 0.8507)^{\mathrm{T}}$	$(0.0825, 1.9330, 0.4390, 1.7947)^{\mathrm{T}}$	14.05	0.2168	0.00000084	20/290
$(0.7104, 0.7055, 0.6849, 0.6957)^{\mathrm{T}}$	$(0.8404, 0.8880, 0.1001, 0.5445)^{\mathrm{T}}$	68.25	1.7492	0.00000051	20/145
$(0.6775, 0.6771, 0.6677, 0.6750)^{\mathrm{T}}$	$(0.3035, 0.6003, 0.4900, 0.7394)^{\mathrm{T}}$	21.75	0.3864	0.00000099	20/216
$(0.9021, 0.8787, 0.8894, 0.8805)^{\mathrm{T}}$	$(1.7119, 0.1941, 2.1384, 0.8396)^{\mathrm{T}}$	15.70	0.2535	0.00000089	20/114
$(0.8104, 0.8007, 0.7908, 0.7841)^{\mathrm{T}}$	$(1.3546, 1.0722, 0.9610, 0.1240)^{\mathrm{T}}$	14.60	0.2121	0.00000071	20/129
$(0.8957, 0.8939, 0.8661, 0.8808)^{\mathrm{T}}$	$(1.4367, 1.9609, 0.1977, 1.2078)^{\mathrm{T}}$	13.60	0.1980	0.00000099	20/114

Table 8 Solutions of TAVE when $\boldsymbol{b} = (-1, 1, 1, 1)$

$\boldsymbol{x}$	$\boldsymbol{b}$	**Iter.**	**Time**	$\max \Vert H(\boldsymbol{x})\Vert$	**Attempts**
$(0.0800, 0.3629, 0.3543, 0.3505)^{\mathrm{T}}$	$(-1, 1, 1, 1)^{\mathrm{T}}$	12.67	0.1644	0.00000070	3/20
$(-0.2593, 0.2948, 0.2891, 0.2903)^{\mathrm{T}}$	$(-1, 1, 1, 1)^{\mathrm{T}}$	13.67	0.1708	0.00000099	3/20
$(0.6258, 0.6600, 0.6522, 0.6537)^{\mathrm{T}}$	$(-1, 1, 1, 1)^{\mathrm{T}}$	11.93	0.1516	0.00000075	14/20

in another way. For each $\mathcal{D}_t$, we generate *type-II* initial points by adding a random number chosen from uniform distribution over $(-0.3, 0.3)$ to $(\mathcal{D}_t \boldsymbol{z}^{*m-1})^{[\frac{1}{m-1}]}$. The results of corresponding TAVE with *type-II* initial points are shown in Table 4. The solutions are exactly in the form of $(\mathcal{D}_t \boldsymbol{z}^{*m-1})^{[\frac{1}{m-1}]}$.

In Tables 3 and 4, t denotes the experiment number corresponding to Table 2, $\boldsymbol{x}_t^*$ denotes the solution vectors returned by Algorithm 1, and $\Vert H(\boldsymbol{x}_t^*)\Vert$ denotes the Euclid norm of $H(\boldsymbol{x}_t^*)$. If the norm $\Vert H(\boldsymbol{x}_t^*)\Vert$ is small enough, we can regard $\boldsymbol{x}_t^*$ as an approximate solution of TAVE. **Iter.** denotes the number of iterations and **Time** denotes the time of the iteration that finds corresponding solution $\boldsymbol{x}_t^*$ by Algorithm 1. In the second experiment, we verify Theorem 3.10 from the instant correctly. Besides, from Tables 3 and 4, we find that under the conditions of Theorem 3.10, the solution $(\mathcal{D}\boldsymbol{z}^{*m-1})^{[\frac{1}{m-1}]}$ may not be the only solution of TAVE $\mathcal{A}\boldsymbol{x}^{m-1} - |\boldsymbol{x}|^{[m-1]} = \boldsymbol{b}$. There might be some other solutions, such as the solution in Table 3. To discuss the uniqueness of the positive solution, we conduct the third experiment.

The third numerical experiment focuses on Theorem 3.5. Here, we first generate a random symmetric non-negative tensor $\mathcal{B}$ whose entries are uniform random numbers in the interval $[0, 1]$. Let

$$c = 1 + (1 + 0.01) \max_{1 \leqslant i \leqslant n} (\mathcal{B}e^3)_i,$$

where $e = (1, 1, 1, 1)^{\mathrm{T}}$. Since $\max_{1\leqslant i\leqslant n} (\mathcal{B}e^3)_i \geqslant \rho(\mathcal{B})$, the choice of c makes sure that $c > \rho(\mathcal{B})+1$. Then let $\mathcal{A} = c\mathcal{I} - \mathcal{B}$, and $\mathcal{A}$ satisfy the conditions of Theorem 3.5, i.e., $\mathcal{A} - \mathcal{I}$ is strong M-tensor. Tensors $\mathcal{B}$ and $\mathcal{A}$ are given in Tables 5 and 6, respectively.

We choose 10 random positive vectors $\boldsymbol{b} \in \mathbb{R}^4_+$. For each random positive vector $\boldsymbol{b}$, we find 20 repeatable solutions of TAVE: $\mathcal{A}\boldsymbol{x}^{m-1} - |\boldsymbol{x}|^{[m-1]} = \boldsymbol{b}$ with random vectors from $N(0, 1)^4$ as initial points repeatedly and summarize the results in Table 7.

In Table 7, $\boldsymbol{x}$ denotes the solution of TAVE, $\boldsymbol{b}$ denotes the random generated positive vector, **Iter.** denotes the average number of iterations that finds the corresponding solution successfully, and **Time** denotes the average time of the iteration that finds the corresponding solution by Algorithm 1. max $\|H(\boldsymbol{x})\|$ is the maximum norm of all $H(\boldsymbol{x})$ returned by Algorithm 1 whose $\boldsymbol{x}$ is the corresponding solution. **Attempts** has the form N/T, where N denotes the number of the corresponding $\boldsymbol{x}$ found by Algorithm 1 and T denotes the number of initial points in all.

In this experiment, we use "while" loop in Matlab program to guarantee that we can get exact 20 solutions (might be repeatable) for each $\boldsymbol{b} \geqslant \mathbf{0}$. According to Table 7, for each $\boldsymbol{b} \geqslant 0$, Algorithm 1 only returns a unique positive solution in all 20 repeatable solutions. This phenomenon fits Theorem 3.5 very well. Besides, in order to get 20 valid solutions, the initial points we attempt is 10 times more than the valid ones. This means that most of the random initial points fail to find a solution by Algorithm 1. The reason might be that the convergence of Newton type method depends on the initial point badly. Theorem 3.5 shows that under the circumstances, there is only one unique positive solution of TAVE. Only if the initial point is in the convergence region of some solution of TAVE, the algorithm will converge. Therefore, it is harder to find valid solutions if $\boldsymbol{b} \geqslant 0$. Table 8 shows the solutions found by Algorithm 1 when $\boldsymbol{b} = (-1, 1, 1, 1)$. The initial points attempted in all is much less.

Moreover, under the circumstances that $\boldsymbol{b} \geqslant 0$ and $\mathcal{A} - \mathcal{I}$ is strong M-tensor, whether the unique positive solution of TAVE is the unique solution of TAVE remains a question. In our experiment we have not found other solutions except for the unique positive ones.

6 Conclusion

We have introduced tensor absolute value equations. The simple definition is a natural generalization of the definition of absolute value equations in the matrix case. We have established some basic properties for tensor absolute value equations and reformulated tensor absolute value equations as a generalized tensor complementarity problem. We have proposed some sufficient conditions for the existence of solutions to the multilinear equations, and an inexact Levenberg-Marquardt-type method (see Algorithm 1) to solve the tensor absolute value equations, and some numerical results have shown that our algorithm performs well.

There are some questions which are still in study. For example, we known that the AVE (2.1) is uniquely solvable for any $\boldsymbol{b} \in \mathbb{R}^n$ if the singular values of A exceed 1 (see [22]). Can we extend the conclusion to TVAE (1.2), i.e., the statement "The TAVE (1.2) is uniquely solvable for any $\boldsymbol{b} \in \mathbb{R}^n$ if the singular values of tensor $\mathcal{A}$ exceed 1" is correct or not? This is still an open question.

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