Dilation theory and analytic model theory for doubly commuting sequences of $C_{\cdot 0}$-contractions

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Abstract It is known that every $C_{\cdot 0}$-contraction has a dilation to a Hardy shift. This leads to an elegant analytic functional model for $C_{\cdot 0}$-contractions, and has motivated lots of further works on the model theory and generalizations to commuting tuples of $C_{\cdot 0}$-contractions. In this paper, we focus on doubly commuting sequences of $C_{\cdot 0}$-contractions, and establish the dilation theory and the analytic model theory for these sequences of operators. These results are applied to generalize the Beurling-Lax theorem and Jordan blocks in the multivariable operator theory to the operator theory in the infinite-variable setting.

Keywords doubly commuting sequence, dilation theory, analytic functional model, Beurling-Lax theorem, Jordan block

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1 Introduction

1.1 Background

Suppose that T is a contraction on a Hilbert space $\mathcal{H}$, i.e., T is a bounded linear operator with $\|T\| \leqslant 1$. An operator V on a larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is said to be a dilation of T if for each $n \in \mathbb{N}$,

$$T^n = P_{\mathcal{H}} V^n \mid_{\mathcal{H}},$$

where $\mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. The Nagy-Foias dilation theory is of great significance in the operator theory, which builds functional models for contractions, not only revealing the structure of these operators, but also giving a way for calculations [50].

For a contraction T, we call

$$D_T = (I - T^*T)^{\frac{1}{2}}$$

the defect operator of T, and call

$$\mathfrak{D}_T = \overline{D_T \mathcal{H}}$$

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the defect space of T. A contraction T is said to be in the class $C_{\cdot 0}$ if $T^{*k} \to 0$ (as $k \to \infty$) in the strong operator topology, in which case $D_T \neq 0$. The functional model for $C_{\cdot 0}$-contractions was built by Rota [40], Rovnyak [41] and Helson [26] in a direct way. To be more specific, let $\mathbb{D}$ denote the open unit disk in the complex plane, and $H^2(\mathbb{D})$ the Hardy space over $\mathbb{D}$. Then a $C_{\cdot 0}$-contraction T is unitarily equivalent to the compression of the multiplication operator M_z, defined on the vector-valued Hardy space $H^2_{\mathfrak{D}_{T^*}}(\mathbb{D}) = H^2(\mathbb{D}) \otimes \mathfrak{D}_{T^*}$, on some invariant subspace $\mathcal{J}$ of $H^2_{\mathfrak{D}_{T^*}}(\mathbb{D})$ for the backward shift M_z^*. In another word, every contraction of the class $C_{\cdot 0}$ has a dilation to a Hardy shift. For the dilation of general contractions, we refer the readers to [50].

The existence of the isometric dilation of any commuting pair of contractions was proved by Andô [4]. However, commuting n-tuples of contractions have no isometric dilations in general when $n \geqslant 3$ [37]. Under some additional conditions, the above graceful analytic model for the single contraction of the class $C_{\cdot 0}$ can be generalized to the situation of a commuting finite-tuple $(T_1, \ldots, T_n)$ of $C_{\cdot 0}$-contractions. That is to say, this tuple $(T_1, \ldots, T_n)$ has a dilation to the tuple $(M_{\zeta_1}, \ldots, M_{\zeta_n})$ of coordinate multiplication operators on an $\mathcal{E}$-valued analytic function space $\mathcal{H}_{\mathcal{E}} = \mathcal{H} \otimes \mathcal{E}$ with $\mathcal{H}$ consisting of holomorphic functions over some domain in $\mathbb{C}^n$ [3,10,15,32,38]. The particular case where the tuple is doubly commuting is rather interesting since in this case the function space $\mathcal{H}$ is exactly the Hardy space $H^2(\mathbb{D}^n)$ over the n-polydisk, and the underlying space $\mathcal{E}$ is the defect space of the tuple $(T_1^*, \ldots, T_n^*)$. Moreover, we can require this dilation to be minimal and regular [10]. Recall that two operators T and S are said to be doubly commuting if $TS = ST$ and $T^*S = ST^*$, and a tuple or a sequence of operators is said to be doubly commuting if any pair of operators in it are doubly commuting. For more about developments on model theory, we refer the readers to [1,31,49,52].

A natural question arises: does a commuting sequence of $C_{\cdot 0}$-contractions, under the conditions analogous to those given in the above-mentioned papers, have a dilation to the tuple of coordinate multiplication operators on some vector-valued Hardy space over the infinite-dimensional polydisk? Except for the assumption of being "doubly commuting", most of these conditions do not carry over to infinitely many operators. For this reason, we focus on doubly commuting sequences of $C_{\cdot 0}$-contractions in this paper, and establish the dilation theory and the analytic model theory for these sequences of operators.

Another motivation for our study of such operator sequences comes from investigations on the structure of some special submodules and quotient modules of the Hardy module over the polydisk in [44,47], which are said to be doubly commuting.

The analytic Hilbert module theory developed by Douglas and Paulsen [19,20] opens a new door for the study of joint invariant subspaces of the Hardy space $H^2(\mathbb{D}^n)$ $(n \in \mathbb{N})$ for the tuple $\boldsymbol{M}_\zeta$ of coordinate multiplication operators. Let $\mathcal{P}_n$ denote the polynomial ring in n-complex variables. It is known that the Hardy space $H^2(\mathbb{D}^n)$ carries a $\mathcal{P}_n$-Hilbert module structure, where the module action is defined by multiplications by polynomials, and a submodule of $H^2(\mathbb{D}^n)$ is just an $\boldsymbol{M}_\zeta$-joint invariant subspace. A quotient module of $H^2(\mathbb{D}^n)$ is defined to be the orthocomplement of some submodule with a $\mathcal{P}_n$-module structure determined by the compression of $\boldsymbol{M}_\zeta$ on it. These notions are defined analogously on vector-valued Hardy spaces $H^2_{\mathcal{E}}(\mathbb{D}^n)$. For more details, we refer the readers to [12,21].

The famous theorem of Beurling [6] states that every nonzero submodule of $H^2(\mathbb{D})$ is of the form $\eta H^2(\mathbb{D})$ for some inner function $\eta \in H^\infty(\mathbb{D})$, where $H^\infty(\mathbb{D})$ denotes the space of bounded holomorphic functions on $\mathbb{D}$. Lax [29] generalized Beurling's theorem to vector-valued Hardy spaces $H^2_{\mathcal{E}}(\mathbb{R})$, and of course there is an equivalent version on $H^2_{\mathcal{E}}(\mathbb{D})$: a nonzero submodule of $H^2_{\mathcal{E}}(\mathbb{D})$ takes the form $\theta H^2_{\mathcal{F}}(\mathbb{D})$ with $\theta \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D})$ being inner, where $H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D})$ is the space of the uniformly bounded holomorphic $\mathcal{B}(\mathcal{F}, \mathcal{E})$-valued functions on $\mathbb{D}$. For the multivariable situation, such a Beurling-Lax type theorem fails in general [42]. Some efforts were made to determine when a submodule enjoys a Beurling-Lax type representation. In [30], Mandrekar considered the case of the scalar-valued Hardy space over the bidisk, and obtained a necessary and sufficient condition that the restriction of the tuple $\boldsymbol{M}_\zeta$ on the submodule is doubly commuting. The same conclusion was also obtained by Nakazi [33]. This characterization was further generalized to the vector-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D}^n)$ for arbitrary positive integer n in [47], and to the Hardy space over the infinite-dimensional polydisk within the language of the Hilbert space of

Dirichlet series with square-summable coefficients in [36].

A proper quotient module of $H^2(\mathbb{D})$ is called a model space, and the compression of the Hardy shift M_z on a model space is called a Jordan block. The notion of the Jordan block plays a central role in the model theory for operators of the class C_0: every C_0-operator is quasi-similar to the direct sum of Jordan blocks [50]. Following previous works on the Hardy quotient module over the bidisk [22, 23, 28], Sarkar [44] proved that if the compression of the tuple $\boldsymbol{M}_\zeta$ on a nonzero quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ is doubly commuting, then

$$\mathcal{Q} = \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_n,$$

where either $\mathcal{J}_i$ is a model space or $\mathcal{J}_i = H^2(\mathbb{D})$ $(1 \leqslant i \leqslant n)$. Thus, for any i so that $\mathcal{J}_i \neq H^2(\mathbb{D})$, the compression $P_\mathcal{Q} M_{\zeta_i}|_\mathcal{Q}$ is the tensor product of a Jordan block and an identity operator. Then the compression of the tuple $\boldsymbol{M}_\zeta$ on $\mathcal{Q}$ can be considered as the Jordan block in finitely many variables. We refer the readers to [27, 39, 46, 53, 54] for the related works and further discussions (see also [17, Remark 4.7]).

In this paper, we establish the dilation theory and the analytic model theory for doubly commuting sequences of $C_{\cdot 0}$-contractions, and then apply them to generalize the Beurling-Lax theorem for doubly commuting submodules and the Jordan block type characterization for doubly commuting quotient modules in the multivariable case to the Hardy module in infinitely many variables.

1.2 Main results

To state our main results, we need to introduce some notations and definitions. Let $\mathbb{D}^\infty$ denote the Cartesian product $\mathbb{D} \times \mathbb{D} \times \cdots$ of countably infinitely many unit disks. The Hilbert's multidisk $\mathbb{D}_2^\infty$ is defined to be

$$\mathbb{D}_2^\infty = \{\zeta = (\zeta_1, \zeta_2, \ldots) \in l^2 : |\zeta_n| < 1 \text{ for all } n \geqslant 1\}.$$

The Hardy space $H^2(\mathbb{D}_2^\infty)$ in infinitely many variables is defined as follows:

$$H^2(\mathbb{D}_2^\infty) = \left\{ F = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} c_\alpha \zeta^\alpha : \|F\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} |c_\alpha|^2 < \infty \right\},$$

where $\mathbb{Z}_+^{(\infty)}$ denotes the set of finitely supported sequences of nonnegative integers, and ζ^α denotes the monomial

$$\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in \mathbb{Z}_+^{(\infty)}$. The space $H^2(\mathbb{D}_2^\infty)$ is a reproducing kernel Hilbert space over Hilbert's multidisk $\mathbb{D}_2^\infty$ with the kernels [35]

$$\mathbf{K}_\lambda(\zeta) = \prod_{n=1}^\infty \frac{1}{1 - \overline{\lambda_n}\zeta_n}, \quad \lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{D}_2^\infty.$$

This space has close connections with the study of Beurling's completeness problem and Dirichlet series [7, 25, 35], and is the expected function space upon which we build analytic models. The vector-valued Hardy space $H_\mathcal{E}^2(\mathbb{D}_2^\infty)$ in infinitely many variables is defined analogously as in the finite-variable situation (see Section 2 for the details).

If $\boldsymbol{T} = (T_1, T_2, \ldots)$ is a doubly commuting sequence of contractions on a Hilbert space $\mathcal{H}$, then the infinite product

$$D_{\boldsymbol{T}} = (\text{SOT}) \lim_{n \to \infty} D_{T_1} \cdots D_{T_n} = (\text{SOT}) \lim_{n \to \infty} (I - T_1^* T_1)^{\frac{1}{2}} \cdots (I - T_n^* T_n)^{\frac{1}{2}}$$

of defect operators of $\{T_n\}_{n \in \mathbb{N}}$ converges [14, Proposition 43.1], and is called the *defect operator* of $\boldsymbol{T}$. Similarly, define the *defect space* $\mathfrak{D}_{\boldsymbol{T}}$ of $\boldsymbol{T}$ to be

$$\mathfrak{D}_{\boldsymbol{T}} = \overline{D_{\boldsymbol{T}}\mathcal{H}}.$$

Let $\mathcal{DC}(\mathcal{H})$ (or simply $\mathcal{DC}$ if no confusion is caused) denote the class of doubly commuting sequences of $C_{\cdot 0}$-contractions on $\mathcal{H}$. For convenience, we identify the class of doubly commuting finite-tuples of $C_{\cdot 0}$-contractions with the class $\mathcal{DCF}$ of sequences in the class $\mathcal{DC}$ with only finitely many nonzero components. Let $\boldsymbol{T}^*$ denote the sequence $(T_1^*, T_2^*, \ldots)$ and $\boldsymbol{T}^\alpha$ the operator $T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ for $\boldsymbol{T} = (T_1, T_2, \ldots)$ and $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in \mathbb{Z}_+^{(\infty)}$ ($n \in \mathbb{N}$). The key ingredient in the dilation theory for doubly commuting finite-tuples of $C_{\cdot 0}$-contractions is the following norm identity [10]:

$$\|x\|^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \|D_{\boldsymbol{T}^*} \boldsymbol{T}^{*\alpha} x\|^2, \quad x \in \mathcal{H}, \tag{1.1}$$

where $n \in \mathbb{N}$ and $\boldsymbol{T} = (T_1, \ldots, T_n) \in \mathcal{DCF}$. It is natural to expect that such an identity holds for sequences in the class $\mathcal{DC}$ in the following sense: for each $x \in \mathcal{H}$ and $\boldsymbol{T} = (T_1, T_2, \ldots) \in \mathcal{DC}$,

$$\|x\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\boldsymbol{T}^*} \boldsymbol{T}^{*\alpha} x\|^2.$$

Unfortunately, the answer is negative in general. We see that there exists a sequence $\boldsymbol{T} \in \mathcal{DC}$ such that the defect operator $D_{\boldsymbol{T}^*}$ of $\boldsymbol{T}^*$ is 0, which cannot occur in the finite-tuple case.

The class of doubly commuting sequences of pure isometries on $\mathcal{H}$ is denoted by $\mathcal{DI}(\mathcal{H})$ (or $\mathcal{DI}$). Since an isometry V is of the class $C_{\cdot 0}$ if and only if V is pure (that is to say, the unitary part in the Wold decomposition of V is 0), $\mathcal{DI}$ is a subclass of $\mathcal{DC}$. Any doubly commuting sequence of contractions has a minimal, doubly commuting, regular isometric dilation, which is unique up to unitary equivalence [48]. Furthermore, we show that the minimal regular isometric dilation of a sequence in the class $\mathcal{DC}$ consists of doubly commuting pure isometries. This therefore provides us with an approach to the question raised in the previous subsection via the study of the sequences in the class $\mathcal{DI}$. These notions concerning the dilation will be explained in Section 2.

Unlike the finite-tuple case, sequences in the class $\mathcal{DI}$ require further classification. Let us start with some notations and definitions. For a family $\mathcal{T}$ of bounded linear operators on $\mathcal{H}$ and a subset E of $\mathcal{H}$, let $[E]_{\mathcal{T}}$ denote the joint invariant subspace for $\mathcal{T}$ generated by E. In particular, if $\boldsymbol{T}$ is a sequence of operators, then one has

$$[E]_{\boldsymbol{T}} = \bigvee_{\alpha \in \mathbb{Z}_+^{(\infty)}} \boldsymbol{T}^\alpha E,$$

where the notation $\bigvee$ denotes the closed linear span of subsets of a Hilbert space. Following [24], if a closed subspace M of $\mathcal{H}$ is orthogonal to $\boldsymbol{T}^\alpha M$ for any $\alpha \in \mathbb{Z}_+^{(\infty)} \setminus \{0\}$, then M is called a *wandering subspace* for the sequence $\boldsymbol{T}$ (see [8] for an analogous definition in the finite-tuple case).

Suppose $\boldsymbol{V} = (V_1, V_2, \ldots) \in \mathcal{DI}$. It is easy to verify that the defect space

$$\mathfrak{D}_{\boldsymbol{V}^*} = \bigcap_{n=1}^{\infty} \operatorname{Ker} V_n^*,$$

and $\mathfrak{D}_{\boldsymbol{V}^*}$ is a wandering subspace for $\boldsymbol{V}$. It follows from Beurling's theorem that if $\mathcal{I}$ is an invariant subspace of $H^2(\mathbb{D})$ for the Hardy shift M_z, then the wandering subspace for the isometry $M_{\mathcal{I}} = M_z|_{\mathcal{I}}$ always generates the entire $\mathcal{I}$, i.e.,

$$[\mathfrak{D}_{M_{\mathcal{I}}^*}]_{M_{\mathcal{I}}} = [\mathcal{I} \ominus z\mathcal{I}]_{M_{\mathcal{I}}} = \mathcal{I}.$$

The conclusion is indeed valid for the vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D})$ by Halmos's observation in [24] or the Beurling-Lax theorem. This suggests the following definition.

Definition 1.1.   Suppose $\boldsymbol{V} \in \mathcal{DI}(\mathcal{H})$. The sequence $\boldsymbol{V}$ is said to be of Beurling type if $[\mathfrak{D}_{\boldsymbol{V}^*}]_{\boldsymbol{V}} = \mathcal{H}$.

Suppose $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$. The sequence $\boldsymbol{T}$ is said to be of Beurling type if its minimal regular isometric dilation is of Beurling type.

It was shown in [45] (also see [11]) that every doubly commuting finite-tuple of pure isometries enjoys a "Beurling type" property. The tuple

$$\boldsymbol{M}_\zeta = (M_{\zeta_1}, M_{\zeta_2}, \ldots)$$

of coordinate multiplication operators on a vector-valued Hardy space

$$H_{\mathcal{E}}^2(\mathbb{D}_2^\infty) = H^2(\mathbb{D}_2^\infty) \otimes \mathcal{E}$$

is clearly of Beurling type. More interestingly, we see that the converse also holds: if a sequence $\boldsymbol{V} \in \mathcal{DI}$ is of Beurling type, then $\boldsymbol{V}$ must be of the above form. Then the question reduces to the characterization for sequences in the class $\mathcal{DC}$ which are of Beurling type.

Let φ_a $(a \in \mathbb{D})$ denote the holomorphic automorphism

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}$$

of $\mathbb{D}$. It is known that the Riesz functional calculus

$$\varphi_a(T) = (aI - T)(I - \bar{a}T)^{-1}, \quad a \in \mathbb{D}$$

of a contraction T on the Hilbert space $\mathcal{H}$ is also a contraction. Furthermore, if $T \in C_{\cdot 0}$, then $\varphi_a(T) \in C_{\cdot 0}$ by the dilation theory for a single $C_{\cdot 0}$-contraction. For $\boldsymbol{T} = (T_1, T_2, \ldots) \in \mathcal{DC}$ and $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{D}^\infty$, set

$$\Phi_\lambda(\boldsymbol{T}) = (\varphi_{\lambda_1}(T_1), \varphi_{\lambda_2}(T_2), \ldots).$$

Obviously, for any $\lambda \in \mathbb{D}^\infty$, Φ_λ defines a bijection from $\mathcal{DC}$ onto itself, and $\Phi_0 = \Phi_\lambda \circ \Phi_\lambda = \mathrm{id}_{\mathcal{DC}}$. Therefore, for a sequence $\boldsymbol{T}$ of operators, $\boldsymbol{T} \in \mathcal{DC}$ if and only if $\Phi_\lambda(\boldsymbol{T}) \in \mathcal{DC}$. It is also easy to see that Φ_λ maps $\mathcal{DI}$ onto itself. We also write $\Phi_\lambda^\alpha(\boldsymbol{T})$ for the operator $\varphi_{\lambda_1}^{\alpha_1}(T_1) \cdots \varphi_{\lambda_n}^{\alpha_n}(T_n)$, where $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in \mathbb{Z}_+^{(\infty)}$ $(n \in \mathbb{N})$.

There are various examples of sequences $\boldsymbol{V}$ in the class $\mathcal{DI}$ satisfying the wandering space $\mathfrak{D}_{\boldsymbol{V}^*} = \{0\}$, while the defect space $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}$ of $\Phi_\lambda(\boldsymbol{V})^*$ is nonzero. For example, let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of nonconstant inner functions in $H^\infty(\mathbb{D})$ and consider the sequence $\boldsymbol{V} = (M_{\widetilde{\eta_1}}, M_{\widetilde{\eta_2}}, \ldots)$ of multiplication operators on the Hardy space $H^2(\mathbb{D}_2^\infty)$, $\widetilde{\eta_n}(\zeta) = \eta_n(\zeta_n)$ $(n \in \mathbb{N}, \zeta \in \mathbb{D}_2^\infty)$. We prove that for "almost all" choices of the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ of inner functions, the wandering space $\mathfrak{D}_{\boldsymbol{V}^*}$ for the $\mathcal{DI}$-sequence $\boldsymbol{V}$ is $\{0\}$, and $\Phi_\lambda(\boldsymbol{V})$ is of Beurling type for some $\lambda \in \mathbb{D}^\infty$ (see Proposition 3.4 and Remark 3.5). Therefore, for those $\mathcal{DI}$-sequences which are not of Beurling type, the family of maps Φ_λ $(\lambda \in \mathbb{D}^\infty)$ could be a powerful tool in building analytic models. We thus obtain important information of such $\mathcal{DI}$-sequences $\boldsymbol{V}$ from the Beurling type sequences $\Phi_\lambda(\boldsymbol{V})$ for some $\lambda \in \mathbb{D}^\infty$.

Inspired by this, we consider the following sequences in the class $\mathcal{DI}$.

Definition 1.2. Suppose $\boldsymbol{V} \in \mathcal{DI}$. The sequence $\boldsymbol{V}$ is said to be of quasi-Beurling type if $\Phi_\lambda(\boldsymbol{V})$ is of Beurling type for some $\lambda \in \mathbb{D}^\infty$.

Suppose $\boldsymbol{T} \in \mathcal{DC}$. The sequence $\boldsymbol{T}$ is said to be of quasi-Beurling type if its minimal regular isometric dilation is of quasi-Beurling type.

Sequences of quasi-Beurling type is relatively tractable in the class $\mathcal{DI}$. In Section 3, we show that a sequence $\boldsymbol{V} \in \mathcal{DI}$ is of quasi-Beurling type if and only if $\boldsymbol{V}$ is jointly unitarily equivalent to the sequence $\Phi_\lambda(\boldsymbol{M}_\zeta)$ on a vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ for some $\lambda \in \mathbb{D}^\infty$. Recall that two sequences $\boldsymbol{T} = (T_1, T_2, \ldots)$ and $\boldsymbol{S} = (S_1, S_2, \ldots)$ of operators, defined on $\mathcal{H}$ and $\mathcal{K}$, respectively, are said to be jointly unitarily equivalent if there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that

$$S_n = UT_nU^*, \quad n \in \mathbb{N}.$$

The first main result in this paper is to give a complete characterization of sequences in the class $\mathcal{DC}$ that can be decomposed into direct sums of sequences of quasi-Beurling type. Note that for a commuting

sequence of operators on $\mathcal{H}$, by using a standard argument involving Zorn's lemma, one can decompose $\mathcal{H}$ into orthogonal direct sums of separable $\boldsymbol{T}$-joint reducing subspaces. It suffices to restrict our study to the case of separable Hilbert spaces. From now on, we only consider separable Hilbert spaces.

A sequence $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ is said to have *a decomposition of quasi-Beurling type* if there exists an orthogonal decomposition $\mathcal{H} = \bigoplus_\gamma \mathcal{H}_\gamma$ of the Hilbert space $\mathcal{H}$, such that each $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing and each $\boldsymbol{T}|_{\mathcal{H}_\gamma}$ is of quasi-Beurling type.

Theorem 1.3.   *Suppose $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$. The following are equivalent*:

(1) *$\boldsymbol{T}$ has a decomposition of quasi-Beurling type*;

(2) *there exists an at most countable subset Λ of $\mathbb{D}^\infty$, such that for each $x \in \mathcal{H}$*,

$$\|x\|^2 = \sum_{\lambda \in \Lambda} \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\Phi_\lambda(\boldsymbol{T})^*} \Phi_\lambda^\alpha(\boldsymbol{T})^* x\|^2;$$

(3) $\bigvee_{\lambda \in \mathbb{D}^\infty} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*} = \mathcal{H}$.

The following particular case of Theorem 1.3 completely answers the question when a doubly commuting sequence of $C_{\cdot 0}$-contractions has a regular dilation to the tuple of coordinate multiplication operators on some vector-valued Hardy space over the infinite-dimensional polydisk.

Corollary 1.4.   *Suppose $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$. The following are equivalent*:

(1) *$\boldsymbol{T}$ is of Beurling type*;

(2) *the minimal regular isometric dilation of $\boldsymbol{T}$ is jointly unitarily equivalent to the tuple $\boldsymbol{M}_\zeta$ of coordinate multiplication operators on a vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$*;

(3) *for each $x \in \mathcal{H}$, $\|x\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\boldsymbol{T}^*} \boldsymbol{T}^{*\alpha} x\|^2$*;

(4) $\bigvee_{\lambda \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*} = \mathcal{H}$.

Note that for $\boldsymbol{T} \in \mathcal{DCF}$, since Corollary 1.4(3) coincides with the identity (1.1), Corollary 1.4 actually generalizes the finite-tuple case.

Here are some remarks for Theorem 1.3. If $\boldsymbol{T} \in \mathcal{DC}$ with a decomposition $\boldsymbol{T} = \bigoplus_\gamma \boldsymbol{T}_\gamma$ of quasi-Beurling type, then there correspond a point $\lambda_\gamma \in \mathbb{D}^\infty$ and a Hilbert space $\mathcal{E}_\gamma$ to each index γ, such that $\boldsymbol{T}_\gamma$ is jointly unitarily equivalent to $P_{\mathcal{Q}_\gamma} \Phi_{\lambda_\gamma}(\boldsymbol{M}_\zeta)|_{\mathcal{Q}_\gamma}$, where the sequence $\Phi_{\lambda_\gamma}(\boldsymbol{M}_\zeta)$ is defined on the vector-valued Hardy space $H_{\mathcal{E}_\gamma}^2(\mathbb{D}_2^\infty)$, and $\mathcal{Q}_\gamma \subseteq H_{\mathcal{E}_\gamma}^2(\mathbb{D}_2^\infty)$ is an $\boldsymbol{M}_\zeta^*$-joint invariant subspace. This gives

$$\boldsymbol{T} \cong \bigoplus_\gamma P_{\mathcal{Q}_\gamma} \Phi_{\lambda_\gamma}(\boldsymbol{M}_\zeta)|_{\mathcal{Q}_\gamma}, \tag{1.2}$$

and we therefore build an analytic model for a sequence $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ under the assumption that the subset

$$\{D_{\Phi_\lambda(\boldsymbol{T})^*} x : \lambda \in \mathbb{D}^\infty, x \in \mathcal{H}\}$$

is complete in $\mathcal{H}$. Note that this assumption always holds for the finite-tuple case (see Lemma 2.7). Condition (2) in Theorem 1.3 further generalizes the identity (1.1) (see Section 3 for more details). Also, the following result illustrates some extreme phenomenon in the infinite-tuple case different from the finite-tuple case (see Lemma 2.7).

Theorem 1.5.   *There exists a sequence $\boldsymbol{V} \in \mathcal{DI}$ such that $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*} = \{0\}$ for each $\lambda \in \mathbb{D}^\infty$.*

We further refine the representation (1.2) by giving a characterization of the subspaces $\mathcal{Q}_\gamma$ involving characterization functions of operators in $\boldsymbol{T}_\gamma$. We give the details in Section 4. This generalizes the results in [10] to the infinite-variable case.

In Section 4, we prove that every sequence in the class $\mathcal{DI}$ is jointly unitarily equivalent to a sequence of multiplication operators induced by operator-valued inner functions each of which involves one different variable (see Theorem 4.1). Thus we establish operator-valued analytic functional models for general sequences in the class $\mathcal{DC}$, which generalize (1.2). We also have the following application of our results.

Corollary 1.6.   *Suppose $\boldsymbol{T} \in \mathcal{DC}$. Then there exists a sequence $\{B_n\}_{n \in \mathbb{N}}$ of finite Blaschke products, such that $\{\prod_{i=1}^n B_i(T_i)\}_{n \in \mathbb{N}}$ converges in the strong operator topology.*

In the rest of this paper, we use the language of Hilbert module by Douglas and Paulsen [20] to state the generalization of the Beurling-Lax theorem and results related with Jordan blocks in the finite-variable case to the case in infinitely many variables.

Let $\mathcal{P}_\infty$ denote the polynomial ring in countably infinitely many complex variables, in which each polynomial only involves finitely many variables. Similar to the finite-variable case, for each commuting sequence $\boldsymbol{T}$ of operators on a Hilbert space $\mathcal{H}$, one defines a $\mathcal{P}_\infty$-module structure on $\mathcal{H}$ by

$$ph = p(\boldsymbol{T})h, \quad p \in \mathcal{P}_\infty, \quad h \in \mathcal{H}.$$

We say that this $\mathcal{P}_\infty$-module $\mathcal{H}$ is *doubly commuting* if the sequence $\boldsymbol{T}$ is doubly commuting. Conversely, any $\mathcal{P}_\infty$-module structure is determined by a commuting sequence of operators in the above way.

We can also define the Hardy module structure on a vector-valued Hardy space $H^2_\mathcal{E}(\mathbb{D}_2^\infty)$ in infinitely many variables via the tuple $\boldsymbol{M}_\zeta$ of coordinate multiplication operators on $H^2_\mathcal{E}(\mathbb{D}_2^\infty)$. The module action is the multiplication by polynomials and submodules of $H^2_\mathcal{E}(\mathbb{D}_2^\infty)$ are exactly joint invariant subspaces for $\boldsymbol{M}_\zeta$. By definition, a submodule $\mathcal{S}$ of $H^2_\mathcal{E}(\mathbb{D}_2^\infty)$ is doubly commuting if the restriction

$$(M_{\zeta_1}|_\mathcal{S}, M_{\zeta_2}|_\mathcal{S}, \dots)$$

of $\boldsymbol{M}_\zeta$ on $\mathcal{S}$ is doubly commuting; a quotient module $\mathcal{Q}$ of $H^2_\mathcal{E}(\mathbb{D}_2^\infty)$ is doubly commuting if the compression

$$(P_\mathcal{Q} M_{\zeta_1}|_\mathcal{Q}, P_\mathcal{Q} M_{\zeta_2}|_\mathcal{Q}, \dots)$$

of $\boldsymbol{M}_\zeta$ on $\mathcal{Q}$ is doubly commuting. We prove that such doubly commuting restrictions and compressions are of Beurling type. As a consequence, we have the following theorem.

Theorem 1.7. *Let $\mathcal{S}$ be a submodule of the vector-valued Hardy module $H^2_\mathcal{E}(\mathbb{D}_2^\infty)$. Then $\mathcal{S}$ is doubly commuting if and only if there exist a Hilbert space $\mathcal{F}$ and an inner function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}_2^\infty)$, such that*

$$\mathcal{S} = \Psi H^2_\mathcal{F}(\mathbb{D}_2^\infty).$$

See Subsection 2.2 for a definition of inner functions in infinitely many variables.

Theorem 1.8. *Every doubly commuting quotient module of $H^2(\mathbb{D}_2^\infty)$ is the tensor product of some sequence of quotient modules of $H^2(\mathbb{D})$.*

Theorem 1.7 is an infinite-variable version of the Beurling-Lax theorem for doubly commuting Hardy submodules. Also from Theorem 1.8, the compression of the tuple of coordinate multiplication operators on a nontrivial doubly commuting quotient module of $H^2(\mathbb{D}_2^\infty)$ is a Jordan block in infinitely many variables (see Section 5 for the details).

Evidences (for example, Theorem 1.5) have shown that compared with the finite-variable case, the infinite-variable case is much more complicated, and there are some essential difficulties in treating this case. For this, we develop new methods and techniques in this paper, which are quite different from those in [10, 36, 44]. Also, our treatments are valid for nonseparable Hilbert spaces.

Finally, we mention that our ideas and techniques used to treat doubly commuting sequences have proven to be powerful in studying the cyclic vector problem in $H^2(\mathbb{D}_2^\infty)$, which is an analytic function space version of the classical and long-standing Beurling-Wintner problem [25, 35]. For example, in [16] we proved that a composition operator defined by some sequence of inner functions with mutually independent variables preserves the cyclicity in $H^2(\mathbb{D}_2^\infty)$.

2 Some preparation

In this section, we first establish a lemma to guarantee the validity of Definitions 1.1 and 1.2. Then we list some basic properties of vector-valued Hardy spaces and operator-valued functions. Lastly, the rest of this section is dedicated to some preparation for the proofs of main results in the subsequent sections.

2.1 The minimal regular isometric dilation

Suppose that the sequence $\boldsymbol{T} = (T_1, T_2, \ldots)$ is a doubly commuting sequence of contractions on $\mathcal{H}$. A sequence $\boldsymbol{V} = (V_1, V_2, \ldots)$ of isometries, defined on a larger Hilbert space $\mathcal{K}_{\boldsymbol{V}} \supseteq \mathcal{H}$, is an isometric dilation of $\boldsymbol{T}$ if for each $\alpha \in \mathbb{Z}_+^{(\infty)}$,

$$\boldsymbol{T}^\alpha = P_{\mathcal{H}} \boldsymbol{V}^\alpha \mid_{\mathcal{H}}.$$

Furthermore, if for $\alpha, \beta \in \mathbb{Z}_+^{(\infty)}$ satisfying $\alpha \wedge \beta = (0, 0, \ldots)$,

$$\boldsymbol{T}^{*\alpha} \boldsymbol{T}^\beta = P_{\mathcal{H}} \boldsymbol{V}^{*\alpha} \boldsymbol{V}^\beta \mid_{\mathcal{H}},$$

then the isometric dilation $\boldsymbol{V}$ of $\boldsymbol{T}$ is said to be regular, where

$$\alpha \wedge \beta = (\min\{\alpha_1, \beta_1\}, \min\{\alpha_2, \beta_2\}, \ldots).$$

An isometric dilation $\boldsymbol{V}$ of $\boldsymbol{T}$ is said to be minimal if the $\boldsymbol{V}$-joint invariant subspace $[\mathcal{H}]_{\boldsymbol{V}}$ generated by $\mathcal{H}$ is $\mathcal{K}_{\boldsymbol{V}}$. It is clear to see that for any regular isometric dilation $\boldsymbol{V}$ of $\boldsymbol{T}$, the restriction $\boldsymbol{V} \mid_{[\mathcal{H}]_{\boldsymbol{V}}}$ of the sequence $\boldsymbol{V}$ on $[\mathcal{H}]_{\boldsymbol{V}}$ is a minimal regular isometric dilation of $\boldsymbol{T}$. The minimal regular isometric dilation is unique up to the joint unitary equivalence in the sense that if both $\boldsymbol{V} = (V_1, V_2, \ldots)$ and $\boldsymbol{W} = (W_1, W_2, \ldots)$ are minimal regular isometric dilations of $\boldsymbol{T}$, then there exists a unitary operator $U : \mathcal{K}_{\boldsymbol{V}} \to \mathcal{K}_{\boldsymbol{W}}$ such that U fixes vectors in $\mathcal{H}$ and

$$U V_n = W_n U, \quad n \in \mathbb{N}.$$

Applying [48, Theorem 4.2] to the semigroup $\mathbb{Z}_+^{(\infty)}$, we see that the minimal regular isometric dilation of $\boldsymbol{T}$ always exists and is also doubly commuting. Note that the existence can also be deduced from [50, pp. 36–37] by restricting the minimal regular unitary dilation $\boldsymbol{U}$ of $\boldsymbol{T}$ to $[\mathcal{H}]_{\boldsymbol{U}}$. For the convenience of the readers, below we prove directly for the case $\boldsymbol{T} \in \mathcal{DC}$ that the minimal regular isometric dilation $\boldsymbol{V}$ of $\boldsymbol{T}$ is in the class $\mathcal{DI}$. Moreover, $\boldsymbol{V}$ is a coextension of $\boldsymbol{T}$, which means that $\mathcal{H}$ is invariant for $\boldsymbol{V}^*$ and $\boldsymbol{T}^* = \boldsymbol{V}^* \mid_{\mathcal{H}}$, the restriction of $\boldsymbol{V}^*$ on $\mathcal{H}$.

Lemma 2.1. *The minimal regular isometric dilation of a sequence $\boldsymbol{T} \in \mathcal{DC}$ is in the class $\mathcal{DI}$ and coextends $\boldsymbol{T}$.*

Proof. It is equivalent to show that if $\boldsymbol{V} = (V_1, V_2, \ldots)$ is the minimal regular isometric dilation of $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$, then for any given $n \in \mathbb{N}$, V_n is pure and doubly commutes with V_m for any $m \neq n$, and $\mathcal{H}$ is invariant for V_n^*. Assume $n = 1$ without loss of generality, and put $\mathcal{E} = \mathfrak{D}_{T_1^*}$ and $\boldsymbol{T}' = (T_2, T_3, \ldots)$. Since $\boldsymbol{T}$ is doubly commuting, we see that $\mathcal{E}$ is $\boldsymbol{T}'$-joint reducing and the restriction $\boldsymbol{T}' \mid_{\mathcal{E}} = (T_2 \mid_{\mathcal{E}}, T_3 \mid_{\mathcal{E}}, \ldots)$ of $\boldsymbol{T}'$ on $\mathcal{E}$ is also doubly commuting. Let the sequence $\boldsymbol{S} = (S_1, S_2, \ldots)$, defined on a Hilbert space $\mathcal{F} \supseteq \mathcal{E}$, be the minimal regular isometric dilation of $\boldsymbol{T}' \mid_{\mathcal{E}}$.

The functional model theory for the single $C_{\cdot 0}$-contraction gives the following isometric embedding:

$$V : \mathcal{H} \to H_{\mathcal{E}}^2(\mathbb{D}) = H^2(\mathbb{D}) \otimes \mathcal{E},$$

$$x \mapsto \sum_{k=0}^{\infty} z^k \cdot D_{T_1^*} T_1^{*k} x.$$

By identifying $\mathcal{H}$ with the $M_z^* \otimes I_{\mathcal{E}}$-invariant subspace $V\mathcal{H}$ via the isometry V, one obtains a minimal isometric dilation $M_z \otimes I_{\mathcal{E}}$ of the contraction T_1, which is also a coextension. Set

$$W_1 = M_z \otimes I_{\mathcal{F}}$$

and

$$W_m = I_{H^2(\mathbb{D})} \otimes S_{m-1}, \quad m \geqslant 2.$$

It is routine to check that the sequence $\boldsymbol{W} = (W_1, W_2, \ldots)$ of isometries is a regular isometric dilation of $\boldsymbol{T}$. We claim that this dilation $\boldsymbol{W}$ is also minimal. Since $\boldsymbol{S} = (S_1, S_2, \ldots)$ is the minimal regular isometric dilation of $\boldsymbol{T}' \mid_{\mathcal{E}}$, we have

$$[H_{\mathcal{E}}^2(\mathbb{D})]_{\boldsymbol{W}'} = H^2(\mathbb{D}) \otimes [\mathcal{E}]_{\boldsymbol{S}} = H_{\mathcal{F}}^2(\mathbb{D}),$$

where $\boldsymbol{W}' = (W_2, W_3, \ldots)$. This together with $[\mathcal{H}]_{W_1} = H_\mathcal{E}^2(\mathbb{D})$ proves the claim. By the uniqueness of the minimal regular isometric dilation, there exists a unitary operator U that intertwines $\boldsymbol{V}$ and $\boldsymbol{W}$ and fixes vectors in $\mathcal{H}$. Therefore, V_1 is pure and doubly commutes with V_m $(m \geqslant 2)$. Moreover,

$$V_1^* \mathcal{H} = U^* W_1^* U \mathcal{H} = U^* W_1^* \mathcal{H} = U^* (M_z \otimes I_\mathcal{E})^* \mathcal{H} \subseteq U^* \mathcal{H} = \mathcal{H}.$$

The proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We also record the following useful lemma concerning the minimal regular isometric dilation.

Lemma 2.2.   *Let $\boldsymbol{T}$ be a sequence in the class $\mathcal{DC}(\mathcal{H})$, and $\boldsymbol{V} \in \mathcal{DI}(\mathcal{K})$ be an isometric coextension of $\boldsymbol{T}$. Put*

$$\mathcal{H}_n = \bigvee_{\substack{\alpha \in \mathbb{Z}_+^{(\infty)} \\ \alpha_n = 0}} \boldsymbol{V}^\alpha \mathcal{H}, \quad n \in \mathbb{N}.$$

Then the following conclusions hold:
   (1) *$\mathcal{H}_n$ $(n \in \mathbb{N})$ is $\boldsymbol{V}^*$-joint invariant and $[\mathcal{H}]_{\boldsymbol{V}}$ is $\boldsymbol{V}$-joint reducing.*
   (2) *$P_\mathcal{H} V_n^* P_{\mathcal{H}_n} = V_n^* P_\mathcal{H}$ for each $n \in \mathbb{N}$.*
   (3) *Furthermore, if $\boldsymbol{V}$ is the minimal isometric dilation of $\boldsymbol{T}$, then*

$$\mathcal{H} = \bigcap_{n=1}^\infty \mathcal{H}_n.$$

Here, for a closed subspace M of $\mathcal{K}$, P_M denotes the orthogonal projection from $\mathcal{K}$ onto M.

Proof.   (1) Since the sequence $\boldsymbol{V}$ is doubly commuting, and $\mathcal{H}$ is joint invariant for $\boldsymbol{V}^*$, one obtains that for $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{Z}_+^{(\infty)}$ with $\alpha_n = 0$,

$$V_m^* \boldsymbol{V}^\alpha \mathcal{H} = \boldsymbol{V}^\alpha V_m^* \mathcal{H} \subseteq \boldsymbol{V}^\alpha \mathcal{H} \subseteq \mathcal{H}_n, \quad \text{if } \alpha_m = 0,$$
$$V_m^* \boldsymbol{V}^\alpha \mathcal{H} = \boldsymbol{V}^{\alpha - 1_m} \mathcal{H} \subseteq \mathcal{H}_n, \quad \text{if } \alpha_m \geqslant 1,$$

where

$$1_m = (0, \ldots, 0, \overset{m\text{-th}}{1}, 0, \ldots).$$

This gives that each $\mathcal{H}_n$ is $\boldsymbol{V}^*$-joint invariant, and then

$$[\mathcal{H}]_{\boldsymbol{V}} = \bigvee_{n=1}^\infty \mathcal{H}_n$$

is $\boldsymbol{V}$-joint reducing.
   (2) Since the sequences $\boldsymbol{T}$ and $\boldsymbol{V}$ are doubly commuting, and $\mathcal{H}$ is joint invariant for $\boldsymbol{V}^*$, we see that for $n \in \mathbb{N}$ and $\alpha \in \mathbb{Z}_+^{(\infty)}$ with $\alpha_n = 0$,

$$P_\mathcal{H} V_n^* \boldsymbol{V}^\alpha |_\mathcal{H} = P_\mathcal{H} \boldsymbol{V}^\alpha P_\mathcal{H} V_n^* |_\mathcal{H} = \boldsymbol{T}^\alpha T_n^* = T_n^* \boldsymbol{T}^\alpha = V_n^* P_\mathcal{H} \boldsymbol{V}^\alpha |_\mathcal{H},$$

forcing

$$\boldsymbol{V}^\alpha \mathcal{H} \subseteq \mathrm{Ker}(P_\mathcal{H} V_n^* - V_n^* P_\mathcal{H}).$$

It follows that for each $n \in \mathbb{N}$,

$$\mathcal{H}_n \subseteq \mathrm{Ker}(P_\mathcal{H} V_n^* - V_n^* P_\mathcal{H}),$$

and then

$$P_\mathcal{H} V_n^* P_{\mathcal{H}_n} = V_n^* P_\mathcal{H} P_{\mathcal{H}_n} = V_n^* P_\mathcal{H}.$$

(3) Write $\widetilde{\mathcal{H}} = \bigcap_{n=1}^\infty \mathcal{H}_n$. It is trivial that $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$. It follows from (1) and (2) that $\widetilde{\mathcal{H}}$ is joint invariant for $\boldsymbol{V}^*$, and for each $n \in \mathbb{N}$,

$$P_\mathcal{H} V_n^* P_{\widetilde{\mathcal{H}}} = P_\mathcal{H} V_n^* P_{\mathcal{H}_n} P_{\widetilde{\mathcal{H}}} = V_n^* P_\mathcal{H} P_{\widetilde{\mathcal{H}}} = V_n^* P_\mathcal{H}.$$

Assume that x is an element in $\widetilde{\mathcal{H}} \ominus \mathcal{H}$. Then for each $n \in \mathbb{N}$,

$$P_{\mathcal{H}} V_n^* x = P_{\mathcal{H}} V_n^* P_{\widetilde{\mathcal{H}}} x = V_n^* P_{\mathcal{H}} x = 0,$$

forcing $V_n^* x \in \widetilde{\mathcal{H}} \ominus \mathcal{H}$. Therefore, $\boldsymbol{V}^{*\alpha} x \in \widetilde{\mathcal{H}} \ominus \mathcal{H}$ for every $\alpha \in \mathbb{Z}_+^{(\infty)}$, i.e., x is orthogonal to the $\boldsymbol{V}$-invariant subspace $[\mathcal{H}]_{\boldsymbol{V}}$ generated by $\mathcal{H}$. Since $\boldsymbol{V}$ is the minimal isometric dilation of the sequence $\boldsymbol{T}$ defined on $\mathcal{H}$, we actually have $[\mathcal{H}]_{\boldsymbol{V}} = \mathcal{K}$. Thus $x = 0$, and this proves $\mathcal{H} = \widetilde{\mathcal{H}}$. $\qquad\square$

2.2   Some basic properties of vector-valued Hardy spaces and operator-valued functions

Here, we list some basic properties of vector-valued Hardy spaces and operator-valued functions, and the notations $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ always denote some Hilbert spaces.

The vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$, over the domain $\mathbb{D}_2^\infty$ (being a connected open subset) in the Hilbert space l^2, consists of all the $\mathcal{E}$-valued functions of the form

$$F(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \zeta^\alpha \cdot x_\alpha, \quad \zeta \in \mathbb{D}_2^\infty$$

with each $x_\alpha \in \mathcal{E}$ and $\|F\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|x_\alpha\|_{\mathcal{E}}^2 < \infty$. By the Cauchy-Schwarz inequality, the above series converges pointwisely on $\mathbb{D}_2^\infty$ in the $\mathcal{E}$-norm. We follow the definition of the holomorphic mapping given in [18, Definitions 3.1 and 3.6] for any vector-valued function $F : \mathbb{D}_2^\infty \to \mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space. Every function $F : \mathbb{D}_2^\infty \to \mathcal{E}$ in $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ is then holomorphic in this sense.

The space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ can be considered as the tensor product of the Hardy space $H^2(\mathbb{D}_2^\infty)$ and the Hilbert space $\mathcal{E}$ by identifying the vector-valued function $F \cdot x$ with the tensor product $F \otimes x$, where $F \in H^2(\mathbb{D}_2^\infty)$ and $x \in \mathcal{E}$. Then the tuple of coordinate multiplication operators on $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ has the form

$$\boldsymbol{M}_\zeta \otimes I_{\mathcal{E}} = (M_{\zeta_1} \otimes I_{\mathcal{E}}, M_{\zeta_2} \otimes I_{\mathcal{E}}, \ldots).$$

For simplicity, we often write only $\boldsymbol{M}_\zeta$ for this tuple. Moreover, one can expanse functions in $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ with respect to any orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of $\mathcal{E}$ as $\sum_{k=1}^\infty F_k \cdot e_k$, where each $F_k \in H^2(\mathbb{D}_2^\infty)$ and $\sum_{k=1}^\infty \|F_k\|_{H^2(\mathbb{D}_2^\infty)}^2 < \infty$.

Let $\boldsymbol{K}_\lambda$ denote the reproducing kernel of $H^2(\mathbb{D}_2^\infty)$ at the point $\lambda \in \mathbb{D}_2^\infty$. Recall that a subset E of a Hilbert space $\mathcal{H}$ is said to be complete in $\mathcal{H}$ if E spans a dense subspace of $\mathcal{H}$, i.e., the orthocomplement $E^\perp$ of E in $\mathcal{H}$ is $\{0\}$.

Lemma 2.3.   *Suppose $F \in H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$. Then the following conclusions hold:*

(1) *$\langle F, \boldsymbol{K}_\lambda \cdot x \rangle = \langle F(\lambda), x \rangle$ for every $\lambda \in \mathbb{D}_2^\infty$ and every $x \in \mathcal{E}$. Consequently, the set*

$$\{\boldsymbol{K}_\lambda \cdot x : \lambda \in \mathbb{D}_2^\infty, x \in \mathcal{E}\}$$

is complete in $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$.

(2) *If $F \neq 0$ and $M_{\zeta_n}^* F = \overline{\lambda_n} F$ for each $n \in \mathbb{N}$ and a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of complex numbers, then $\lambda \in \mathbb{D}_2^\infty$ and $F = \boldsymbol{K}_\lambda \cdot x$ for some $x \in \mathcal{E}$.*

Proof.   (1) Assume $\|x\| = 1$ without loss of generality and take any orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of the subspace $\{x\}^\perp$ of $\mathcal{E}$. Expand the function F with respect to the orthonormal basis $\{x\} \cup \{e_k\}_{k \in \mathbb{N}}$ of $\mathcal{E}$ as

$$F = F_x \cdot x + \sum_{k=1}^\infty F_k \cdot e_k.$$

Then we have

$$\langle F, \boldsymbol{K}_\lambda \cdot x \rangle = \langle F_x \cdot x, \boldsymbol{K}_\lambda \cdot x \rangle + \sum_{k=1}^\infty \langle F_k \cdot e_k, \boldsymbol{K}_\lambda \cdot x \rangle = F_x(\lambda)$$

and

$$\langle F(\lambda), x \rangle = \langle F_x(\lambda) x, x \rangle + \sum_{k=1}^\infty \langle F_k(\lambda) e_k, x \rangle = F_x(\lambda).$$

This proves (1).

(2) Expand the function F with respect to an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of $\mathcal{E}$ as $F = \sum_{k=1}^{\infty} F_k \cdot e_k$, and then

$$M_{\zeta_n}^* F = M_{\zeta_n}^* \left(\sum_{k=1}^{\infty} F_k \cdot e_k \right) = \sum_{k=1}^{\infty} (M_{\zeta_n}^* F_k) \cdot e_k, \quad n \in \mathbb{N}.$$

It follows that $M_{\zeta_n}^* F_k = \overline{\lambda_n} F_k$ for every $k, n \in \mathbb{N}$. This implies that all F_k's are in the orthocomplement of the invariant subspace generated by $\zeta_1 - \lambda_1, \zeta_2 - \lambda_2, \ldots$ Since $F \neq 0$, it follows that this invariant subspace is proper, and its orthocomplement is $\mathbb{C} \boldsymbol{K}_\lambda$. Therefore, we have $\lambda \in \mathbb{D}_2^\infty$ and $F_k = c_k \boldsymbol{K}_\lambda$ with $\sum_{k=1}^{\infty} |c_k|^2 < \infty$, and thus

$$F = \sum_{k=1}^{\infty} F_k \cdot e_k = \boldsymbol{K}_\lambda \cdot \left(\sum_{k=1}^{\infty} c_k e_k \right).$$

The proof is completed. $\qquad \square$

The space $H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D}_2^\infty)$ consists of all the uniformly bounded holomorphic operator-valued functions $\Psi : \mathbb{D}_2^\infty \to \mathcal{B}(\mathcal{F}, \mathcal{E})$. Here, by Ψ uniformly bounded we mean $\|\Psi\|_\infty = \sup_{\zeta \in \mathbb{D}_2^\infty} \|\Psi(\zeta)\| < \infty$, and $\mathcal{B}(\mathcal{F}, \mathcal{E})$ denotes the Banach space of bounded linear operators from $\mathcal{F}$ to $\mathcal{E}$. Every function Ψ in $H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D}_2^\infty)$ naturally induces a multiplication operator M_Ψ as follows:

$$M_\Psi : H_{\mathcal{F}}^2(\mathbb{D}_2^\infty) \to H_{\mathcal{E}}^2(\mathbb{D}_2^\infty),$$
$$F \mapsto \Psi F,$$

where $\Psi F(\zeta) = \Psi(\zeta) F(\zeta)$ ($\zeta \in \mathbb{D}_2^\infty$). It is clear that M_Ψ is bounded by $\|\Psi\|_\infty$. We now claim that

$$M_\Psi^*(\boldsymbol{K}_\lambda \cdot x) = \boldsymbol{K}_\lambda \cdot \Psi(\lambda)^* x \tag{2.1}$$

for every $\lambda \in \mathbb{D}_2^\infty$ and every $x \in \mathcal{E}$. By Lemma 2.3(1), it suffices to show that for any fixed $\mu \in \mathbb{D}_2^\infty$ and $y \in \mathcal{F}$,

$$\langle \boldsymbol{K}_\mu \cdot y, M_\Psi^*(\boldsymbol{K}_\lambda \cdot x) \rangle = \langle \boldsymbol{K}_\mu \cdot y, \boldsymbol{K}_\lambda \cdot \Psi(\lambda)^* x \rangle.$$

Again by Lemma 2.3(1), we have

$$\langle \boldsymbol{K}_\mu \cdot y, M_\Psi^*(\boldsymbol{K}_\lambda \cdot x) \rangle = \langle \Psi(\boldsymbol{K}_\mu \cdot y), \boldsymbol{K}_\lambda \cdot x \rangle = \langle \boldsymbol{K}_\mu(\lambda) \Psi(\lambda) y, x \rangle.$$

On the other hand,

$$\langle \boldsymbol{K}_\mu \cdot y, \boldsymbol{K}_\lambda \cdot \Psi(\lambda)^* x \rangle = \langle \boldsymbol{K}_\mu, \boldsymbol{K}_\lambda \rangle \cdot \langle \Psi(\lambda) y, x \rangle = \boldsymbol{K}_\mu(\lambda) \langle \Psi(\lambda) y, x \rangle.$$

This proves the claim.

We are ready to prove the following proposition.

Proposition 2.4. *Let $\boldsymbol{M}_\zeta$ and $\boldsymbol{M}_\xi$ be the tuple of coordinate multiplication operators on $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ and $H_{\mathcal{F}}^2(\mathbb{D}_2^\infty)$, respectively. If $T : H_{\mathcal{F}}^2(\mathbb{D}_2^\infty) \to H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ is a bounded linear operator satisfying*

$$T M_{\xi_n} = M_{\zeta_n} T, \quad n \in \mathbb{N},$$

then there exists an operator-valued function $\Psi \in H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D}_2^\infty)$ such that $T = M_\Psi$.

Proof. By (2.1), for each $n \in \mathbb{N}$, each $\lambda \in \mathbb{D}_2^\infty$ and each $x \in \mathcal{E}$,

$$M_{\xi_n}^* T^*(\boldsymbol{K}_\lambda \cdot x) = T^* M_{\zeta_n}^*(\boldsymbol{K}_\lambda \cdot x) = \overline{\lambda_n} T^*(\boldsymbol{K}_\lambda \cdot x).$$

Then by Lemma 2.3(2), to every pair λ, x, there corresponds an element $y_{\lambda,x} \in \mathcal{F}$ such that

$$T^*(\boldsymbol{K}_\lambda \cdot x) = \boldsymbol{K}_\lambda \cdot y_{\lambda,x},$$

and therefore $x \mapsto y_{\lambda,x}$ defines a linear operator $S_\lambda \in \mathcal{B}(\mathcal{E}, \mathcal{F})$ for each $\lambda \in \mathbb{D}_2^\infty$. Now put $\Psi(\lambda) = S_\lambda^*$ ($\lambda \in \mathbb{D}_2^\infty$). It is routine to check that $\Psi : \mathbb{D}_2^\infty \to \mathcal{B}(\mathcal{F}, \mathcal{E})$ is uniformly bounded and holomorphic with $\|\Psi\|_\infty \leqslant \|T\|$. Since T^* and M_Ψ^* coincide on the set $\{\boldsymbol{K}_\lambda \cdot x : \lambda \in \mathbb{D}_2^\infty, x \in \mathcal{E}\}$ by (2.1), it follows from Lemma 2.3(1) that $T = M_\Psi$. $\qquad \square$

In [5], the infinite tensor product of a sequence of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ with the stabilizing sequence $\{e^{(n)}\}_{n\in\mathbb{N}}$ is introduced, where $e^{(n)}$ is a unit vector in $\mathcal{H}_n$ for each $n \in \mathbb{N}$. We remark that the Hilbert space structure of $H^2(\mathbb{D}_2^\infty)$ coincides with the infinite tensor product

$$H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots$$

of countably infinitely many Hardy spaces over $\mathbb{D}$ with the stabilizing sequence $\{1\}_{n\in\mathbb{N}}$. So for any closed subspace M of the Hardy space

$$H^2(\mathbb{D}^n) = \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{n \text{ times}}, \quad n \in \mathbb{N},$$

the infinite tensor product

$$M \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots$$

is exactly the $(M_{\zeta_{n+1}}, M_{\zeta_{n+2}}, \ldots)$-joint invariant subspace of $H^2(\mathbb{D}_2^\infty)$ generated by M $(= M \otimes \mathbb{C} \otimes \mathbb{C} \otimes \cdots)$. For later use, we also note that the vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ can be written as

$$H_{\mathcal{E}}^2(\mathbb{D}_2^\infty) = \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{n-1 \text{ times}} \otimes H_{\mathcal{E}}^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots$$

$$= H_{\mathcal{E}}^2(\mathbb{D}^n) \otimes \mathcal{L}_n, \tag{2.2}$$

where

$$\mathcal{L}_n = \underbrace{\mathbb{C} \otimes \cdots \otimes \mathbb{C}}_{n \text{ times}} \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots$$

$$= \overline{\operatorname{span}\{\zeta^\alpha : \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)} \text{ with } \alpha_1 = \cdots = \alpha_n = 0\}}.$$

One simplest class of vector-valued or operator-valued functions on $\mathbb{D}_2^\infty$ is the functions induced by those of one variable. More precisely, put $\tilde{f}(\zeta) = f(\zeta_n)$ and $\widetilde{\theta}(\zeta) = \theta(\zeta_n)$ $(\zeta \in \mathbb{D}_2^\infty)$ for every $f \in H_{\mathcal{E}}^2(\mathbb{D})$, every $\theta \in H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D})$ and every $n \in \mathbb{N}$. Then we have $\tilde{f} \in H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ and $\widetilde{\theta} \in H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D}_2^\infty)$. The multiplication operator $M_{\widetilde{\theta}}$ has the form

$$M_{\widetilde{\theta}} = \underbrace{I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}}_{n-1 \text{ times}} \otimes M_\theta \otimes I_{H^2(\mathbb{D})} \otimes \cdots$$

with respect to the representation (2.2). The property of $M_{\widetilde{\theta}}$ thus relies heavily on that of θ. For example, $M_{\widetilde{\theta}}$ is isometric if and only if M_θ is isometric, if and only if θ is inner. Recall that $\theta \in H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D})$ is said to be inner if the boundary values of θ, on some subset E of the unit circle $\mathbb{T}$ with full measure, are isometries, which is defined to be the radial limit

$$(\text{SOT}) \lim_{r\to1^-} \theta(rz), \quad z \in E$$

(see, for example, [34, 50]). On the other hand, since $r\mathbb{D}^\infty \nsubseteq \mathbb{D}_2^\infty$ for every $0 < r < 1$, the radial limits for bounded holomorphic functions on $\mathbb{D}_2^\infty$ do not make sense in general. Let $\mathbb{T}^\infty$ be the infinite torus $\mathbb{T} \times \mathbb{T} \times \cdots$, and σ be the Haar measure of $\mathbb{T}^\infty$. In [43], Saksman and Seip defined boundary values, on the distinguished boundary $\mathbb{T}^\infty$, for bounded holomorphic functions on $\mathbb{D}_2^\infty$ by taking quasi-radial limits instead (see also [2]). Another way to define the boundary value function can be derived from the work of Cole and Gamelin [13]. It was shown in [13] that each $F \in H^2(\mathbb{D}_2^\infty)$ is the Poisson integral of a unique function $F^* \in L^2(\mathbb{T}^\infty, \sigma)$. Thus one naturally defines the boundary value function of F to be F^*. A bounded holomorphic function η on $\mathbb{D}_2^\infty$ is said to be *inner* if $|\eta^*| = 1$, a.e. (see [16, Subsection 2.2]). However, to avoid more discussion about the boundary behavior of operator-valued functions on $\mathbb{D}_2^\infty$, we introduce an alternative definition that $\Psi \in H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D}_2^\infty)$ is said to be *inner* if M_Ψ is an isometry. Finally, applying Lemma 2.3(1) to the function $\tilde{f}$ and (2.1) to the function $\widetilde{\theta}$, we have

(i) $\langle f, K_a \cdot x \rangle = \langle f(a), x \rangle$ for every $a \in \mathbb{D}$ and every $x \in \mathcal{E}$; consequently, the set

$$\{K_a \cdot x : a \in \mathbb{D}, x \in \mathcal{E}\}$$

is complete in $H_{\mathcal{E}}^2(\mathbb{D})$;

(ii) $M_\theta^*(K_a \cdot x) = K_a \cdot \theta(a)^* x$ for every $a \in \mathbb{D}$ and every $x \in \mathcal{E}$.

Here, K_a denotes the reproducing kernel of $H^2(\mathbb{D})$ at the point $a \in \mathbb{D}$. Note that by (ii), for $a, b \in \mathbb{D}$ and $x, y \in \mathcal{E}$, we have

$$\begin{aligned}
\langle M_\theta M_\theta^*(K_a \cdot x), K_b \cdot y \rangle &= \langle M_\theta^*(K_a \cdot x), M_\theta^*(K_b \cdot y) \rangle \\
&= \langle K_a \cdot \theta(a)^* x, K_b \cdot \theta(b)^* y \rangle \\
&= \frac{1}{1 - \bar{a}b} \langle \theta(b)\theta(a)^* x, y \rangle.
\end{aligned} \tag{2.3}$$

The following lemma is also needed in the sequel.

Lemma 2.5.   *Suppose $\theta \in H_{\mathcal{B}(\mathcal{F},\mathcal{E})}^\infty(\mathbb{D})$ and $\vartheta \in H_{\mathcal{B}(\mathcal{G},\mathcal{E})}^\infty(\mathbb{D})$. Then the following conclusions hold*:

(1) *Let $\mathcal{E}_0$ be a closed subspace of $\mathcal{E}$. Then $H_{\mathcal{E}_0}^2(\mathbb{D})$ is invariant for $M_\theta M_\theta^*$ if and only if $\mathcal{E}_0$ is invariant for $\theta(b)\theta(a)^*$ for every $a, b \in \mathbb{D}$. In this case, if θ is inner, we have*

$$M_\theta^* H_{\mathcal{E}_0}^2(\mathbb{D}) = H_{\mathcal{F}_0}^2(\mathbb{D}),$$

where $\mathcal{F}_0 = \bigvee_{a \in \mathbb{D}} \theta(a)^* \mathcal{E}_0$.

(2) *$M_\theta M_\theta^* = M_\vartheta M_\vartheta^*$ if and only if*

$$\theta(b)\theta(a)^* = \vartheta(b)\vartheta(a)^*$$

for every $a, b \in \mathbb{D}$.

Proof.   (1) By (i), $H_{\mathcal{E}_0}^2(\mathbb{D})$ is reducing for $M_\theta M_\theta^*$ if and only if

$$\langle M_\theta M_\theta^*(K_a \cdot x), K_b \cdot y \rangle = 0$$

for every $a, b \in \mathbb{D}$, every $x \in \mathcal{E}_0$ and every $y \in \mathcal{E} \ominus \mathcal{E}_0$. This together with (2.3) gives that $H_{\mathcal{E}_0}^2(\mathbb{D})$ is reducing for $M_\theta M_\theta^*$ if and only if $\theta(b)\theta(a)^* \mathcal{E}_0 \subseteq \mathcal{E}_0$ for every $a, b \in \mathbb{D}$. For the latter conclusion, note that by (ii), $M_\theta^* H_{\mathcal{E}_0}^2(\mathbb{D})$ is a dense subspace of $H_{\mathcal{F}_0}^2(\mathbb{D})$. Now suppose in addition that θ is inner. Then $M_\theta M_\theta^*$ is an orthogonal projection on $H_{\mathcal{E}}^2(\mathbb{D})$. Since $H_{\mathcal{E}_0}^2(\mathbb{D})$ is reducing for $M_\theta M_\theta^*$, $M_\theta M_\theta^* H_{\mathcal{E}_0}^2(\mathbb{D})$ is closed, forcing $M_\theta^* H_{\mathcal{E}_0}^2(\mathbb{D})$ to be also closed. This proves (1).

(2) By (i), $M_\theta M_\theta^* = M_\vartheta M_\vartheta^*$ if and only if

$$\langle M_\theta M_\theta^*(K_a \cdot x), K_b \cdot y \rangle = \langle M_\vartheta M_\vartheta^*(K_a \cdot x), K_b \cdot y \rangle$$

for every $a, b \in \mathbb{D}$ and every $x, y \in \mathcal{E}$. Then (2) follows from (2.3). $\square$

2.3   Some preparation for proofs

Lemma 2.6.   *If $\boldsymbol{T}$ is a doubly commuting sequence of contractions on $\mathcal{H}$ and $\boldsymbol{V}$ is a doubly commuting isometric coextension of $\boldsymbol{T}$, then for each $\lambda \in \mathbb{D}^\infty$ and each $x \in \mathcal{H}$,*

$$\|D_{\Phi_\lambda(\boldsymbol{T})^*} x\| = \|D_{\Phi_\lambda(\boldsymbol{V})^*} x\|.$$

Proof.   For any fixed $\lambda \in \mathbb{D}^\infty$, set

$$\boldsymbol{S} = (S_1, S_2, \ldots) = \Phi_\lambda(\boldsymbol{T})$$

and

$$\boldsymbol{W} = (W_1, W_2, \ldots) = \Phi_\lambda(\boldsymbol{V}).$$

It is clear that $\boldsymbol{W}$ is an isometric coextension of $\boldsymbol{S}$. Since for $x \in \mathcal{H}$,

$$\|D_{\Phi_\lambda(\boldsymbol{T})^*}x\| = \|D_{\boldsymbol{S}^*}x\| = \lim_{n\to\infty}\|D_{S_1^*}\cdots D_{S_n^*}x\|$$

and

$$\|D_{\Phi_\lambda(\boldsymbol{V})^*}x\| = \|D_{\boldsymbol{W}^*}x\| = \lim_{n\to\infty}\|D_{W_1^*}\cdots D_{W_n^*}x\|,$$

it suffices to prove that for $n \in \mathbb{N}$,

$$\|D_{S_1^*}\cdots D_{S_n^*}x\| = \|D_{W_1^*}\cdots D_{W_n^*}x\|, \quad x \in \mathcal{H}. \tag{2.4}$$

We show this by induction on n. For $n = 1$, one has

$$\|D_{S_1^*}x\|^2 = \|x\|^2 - \|S_1^*x\|^2 = \|x\|^2 - \|W_1^*x\|^2 = \|D_{W_1^*}x\|^2, \quad x \in \mathcal{H}.$$

Assume that (2.4) holds for $n = k$. Since $\boldsymbol{S}$ and $\boldsymbol{W}$ are doubly commuting, we have that for $x \in \mathcal{H}$,

$$\begin{aligned}
\|D_{S_1^*}\cdots D_{S_{k+1}^*}x\|^2 &= \|D_{S_1^*}\cdots D_{S_k^*}x\|^2 - \|S_{k+1}^* D_{S_1^*}\cdots D_{S_k^*}x\|^2 \\
&= \|D_{S_1^*}\cdots D_{S_k^*}x\|^2 - \|D_{S_1^*}\cdots D_{S_k^*}S_{k+1}^*x\|^2 \\
&= \|D_{W_1^*}\cdots D_{W_k^*}x\|^2 - \|D_{W_1^*}\cdots D_{W_k^*}S_{k+1}^*x\|^2 \\
&= \|D_{W_1^*}\cdots D_{W_k^*}x\|^2 - \|D_{W_1^*}\cdots D_{W_k^*}W_{k+1}^*x\|^2 \\
&= \|D_{W_1^*}\cdots D_{W_k^*}x\|^2 - \|W_{k+1}^* D_{W_1^*}\cdots D_{W_k^*}x\|^2 \\
&= \|D_{W_1^*}\cdots D_{W_{k+1}^*}x\|^2.
\end{aligned}$$

Thus, (2.4) also holds for $n = k + 1$. This completes the proof. $\square$

Lemma 2.7.   *If $(T_1,\ldots,T_n)$ is a doubly commuting n-tuple of $C_{\cdot 0}$ contractions on a Hilbert space $\mathcal{H}$, then*

$$\bigcap_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{D}^n} \mathrm{Ker}D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_n}(T_n)^*} = \{0\}.$$

Equivalently,

$$\bigvee_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{D}^n} D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_n}(T_n)^*}\mathcal{H} = \mathcal{H}.$$

Proof.   The equivalence is guaranteed by the fact that $(T_1,\ldots,T_n)$ is doubly commuting and then the operator

$$D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_n}(T_n)^*}$$

is self-adjoint. We prove this lemma by induction on n. For $n = 1$, let T be a contraction of the class $C_{\cdot 0}$, and assume $f \in \mathcal{H}$ so that $D_{\varphi_a(T)^*}f = 0$ for any $a \in \mathbb{D}$. As in the proof of Lemma 2.1, T has a coextension to the Hardy shift M_z on the vector-valued Hardy space $H^2_{\mathfrak{D}_{T^*}}(\mathbb{D})$. Since M_{φ_a} $(a \in \mathbb{D})$ is isometric on $H^2_{\mathfrak{D}_{T^*}}(\mathbb{D})$, we have

$$\|f\| = \|\varphi_a(T)^*f\| = \|\varphi_a(M_z)^*f\| = \|M_{\varphi_a}^*f\| = \|M_{\varphi_a}M_{\varphi_a}^*f\|.$$

Note that $M_{\varphi_a}M_{\varphi_a}^*$ is the orthogonal projection onto $\mathrm{Ran}M_{\varphi_a}$ for each $a \in \mathbb{D}$, and the above identity yields $f = M_{\varphi_a}g_a = \varphi_a \cdot g_a$ for some $g_a \in H^2_{\mathfrak{D}_{T^*}}(\mathbb{D})$. This gives

$$f(a) = \varphi_a(a) \cdot g_a(a) = 0, \quad a \in \mathbb{D},$$

forcing $f = 0$, which proves the case $n = 1$. Now assume that the conclusion holds for $n = k$, and let $(T_1,\ldots,T_{k+1})$ be a doubly commuting tuple of $C_{\cdot 0}$ contractions. Then

$$\bigvee_{(\lambda_1,\ldots,\lambda_{k+1})\in\mathbb{D}^{k+1}} D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_{k+1}}(T_{k+1})^*}\mathcal{H}$$

$$
\begin{aligned}
&= \bigvee_{(\lambda_1,\ldots,\lambda_k)\in\mathbb{D}^k} \left(\bigvee_{a\in\mathbb{D}} D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_k}(T_k)^*} D_{\varphi_{\lambda_a}(T_{k+1})^*}\mathcal{H} \right) \\
&= \bigvee_{(\lambda_1,\ldots,\lambda_k)\in\mathbb{D}^k} D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_k}(T_k)^*} \left(\bigvee_{a\in\mathbb{D}} D_{\varphi_{\lambda_a}(T_{k+1})^*}\mathcal{H} \right) \\
&= \bigvee_{(\lambda_1,\ldots,\lambda_k)\in\mathbb{D}^k} D_{\varphi_{\lambda_1}(T_1)^*}\cdots D_{\varphi_{\lambda_k}(T_k)^*}\mathcal{H} \\
&= \mathcal{H}.
\end{aligned}
$$

By induction, the proof is completed. $\qquad\square$

Suppose that $\boldsymbol{V}=(V_1,V_2,\ldots)$ is a sequence in the class $\mathcal{DI}$. Recall that for any $\lambda=(\lambda_1,\lambda_2,\ldots)\in\mathbb{D}^\infty$, the defect space $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}$ of the sequence $\Phi_\lambda(\boldsymbol{V})^*$ is the closure of the range of the defect operator

$$
D_{\Phi_\lambda(\boldsymbol{V})^*} = \prod_{n=1}^{\infty}(I - \varphi_{\lambda_n}(V_n)\varphi_{\lambda_n}(V_n)^*).
$$

Since $\boldsymbol{V}$ is doubly commuting, $\{I - \varphi_{\lambda_n}(V_n)\varphi_{\lambda_n}(V_n)^*\}_{n\in\mathbb{N}}$ is a commuting sequence of orthogonal projections, and then $D_{\Phi_\lambda(\boldsymbol{V})^*}$ is also an orthogonal projection onto the subspace

$$
\bigcap_{n=1}^{\infty}\mathrm{Ran}(I - \varphi_{\lambda_n}(V_n)\varphi_{\lambda_n}(V_n)^*) = \bigcap_{n=1}^{\infty}(\mathrm{Ran}\varphi_{\lambda_n}(V_n))^\perp = \bigcap_{n=1}^{\infty}\mathrm{Ker}\varphi_{\lambda_n}(V_n)^*.
$$

Note that $x\in\mathrm{Ker}\varphi_{\lambda_n}(V_n)^*$ if and only if $V_n^*x=\overline{\lambda_n}x$ $(n\in\mathbb{N})$, and nonzero elements in the defect space $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}$ exactly coincide with the set of joint eigenvectors of the sequence $\boldsymbol{V}^*$ corresponding to the joint eigenvalue $\overline{\lambda}=(\overline{\lambda_1},\overline{\lambda_2},\ldots)$.

Lemma 2.8. *Suppose $\boldsymbol{V}\in\mathcal{DI}$. If x is a nonzero element in the defect space $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}$ of the sequence $\Phi_\lambda(\boldsymbol{V})^*$ for some $\lambda\in\mathbb{D}^\infty$, then $[x]_{\boldsymbol{V}}(=[\{x\}]_{\boldsymbol{V}})$ is $\boldsymbol{V}$-joint reducing, and $\boldsymbol{V}\mid_{[x]_{\boldsymbol{V}}}$ is jointly unitarily equivalent to the sequence $\Phi_\lambda(\boldsymbol{M}_\zeta)$, where $\boldsymbol{M}_\zeta$ is the tuple of coordinate multiplication operators on the Hardy space $H^2(\mathbb{D}_2^\infty)$.*

Proof. Assume $\|x\|=1$ without loss of generality, and we first prove the desired conclusion for $\lambda=\boldsymbol{0}=(0,0,\ldots)$. Since $x\in\mathfrak{D}_{\boldsymbol{V}^*}$, we have that for $n,m\in\mathbb{N}$ and $\alpha\in\mathbb{Z}_+^{(\infty)}$,

$$
\begin{aligned}
V_m^*\boldsymbol{V}^\alpha x &= \boldsymbol{V}^\alpha V_m^*x = 0\in[x]_{\boldsymbol{V}}, \quad &&\text{if } \alpha_m=0, \\
V_m^*\boldsymbol{V}^\alpha x &= \boldsymbol{V}^{\alpha-1_m}x\in[x]_{\boldsymbol{V}}, \quad &&\text{if } \alpha_m\geqslant 1.
\end{aligned}
$$

This implies that $[x]_{\boldsymbol{V}}$ is $\boldsymbol{V}$-joint reducing. The rest of the proof is given by defining a linear map U from $\mathcal{P}_\infty$ to $[x]_{\boldsymbol{V}}$ as follows:
$$
Up = p(\boldsymbol{V})x, \quad p\in\mathcal{P}_\infty,
$$
where $\mathcal{P}_\infty=\mathbb{C}[\zeta_1,\zeta_2,\ldots]$, the polynomial ring in countably infinitely many variables, which is dense in $H^2(\mathbb{D}_2^\infty)$ [35]. It is routine to check that U can be extended to a unitary operator from $H^2(\mathbb{D}_2^\infty)$ onto $[x]_{\boldsymbol{V}}$, and $\boldsymbol{M}_\zeta$ is jointly unitarily equivalent to $\boldsymbol{V}\mid_{[x]_{\boldsymbol{V}}}$ via this unitary operator.

For general $\lambda\in\mathbb{D}^\infty$, put $\boldsymbol{W}=\Phi_\lambda(\boldsymbol{V})$. Then the above argument shows that $\boldsymbol{W}\mid_{[x]_{\boldsymbol{V}}}=\boldsymbol{W}\mid_{[x]_{\boldsymbol{W}}}$ is jointly unitarily equivalent to $\boldsymbol{M}_\zeta$; equivalently, $\boldsymbol{V}\mid_{[x]_{\boldsymbol{V}}}=\Phi_\lambda(\boldsymbol{W}\mid_{[x]_{\boldsymbol{V}}})$ is jointly unitarily equivalent to the sequence $\Phi_\lambda(\boldsymbol{M}_\zeta)$. The proof is completed. $\qquad\square$

Lemma 2.9. *Let λ be a point in $\mathbb{D}^\infty$ and $\boldsymbol{M}_\zeta$ be the tuple of coordinate multiplication operators on $H^2(\mathbb{D}_2^\infty)$. Then $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{M}_\zeta)^*}=\{0\}$ if and only if $\lambda\notin\mathbb{D}_2^\infty$.*

Proof. By comments before the previous lemma, we have

$$
\mathfrak{D}_{\Phi_\lambda(\boldsymbol{M}_\zeta)^*} = \bigcap_{n=1}^{\infty}(\widetilde{\varphi_{\lambda_n}}H^2(\mathbb{D}_2^\infty))^\perp,
$$

and hence

$$\mathfrak{D}_{\Phi_\lambda(\boldsymbol{M}_\zeta)^*}^{\perp} = \bigvee_{n=1}^{\infty} \widetilde{\varphi_{\lambda_n}} H^2(\mathbb{D}_2^\infty) = [\{\widetilde{\varphi_{\lambda_n}}\}_{n\in\mathbb{N}}]_{\boldsymbol{M}_\zeta},$$

where $\widetilde{\varphi_{\lambda_n}}(\zeta) = \varphi_{\lambda_n}(\zeta_n)$ $(\zeta \in \mathbb{D}_2^\infty)$. By [17, Proposition 4.5],

$$[\{\widetilde{\varphi_{\lambda_n}}\}_{n\in\mathbb{N}}]_{\boldsymbol{M}_\zeta} = H^2(\mathbb{D}_2^\infty)$$

if and only if $\lambda \notin \mathbb{D}_2^\infty$. This completes the proof. $\qquad\square$

By an irreducible family of operators, we mean that these operators have no nontrivial joint reducing subspaces.

Lemma 2.10. *The tuple $\boldsymbol{M}_\zeta$ of coordinate multiplication operators on $H^2(\mathbb{D}_2^\infty)$ is irreducible.*

Proof. It suffices to prove that for any orthogonal projection P commuting with every coordinate multiplication operator, we have $P1 = 0$ or $P1 = 1$. Put $Q = I - P$ and $Q1 = \sum_{\alpha\in\mathbb{Z}_+^{(\infty)}} c_\alpha \zeta^\alpha$. Since

$$P\zeta^\alpha = PM_{\zeta^\alpha}1 = M_{\zeta^\alpha}P1 = \zeta^\alpha \cdot P1,$$

we have

$$0 = P(Q1) = P\bigg(\sum_{\alpha\in\mathbb{Z}_+^{(\infty)}} c_\alpha \zeta^\alpha\bigg) = \sum_{\alpha\in\mathbb{Z}_+^{(\infty)}} c_\alpha P\zeta^\alpha = \sum_{\alpha\in\mathbb{Z}_+^{(\infty)}} (c_\alpha \zeta^\alpha \cdot P1) = Q1 \cdot P1.$$

Note that $Q1 = (I - P)1 = 1 - P1$, and $P1$ only takes the value in $\{0, 1\}$. Then the proof is completed due to the continuity of $P1$. $\qquad\square$

Lemma 2.11. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, and $\mathcal{T}$ be an irreducible family of bounded linear operators on $\mathcal{H}$. Then any bounded linear operator on $\mathcal{H} \otimes \mathcal{K}$ that doubly commutes with the family $\{T \otimes I_\mathcal{K} : T \in \mathcal{T}\}$ is of the form $I_\mathcal{H} \otimes S$ for some $S \in \mathcal{B}(\mathcal{K})$.*

In particular, any joint reducing subspace for the family $\{T \otimes I_\mathcal{K} : T \in \mathcal{T}\}$ is of the form $\mathcal{H} \otimes M$ for some closed subspace M of $\mathcal{K}$.

Proof. Since the family $\mathcal{T}$ of operators has no nontrivial joint reducing subspace, the von Neumann algebra generated by $\mathcal{T}$ is the entire $\mathcal{B}(\mathcal{H})$ by the double commutant theorem [14]. This gives

$$\mathcal{V}^*(\{T \otimes I_\mathcal{K} : T \in \mathcal{T}\}) = \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}I_\mathcal{K} = \{T \otimes I_\mathcal{K} : T \in \mathcal{B}(\mathcal{H})\},$$

where $\mathcal{V}^*(\{T \otimes I_\mathcal{K} : T \in \mathcal{T}\})$ denotes the von Neumann algebra generated by the family $\{T \otimes I_\mathcal{K} : T \in \mathcal{T}\}$. If a bounded linear operator on $\mathcal{H} \otimes \mathcal{K}$ doubly commutes with the family $\{T \otimes I_\mathcal{K} : T \in \mathcal{T}\}$, then it also commutes with the algebra $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}I_\mathcal{K}$, and therefore has the form $I_\mathcal{H} \otimes S$ for some bounded linear operator S on $\mathcal{K}$ (see [51, p. 184]). $\qquad\square$

Lemma 2.12. *Suppose $\boldsymbol{V} \in \mathcal{DI}(\mathcal{H})$. Then for any $\varepsilon > 0$ and $x \in \mathcal{H}$, there exists a sequence $(k_1, k_2, \ldots)$ of positive integers such that*

$$\|D_{\boldsymbol{W}^*}x\| \geqslant (1 - \varepsilon)\|x\|,$$

where $\boldsymbol{W} = (V_1^{k_1}, V_2^{k_2}, \ldots)$.

Proof. Take an arbitrary positive number ε and assume $\|x\| = 1$ without loss of generality. Since each V_n $(n \in \mathbb{N})$ is pure, there exists a sequence $(k_1, k_2, \ldots)$ of positive integers, such that

$$\|V_n^{*k_n}x\| < \frac{1}{2^n}\varepsilon, \quad n \in \mathbb{N}. \tag{2.5}$$

Rewrite $(W_1, W_2, \ldots) = (V_1^{k_1}, V_2^{k_2}, \ldots)$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|(I - W_1W_1^*)\cdots(I - W_nW_n^*)x\| \\
&\qquad \geqslant \|(I - W_2W_2^*)\cdots(I - W_nW_n^*)x\| - \|W_1W_1^*(I - W_2W_2^*)\cdots(I - W_nW_n^*)x\| \\
&\qquad \geqslant \|(I - W_2W_2^*)\cdots(I - W_nW_n^*)x\| - \|W_1(I - W_2W_2^*)\cdots(I - W_nW_n^*)W_1^*x\|
\end{aligned}$$

$$\geqslant \|(I - W_2 W_2^*) \cdots (I - W_n W_n^*) x\| - \|W_1^* x\|,$$

and then by induction and (2.5),

$$\|(I - W_1 W_1^*) \cdots (I - W_n W_n^*) x\| \geqslant \|x\| - \sum_{i=1}^{n} \|W_i^* x\| > 1 - \left(1 - \frac{1}{2^n}\right)\varepsilon.$$

Setting $n \to \infty$ in the above inequality, we obtain the desired conclusion. $\qquad\square$

3 Dilation theory

In this section, we give some operator-theoretical characterization for sequences in the class $\mathcal{DC}$. Recall that a sequence $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ is said to have a decomposition of quasi-Beurling type if there exists an orthogonal decomposition $\mathcal{H} = \bigoplus_\gamma \mathcal{H}_\gamma$ of the Hilbert space $\mathcal{H}$, such that each $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing and $\boldsymbol{T}|_{\mathcal{H}_\gamma}$ is of quasi-Beurling type.

Proof of Theorem 1.3.   Consider the following statement:

(2′) there exist an orthogonal decomposition $\mathcal{H} = \bigoplus_\gamma \mathcal{H}_\gamma$ of the Hilbert space $\mathcal{H}$ and $\lambda_\gamma \in \mathbb{D}^\infty$ corresponding to each index γ, such that each $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing and

$$\|x\|^2 = \sum_\gamma \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|P_{\mathcal{H}_\gamma} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2.$$

Here, $P_{\mathcal{H}_\gamma}$ denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_\gamma$.

Our strategy of the proof is to show that (1), (2′) and (3) are equivalent firstly, and then show that (2) and (3) are equivalent.

(1) $\Rightarrow$ (2′). Suppose that there exists an (at most countable) orthogonal decomposition $\mathcal{H} = \bigoplus_\gamma \mathcal{H}_\gamma$ of the Hilbert space $\mathcal{H}$, such that each $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing and $\boldsymbol{T}|_{\mathcal{H}_\gamma}$ is of quasi-Beurling type. For each γ, choose $\lambda_\gamma \in \mathbb{D}^\infty$ such that $\Phi_{\lambda_\gamma}(\boldsymbol{T})$ is of Beurling type. Now take an arbitrary element x in $\mathcal{H}$, and let x_γ be the orthogonal projection of x into the subspace $\mathcal{H}_\gamma$. Then $\|x\|^2 = \sum_\gamma \|x_\gamma\|^2$. The implication (1) $\Rightarrow$ (2′) is proved once we show

$$\|x_\gamma\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|P_{\mathcal{H}_\gamma} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2. \tag{3.1}$$

Since each $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing, we see $\boldsymbol{T}|_{\mathcal{H}_\gamma} \in \mathcal{DC}(\mathcal{H}_\gamma)$. Fix an index γ and let $\boldsymbol{V} \in \mathcal{DI}(\mathcal{K})$ be the minimal regular isometric dilation of $\boldsymbol{T}|_{\mathcal{H}_\gamma}$. Then by Lemma 2.1, $\boldsymbol{V}$ is a coextension of $\boldsymbol{T}|_{\mathcal{H}_\gamma}$. Since $\boldsymbol{T}|_{\mathcal{H}_\gamma}$ is of quasi-Beurling type, there is a point $\lambda \in \mathbb{D}^\infty$ such that $\boldsymbol{W} = (W_1, W_2, \ldots) = \Phi_\lambda(\boldsymbol{V})$ is of Beurling type, i.e., $[\mathfrak{D}_{\boldsymbol{W}^*}]_{\boldsymbol{W}} = \mathcal{K}$. Rewrite $\mathcal{E} = \mathfrak{D}_{\boldsymbol{W}^*}$. Since $\boldsymbol{W}$ is doubly commuting and $\mathcal{E} = \bigcap_{n=1}^\infty \mathrm{Ker} W_n^*$, we have that the family $\{\boldsymbol{W}^\alpha \mathcal{E}\}_{\alpha \in \mathbb{Z}_+^{(\infty)}}$ of subspaces is pairwise orthogonal, and therefore,

$$\mathcal{K} = \bigoplus_{\alpha \in \mathbb{Z}_+^{(\infty)}} \boldsymbol{W}^\alpha \mathcal{E}.$$

It is easy to see that for each $\alpha \in \mathbb{Z}_+^{(\infty)}$, the operator $\boldsymbol{W}^\alpha D_{\boldsymbol{W}^*} \boldsymbol{W}^{*\alpha}$ is exactly the orthogonal projection onto $\boldsymbol{W}^\alpha \mathcal{E}$, and then

$$\|x_\gamma\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|\boldsymbol{W}^\alpha D_{\boldsymbol{W}^*} \boldsymbol{W}^{*\alpha} x_\gamma\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\boldsymbol{W}^*} \boldsymbol{W}^{*\alpha} x_\gamma\|^2.$$

Also noting that $\boldsymbol{W}$ is an isometrical coextension of $\Phi_\lambda(\boldsymbol{T})$, by Lemma 2.6 we have

$$
\begin{aligned}
\|x_\gamma\|^2 &= \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\boldsymbol{W}^*} \boldsymbol{W}^{*\alpha} x_\gamma\|^2 \\
&= \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\boldsymbol{W}^*} \Phi_\lambda^\alpha(\boldsymbol{T}|_{\mathcal{H}_\gamma})^* x_\gamma\|^2 \\
&= \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\Phi_\lambda(\boldsymbol{T}|_{\mathcal{H}_\gamma})^*} \Phi_\lambda^\alpha(\boldsymbol{T}|_{\mathcal{H}_\gamma})^* x_\gamma\|^2 \\
&= \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\Phi_\lambda(\boldsymbol{T})^*}|_{\mathcal{H}_\gamma} \Phi_\lambda^\alpha(\boldsymbol{T})^*|_{\mathcal{H}_\gamma} x_\gamma\|^2 \\
&= \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{\Phi_\lambda(\boldsymbol{T})^*} \Phi_\lambda^\alpha(\boldsymbol{T})^* P_{\mathcal{H}_\gamma} x\|^2 \\
&= \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|P_{\mathcal{H}_\gamma} D_{\Phi_\lambda(\boldsymbol{T})^*} \Phi_\lambda^\alpha(\boldsymbol{T})^* x\|^2.
\end{aligned}
$$

$(2') \Rightarrow (3)$. Fix $x \in \mathcal{H} \ominus (\bigvee_{\lambda \in \mathbb{D}^\infty} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*})$. We show that for any given $\mu = (\mu_1, \mu_2, \ldots) \in \mathbb{D}^\infty$ and $\alpha = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in \mathbb{Z}_+^{(\infty)}$ ($n \in \mathbb{N}$),

$$
D_{\Phi_\mu(\boldsymbol{T})^*} \Phi_\mu^\alpha(\boldsymbol{T})^* x = 0
$$

so that $x = 0$ by the assumption in $(2')$.

Put $\widetilde{\boldsymbol{T}} = (T_{n+1}, T_{n+2}, \ldots)$ and $\tilde{\mu} = (\mu_{n+1}, \mu_{n+2}, \ldots)$, and let $(\lambda, \tilde{\mu})$ denote the sequence

$$
(\lambda_1, \ldots, \lambda_n, \mu_{n+1}, \mu_{n+2}, \ldots)
$$

for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$. Since x is orthogonal to

$$
\mathfrak{D}_{\Phi_{(\lambda,\tilde{\mu})}(\boldsymbol{T})^*} = \overline{\mathrm{Ran} D_{\Phi_{(\lambda,\tilde{\mu})}(\boldsymbol{T})^*}}
$$

for each $\lambda \in \mathbb{D}^n$, we have

$$
x \in \bigcap_{\lambda \in \mathbb{D}^n} \mathrm{Ker} D^*_{\Phi_{(\lambda,\tilde{\mu})}(\boldsymbol{T})^*} = \bigcap_{\lambda \in \mathbb{D}^n} \mathrm{Ker} D_{\Phi_{(\lambda,\tilde{\mu})}(\boldsymbol{T})^*}. \tag{3.2}
$$

Note that for each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}^n$,

$$
D_{\Phi_{(\lambda,\tilde{\mu})}(\boldsymbol{T})^*} = D_{\varphi_{\lambda_1}(T_1)^*} \cdots D_{\varphi_{\lambda_n}(T_n)^*} D_{\Phi_{\tilde{\mu}}(\widetilde{\boldsymbol{T}})^*},
$$

and therefore (3.2) gives $D_{\Phi_{\tilde{\mu}}(\widetilde{\boldsymbol{T}})^*} x \in \mathrm{Ker} D_{\varphi_{\lambda_1}(T_1)^*} \cdots D_{\varphi_{\lambda_n}(T_n)^*}$. This together with Lemma 2.7 implies $D_{\Phi_{\tilde{\mu}}(\widetilde{\boldsymbol{T}})^*} x = 0$. Since the sequence $\boldsymbol{T}$ is doubly commuting, $D_{\Phi_{\tilde{\mu}}(\widetilde{\boldsymbol{T}})^*}$ commutes with $\Phi_\mu^\alpha(\boldsymbol{T})^*$ on $\mathcal{H}$, which gives

$$
\begin{aligned}
D_{\Phi_\mu(\boldsymbol{T})^*} \Phi_\mu^\alpha(\boldsymbol{T})^* x &= D_{\varphi_{\mu_1}(T_1)^*} \cdots D_{\varphi_{\mu_n}(T_n)^*} D_{\Phi_{\tilde{\mu}}(\widetilde{\boldsymbol{T}})^*} \Phi_\mu^\alpha(\boldsymbol{T})^* x \\
&= D_{\varphi_{\mu_1}(T_1)^*} \cdots D_{\varphi_{\mu_n}(T_n)^*} \Phi_\mu^\alpha(\boldsymbol{T})^* D_{\Phi_{\tilde{\mu}}(\widetilde{\boldsymbol{T}})^*} x \\
&= 0.
\end{aligned}
$$

$(3) \Rightarrow (1)$. In order to make this part of the proof more accessible, we divide it into several steps.

Step I.   We give the construction of the subspaces $\mathcal{H}_\gamma$ of the Hilbert space $\mathcal{H}$, and prove that each $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing.

Define a binary relation $\sim$ on $\mathbb{D}^\infty$ as follows: for two points $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ in $\mathbb{D}^\infty$, set

$$
\lambda \sim \mu \Leftrightarrow \sum_{n=1}^\infty \left| \frac{\lambda_n - \mu_n}{1 - \overline{\lambda_n} \mu_n} \right|^2 < \infty.
$$

It follows from the triangle inequality of the pseudo-hyperbolic distance

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in \mathbb{D}$$

that the binary relation $\sim$ is transitive, and therefore is an equivalence relation on $\mathbb{D}^\infty$. The set of $\sim$-equivalence classes is denoted by Δ. For $\gamma \in \Delta$, put

$$\mathcal{H}_\gamma = \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*}.$$

Now we show that $\mathcal{H}_\gamma$ is $\boldsymbol{T}$-joint reducing for any given $\gamma \in \Delta$. Put $\boldsymbol{T}' = (T_2, T_3, \ldots)$ and $\gamma' = \{(\lambda_2, \lambda_3, \ldots) : \lambda = (\lambda_1, \lambda_2, \ldots) \in \gamma\}$. Since the sequence $\boldsymbol{T}$ is doubly commuting, we have that the defect operators $D_{\varphi_a(T_1)^*}$ and $D_{\Phi_\lambda(\boldsymbol{T}')^*}$ commute with each other for every $a \in \mathbb{D}$ and every $\lambda \in \mathbb{D}^\infty$. Also, it is easy to see that $\gamma = \mathbb{D} \times \gamma'$, i.e.,

$$\gamma = \{(a, \mu) : a \in \mathbb{D}, \mu \in \gamma'\}.$$

It follows from Lemma 2.7 that

$$\mathcal{H} = \bigvee_{a \in \mathbb{D}} D_{\varphi_a(T_1)^*} \mathcal{H},$$

and therefore,

$$
\begin{aligned}
\mathcal{H}_\gamma &= \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*} \\
&= \bigvee_{\mu \in \gamma'} \left(\bigvee_{a \in \mathbb{D}} D_{\varphi_a(T_1)^*} D_{\Phi_\mu(\boldsymbol{T}')^*} \mathcal{H} \right) \\
&= \bigvee_{\mu \in \gamma'} D_{\Phi_\mu(\boldsymbol{T}')^*} \left(\bigvee_{a \in \mathbb{D}} D_{\varphi_a(T_1)^*} \mathcal{H} \right) \\
&= \bigvee_{\mu \in \gamma'} \mathfrak{D}_{\Phi_\mu(\boldsymbol{T}')^*}.
\end{aligned}
$$

This implies that $\mathcal{H}_\gamma$ is reducing for T_1. Similarly, $\mathcal{H}_\gamma$ is T_n-reducing for each $n \geqslant 2$.

Step II.   We prove $\mathcal{H} = \bigoplus_{\gamma \in \Delta} \mathcal{H}_\gamma$.

By the assumption in (3), we have

$$\mathcal{H} = \bigvee_{\gamma \in \Delta} \mathcal{H}_\gamma.$$

It remains to show that the subspaces $\mathcal{H}_\gamma$ ($\gamma \in \Delta$) are pairwise orthogonal.

Let $\boldsymbol{V} = (V_1, V_2, \ldots) \in \mathcal{DI}(\mathcal{K})$ be the minimal regular isometric dilation of $\boldsymbol{T}$. Put

$$\mathcal{K}_\gamma = \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}$$

for $\gamma \in \Delta$. Applying the argument in Step I, we see that each $\mathcal{K}_\gamma$ is $\boldsymbol{V}$-joint reducing.

For any closed subspace M of the Hilbert space $\mathcal{K}$, let P_M denote the orthogonal projection from $\mathcal{K}$ onto M. We first prove the following claims:

(a) For each $\gamma \in \Delta$, $\mathcal{H}_\gamma \subseteq \overline{P_{\mathcal{H}} \mathcal{K}_\gamma}$.

Assume that $x \in \mathcal{H}$ is orthogonal to $P_{\mathcal{H}} \mathcal{K}_\gamma$. Then x is orthogonal to $\mathcal{K}_\gamma$, which implies

$$x \in \mathrm{Ker} D_{\Phi_\lambda(\boldsymbol{V})^*} = \mathrm{Ker} D^*_{\Phi_\lambda(\boldsymbol{V})^*}, \quad \lambda \in \gamma.$$

It follows from Lemma 2.1 that $\boldsymbol{V}$ is a coextension of $\boldsymbol{T}$. Then by Lemma 2.6, we have

$$x \in \mathrm{Ker} D_{\Phi_\lambda(\boldsymbol{T})^*} = \mathrm{Ker} D^*_{\Phi_\lambda(\boldsymbol{T})^*}, \quad \lambda \in \gamma.$$

Therefore, x is orthogonal to $\mathcal{H}_\gamma$. This gives

$$\mathcal{H} \ominus \overline{P_\mathcal{H} \mathcal{K}_\gamma} \subseteq \mathcal{H} \ominus \mathcal{H}_\gamma,$$

and then the claim (a) is proved.

(b) The subspaces $\mathcal{K}_\gamma$ ($\gamma \in \Delta$) of $\mathcal{K}$ are pairwise orthogonal.

Assume that $x \in \mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*} \neq \{0\}$ for some $\lambda \in \mathbb{D}^\infty$. It suffices to show that for any $\mu \in \mathbb{D}^\infty$ not equivalent to λ,

$$x \in \mathrm{Ker} D_{\Phi_\mu(\boldsymbol{V})^*} = \mathrm{Ker} D^*_{\Phi_\mu(\boldsymbol{V})^*}.$$

We first consider the case $\lambda = \boldsymbol{0}$. Then we have $\mu \notin \mathbb{D}_2^\infty$ in this case. By Lemma 2.8, $[x]_{\boldsymbol{V}}$ is $\boldsymbol{V}$-joint reducing, and $\boldsymbol{V}|_{[x]_{\boldsymbol{V}}}$ is jointly unitarily equivalent to the tuple $\boldsymbol{M}_\zeta$ of coordinate multiplication operators on the Hardy space $H^2(\mathbb{D}_2^\infty)$. Then $[x]_{\boldsymbol{V}}$ is reducing for $D_{\Phi_\mu(\boldsymbol{V})^*}$, and $D_{\Phi_\mu(\boldsymbol{V})^*}|_{[x]_{\boldsymbol{V}}}$ is jointly unitarily equivalent to $D_{\Phi_\mu(\boldsymbol{M}_\zeta)^*}$. It follows from Lemma 2.9 that $D_{\Phi_\mu(\boldsymbol{M}_\zeta)^*} = 0$, and hence $D_{\Phi_\mu(\boldsymbol{V})^*}$ vanishes on $[x]_{\boldsymbol{V}}$. The case for general $\lambda \in \mathbb{D}^\infty$ is proved by replacing the sequence $\boldsymbol{V}$ with $\Phi_\lambda(\boldsymbol{V})$ in the previous argument.

By the claims (a) and (b), it suffices to show that $P_\mathcal{H} P_{\mathcal{K}_\gamma} = P_{\mathcal{K}_\gamma} P_\mathcal{H}$ for every $\gamma \in \Delta$. Put

$$\mathcal{H}_n = \bigvee_{\substack{\alpha \in \mathbb{Z}_+^{(\infty)} \\ \alpha_n = 0}} \boldsymbol{V}^\alpha \mathcal{H}$$

for $n \in \mathbb{N}$. We also claim that

(c) for each $n \in \mathbb{N}$ and $\gamma \in \Delta$, $P_{\mathcal{H}_n} P_{\mathcal{K}_\gamma} = P_{\mathcal{K}_\gamma} P_{\mathcal{H}_n}$.

Assuming this claim, we see that the desired conclusion $P_\mathcal{H} P_{\mathcal{K}_\gamma} = P_{\mathcal{K}_\gamma} P_\mathcal{H}$ ($\gamma \in \Delta$) immediately follows from Lemma 2.2(3). Below we prove the claim (c).

Without loss of generality, we only prove the case for $n = 1$. Put $\boldsymbol{V}' = (V_2, V_3, \ldots)$. By Lemma 2.2(1), $\mathcal{H}_1$ is joint invariant for $\boldsymbol{V}^*$. This gives that $\mathcal{H}_1$ is $\boldsymbol{V}'$-joint reducing, and then for each $\lambda \in \mathbb{D}^\infty$,

$$P_{\mathcal{H}_1} D_{\Phi_\lambda(\boldsymbol{V}')^*} = D_{\Phi_\lambda(\boldsymbol{V}')^*} P_{\mathcal{H}_1}.$$

Applying a similar argument in Step I, one obtains

$$\mathcal{K}_\gamma = \bigvee_{\mu \in \gamma'} \mathfrak{D}_{\Phi_\mu(\boldsymbol{V}')^*},$$

where $\gamma' = \{(\lambda_2, \lambda_3, \ldots) : \lambda = (\lambda_1, \lambda_2, \ldots) \in \gamma\}$. Thus, the claim (c) follows.

Step III.   We prove that the restriction $\boldsymbol{T}|_{\mathcal{H}_\gamma}$ on each nonzero $\mathcal{H}_\gamma$ ($\gamma \in \Delta$) is of quasi-Beurling type, and this completes the proof.

Suppose that $\mathcal{H}_\gamma$ is nonzero for some $\gamma \in \Delta$. By the claims in Step II of the proof, we see

$$\mathcal{H}_\gamma \subseteq \overline{P_\mathcal{H} \mathcal{K}_\gamma} = \mathcal{H} \cap \mathcal{K}_\gamma.$$

Then the minimal regular isometric dilation of the sequence $\boldsymbol{T}|_{\mathcal{H}_\gamma}$ is $\boldsymbol{V}|_{\mathcal{K}_\gamma}$. It remains to prove that $\boldsymbol{V}|_{\mathcal{K}_\gamma}$ is of quasi-Beurling type, i.e.,

$$[\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}]_{\boldsymbol{V}} = [\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}]_{\Phi_\lambda(\boldsymbol{V})} = \mathcal{K}_\gamma$$

for some $\lambda \in \gamma$. Actually, we show that the above identity holds for every $\lambda \in \gamma$.

Fix $\lambda = (\lambda_1, \lambda_2, \ldots) \in \gamma$. Since $\mathcal{K}_\gamma$ is $\boldsymbol{V}$-joint reducing (see Step II), one has $[\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}]_{\boldsymbol{V}} \subseteq \mathcal{K}_\gamma$. For the converse inclusion, we show $\mathfrak{D}_{\Phi_\mu(\boldsymbol{V})^*} \subseteq [\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}]_{\boldsymbol{V}}$ for every $\mu = (\mu_1, \mu_2, \ldots) \in \gamma$ with $\mathfrak{D}_{\Phi_\mu(\boldsymbol{V})^*} \neq \{0\}$. Now take an arbitrary nonzero element $x \in \mathfrak{D}_{\Phi_\mu(\boldsymbol{V})^*}$. Lemma 2.8 implies that $[x]_{\boldsymbol{V}}$ is $\boldsymbol{V}$-joint reducing, and $\boldsymbol{V}|_{[x]_{\boldsymbol{V}}}$ is jointly unitarily equivalent to the sequence $\Phi_\mu(\boldsymbol{M}_\zeta)$, where $\boldsymbol{M}_\zeta$ is the tuple of coordinate multiplication operators on the Hardy space $H^2(\mathbb{D}_2^\infty)$. Then $[x]_{\boldsymbol{V}}$ is reducing for $D_{\Phi_\lambda(\boldsymbol{V})^*}$, and $D_{\Phi_\lambda(\boldsymbol{V})^*}|_{[x]_{\boldsymbol{V}}}$ is jointly unitarily equivalent to

$$D_{(\Phi_\lambda \circ \Phi_\mu(\boldsymbol{M}_\zeta))^*} = \prod_{n=1}^\infty (I - (\varphi_{\lambda_n} \circ \varphi_{\mu_n}(M_{\zeta_n}))(\varphi_{\lambda_n} \circ \varphi_{\mu_n}(M_{\zeta_n}))^*).$$

Note that for each $n \in \mathbb{N}$, $\varphi_{\lambda_n} \circ \varphi_{\mu_n} = c_n \varphi_{\eta_n}$ for some unimodular constant c_n and $\eta_n \in \mathbb{D}$, and we have

$$
\begin{aligned}
D_{(\Phi_\lambda \circ \Phi_\mu(\boldsymbol{M}_\zeta))^*} &= \prod_{n=1}^{\infty} (I - (c_n \varphi_{\eta_n}(M_{\zeta_n}))(c_n \varphi_{\eta_n}(M_{\zeta_n}))^*) \\
&= \prod_{n=1}^{\infty} (I - \varphi_{\eta_n}(M_{\zeta_n}) \varphi_{\eta_n}(M_{\zeta_n})^*) \\
&= D_{\Phi_\eta(\boldsymbol{M}_\zeta)^*}.
\end{aligned}
$$

The fact that $\lambda \sim \mu$ yields $\eta = (\eta_1, \eta_2, \ldots) \in \mathbb{D}_2^\infty$, and then Lemma 2.9 gives $D_{\Phi_\eta(\boldsymbol{M}_\zeta)^*} \neq 0$. Thus, there exists an element $x' \in [x]_{\boldsymbol{V}}$ such that

$$
y = D_{\Phi_\lambda(\boldsymbol{V})^*} x' \in [x]_{\boldsymbol{V}} \cap \mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}
$$

is nonzero (otherwise, one would have $D_{\Phi_\lambda(\boldsymbol{V})^*} |_{[x]_{\boldsymbol{V}}} = 0$). Again by Lemma 2.8, $[y]_{\boldsymbol{V}}$ is a joint reducing subspace of $[x]_{\boldsymbol{V}}$ for the sequence $\boldsymbol{V} |_{[x]_{\boldsymbol{V}}}$, and of course for the sequence $\Phi_\mu(\boldsymbol{V} |_{[x]_{\boldsymbol{V}}})$. Since $\Phi_\mu(\boldsymbol{V} |_{[x]_{\boldsymbol{V}}})$ is jointly unitarily equivalent to $\boldsymbol{M}_\zeta$, Lemma 2.10 implies that $\Phi_\mu(\boldsymbol{V} |_{[x]_{\boldsymbol{V}}})$ is also irreducible, which gives

$$
[x]_{\boldsymbol{V}} = [y]_{\boldsymbol{V}} \subseteq [\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}]_{\boldsymbol{V}}.
$$

In particular, $x \in [\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}]_{\boldsymbol{V}}$. This completes the proof of Step III.

$(2) \Rightarrow (3)$. In the proof of the implication $(2') \Rightarrow (3)$, we have shown that if

$$
x \in \mathcal{H} \ominus \left(\bigvee_{\lambda \in \mathbb{D}^\infty} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*} \right),
$$

then

$$
D_{\Phi_\mu(\boldsymbol{T})^*} \Phi_\mu^\alpha(\boldsymbol{T})^* x = 0
$$

for every $\mu \in \mathbb{D}^\infty$ and every $\alpha \in \mathbb{Z}_+^{(\infty)}$. Thus for such an x, we still have $x = 0$ under the assumption in (2).

$(3) \Rightarrow (2)$. Now assume that (3) holds. In the proof of the implication $(3) \Rightarrow (1)$, we actually establish a concrete orthogonal decomposition of $\mathcal{H}$, i.e., there exists an at most countable subset Δ_0 of Δ, such that $\mathcal{H} = \bigoplus_{\gamma \in \Delta_0} \mathcal{H}_\gamma$, where

$$
\mathcal{H}_\gamma = \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*}.
$$

Moreover, $\Phi_\lambda(\boldsymbol{T} |_{\mathcal{H}_\gamma})$ is of Beurling-type for every $\lambda \in \gamma$.

For each $\gamma \in \Delta_0$, we fix $\lambda_\gamma \in \gamma$. As in the proof of the implication $(1) \Rightarrow (2')$, one has

$$
\|P_{\mathcal{H}_\gamma} x\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|P_{\mathcal{H}_\gamma} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2
$$

for each $\gamma \in \Delta_0$. Since $\|x\|^2 = \sum_{\gamma \in \Delta_0} \|P_{\mathcal{H}_\gamma} x\|^2$ for every $x \in \mathcal{H}$, it remains to show

$$
\|P_{\mathcal{H}_\gamma} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2 = \|D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2
$$

for any given $\gamma \in \Delta_0$, $\alpha \in \mathbb{Z}_+^{(\infty)}$ and $x \in \mathcal{H}$. Since for any $\delta \in \Delta_0$ with $\delta \neq \gamma$, $\mathcal{H}_\delta$ is orthogonal to

$$
\mathcal{H}_\gamma = \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*},
$$

we have $P_{\mathcal{H}_\delta} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} = 0$. It follows that

$$
\begin{aligned}
\|D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2 &= \|P_{\mathcal{H}_\gamma} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2 + \sum_{\delta \neq \gamma} \|P_{\mathcal{H}_\delta} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2 \\
&= \|P_{\mathcal{H}_\gamma} D_{\Phi_{\lambda_\gamma}(\boldsymbol{T})^*} \Phi_{\lambda_\gamma}^\alpha(\boldsymbol{T})^* x\|^2,
\end{aligned}
$$

which completes the proof. $\qquad \square$

From the proof of Theorem 1.3, we actually obtain the following collection of results, which will be used later.

Corollary 3.1. *Let T be a sequence in the class $\mathcal{DC}(\mathcal{H})$, and $V \in \mathcal{DI}(\mathcal{K})$ be the minimal regular isometric dilation of T.*

(1) *If T is of Beurling type, then $\mathcal{K}$ has an orthogonal decomposition*

$$\mathcal{K} = \bigoplus_{\alpha \in \mathbb{Z}_+^{(\infty)}} V^\alpha \mathfrak{D}_{V^*},$$

and for each $x \in \mathcal{H}$,

$$\|x\|^2 = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \|D_{T^*} T^{*\alpha} x\|^2.$$

(2) *If $\lambda, \mu \in \mathbb{D}^\infty$ are not $\sim$-equivalent, then the defect spaces $\mathfrak{D}_{\Phi_\lambda(T)^*}$ and $\mathfrak{D}_{\Phi_\mu(T)^*}$ are mutually orthogonal.*

(3) *For any $\sim$-equivalence class γ, put*

$$\mathcal{H}_\gamma = \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(T)^*}, \quad \mathcal{K}_\gamma = \bigvee_{\lambda \in \gamma} \mathfrak{D}_{\Phi_\lambda(V)^*}.$$

Then $\mathcal{K}_\gamma = [\mathfrak{D}_{\Phi_\mu(V)^*}]_V$ for each $\mu \in \gamma$, $V|_{\mathcal{K}_\gamma}$ is the minimal regular isometric dilation of $T|_{\mathcal{H}_\gamma}$, and $\mathcal{H}_\gamma, \mathcal{K}_\gamma$ are joint reducing for T and V, respectively.

We are ready to prove Corollary 1.4.

Proof of Corollary 1.4.   (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). See Corollary 3.1(1).

(2) $\Rightarrow$ (1). This is obvious.

(3) $\Rightarrow$ (4). This implication follows from the proof of Theorem 1.3 and the fact that $\mathbb{D}_2^\infty$ coincides with the Cartesian product

$$\{(\lambda, \mu) : \lambda \in \mathbb{D}^n, \mu \in \mathbb{D}_2^\infty\}.$$

(4) $\Rightarrow$ (1). Let $V \in \mathcal{DI}(\mathcal{K})$ be the minimal regular isometric dilation of T. Note that $\mathbb{D}_2^\infty$ is a $\sim$-equivalence class that contains $\mathbf{0}$ and $\mathcal{H} = \mathcal{H}_{\mathbb{D}_2^\infty}$. It follows from Corollary 3.1(3) that

$$\mathcal{K} = \mathcal{K}_{\mathbb{D}_2^\infty} = [\mathfrak{D}_{V^*}]_V.$$

The proof is completed. $\square$

It follows immediately from Corollary 1.4 that a sequence $T \in \mathcal{DC}$ is of quasi-Beurling type if and only if the minimal regular isometric dilation of T is jointly unitarily equivalent to the sequence $\Phi_\lambda(M_\zeta)$ on a vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ for some $\lambda \in \mathbb{D}^\infty$.

Below we give an example to illustrate that Theorem 1.3(3) is nontrivial for sequences in the class $\mathcal{DC}$. By comparing this with Lemma 2.7, we see that the infinite-tuple case diverges considerably from the finite-tuple case.

Theorem 3.2. *There exists a sequence $V \in \mathcal{DI}$ such that $\mathfrak{D}_{\Phi_\lambda(V)^*} = \{0\}$ for each $\lambda \in \mathbb{D}^\infty$.*

Proof.   Let T^2 denote $(T_1^2, T_2^2, \ldots)$ for a sequence $T = (T_1, T_2, \ldots)$ of operators, and $M_\zeta = (M_{\zeta_1}, M_{\zeta_2}, \ldots)$ be the tuple of coordinate multiplication operators on $H^2(\mathbb{D}_2^\infty)$. Put

$$\mathcal{E}_n = \operatorname{Ker} M_{\zeta_n}^* = H^2(\mathbb{D}_2^\infty) \ominus \zeta_n H^2(\mathbb{D}_2^\infty), \quad n \in \mathbb{N}.$$

It is clear that

$$H^2(\mathbb{D}_2^\infty) = \bigoplus_{k=0}^\infty \zeta_n^k \mathcal{E}_n$$

for each $n \in \mathbb{N}$. Define a sequence V of isometries on $H^2(\mathbb{D}_2^\infty)$ by setting

$$V_n(\zeta_n^k F) = \begin{cases} \zeta_n^{k+3} F, & \text{if } k \text{ is even}, \\ \zeta_n^{k-1} F, & \text{if } k \text{ is odd} \end{cases}$$

for $n \in \mathbb{N}$ and $F \in \mathcal{E}_n$. It is routine to check that $\boldsymbol{V} \in \mathcal{DI}$ and $\boldsymbol{V}^2 = \boldsymbol{M}_\zeta^2$.

Now we show $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*} = \{0\}$ for each $\lambda \in \mathbb{D}^\infty$. Assume conversely that there exists a point $\lambda \in \mathbb{D}^\infty$ such that $\mathfrak{D}_{\Phi_\lambda(\boldsymbol{V})^*}$ contains a function $F \neq 0$. By the comments above Lemma 2.8, F is exactly an eigenvector of the sequence $\boldsymbol{V}^*$ corresponding to the joint eigenvalue $\overline{\lambda} = (\overline{\lambda_1}, \overline{\lambda_2}, \ldots)$, and therefore,

$$F \in \mathfrak{D}_{\Phi_{\lambda^2}(\boldsymbol{V}^2)^*} = \mathfrak{D}_{\Phi_{\lambda^2}(\boldsymbol{M}_\zeta^2)^*},$$

where $\lambda^2 = (\lambda_1^2, \lambda_2^2, \ldots)$. Let $\boldsymbol{K}_\mu$ denote the reproducing kernel of $H^2(\mathbb{D}_2^\infty)$ at the point $\mu \in \mathbb{D}_2^\infty$. Then

$$\boldsymbol{K}_\mu \in \mathfrak{D}_{\Phi_{\mu^2}(\boldsymbol{M}_\zeta^2)^*} \subseteq \bigvee_{\xi \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\xi(\boldsymbol{M}_\zeta^2)^*},$$

which gives

$$\bigvee_{\xi \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\xi(\boldsymbol{M}_\zeta^2)^*} = H^2(\mathbb{D}_2^\infty),$$

since the set $\{\boldsymbol{K}_\mu : \mu \in \mathbb{D}_2^\infty\}$ is complete in $H^2(\mathbb{D}_2^\infty)$. Then by Corollary 3.1(2), $\mathfrak{D}_{\Phi_\xi(\boldsymbol{V}^2)^*} = \{0\}$ for $\xi \notin \mathbb{D}_2^\infty$, forcing $\lambda^2 \in \mathbb{D}_2^\infty$. In particular, $\lambda_n \to 0$ $(n \to \infty)$.

Write $\mathcal{F} = \mathfrak{D}_{\boldsymbol{V}^{*2}} = \mathfrak{D}_{\boldsymbol{M}_\zeta^{*2}}$. Then

$$\mathcal{F} = \bigcap_{n=1}^\infty \mathrm{Ker} M_{\zeta_n}^{*2} = \overline{\mathrm{span}\{\zeta^\alpha : \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)} \text{ with each } \alpha_n \leqslant 1\}}$$

and

$$P_\mathcal{F} = \prod_{n=1}^\infty (I - V_n^2 V_n^{*2}) = \prod_{n=1}^\infty (I - M_{\zeta_n}^2 M_{\zeta_n}^{*2}),$$

where $P_\mathcal{F}$ is the orthogonal projection from $H^2(\mathbb{D}_2^\infty)$ onto $\mathcal{F}$. This implies that

$$\prod_{i=1}^n (I - M_{\zeta_i}^2 M_{\zeta_i}^{*2})F = \prod_{i=1}^n (I - M_{\zeta_i}^2 V_i^{*2})F = F \cdot \prod_{i=1}^n (1 - \overline{\lambda_n^2} \zeta_n^2)$$

converges to $G = P_\mathcal{F} F$ $(n \to \infty)$ in the $H^2(\mathbb{D}_2^\infty)$-norm. Note that the reproducing kernel $\boldsymbol{K}_{\lambda^2}$ vanishes nowhere on $\mathbb{D}_2^\infty$, $\prod_{i=1}^n (1 - \overline{\lambda_n^2} \zeta_n^2)$ converges pointwisely to the function $\frac{1}{\boldsymbol{K}_{\lambda^2}}$ on $\mathbb{D}_2^\infty$ as $n \to \infty$, and G must coincide with the function $\frac{F}{\boldsymbol{K}_{\lambda^2}}$ on $\mathbb{D}_2^\infty$, forcing $G \neq 0$. Since $G \in \mathcal{F}$, there exists some $\alpha = (\alpha_1, \alpha_2, \ldots)$ $\in \mathbb{Z}_+^{(\infty)}$ with each $\alpha_n \leqslant 1$ such that $\langle G, \zeta^\alpha \rangle \neq 0$.

Below we prove $\langle G, \zeta^\alpha \rangle = 0$ to reach a contradiction. Since for each $n \in \mathbb{N}$,

$$G \in \mathrm{Ker} M_{\zeta_n}^{*2} = \mathrm{Ker} V_n^{*2},$$

there corresponds a decomposition $G = G_n + H_n$ of G such that $G_n \in \mathrm{Ker} V_n^*$ and $H_n \in \mathrm{Ker} V_n^{*2} \ominus \mathrm{Ker} V_n^*$. Since $\boldsymbol{V}$ is doubly commuting, for any $n \in \mathbb{N}$,

$$G_n = (I - V_n V_n^*)G = (I - V_n V_n^*) \cdot \prod_{m \neq n}^\infty (I - V_m^2 V_m^{*2})F$$

and

$$\begin{aligned}
H_n &= (V_n V_n^* - V_n^2 V_n^{*2})G \\
&= (V_n V_n^* - V_n^2 V_n^{*2}) \cdot \prod_{m \neq n}^\infty (I - V_m^2 V_m^{*2})F \\
&= V_n(I - V_n V_n^*)V_n^* \cdot \prod_{m \neq n}^\infty (I - V_m^2 V_m^{*2})F \\
&= V_n(I - V_n V_n^*) \cdot \prod_{m \neq n}^\infty (I - V_m^2 V_m^{*2})V_n^* F \\
&= \overline{\lambda_n} V_n(I - V_n V_n^*) \cdot \prod_{m \neq n}^\infty (I - V_m^2 V_m^{*2})F,
\end{aligned}$$

which gives

$$\|H_n\| = |\lambda_n| \|V_n G_n\| = |\lambda_n| \|G_n\|.$$

By the fact that $\lambda_n \to 0$ $(n \to \infty)$ and $\|G\|^2 = \|G_n\|^2 + \|H_n\|^2$ $(n \in \mathbb{N})$, we see $\|H_n\| \to 0$ $(n \to \infty)$. This gives that $G_n \to G$ $(n \to \infty)$ in the norm, and then $\langle G_n, \zeta^\alpha \rangle \to \langle G, \zeta^\alpha \rangle$ $(n \to \infty)$. Since $G_n \in \zeta_n H^2(\mathbb{D}_2^\infty)$ $(n \in \mathbb{N})$, $\langle G_n, \zeta^\alpha \rangle = 0$ for n large sufficiently, forcing $\langle G, \zeta^\alpha \rangle = 0$. This completes the proof. $\qquad\square$

We also have the following application of Theorem 1.3 and Corollary 1.4. Set Γ to be the set of points

$$\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{D}^\infty$$

satisfying

$$\lim_{m \to \infty} \prod_{n=m}^\infty \lambda_m = 1.$$

For the equivalence relation $\sim$ on $\mathbb{D}^\infty$ given in the proof of Theorem 1.3, Γ is a union of some $\sim$-equivalence classes, namely for a pair λ, μ of $\sim$-equivalent points in $\mathbb{D}^\infty$, $\lambda \in \Gamma$ if and only if $\mu \in \Gamma$. In fact, if $\lambda \in \Gamma$ and $\lambda \sim \mu$, then

$$\sum_{n=1}^\infty (1 - |\lambda_n|) < \infty$$

and

$$\sum_{n=1}^\infty |\lambda_n - \mu_n|^2 < \infty,$$

which give

$$\sum_{n=1}^\infty |1 - \overline{\lambda_n} \mu_n|^2 < \infty,$$

since

$$|1 - \overline{\lambda_n} \mu_n| \leqslant |1 - \overline{\lambda_n} \lambda_n| + |\overline{\lambda_n} \lambda_n - \overline{\lambda_n} \mu_n| \leqslant 2(1 - |\lambda_n|) + |\lambda_n - \mu_n|, \quad n \in \mathbb{N}.$$

Hence by the Cauchy-Schwarz inequality,

$$\sum_{n=1}^\infty |\lambda_n - \mu_n| \leqslant \left(\sum_{n=1}^\infty \left| \frac{\lambda_n - \mu_n}{1 - \overline{\lambda_n} \mu_n} \right|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^\infty |1 - \overline{\lambda_n} \mu_n|^2 \right)^{\frac{1}{2}} < \infty,$$

forcing $\sum_{n=1}^\infty (1 - |\mu_n|) < \infty$ and $\sum_{n=1}^\infty |\arg \lambda_n - \arg \mu_n| < \infty$. It follows immediately that $\mu \in \Gamma$.

Corollary 3.3.   *Suppose* $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ *and put* $A_n = T_1 \cdots T_n$ $(n \in \mathbb{N})$. *If* $\bigvee_{\lambda \in \Gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*} = \mathcal{H}$, *then* $\{A_n\}_{n \in \mathbb{N}}$ *converges in the strong operator topology.*

Proof.   Assume $\bigvee_{\lambda \in \Gamma} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{T})^*} = \mathcal{H}$. By Theorem 1.3, $\boldsymbol{T}$ can be decomposed into the direct sum of $\mathcal{DC}$-sequences of quasi-Beurling type. Without loss of generality, we may further assume that $\boldsymbol{T}$ itself is of quasi-Beurling type. Take $\lambda \in \Gamma$ so that $\Phi_\lambda(\boldsymbol{T})$ is of Beurling type. Then Corollary 1.4 implies that the minimal regular isometric dilation $\boldsymbol{V}$ of $\boldsymbol{T}$ is jointly unitarily equivalent to the sequence $(M_{\widetilde{\varphi_{\lambda_1}}}, M_{\widetilde{\varphi_{\lambda_2}}}, \ldots)$ of multiplication operators on a vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$, where $\widetilde{\varphi_{\lambda_n}}(\zeta) = \varphi_{\lambda_n}(\zeta_n)$ $(\zeta \in \mathbb{D}_2^\infty)$.

Put $F_n = \prod_{i=1}^n \widetilde{\varphi_{\lambda_i}}$ $(n \in \mathbb{N})$. It suffices to show that $\{M_{F_n}\}_{n \in \mathbb{N}}$ converges in the strong operator topology. Since this sequence has uniformly bounded operator norms and

$$\{p \cdot x : p \in \mathcal{P}_\infty, x \in \mathcal{E}\}$$

is complete in $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$, we only need to prove that for every $p \in \mathcal{P}_\infty$ and every $x \in \mathcal{E}$, $\{F_n p \cdot x\}_{n \in \mathbb{N}}$ converges in $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$, where $\mathcal{P}_\infty$ is the polynomial ring in countably infinitely many variables. This is clearly equivalent to the fact that $\{F_n\}_{n \in \mathbb{N}}$ converges in $H^2(\mathbb{D}_2^\infty)$.

For $n > m$, since the norm of a function in $H^2(\mathbb{D}_2^\infty)$ equals the norm of the function on a polydisk if the function depends only on finitely many variables, we have[1]

$$
\begin{aligned}
\|F_n - F_m\|^2 &= \left\| \prod_{i=m+1}^n \widetilde{\varphi_{\lambda_i}} - 1 \right\|^2 \\
&= \int_{\mathbb{T}^{n-m}} \left| \prod_{i=m+1}^n \varphi_{\lambda_i}(\xi_i) - 1 \right|^2 d\xi_{m+1} \cdots d\xi_n \\
&= \int_{\mathbb{T}^{n-m}} \left| \prod_{i=m+1}^n \varphi_{\lambda_i}(\xi_i) \right|^2 d\xi_{m+1} \cdots d\xi_n \\
&\quad - 2\mathrm{Re} \int_{\mathbb{T}^{n-m}} \prod_{i=m+1}^n \varphi_{\lambda_i}(\xi_i) d\xi_{m+1} \cdots d\xi_n + 1 \\
&= 2 - 2\mathrm{Re}\left(\prod_{i=m+1}^n \lambda_i \right) \\
&= 2\mathrm{Re}\left(1 - \prod_{i=m+1}^n \lambda_i \right).
\end{aligned}
$$

Since $\lim_{m \to \infty} \prod_{n=m}^\infty \lambda_m = 1$, $\{F_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, which completes the proof. □

To end this section, we record an example of the $\mathcal{DC}$-sequence of quasi-Beurling type.

Proposition 3.4.   *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of bounded analytic functions on the unit disk $\mathbb{D}$. If for each $n \in \mathbb{N}$, $\|f_n\|_\infty \leqslant 1$ and f_n is not a unimodular constant, then the sequence $(M_{\widetilde{f_1}}, M_{\widetilde{f_2}}, \ldots)$ of multiplication operators on the Hardy space $H^2(\mathbb{D}_2^\infty)$ is a $\mathcal{DC}$-sequence of quasi-Beurling type, where $\widetilde{f_n}(\zeta) = f_n(\zeta_n)$ ($n \in \mathbb{N}$, $\zeta \in \mathbb{D}_2^\infty$).*

Proof.   It is routine to check that $(M_{\widetilde{f_1}}, M_{\widetilde{f_2}}, \ldots)$ is a sequence in the class $\mathcal{DC}$. Put

$$
g_n = \varphi_{f_n(0)} \circ f_n, \quad n \in \mathbb{N}.
$$

Then for each $n \in \mathbb{N}$, $g_n(0) = 0$ and $M_{\widetilde{g_n}} = \varphi_{f_n(0)}(M_{\widetilde{f_n}})$, where $\widetilde{g_n}(\zeta) = g_n(\zeta_n)$. We show that the sequence $\boldsymbol{M} = (M_{\widetilde{g_1}}, M_{\widetilde{g_2}}, \ldots)$ is of Beurling type, which implies the desired conclusion.

Note that for each $n \in \mathbb{N}$ and $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{D}_2^\infty$,

$$
\varphi_{g_n(\lambda_n)}(M_{\widetilde{g_n}})^* \boldsymbol{K}_\lambda = M^*_{\varphi_{g_n(\lambda_n)} \circ \widetilde{g_n}} \boldsymbol{K}_\lambda = \varphi_{g_n(\lambda_n)}(\widetilde{g_n}(\lambda)) \boldsymbol{K}_\lambda = 0,
$$

which gives

$$
(I - \varphi_{g_n(\lambda_n)}(M_{\widetilde{g_n}}) \varphi_{g_n(\lambda_n)}(M_{\widetilde{g_n}})^*) \boldsymbol{K}_\lambda = \boldsymbol{K}_\lambda.
$$

It follows that $D_{\Phi_{\mu(\lambda)}(\boldsymbol{M})^*} \boldsymbol{K}_\lambda = \boldsymbol{K}_\lambda$ for every $\lambda \in \mathbb{D}_2^\infty$, where

$$
\mu(\lambda) = (g_1(\lambda_1), g_2(\lambda_2), \ldots).
$$

Also by Schwarz's lemma, we have that for each $\lambda \in \mathbb{D}_2^\infty$,

$$
\sum_{n=1}^\infty |g_n(\lambda_n)|^2 \leqslant \sum_{n=1}^\infty |\lambda_n|^2 < \infty,
$$

and then

$$
\boldsymbol{K}_\lambda \in \mathfrak{D}_{\Phi_{\mu(\lambda)}(\boldsymbol{M})^*} \subseteq \bigvee_{\mu \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\mu(\boldsymbol{M})^*}.
$$

This gives

$$
\bigvee_{\mu \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\mu(\boldsymbol{M})^*} = H^2(\mathbb{D}_2^\infty).
$$

It follows from Corollary 1.4 that the sequence $\boldsymbol{M} = (M_{\widetilde{g_1}}, M_{\widetilde{g_2}}, \ldots)$ is of Beurling type. □

---

[1] The authors thank the referees for suggesting this calculation to make things more clear.

Remark 3.5. From the proof of Proposition 3.4, we see that the $\sim$-equivalence class of the point $(f_1(0), f_2(0), \ldots) \in \mathbb{D}^\infty$ (see Corollary 3.1) is an invariant for the sequence $(M_{\widetilde{f_1}}, M_{\widetilde{f_2}}, \ldots)$. More precisely, let $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be two sequences of functions that satisfy the conditions given in the proposition, and put $\widetilde{f_n}(\zeta) = f_n(\zeta_n)$ and $\widetilde{g_n}(\zeta) = g_n(\zeta_n)$ ($n \in \mathbb{N}$, $\zeta \in \mathbb{D}_2^\infty$). If the multiplication operators $(M_{\widetilde{f_1}}, M_{\widetilde{f_2}}, \ldots)$ and $(M_{\widetilde{g_1}}, M_{\widetilde{g_2}}, \ldots)$ are jointly unitarily equivalent, then $(f_1(0), f_2(0), \ldots)$ and $(g_1(0), g_2(0), \ldots)$ belong to the same $\sim$-equivalence class, i.e.,

$$\sum_{n=1}^{\infty} \left| \frac{f_n(0) - g_n(0)}{1 - \overline{f_n(0)} g_n(0)} \right|^2 < \infty.$$

Since the number of $\sim$-equivalence classes is uncountable, $\mathbb{D}_2^\infty$ is a very "small" part in $\mathbb{D}^\infty$. Hence, for "almost all" choices of the sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions, the defect space of $(M_{\widetilde{f_1}}^*, M_{\widetilde{f_2}}^*, \ldots)$ is $\{0\}$ by Proposition 3.4 and Corollary 3.1, and then $(M_{\widetilde{f_1}}, M_{\widetilde{f_2}}, \ldots)$ is not of Beurling type.

4 Analytic model

In this section, we prove that every sequence in the class $\mathcal{DI}$ is jointly unitarily equivalent to a sequence of multiplication operators induced by operator-valued inner functions each of which involves one different variable. We thus establish an operator-valued analytic functional model for general $\mathcal{DC}$-sequences.

Theorem 4.1. *Let $\mathbf{T}$ be a sequence in the class $\mathcal{DC}(\mathcal{H})$, and $\mathbf{V} \in \mathcal{DI}(\mathcal{K})$ be the minimal regular isometric dilation of $\mathbf{T}$. Then there exist a Hilbert space $\mathcal{E}$, a unitary operator $U : \mathcal{K} \to H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ and a sequence $\Theta = (\theta_1, \theta_2, \ldots)$ of inner functions in $H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$, such that*

(1) *for each $n \in \mathbb{N}$, $U V_n U^* = M_{\widetilde{\theta_n}}$, where $\widetilde{\theta_n}(\zeta) = \theta_n(\zeta_n)$ ($\zeta \in \mathbb{D}_2^\infty$);*
(2) *$\mathcal{Q} = U\mathcal{H}$ is a quotient module of $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$.*

The tuple $(\mathcal{Q}, \Theta)$ in Theorem 4.1 is said to be an *analytic model* for the $\mathcal{DC}$-sequence $\mathbf{T}$, and the Hilbert space $\mathcal{E}$ is called the *underlying space* of the analytic model $(\mathcal{Q}, \Theta)$. The sequence $(M_{\widetilde{\theta_1}}, M_{\widetilde{\theta_2}}, \ldots)$ is denoted by $\mathbf{M}_\Theta$ for simplicity. Also, for the trivial case

$$\Theta = (z \cdot I_\mathcal{E}, z \cdot I_\mathcal{E}, \ldots),$$

we simply write $\mathcal{Q}$ for $(\mathcal{Q}, \Theta)$. It is clear that $\mathbf{T}$ is jointly unitarily equivalent to the sequence $P_\mathcal{Q} \mathbf{M}_\Theta |_\mathcal{Q}$, the compression of the sequence $\mathbf{M}_\Theta$ on $\mathcal{Q}$.

To prove Theorem 4.1, we need the following proposition.

Proposition 4.2. *Suppose $\mathbf{T} \in \mathcal{DC}$. Then there exists a sequence $(k_1, k_2, \ldots)$ of positive integers, such that $(T_1^{k_1}, T_2^{k_2}, \ldots)$ is of Beurling type.*

Proof. Assume that $\mathbf{T} \in \mathcal{DC}(\mathcal{H})$ and $\mathbf{V} \in \mathcal{DI}(\mathcal{K})$ is the minimal regular isometric dilation of $\mathbf{T}$. For a sequence $\mathbf{k} = (k_1, k_2, \ldots)$ of positive integers, put

$$\mathbf{V_k} = (V_1^{k_1}, V_2^{k_2}, \ldots)$$

and $M_{\mathbf{k}} = [\mathfrak{D}_{\mathbf{V_k}^*}]_{\mathbf{V_k}}$. It suffices to prove that there exists a sequence $\mathbf{k}$ of positive integers such that $\mathcal{K} = M_{\mathbf{k}}$.

Note that we have made the convention that $\mathcal{H}$ is a separable Hilbert space in Subsection 1.2. Then $\mathcal{K} = [\mathcal{H}]_{\mathbf{V}}$ is also separable. Take a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathcal{K}$ that constitutes a dense subset of $\mathcal{K}$ and each element appears infinitely many times in the sequence. It follows from Lemma 2.12 that for each $n \in \mathbb{N}$, there exists a sequence $\mathbf{k}^{(n)} = (k_1^{(n)}, k_2^{(n)}, \ldots)$ of positive integers, such that

$$\|D_{\mathbf{V}_{\mathbf{k}^{(n)}}^*} x_n\| \geqslant \left(1 - \frac{1}{2^n}\right) \|x_n\|. \tag{4.1}$$

Set $k_n = \max\{k_n^{(1)}, \ldots, k_n^{(n)}\}$ ($n \in \mathbb{N}$) and put $\mathbf{k} = (k_1, k_2, \ldots)$. Note that

$$[\mathfrak{D}_{\mathbf{V_k}^*}]_{V_1^{k_1}} = [(I - V_1^{k_1} V_1^{*k_1}) \mathfrak{D}_{\mathbf{V}_{(0,\mathbf{k}')}^*}]_{V_1^{k_1}} = \mathfrak{D}_{\mathbf{V}_{(0,\mathbf{k}')}^*},$$

where $\boldsymbol{k}' = (k_2, k_3, \ldots)$. It follows that

$$M_{\boldsymbol{k}} = [\mathfrak{D}_{\boldsymbol{V}_{\boldsymbol{k}}^*}]_{\boldsymbol{V}_{\boldsymbol{k}}} = [\mathfrak{D}_{\boldsymbol{V}_{(0,\boldsymbol{k}')}^*}]_{\boldsymbol{V}_{(0,\boldsymbol{k}')}} = M_{(0,\boldsymbol{k}')},$$

and thus by induction,

$$M_{\boldsymbol{k}} = M_{(0,\boldsymbol{k}')} = \cdots = M_{(0,\ldots,0,k_n,k_{n+1},\ldots)}$$
$$\supseteq \mathfrak{D}_{\boldsymbol{V}_{(0,\ldots,0,k_n,k_{n+1},\ldots)}^*} \supseteq \mathfrak{D}_{\boldsymbol{V}_{(0,\ldots,0,k_n^{(n)},k_{n+1}^{(n)},\ldots)}^*} \supseteq \mathfrak{D}_{\boldsymbol{V}_{\boldsymbol{k}^{(n)}}^*}$$

for each $n \in \mathbb{N}$. By (4.1), one obtains

$$\|P_{M_{\boldsymbol{k}}} x_n\| \geqslant \|D_{\boldsymbol{V}_{\boldsymbol{k}^{(n)}}^*} x_n\| \geqslant \left(1 - \frac{1}{2^n}\right)\|x_n\|, \quad n \in \mathbb{N},$$

forcing $\|P_{M_{\boldsymbol{k}}} x\| = \|x\|$ for every $x \in \mathcal{K}$, where $P_{M_{\boldsymbol{k}}}$ is the orthogonal projection from $\mathcal{K}$ onto $M_{\boldsymbol{k}}$. This completes the proof. $\qquad\square$

Proof of Theorem 4.1. Assume that $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ and $\boldsymbol{V} \in \mathcal{DI}(\mathcal{K})$ is the minimal regular isometric dilation of $\boldsymbol{T}$. It follows from Proposition 4.2 that there exists a sequence $(k_1, k_2, \ldots)$ of positive integers, such that $(T_1^{k_1}, T_2^{k_2}, \ldots)$ is of Beurling type. Then by Corollary 1.4, the sequence $(V_1^{k_1}, V_2^{k_2}, \ldots)$ is jointly unitarily equivalent to the tuple $\boldsymbol{M}_\zeta = (M_{\zeta_1}, M_{\zeta_2}, \ldots)$ of coordinate multiplication operators on a vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ via a unitary operator $U : \mathcal{K} \to H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$. This implies that for each $n \in \mathbb{N}$,

$$M_{\zeta_n} = UV_n^{k_n}U^* = (UV_nU^*)^{k_n},$$

and hence $\widetilde{V_n} = UV_nU^*$ commutes with M_{ζ_n} and doubly commutes with M_{ζ_m} for every $m \neq n$.

It remains to show that $\widetilde{V_n}$ ($n \in \mathbb{N}$) is a multiplication operator induced by an operator-valued inner function $\widetilde{\theta_n} \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D}_2^\infty)$, which depends only on the n-th variable ζ_n. We first prove the case for $n = 1$. Put $\boldsymbol{M}_\zeta' = (M_{\zeta_2}, M_{\zeta_3}, \ldots)$ and set

$$\mathcal{L} = \overline{\mathrm{span}\{\zeta^\alpha : \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)} \text{ with } \alpha_1 = 0\}}.$$

Then we have $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty) = H_{\mathcal{E}}^2(\mathbb{D}) \otimes \mathcal{L}$ (see Subsection 2.2), $M_{\zeta_1} = M_z \otimes I_{\mathcal{L}}$ and $\boldsymbol{M}_\zeta'$ has the form $(I_{H_{\mathcal{E}}^2(\mathbb{D})} \otimes T_1, I_{H_{\mathcal{E}}^2(\mathbb{D})} \otimes T_2, \ldots)$ for a sequence $(T_1, T_2, \ldots)$ of operators on $\mathcal{L}$ jointly unitarily equivalent to the tuple of coordinate multiplication operators on $H^2(\mathbb{D}_2^\infty)$. Since $\widetilde{V_1}$ doubly commutes with $\boldsymbol{M}_\zeta'$, it follows from Lemmas 2.10 and 2.11 that $\widetilde{V_1} = S \otimes I_{\mathcal{L}}$ for some isometry S on $H_{\mathcal{E}}^2(\mathbb{D})$. Therefore,

$$M_z \otimes I_{\mathcal{L}} = M_{\zeta_1} = \widetilde{V_1}^{k_1} = S^{k_1} \otimes I_{\mathcal{L}},$$

forcing $M_z = S^{k_1}$. In particular, S commutes with M_z. Then there is a $\mathcal{B}(\mathcal{E})$-valued inner function θ_1 in the single variable $z \in \mathbb{D}$, such that $S = M_{\theta_1}$ (see, for example, [50, pp. 200–201]), which gives

$$\widetilde{V_1} = M_{\theta_1} \otimes I_{\mathcal{L}} = M_{\widetilde{\theta_1}},$$

where $\widetilde{\theta_1}(\zeta) = \theta_1(\zeta_1)$ ($\zeta \in \mathbb{D}_2^\infty$). Similarly, for each $n \geqslant 2$, $\widetilde{V_n} = M_{\widetilde{\theta_n}}$, where $\widetilde{\theta_n}(\zeta) = \theta_n(\zeta_n)$ ($\zeta \in \mathbb{D}_2^\infty$) for some inner function $\theta_n \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$. The proof is completed. $\qquad\square$

Combining Proposition 4.2 with Corollary 3.3, we have the following corollary.

Corollary 4.3. *Suppose $\boldsymbol{T} \in \mathcal{DC}$. Then there exists a sequence $\{B_n\}_{n \in \mathbb{N}}$ of finite Blaschke products, such that $\{\prod_{i=1}^n B_i(T_i)\}_{n \in \mathbb{N}}$ converges in the strong operator topology.*

Now we establish an analytic model for sequences in the class $\mathcal{DC}$ with a decomposition of quasi-Beurling type.

Suppose that $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ is of Beurling type, and $\boldsymbol{V} \in \mathcal{DI}(\mathcal{K})$ is the minimal regular isometric dilation of $\boldsymbol{T}$. By Corollary 3.1(1), $\boldsymbol{V}$ is jointly unitarily equivalent to the tuple $\boldsymbol{M}_\zeta$ of coordinate multiplication operators on a vector-valued Hardy space $H_{\mathfrak{D}_{\boldsymbol{V}^*}}^2(\mathbb{D}_2^\infty)$. We claim that the map

$$D_{\boldsymbol{V}^*} x \mapsto D_{\boldsymbol{T}^*} x, \quad x \in \mathcal{H} \tag{4.2}$$

can be extended to a unitary operator from $\mathfrak{D}_{\boldsymbol{V}^*}$ onto $\mathfrak{D}_{\boldsymbol{T}^*}$. By Lemma 2.6, it remains to prove that $D_{\boldsymbol{V}^*}\mathcal{H}$ is dense in $\mathfrak{D}_{\boldsymbol{V}^*}$. Assume that $x \in \mathfrak{D}_{\boldsymbol{V}^*}$ is orthogonal to $D_{\boldsymbol{V}^*}\mathcal{H}$. Then x is orthogonal to $\mathcal{H}$ and $\mathrm{Ran}V_n$ for all $n \in \mathbb{N}$. In particular, x is orthogonal to $[\mathcal{H}]_{\boldsymbol{V}} = \mathcal{K}$, forcing $x = 0$. This proves the claim. Therefore, the above unitary operator from $\mathfrak{D}_{\boldsymbol{V}^*}$ onto $\mathfrak{D}_{\boldsymbol{T}^*}$ naturally induces a unitary operator $U_{\boldsymbol{T}}$ from $\mathcal{K}$ onto $H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty)$. It is easy to see that $\mathcal{Q}_{\boldsymbol{T}} = U_{\boldsymbol{T}}\mathcal{H}$ is an analytic model for the sequence $\boldsymbol{T}$. We call $\mathcal{Q}_{\boldsymbol{T}}$ the *canonical analytic model* for $\boldsymbol{T}$.

As a consequence, for a sequence $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ which is of quasi-Beurling type, one can find some $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathbb{D}^\infty$ so that $\Phi_\lambda(\boldsymbol{T})$ is of Beurling type, and then $(\mathcal{Q}_{\Phi_\lambda(\boldsymbol{T})}, (\varphi_{\lambda_1} \cdot I, \varphi_{\lambda_2} \cdot I, \ldots))$ is an analytic model for the sequence $\boldsymbol{T}$.

Suppose that $\{\mathcal{E}_\gamma\}_\gamma$ is a family of Hilbert spaces, and for each index γ, Θ_γ is a sequence in $H^\infty_{\mathcal{B}(\mathcal{E}_\gamma)}(\mathbb{D})$. In a natural way, we can define the direct sum $\bigoplus_\gamma \Theta_\gamma$ of $\{\Theta_\gamma\}_\gamma$, which is then a sequence in $H^\infty_{\mathcal{B}(\bigoplus_\gamma \mathcal{E}_\gamma)}(\mathbb{D})$. Now let $\boldsymbol{T} = \bigoplus_\gamma \boldsymbol{T}_\gamma$ be the direct sum of $\mathcal{DC}$-sequences of quasi-Beurling type. For each index γ, we have previously established an analytic model $(\mathcal{Q}_\gamma, \Theta_\gamma)$ for $\boldsymbol{T}_\gamma$. Put $\mathcal{Q} = \bigoplus_\gamma \mathcal{Q}_\gamma$ and $\Theta = \bigoplus_\gamma \Theta_\gamma$. Then $(\mathcal{Q}, \Theta)$ is an analytic model for $\boldsymbol{T}$.

In fact, we have the following theorem.

Theorem 4.4. *Suppose $\boldsymbol{T} \in \mathcal{DC}$. Then $\boldsymbol{T}$ has a decomposition of quasi-Beurling type if and only if $\boldsymbol{T}$ has an analytic model $(\mathcal{Q}, \Theta)$ so that all $\theta_n(z)$'s ($n \in \mathbb{N}$, $z \in \mathbb{D}$) are simultaneously diagonalizable with respect to some orthonormal basis of the underlying space $\mathcal{E}$ of $(\mathcal{Q}, \Theta)$.*

Proof. The necessity follows from the construction in the previous paragraphs. Now assume that $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$ has an analytic model $(\mathcal{Q}, \Theta)$ so that all $\theta_n(z)$'s ($n \in \mathbb{N}$, $z \in \mathbb{D}$) are simultaneously diagonalizable with respect to some orthonormal basis $\{e_i\}_{i \in \Lambda}$ of the underlying space $\mathcal{E}$ of $(\mathcal{Q}, \Theta)$. We prove that $\boldsymbol{T}$ has a decomposition of quasi-Beurling type. By assumption, we have

$$\theta_n = \sum_{i \in \Lambda} \eta_{ni} \cdot e_i \widehat{\otimes} e_i, \quad n \in \mathbb{N},$$

where each η_{ni} is an $H^\infty(\mathbb{D})$-inner function, and $e_i \widehat{\otimes} e_i$ denotes the 1-rank projection

$$e_i \widehat{\otimes} e_i(x) = \langle x, e_i \rangle e_i, \quad x \in \mathcal{E}.$$

Then for each $i \in \Lambda$, $H^2_{\mathbb{C}e_i}(\mathbb{D}_2^\infty)$, as a subspace of $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$, is joint reducing for $\boldsymbol{M}_\Theta$, and the restriction of $\boldsymbol{M}_\Theta$ on $H^2_{\mathbb{C}e_i}(\mathbb{D}_2^\infty)$ is jointly unitarily equivalent to the sequence $(M_{\widetilde{\eta_{1i}}}, M_{\widetilde{\eta_{2i}}}, \ldots)$ of multiplication operators on the Hardy space $H^2(\mathbb{D}_2^\infty)$, where $\widetilde{\eta_{ni}}(\zeta) = \eta_{ni}(\zeta_n)$ ($n \in \mathbb{N}, \zeta \in \mathbb{D}_2^\infty$). It follows from Proposition 3.4 that the sequence $\boldsymbol{M}_\Theta$ has a decomposition of quasi-Beurling type, and therefore so does the minimal isometric dilation $\boldsymbol{V}$ of $\boldsymbol{T}$ by the definition of the analytic model. Then Theorem 1.3 together with Corollary 3.1(3) implies that $\boldsymbol{T}$ also has a decomposition of quasi-Beurling type. $\square$

To conclude this section, we give a characterization of the canonical analytic model $\mathcal{Q}_{\boldsymbol{T}}$ for a sequence $\boldsymbol{T} \in \mathcal{DC}$ which is of Beurling type. One approach, inspired by the single $C_{\cdot 0}$-contraction case, is utilizing the characteristic functions of contractions.

Recall that the characteristic function θ_T of a contraction $T \in \mathcal{B}(\mathcal{H})$ is a $\mathcal{B}(\mathfrak{D}_T, \mathfrak{D}_{T^*})$-valued function defined by

$$\theta_T(z) = [-T + zD_{T^*}(1 - zT^*)^{-1}D_T]|_{\mathfrak{D}_T}, \quad z \in \mathbb{D}.$$

Suppose in addition that $T \in C_{\cdot 0}$, and θ_T is further an inner function (see [50]). Then the multiplication operator M_{θ_T} is an isometry from $H^2_{\mathfrak{D}_T}(\mathbb{D})$ to $H^2_{\mathfrak{D}_{T^*}}(\mathbb{D})$, and thus

$$M^*_{\theta_T} H^2_{\mathfrak{D}_{T^*}}(\mathbb{D}) = H^2_{\mathfrak{D}_T}(\mathbb{D}).$$

Since the minimal isometric dilation $V \in \mathcal{B}(\mathcal{K})$ of T is pure, $\mathcal{K}$ has a decomposition as $\mathcal{K} = \bigoplus_{k=0}^\infty V^k \mathfrak{D}_{V^*}$, and therefore can be identified with the vector-valued Hardy space $H^2_{\mathfrak{D}_{V^*}}(\mathbb{D})$. Similar to (4.2), the map

$$D_{V^*}x \mapsto D_{T^*}x, \quad x \in \mathcal{H}$$

can be extended to a unitary operator from $\mathfrak{D}_{V^*}$ onto $\mathfrak{D}_{T^*}$, and then induces a unitary operator U_T from $\mathcal{K}$ onto $H^2_{\mathfrak{D}_{T^*}}(\mathbb{D})$. It is easy to see that U_T is actually an extension of the isometry

$$V : \mathcal{H} \to H^2_{\mathfrak{D}_{T^*}}(\mathbb{D}),$$

$$x \mapsto \sum_{k=1}^{\infty} z^k \cdot D_{T^*} T^{*k} x.$$

It follows from [50, pp. 244–245] that for $a, b \in \mathbb{D}$,

$$I_{\mathfrak{D}_{T^*}} - \theta_T(b)\theta_T(a)^* = (1 - \bar{a}b)[D_{T^*}(1 - bT^*)^{-1}(1 - \bar{a}T)^{-1}D_{T^*}]|_{\mathfrak{D}_{T^*}}. \tag{4.3}$$

This gives

$$U_T P_{\mathcal{H}} U_T^* = VV^* = I - M_{\theta_T} M_{\theta_T}^* \tag{4.4}$$

(see [9] or [10]), i.e.,

$$H^2_{\mathfrak{D}_{T^*}}(\mathbb{D}) \ominus U_T \mathcal{H} = M_{\theta_T} M_{\theta_T}^* H^2_{\mathfrak{D}_{T^*}}(\mathbb{D}).$$

Let us return to the characterization of the canonical analytic model $\mathcal{Q}_{\boldsymbol{T}}$ for the sequence $\boldsymbol{T} = (T_1, T_2, \ldots)$. Now consider the following spaces:

$$M_{\theta_{T_n}} M_{\theta_{T_n}}^* H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}), \quad n \in \mathbb{N}.$$

By (4.3), for $a, b \in \mathbb{D}$,

$$\begin{aligned}
\theta_{T_1}(b)\theta_{T_1}(a)^* D_{\boldsymbol{T}^*} x &= D_{\boldsymbol{T}'^*} \theta_{T_1}(b)\theta_{T_1}(a)^* D_{T_1^*} x \\
&= D_{\boldsymbol{T}^*} x - (1 - \bar{a}b)[D_{\boldsymbol{T}^*}(1 - bT^*)^{-1}(1 - \bar{a}T)^{-1}D_{T_1^*}^2]x \\
&= D_{\boldsymbol{T}^*}[1 - (1 - \bar{a}b)(1 - bT_1^*)^{-1}(1 - \bar{a}T_1)^{-1}D_{T_1^*}^2]x,
\end{aligned}$$

where $\boldsymbol{T}' = (T_2, T_3, \ldots)$. Similarly, for every $n \in \mathbb{N}$ and $a, b \in \mathbb{D}$,

$$\theta_{T_n}(b)\theta_{T_n}(a)^* D_{\boldsymbol{T}^*} x = D_{\boldsymbol{T}^*}[1 - (1 - \bar{a}b)(1 - bT_n^*)^{-1}(1 - \bar{a}T_n)^{-1}D_{T_n^*}^2]x, \tag{4.5}$$

and then by Lemma 2.5 (1), $H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})$ is reducing for $M_{\theta_{T_n}} M_{\theta_{T_n}}^*$, and

$$M_{\theta_{T_n}}^* H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}) = H^2_{\mathcal{F}_n}(\mathbb{D}), \tag{4.6}$$

where $\mathcal{F}_n$ is the defect space of the sequence $(T_1^*, \ldots, T_{n-1}^*, T_n, T_{n+1}^*, \ldots)$. In particular,

$$M_{\theta_{T_n}} M_{\theta_{T_n}}^* H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}) = \theta_{T_n} H^2_{\mathcal{F}_n}(\mathbb{D}), \quad n \in \mathbb{N}$$

is an M_z-invariant subspace of $H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})$.

Put $\widetilde{\theta_{T_n}}(\zeta) = \theta_{T_n}(\zeta_n)$ $(n \in \mathbb{N}, \zeta \in \mathbb{D}_2^\infty)$. Then $M_{\widetilde{\theta_{T_n}}} M_{\widetilde{\theta_{T_n}}}^* H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty)$ $(n \in \mathbb{N})$ is of the form

$$\underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{n-1 \text{ times}} \otimes M_{\theta_{T_n}} M_{\theta_{T_n}}^* H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots, \tag{4.7}$$

forcing it to be a joint invariant subspace of $H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty)$ for the tuple of coordinate multiplication operators.

Our result presented below looks somehow similar to the above single $C_{\cdot 0}$-contraction case, or the finite-tuple case considered in [10]. However, instead of following the proof in [10], we give an original proof.

Theorem 4.5. Let $\boldsymbol{T} = (T_1, T_2, \ldots) \in \mathcal{DC}$ be of Beurling type, and $\mathcal{Q}_{\boldsymbol{T}}$ be the canonical analytic model for $\boldsymbol{T}$. Then

$$H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty) \ominus \mathcal{Q}_{\boldsymbol{T}} = \bigvee_{n=1}^{\infty} M_{\widetilde{\theta_{T_n}}} M_{\widetilde{\theta_{T_n}}}^* H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty),$$

where $\widetilde{\theta_{T_n}}(\zeta) = \theta_{T_n}(\zeta_n)$ $(n \in \mathbb{N}, \zeta \in \mathbb{D}_2^\infty)$.

Proof.   Let $\boldsymbol{V} \in \mathcal{DI}(\mathcal{K})$ be the minimal regular isometric dilation of $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$, and put

$$\mathcal{H}_n = \bigvee_{\substack{\alpha \in \mathbb{Z}_+^{(\infty)} \\ \alpha_n = 0}} \boldsymbol{V}^\alpha \mathcal{H}, \quad n \in \mathbb{N}.$$

By Lemma 2.2(3), one has

$$\mathcal{Q}_{\boldsymbol{T}} = U_{\boldsymbol{T}} \mathcal{H} = \bigcap_{n=1}^{\infty} U_{\boldsymbol{T}} \mathcal{H}_n.$$

It thus suffices to prove that for each $n \in \mathbb{N}$,

$$H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty) \ominus U_{\boldsymbol{T}} \mathcal{H}_n = M_{\widetilde{\theta_{T_n}}} M^*_{\widetilde{\theta_{T_n}}} H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty);$$

equivalently,

$$I - U_{\boldsymbol{T}} P_{\mathcal{H}_n} U_{\boldsymbol{T}}^* = M_{\widetilde{\theta_{T_n}}} M^*_{\widetilde{\theta_{T_n}}} |_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}_2^\infty)}. \tag{4.8}$$

Assume $n = 1$ without loss of generality, and put $\boldsymbol{S} = (S_1, S_2, \ldots) = P_{\mathcal{H}_1} \boldsymbol{V} |_{\mathcal{H}_1}$, the compression of the sequence $\boldsymbol{V}$ on the subspace $\mathcal{H}_1$. Rewrite $T = T_1$ and $S = S_1$ for simplicity. One can define the unitary operators $U_{\boldsymbol{S}} : \mathcal{K} \to H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}_2^\infty)$ and $U_S : \mathcal{K} \to H^2_{\mathfrak{D}_{S^*}}(\mathbb{D})$ as done previously in this section. Since (4.4) remains valid for S, and (4.5) and (4.6) remain valid for $\boldsymbol{S}$, it follows that

$$P_{\mathcal{H}_1} = I - U_S^* M_{\theta_S} M^*_{\theta_S} U_S, \tag{4.9}$$

$H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})$ is reducing for $M_{\theta_S} M^*_{\theta_S}$ and

$$M^*_{\theta_S} H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}) = H^2_{\mathcal{F}}(\mathbb{D}), \tag{4.10}$$

where $\mathcal{F}$ is the defect space of the sequence $(S, S_2^*, S_3^*, \ldots)$.

We now claim that

$$U_S^* M_{\theta_S} M^*_{\theta_S} U_S = U_{\boldsymbol{S}}^* (M_{\theta_S} M^*_{\theta_S} |_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})} \otimes I_{\mathcal{L}}) U_{\boldsymbol{S}}, \tag{4.11}$$

where

$$\mathcal{L} = \overline{\operatorname{span}\{\zeta^\alpha : \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)} \text{ with } \alpha_1 = 0\}}.$$

Since operators on both sides of (4.11) are orthogonal projections, the claim is equivalent to

$$U_S^* M_{\theta_S} M^*_{\theta_S} H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}) = U_{\boldsymbol{S}}^* (M_{\theta_S} M^*_{\theta_S} |_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})} \otimes I_{\mathcal{L}}) (H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}) \otimes \mathcal{L}). \tag{4.12}$$

The left-hand side of (4.12) is

$$U_S^* M_{\theta_S} H^2_{\mathfrak{D}_S}(\mathbb{D}),$$

and by (4.10), the right-hand side of (4.12) is

$$U_{\boldsymbol{S}}^* (M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}) \otimes \mathcal{L}).$$

Put $\boldsymbol{V}' = (V_2, V_3, \ldots)$ and $\boldsymbol{S}' = (S_2, S_3, \ldots)$. Then by Lemma 2.2(1), $\mathcal{H}_1$ is reducing for $\boldsymbol{V}'$, and hence $\boldsymbol{S}' = \boldsymbol{V}' |_{\mathcal{H}_1}$ is a $\mathcal{DI}$-sequence of Beurling type, and so is $\boldsymbol{S}' |_{\mathfrak{D}_S}$, the restriction of $\boldsymbol{S}'$ on the joint reducing subspace $\mathfrak{D}_S$. This together with Corollary 3.1(1) gives

$$\mathfrak{D}_S = \bigoplus_{\alpha \in \mathbb{Z}_+^{(\infty)}} \boldsymbol{S}'^\alpha \mathfrak{D}_{(\boldsymbol{S}' |_{\mathfrak{D}_S})^*} = \bigoplus_{\alpha \in \mathbb{Z}_+^{(\infty)}} \boldsymbol{S}'^\alpha \overline{D_S \mathfrak{D}_{\boldsymbol{S}'^*}} = \bigoplus_{\alpha \in \mathbb{Z}_+^{(\infty)}} \boldsymbol{S}'^\alpha \mathcal{F} = \bigoplus_{\alpha_1 = 0} \boldsymbol{S}^\alpha \mathcal{F},$$

forcing

$$H^2_{\mathfrak{D}_S}(\mathbb{D}) = \bigoplus_{\alpha_1 = 0} H^2_{\boldsymbol{S}^\alpha \mathcal{F}}(\mathbb{D}) = \bigoplus_{\alpha_1 = 0} (I_{H^2(\mathbb{D})} \otimes \boldsymbol{S}^\alpha) H^2_{\mathcal{F}}(\mathbb{D}).$$

Then for each $f \in H^2_{\mathfrak{D}_S}(\mathbb{D})$, there exists a unique sequence $\{f_\alpha\}_{\alpha_1=0}$ in $H^2_{\mathcal{F}}(\mathbb{D})$, such that

$$\sum_{\alpha_1=0} \|f_\alpha\|^2 < \infty$$

and

$$f(z) = \sum_{\alpha_1=0} \boldsymbol{S}^\alpha(f_\alpha(z)), \quad z \in \mathbb{D}.$$

Since S doubly commutes with S_n $(n \geqslant 2)$, one has

$$(M_{\theta_S} f)(z) = \theta_S(z) f(z) = \sum_{\alpha_1=0} \boldsymbol{S}^\alpha(\theta_S(z) f_\alpha(z)) = \sum_{\alpha_1=0} \boldsymbol{S}^\alpha((M_{\theta_S} f_\alpha)(z)).$$

That is to say,

$$M_{\theta_S} H^2_{\mathfrak{D}_S}(\mathbb{D}) = \bigoplus_{\alpha_1=0} (I_{H^2(\mathbb{D})} \otimes \boldsymbol{S}^\alpha) M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}).$$

On the other hand, it is easy to see that

$$M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}) \otimes \mathcal{L} = \bigoplus_{\alpha_1=0} M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}) \otimes \mathbb{C}\zeta^\alpha.$$

Thus we can further reduce (4.12) to proving the following:

$$U_S U_{\boldsymbol{S}}^*(M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}) \otimes \mathbb{C}\zeta^\alpha) = (I_{H^2(\mathbb{D})} \otimes \boldsymbol{S}^\alpha) M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}), \quad \alpha_1 = 0.$$

A calculation gives that for every $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)}$ and $x \in \mathcal{H}_1$,

$$U_S U_{\boldsymbol{S}}^*(\zeta^\alpha \cdot D_{\boldsymbol{S}^*} x) = U_S \boldsymbol{V}^\alpha D_{\boldsymbol{V}^*} x = U_S V_1^{\alpha_1} D_{V_1^*} \boldsymbol{V}'^{\alpha'} D_{\boldsymbol{V}'^*} x = z^{\alpha_1} \cdot D_{S^*} \boldsymbol{V}'^{\alpha'} D_{\boldsymbol{V}'^*} x,$$

where $\alpha' = (\alpha_2, \alpha_3, \ldots)$. Here, we used the fact that the restriction of $\boldsymbol{V}'^{\alpha'} D_{\boldsymbol{V}'^*}$ on $\mathcal{H}_1$ is $\boldsymbol{S}'^{\alpha'} D_{\boldsymbol{S}'^*}$, which implies

$$\boldsymbol{V}'^\alpha D_{\boldsymbol{V}'^*} x = \boldsymbol{S}'^\alpha D_{\boldsymbol{S}'^*} x \in \mathcal{H}_1.$$

Moreover, since S doubly commutes with S_n $(n \geqslant 2)$, we further have

$$U_S U_{\boldsymbol{S}}^*(\zeta^\alpha \cdot D_{\boldsymbol{S}^*} x) = z^{\alpha_1} \cdot D_{S^*} \boldsymbol{V}'^{\alpha'} D_{\boldsymbol{V}'^*} x = z^{\alpha_1} \cdot D_{S^*} \boldsymbol{S}'^\alpha D_{\boldsymbol{S}'^*} x = z^{\alpha_1} \cdot \boldsymbol{S}'^\alpha D_{\boldsymbol{S}^*} x.$$

Note that

$$M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D}) = M_{\theta_S} M_{\theta_S}^* H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}) \subseteq H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}),$$

and each $f \in M_{\theta_S} H^2_{\mathcal{F}}(\mathbb{D})$ can be written as $f = \sum_{k=0}^\infty z^k \cdot x_k$ for some sequence $\{x_k\}_{k \in \mathbb{N}}$ in $\mathfrak{D}_{\boldsymbol{S}^*}$. Then for any $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)}$ with $\alpha_1 = 0$,

$$f \otimes \zeta^\alpha \in H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}) \otimes \mathcal{L} = H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}_2^\infty)$$

can also be written as

$$f \otimes \zeta^\alpha = \sum_{k=0}^\infty \zeta_1^{k_1} \zeta^\alpha \cdot x_k,$$

and hence,

$$U_S U_{\boldsymbol{S}}^*(f \otimes \zeta^\alpha) = \sum_{k=0}^\infty U_S U_{\boldsymbol{S}}^*(\zeta_1^{k_1} \zeta^\alpha \cdot x_k) = \sum_{k=0}^\infty z^k \cdot \boldsymbol{S}^\alpha x_k = (I_{H^2(\mathbb{D})} \otimes \boldsymbol{S}^\alpha) f.$$

This proves the claim.

The operator $U_{\boldsymbol{T}} U_{\boldsymbol{S}}^*$ has the form $I_{H^2(\mathbb{D})} \otimes \Pi \otimes I_{\mathcal{L}}$ with respect to the representation

$$H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D}_2^\infty) = H^2(\mathbb{D}) \otimes \mathfrak{D}_{\boldsymbol{S}^*} \otimes \mathcal{L},$$

where $\Pi : \mathfrak{D}_{\boldsymbol{S}^*} \to \mathfrak{D}_{\boldsymbol{T}^*}$ is a unitary operator satisfying

$$\Pi(D_{\boldsymbol{S}^*}x) = D_{\boldsymbol{T}^*}x, \quad x \in \mathcal{H}_1. \tag{4.13}$$

Let $\Pi \cdot \theta_S$ denote the operator-valued function defined by $z \mapsto \Pi(\theta_S(z))$ ($z \in \mathbb{D}$). Then

$$M^*_{\Pi \cdot \theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})} = M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})}(I_{H^2(\mathbb{D})} \otimes \Pi^*),$$

since two operators coincide on the set $\{K_a \cdot x : a \in \mathbb{D}, x \in \mathfrak{D}_{\boldsymbol{T}^*}\}$, and thus,

$$\begin{aligned}
M_{\Pi \cdot \theta_S} M^*_{\Pi \cdot \theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})} &= (M^*_{\Pi \cdot \theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})})^* M^*_{\Pi \cdot \theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})} \\
&= (I_{H^2(\mathbb{D})} \otimes \Pi)(M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})})^* M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})}(I_{H^2(\mathbb{D})} \otimes \Pi^*) \\
&= (I_{H^2(\mathbb{D})} \otimes \Pi)M_{\theta_S} M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})}(I_{H^2(\mathbb{D})} \otimes \Pi^*).
\end{aligned} \tag{4.14}$$

Therefore, we have

$$\begin{aligned}
I - U_{\boldsymbol{T}}P_{\mathcal{H}_1}U^*_{\boldsymbol{T}} &= I - U_{\boldsymbol{T}}(I - U^*_S M_{\theta_S} M^*_{\theta_S} U_S)U^*_{\boldsymbol{T}} \\
&= U_{\boldsymbol{T}}U^*_S M_{\theta_S} M^*_{\theta_S} U_S U^*_{\boldsymbol{T}} \\
&= U_{\boldsymbol{T}}U^*_{\boldsymbol{S}}(M_{\theta_S} M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})} \otimes I_{\mathcal{L}})U_S U^*_{\boldsymbol{T}} \\
&= (I_{H^2(\mathbb{D})} \otimes \Pi \otimes I_{\mathcal{L}})(M_{\theta_S} M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})} \otimes I_{\mathcal{L}})(I_{H^2(\mathbb{D})} \otimes \Pi^* \otimes I_{\mathcal{L}}) \\
&= ((I_{H^2(\mathbb{D})} \otimes \Pi)M_{\theta_S} M^*_{\theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{S}^*}}(\mathbb{D})}(I_{H^2(\mathbb{D})} \otimes \Pi^*)) \otimes I_{\mathcal{L}} \\
&= (M_{\Pi \cdot \theta_S} M^*_{\Pi \cdot \theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})}) \otimes I_{\mathcal{L}},
\end{aligned}$$

where the first identity follows from (4.9), the third identity follows from (4.11), and the fifth identity follows from (4.14). On the other hand, (4.7) gives

$$M_{\widetilde{\theta_T}} M^*_{\widetilde{\theta_T}}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D}^\infty_2)} = M_{\theta_T} M^*_{\theta_T}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})} \otimes I_{\mathcal{L}},$$

which reduces (4.8) to

$$M_{\Pi \cdot \theta_S} M^*_{\Pi \cdot \theta_S}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})} = M_{\theta_T} M^*_{\theta_T}\,|_{H^2_{\mathfrak{D}_{\boldsymbol{T}^*}}(\mathbb{D})}.$$

By Lemma 2.5(2), it remains to prove that for any fixed $a, b \in \mathbb{D}$ and any fixed $x \in \mathcal{H} \subseteq \mathcal{H}_1$,

$$(\Pi \cdot \theta_S)(b)(\Pi \cdot \theta_S)(a)^* D_{\boldsymbol{T}^*}x = \Pi\theta_S(b)\theta_S(a)^*\Pi^* D_{\boldsymbol{T}^*}x = \theta_T(b)\theta_T(a)^* D_{\boldsymbol{T}^*}x.$$

Note that Lemma 2.2(2) implies that $\mathcal{H}$ is reducing for S and $T = S\,|_{\mathcal{H}}$. By (4.5), we have

$$\theta_S(b)\theta_S(a)^* D_{\boldsymbol{S}^*}x = D_{\boldsymbol{S}^*}y$$

and

$$\theta_T(b)\theta_T(a)^* D_{\boldsymbol{T}^*}x = D_{\boldsymbol{T}^*}y,$$

where

$$\begin{aligned}
y &= [1 - (1 - \bar{a}b)(1 - bS^*)^{-1}(1 - \bar{a}S)^{-1}D^2_{S^*}]x \\
&= [1 - (1 - \bar{a}b)(1 - bT^*)^{-1}(1 - \bar{a}T)^{-1}D^2_{T^*}]x \in \mathcal{H}.
\end{aligned}$$

It follows from (4.13) that

$$\Pi\theta_S(b)\theta_S(a)^*\Pi^* D_{\boldsymbol{T}^*}x = \Pi\theta_S(b)\theta_S(a)^* D_{\boldsymbol{S}^*}x = \Pi D_{\boldsymbol{S}^*}y = D_{\boldsymbol{T}^*}y.$$

This completes the proof.　　□

Now we are ready to refine the representation (1.2) in Subsection 1.2. Note that for each index γ, $\boldsymbol{S}_\gamma = (S_{\gamma 1}, S_{\gamma 2}, \ldots) = \Phi_{\lambda_\gamma}(\boldsymbol{T}_\gamma)$ is of Beurling type. Without loss of generality, we may assume that $\mathcal{E}_\gamma = \mathfrak{D}_{\boldsymbol{S}_\gamma^*}$ and $\mathcal{Q}_\gamma$ is the canonical analytic model for $\boldsymbol{S}_\gamma$ in (1.2). Put $\lambda_\gamma = (\lambda_{\gamma 1}, \lambda_{\gamma 2}, \ldots)$ and let $\theta_{\gamma n}$ and $\vartheta_{\gamma n}$ $(n \in \mathbb{N})$ denote the characteristic functions of $T_{\gamma n}$ and $S_{\gamma n}$, respectively. Then for each γ, $\vartheta_{\gamma n}$ coincides with $\theta_{\gamma n} \circ \varphi_{\lambda_{\gamma n}}$ (see [50, pp. 246–247]), and it follows from Theorem 4.5 that

$$H_{\mathcal{E}_\gamma}^2(\mathbb{D}_2^\infty) \ominus \mathcal{Q}_\gamma = \bigvee_{n=1}^\infty M_{\widetilde{\vartheta_{\gamma n}}} M_{\widetilde{\vartheta_{\gamma n}}}^* H_{\mathcal{E}_\gamma}^2(\mathbb{D}_2^\infty),$$

where $\widetilde{\vartheta_{\gamma n}}(\zeta) = \vartheta_{\gamma n}(\zeta_n)$ $(n \in \mathbb{N}, \zeta \in \mathbb{D}_2^\infty)$.

5   Doubly commuting submodules and quotient modules of $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$

For submodules and quotient modules of the Hardy module, we are interested in the module actions on them, i.e., the restrictions of the tuple of coordinate multiplication operators on submodules and the compressions of the tuple of coordinate multiplication operators on quotient modules. In this section, we mainly consider such restrictions and compressions that are doubly commuting.

Recall that a submodule $\mathcal{S}$ of $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ is said to be doubly commuting if the restriction

$$(M_{\zeta_1}|_{\mathcal{S}}, M_{\zeta_2}|_{\mathcal{S}}, \ldots)$$

of $\boldsymbol{M}_\zeta$ on $\mathcal{S}$ is doubly commuting.

Theorem 5.1.   *Let $\boldsymbol{M}_\zeta$ be the tuple of coordinate multiplication operators on the vector-valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$. Then the restriction of $\boldsymbol{M}_\zeta$ on a doubly commuting submodule of $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$ is of Beurling type.*

Before giving the proof, we introduce the notion of homogeneous components of functions in $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$. Suppose $F \in H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$, and let $F = \sum_{\alpha \in \mathbb{Z}_+^{(\infty)}} \zeta^\alpha \cdot x_\alpha$ be the power series expansion of F. The sum $\sum_{|\alpha|=k} \zeta^\alpha \cdot x_\alpha$ $(k = 0, 1, 2, \ldots)$ is called the *k-th homogeneous component* of F, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots$. It is clear that $\|F\|^2$ is equal to the quadratic sum of norms of all the homogeneous components of F.

Proof of Theorem 5.1.   Let $\mathcal{S}$ be a doubly commuting submodule of $H_{\mathcal{E}}^2(\mathbb{D}_2^\infty)$, and set $\boldsymbol{R} = \boldsymbol{M}_\zeta|_{\mathcal{S}}$, the restriction of $\boldsymbol{M}_\zeta$ on $\mathcal{S}$. Assume conversely that $\boldsymbol{R}$ is not of Beurling type to reach a contradiction. Then by Corollaries 1.4 and 3.1(3),

$$\widetilde{\mathcal{S}} = \mathcal{S} \ominus \bigvee_{\lambda \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\lambda(\boldsymbol{R})^*}$$

is a nonzero $\boldsymbol{R}$-joint reducing subspace of $\mathcal{S}$. Moreover, since $D_{\boldsymbol{R}^*}$ is self-adjoint, we have

$$D_{\widetilde{\boldsymbol{R}}^*} = D_{\boldsymbol{R}^*}|_{\widetilde{\mathcal{S}}} = 0,$$

where $\widetilde{\boldsymbol{R}} = \boldsymbol{R}|_{\widetilde{\mathcal{S}}}$. So without loss of generality, we may assume $\mathfrak{D}_{\boldsymbol{R}^*} = \{0\}$.

Set k_0 to the minimal nonnegative integer among those k's such that a nonzero k-th homogeneous component appears in some functions belonging to $\mathcal{S}$. Now choose a function $F \in \mathcal{S}$ so that the norm of the k_0-th homogeneous component of F is 1. Since $\mathfrak{D}_{\boldsymbol{R}^*} = \{0\}$, one has

$$\mathcal{S} = \bigvee_{n=1}^\infty \operatorname{Ran} R_n = \bigvee_{n=1}^\infty \zeta_n \mathcal{S},$$

and then there exist $n \in \mathbb{N}$ and n functions $F_1, \ldots, F_n \in \mathcal{S}$, such that

$$\left\| F - \sum_{i=1}^n \zeta_i F_i \right\| < 1.$$

This implies that the norm of the k_0-th homogeneous component of the function $F - \sum_{i=1}^n \zeta_i F_i$ is less than 1. However, it is clear that the k_0-th homogeneous component of $\sum_{i=1}^n \zeta_i F_i$ is 0, which contradicts the choice of F. The proof is completed. $\qquad\square$

We also recall that a function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}_2^\infty)$ is inner if the multiplication operator M_Ψ induced by Ψ is an isometry. It is clear that for any inner function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}_2^\infty)$, $\Psi H^2_{\mathcal{F}}(\mathbb{D}_2^\infty)$ is a doubly commuting submodule of $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$. The following is a Beurling-Lax type theorem for the vector-valued Hardy space in infinitely many variables.

Corollary 5.2. *Let $\mathcal{S}$ be a submodule of the vector-valued Hardy module $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$. Then $\mathcal{S}$ is doubly commuting if and only if there exist a Hilbert space $\mathcal{F}$ and an inner function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}_2^\infty)$, such that*

$$\mathcal{S} = \Psi H^2_{\mathcal{F}}(\mathbb{D}_2^\infty). \tag{5.1}$$

Proof. Let $\boldsymbol{M}_\zeta$ be the tuple of coordinate multiplication operators on $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$, and $\boldsymbol{R}$ be the restriction of $\boldsymbol{M}_\zeta$ on $\mathcal{S}$. It follows from Theorem 5.1 and Corollary 1.4 that $\boldsymbol{R}$ is jointly unitarily equivalent to the tuple $\boldsymbol{M}_\xi$ of coordinate multiplication operators on a vector-valued Hardy space $H^2_{\mathcal{F}}(\mathbb{D}_2^\infty)$. This naturally induces an isometry

$$V : H^2_{\mathcal{F}}(\mathbb{D}_2^\infty) \to H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$$

satisfying $\mathrm{Ran} V = \mathcal{S}$ and

$$V M_{\xi_n} = R_n V = M_{\zeta_n} V, \quad n \in \mathbb{N}.$$

Then by Proposition 2.4, $V = M_\Psi$ for some operator-valued inner function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}_2^\infty)$, where M_Ψ is the multiplication operator induced by Ψ. This completes the proof. $\qquad\square$

In particular, we reprove the known result that every doubly commuting submodule of $H^2(\mathbb{D}_2^\infty)$ is generated by a single function (see [36]).

Corollary 5.3. *Let $\mathcal{S}$ be a nonzero submodule of $H^2(\mathbb{D}_2^\infty)$. Then the following are equivalent*:
(1) $\mathcal{S}$ *is doubly commuting*;
(2) $\mathcal{S}$ *is generated by a single inner function in $H^\infty(\mathbb{D}_2^\infty)$*;
(3) $\mathcal{S}$ *as a $\mathcal{P}_\infty$-module is unitarily equivalent to $H^2(\mathbb{D}_2^\infty)$.*

We say that two $\mathcal{P}_\infty$-modules $(\mathcal{H}, \boldsymbol{T})$ and $(\mathcal{K}, \boldsymbol{S})$ are *unitarily equivalent* if $\boldsymbol{T}$ and $\boldsymbol{S}$ are jointly unitarily equivalent, and the unitary operator $U : \mathcal{H} \to \mathcal{K}$ intertwining $\boldsymbol{T}$ and $\boldsymbol{S}$ is called a *unitary module map*.

The classification of doubly commuting Hardy submodules up to unitary equivalence of $\mathcal{P}_\infty$-modules is trivial, since by Corollary 5.2, it is completely determined by the dimension of $\mathcal{F}$ in the representation (5.1).

Now we turn to the situation of quotient modules. Recall that a quotient module $\mathcal{Q}$ of $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$ is said to be doubly commuting if the compression

$$(P_{\mathcal{Q}} M_{\zeta_1}|_{\mathcal{Q}}, P_{\mathcal{Q}} M_{\zeta_2}|_{\mathcal{Q}}, \ldots)$$

of $\boldsymbol{M}_\zeta$ on $\mathcal{Q}$ is doubly commuting.

Theorem 5.4. *Let $\boldsymbol{M}_\zeta$ be the tuple of coordinate multiplication operators on the vector-valued Hardy module $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$. Then the compression of $\boldsymbol{M}_\zeta$ on a doubly commuting quotient module of $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$ is of Beurling type.*

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2_{\mathcal{E}}(\mathbb{D}_2^\infty)$, and set $\boldsymbol{C} = P_{\mathcal{Q}} \boldsymbol{M}_\zeta|_{\mathcal{Q}}$, the compression of $\boldsymbol{M}_\zeta$ on $\mathcal{Q}$. Since both sequences $\boldsymbol{M}_\zeta$ and $\boldsymbol{C}$ are doubly commuting, for $\alpha, \beta \in \mathbb{Z}_+^{(\infty)}$ satisfying $\alpha \wedge \beta = (0, 0, \ldots)$, we have

$$P_{\mathcal{Q}} \boldsymbol{M}_\zeta^{*\alpha} \boldsymbol{M}_\zeta^\beta|_{\mathcal{Q}} = P_{\mathcal{Q}} \boldsymbol{M}_\zeta^\beta \boldsymbol{M}_\zeta^{*\alpha}|_{\mathcal{Q}} = P_{\mathcal{Q}} \boldsymbol{M}_\zeta^\beta P_{\mathcal{Q}} \boldsymbol{M}_\zeta^{*\alpha}|_{\mathcal{Q}} = \boldsymbol{C}^\beta \boldsymbol{C}^{*\alpha} = \boldsymbol{C}^{*\alpha} \boldsymbol{C}^\beta,$$

where $\alpha \wedge \beta = (\min\{\alpha_1, \beta_1\}, \min\{\alpha_2, \beta_2\}, \ldots)$. That is to say, $\boldsymbol{M}_\zeta$ is a regular isometric dilation of $\boldsymbol{C}$. So the minimal regular isometric dilation of $\boldsymbol{C}$ is the restriction of $\boldsymbol{M}_\zeta$ on the subspace $[\mathcal{Q}]_{\boldsymbol{M}_\zeta}$. It follows from Lemma 2.2(1) that $[\mathcal{Q}]_{\boldsymbol{M}_\zeta}$ is joint reducing for $\boldsymbol{M}_\zeta$, and then by Lemma 2.11, one has

$$[\mathcal{Q}]_{\boldsymbol{M}_\zeta} = H^2_{\mathcal{E}_0}(\mathbb{D}_2^\infty)$$

for some closed subspace $\mathcal{E}_0$ of $\mathcal{E}$. Therefore, Corollary 1.4 gives that $\boldsymbol{C}$ is of Beurling type. $\qquad\square$

Follow the notations in the proof of Theorem 5.4. Now we can use the characterization of the canonical analytic model $\mathcal{Q}_{\boldsymbol{C}}$ for $\boldsymbol{C}$ (see Theorem 4.5) to study the structure of the doubly commuting quotient module $\mathcal{Q}$ since as $\mathcal{P}_\infty$-modules, $\mathcal{Q}$ and $\mathcal{Q}_{\boldsymbol{C}}$ are unitarily equivalent. The unitary module map is of the form $I_{H^2(\mathbb{D}_2^\infty)} \otimes U$, where $U : \mathcal{E}_0 \to \mathfrak{D}_{\boldsymbol{C}^*}$ is given as in (4.2). Then there exists a sequence of quotient modules $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ of $H^2_{\mathcal{E}_0}(\mathbb{D})$, such that

$$\mathcal{Q} = \bigcap_{n=1}^{\infty} \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{n-1 \text{ times}} \otimes \mathcal{J}_n \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots . \tag{5.2}$$

Note that the joint unitary equivalence is an equivalence relation on the class $\mathcal{DC}$. We can establish a one-to-one correspondence between the equivalence classes of doubly commuting quotient Hardy modules and the equivalence classes of $\mathcal{DC}$-sequences of Beurling type as illustrated below:

$$\left\{\begin{array}{c} \text{equivalence classes of} \\ \text{doubly commuting} \\ \text{quotient Hardy modules} \end{array}\right\} \underset{\text{canonical analytic models}}{\overset{\text{module actions}}{\rightleftharpoons}} \left\{\begin{array}{c} \text{equivalence classes of} \\ \mathcal{DC}\text{-sequences} \\ \text{of Beurling type} \end{array}\right\}$$

In another word, the classification of doubly commuting Hardy quotient modules is equivalent to the classification of $\mathcal{DC}$-sequences of Beurling type.

Finally, we consider the particular case $\mathcal{E} = \mathbb{C}$. By (5.2), we have

$$\mathcal{Q} = \bigcap_{n=1}^{\infty} \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_n \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots . \tag{5.3}$$

It follows that $\mathcal{Q} = \mathcal{J}_1 \otimes \mathcal{Q}'$ for some closed subspace $\mathcal{Q}'$ of

$$\mathcal{L} = \overline{\operatorname{span}\{\zeta^\alpha : \alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)} \text{ with } \alpha_1 = 0\}},$$

and thus,

$$P_{\mathcal{Q}} M_{\zeta_1} |_{\mathcal{Q}} = P_{\mathcal{J}_1} M_z |_{\mathcal{J}_1} \otimes I_{\mathcal{Q}'}.$$

Similarly, $P_{\mathcal{Q}} M_{\zeta_n} |_{\mathcal{Q}}$ is the tensor product of $P_{\mathcal{J}_n} M_z |_{\mathcal{J}_n}$ and an identity operator for each $n \in \mathbb{N}$. Recall that a model space is a proper quotient module of $H^2(\mathbb{D})$, and the compression of the Hardy shift M_z on a model space is called a Jordan block. Then for every $n \in \mathbb{N}$ so that $\mathcal{J}_n \neq H^2(\mathbb{D})$, the compression $P_{\mathcal{Q}} M_{\zeta_n} |_{\mathcal{Q}}$ is the tensor product of a Jordan block and an identity operator. So the compression of the tuple $\boldsymbol{M}_\zeta$ on $\mathcal{Q}$ can be considered as a Jordan block in the infinite-variable setting.

Recall that C_0 is the class of those completely nonunitary contractions T for which there exists a nonzero function $f \in H^\infty(\mathbb{D})$ such that $f(T) = 0$ [50]. The following application to the operator theory is motivated by [23, Proposition 4.1].

Corollary 5.5.   *Suppose $\boldsymbol{T} \in \mathcal{DC}$. If there exists a cyclic vector x for $\boldsymbol{T}$ such that $T_n^* x = \overline{\lambda_n} x$ ($n \in \mathbb{N}$) for some $\lambda \in \mathbb{D}^\infty$, then for each $n \in \mathbb{N}$, T_n is either a C_0-contraction or a pure isometry.*

Proof.   Assume $\boldsymbol{T} \in \mathcal{DC}(\mathcal{H})$. Since for each $n \in \mathbb{N}$, T_n is a C_0-contraction (pure isometry) if and only if $\varphi_{\lambda_n}(T_n)$ is a C_0-contraction (pure isometry), we may assume $\lambda = \boldsymbol{0}$ without loss of generality.

Put

$$\widetilde{\mathcal{H}} = \bigvee_{\mu \in \mathbb{D}_2^\infty} \mathfrak{D}_{\Phi_\mu(\boldsymbol{T})^*}.$$

Then by Corollary 3.1(3), $\widetilde{\mathcal{H}}$ is joint reducing for $\boldsymbol{T}$. Therefore,

$$\widetilde{\mathcal{H}} = [\widetilde{\mathcal{H}}]_{\boldsymbol{T}} \supseteq [\mathfrak{D}_{\boldsymbol{T}^*}]_{\boldsymbol{T}} \supseteq [x]_{\boldsymbol{T}} = \mathcal{H},$$

forcing $\widetilde{\mathcal{H}} = \mathcal{H}$. This together with Corollary 1.4 implies that $\boldsymbol{T}$ is of Beurling type. Since $\dim \mathfrak{D}_{\boldsymbol{T}^*} = 1$ (otherwise x is not cyclic for $\boldsymbol{T}$), the canonical analytic model $\mathcal{Q}_{\boldsymbol{T}}$ for $\boldsymbol{T}$ can be viewed as a doubly

commuting quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}_2^\infty)$, and then $\boldsymbol{T}$ is jointly unitarily equivalent to the compression $P_\mathcal{Q} \boldsymbol{M}_\zeta |_\mathcal{Q}$ of $\boldsymbol{M}_\zeta$ on $\mathcal{Q}$, where $\boldsymbol{M}_\zeta$ is the tuple of coordinate multiplication operators on $H^2(\mathbb{D}_2^\infty)$. This completes the proof. □

Also from (5.3), it seems plausible to view every doubly commuting quotient module of $H^2(\mathbb{D}_2^\infty)$ as the tensor product of infinitely many model spaces or $H^2(\mathbb{D})$'s. This can be realized after giving an appropriate definition for the infinite tensor product.

Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of closed subspaces of $H^2(\mathbb{D})$. *The tensor product of* $\{M_n\}_{n\in\mathbb{N}}$ *in* $H^2(\mathbb{D}_2^\infty)$, denoted by $\bigotimes_{n=1}^\infty M_n$, is defined to be the closed subspace of $H^2(\mathbb{D}_2^\infty)$ spanned by the functions in $H^2(\mathbb{D}_2^\infty)$ of the form $\prod_{n=1}^\infty f_n(\zeta_n)$ (in pointwise convergence for $\zeta \in \mathbb{D}_2^\infty$) with $f_n \in M_n$ for each $n \in \mathbb{N}$. Note that for infinite tensor products of the form

$$M_1 \otimes \cdots \otimes M_n \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots,$$

this new definition coincides with the original one (see Subsection 2.2).

Corollary 5.6. *Every doubly commuting quotient module of* $H^2(\mathbb{D}_2^\infty)$ *is the tensor product of some sequence of quotient modules of* $H^2(\mathbb{D})$.

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2(\mathbb{D}_2^\infty)$. Then there exists a sequence $\{\mathcal{J}_n\}_{n\in\mathbb{N}}$ of quotient modules of $H^2(\mathbb{D})$, such that

$$\mathcal{Q} = \bigcap_{n=1}^\infty \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_n \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots.$$

Now we prove

$$\mathcal{Q} = \bigotimes_{n=1}^\infty \mathcal{J}_n.$$

For simplicity, rewrite

$$\mathcal{Q}_n = \mathcal{J}_1 \otimes \cdots \otimes \mathcal{J}_n \otimes H^2(\mathbb{D}) \otimes H^2(\mathbb{D}) \otimes \cdots.$$

The inclusion

$$\bigotimes_{n=1}^\infty \mathcal{J}_n \subseteq \bigcap_{n=1}^\infty \mathcal{Q}_n = \mathcal{Q}$$

is trivial to see. For the reverse inclusion, note that the set $\{P_\mathcal{Q}\zeta^\alpha : \alpha \in \mathbb{Z}_+^{(\infty)}\}$ is complete in $\mathcal{Q}$, where $P_\mathcal{Q}$ is the orthogonal projection from $H^2(\mathbb{D}_2^\infty)$ onto $\mathcal{Q}$. It suffices to show that for any fixed $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_+^{(\infty)}$, $P_\mathcal{Q}\zeta^\alpha$ belongs to the infinite tensor product $\bigotimes_{n=1}^\infty \mathcal{J}_n$. Let $P_{\mathcal{Q}_n}$ $(n \in \mathbb{N})$ denote the orthogonal projection from $H^2(\mathbb{D}_2^\infty)$ onto $\mathcal{Q}_n$, and $P_{\mathcal{J}_n}$ denote the orthogonal projection from $H^2(\mathbb{D})$ onto $\mathcal{J}_n$. Then for each $n \in \mathbb{N}$,

$$P_{\mathcal{Q}_n} = P_{\mathcal{J}_1} \otimes \cdots \otimes P_{\mathcal{J}_n} \otimes I_{H^2(\mathbb{D})} \otimes I_{H^2(\mathbb{D})} \otimes \cdots.$$

Taking $m \in \mathbb{N}$ so that $\alpha_{m+1} = \alpha_{m+2} = \cdots = 0$ and setting $f_n = P_{\mathcal{J}_n}1$ $(n \in \mathbb{N})$, we further have

$$(P_{\mathcal{Q}_n}\zeta^\alpha)(\zeta) = (P_{\mathcal{J}_1}z^{\alpha_1})(\zeta_1)\cdots(P_{\mathcal{J}_m}z^{\alpha_m})(\zeta_m)f_{m+1}(\zeta_{m+1})\cdots f_n(\zeta_n)$$

for every $n \geqslant m+1$. On the other hand, since $\{P_{\mathcal{Q}_n}\}_{n\in\mathbb{N}}$ converges to $P_\mathcal{Q}$ $(n \to \infty)$ in the strong operator topology, $P_{\mathcal{Q}_n}\zeta^\alpha$ converges to $P_\mathcal{Q}\zeta^\alpha$ $(n \to \infty)$ in the $H^2(\mathbb{D}_2^\infty)$-norm. In particular, $P_{\mathcal{Q}_n}\zeta^\alpha$ converges pointwisely to $P_\mathcal{Q}\zeta^\alpha$ as $n \to \infty$, and then $P_\mathcal{Q}\zeta^\alpha$ is of form

$$(P_{\mathcal{J}_1}z^{\alpha_1})(\zeta_1)\cdots(P_{\mathcal{J}_m}z^{\alpha_m})(\zeta_m)\prod_{n=m+1}^\infty f_n(\zeta_n), \quad \zeta \in \mathbb{D}_2^\infty.$$

This gives $P_\mathcal{Q}\zeta^\alpha \in \bigotimes_{n=1}^\infty \mathcal{J}_n$, and the proof is completed. □

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