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Problem of invariant subspaces in C^*-algebras

ZHANG Lunchuan (张伦传)

School of Information Science, Renmin University of China, Beijing 100080, China
(email: zlc@math03.math.ac.cn)

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Abstract Let A be a separable simple C^*-algebra. For each $a(\neq 0)$ in A, there exists a separable faithful and irreducible $*$ representation (π, H_π) on A such that $\pi(a)$ has a non-trivial invariant subspace in H_π.

Keywords: C^*-algebras, irreducible $*$ representation, invariant subspace.

Given a separable Hilbert space H, let $B(H)$ denote the set of all bounded linear operators on H, and $K(H)$ be the subset of all compact operators in $B(H)$. It is well known that $B(H)$ is a C^*-algebra, and $K(H)$ is its closed two-sided ideal. In the study of linear operator in $B(H)$, there is a famous open problem, that is, the invariant subspaces problem as below: Whether each bounded linear operator in $B(H)$ has a non-trivial invariant subspace?

For some special cases, such as compact operators, normal operators, and subnormal operators, it is well known that they have non-trivial invariant subspaces respectively. But the general case has not been solved by now. Discussing the above invariant subspaces problem has two basic methods: one way is only to use single operator theory, another way is by the aid of operator algebras. The details of invariant subspace problem can refer the book by Beauzamy[1].

In this paper we consider the invariant subspace problem in a separable simple C^*-algebra in the following:

Problem 1. Let A be a separable simple C^*-algebra, for each $a(\neq 0)$ in A, whether there exists a separable and faithful irreducible $*$ representation (π, H_π) on A such that $\pi(a)$ has a non-trivial invariant subspace in H_π or not?

In 1992 Lin[2] showed that for finite separable matroid C^*-algebra, the above Problem 1 has a positive solution. But the general case has not been settled from then. In this paper we use C^*-algebra representation theory and hereditary C^*-subalgebras as main tools to answer Problem 1 positively.

1 Basic lemmas

First we recall the definitions of liminal and antiliminal C^*-algebras. We say that if a C^*-algebra A is liminal, for each irreducible representation (π, H_π) on A and each x in A, then $\pi(x)$ is a compact operator in $B(H_\pi)$. For example, $K(H)$ is a liminal C^*-algebra, and each commutative C^*-algebra is liminal. For convenience we introduce a new concept of liminal element as follows: Given a C^*-algebra A, and an element a in A, if for each irreducible reperentation (π, H_π) on A, it follows that $\pi(a)$ is a compact operator in $B(H_\pi)$, then a is said to be a liminal element in A. It is easy to check that if a simple C^*-algebra A contains a nonzero liminal element, then A is liminal itself. Contrary to the liminal C^*-algebras, if a C^*-algebra does not contain any nonzero liminal element, then it is called antiliminal. For instance, the Calkin algebra is an antiliminal C^*-algebra. We refer the reader to see the book "C^*-algebra" by Dixmier (ref. [3], Chapter 4).

Another useful concept is hereditary C^*-subalgebras. A C^*-subalgebra B of a C^*-algebra A is said to be hereditary if, for each a in A and b in B with the inequality $a < b$ implies that a in B. From ref. [4] Proposition 3.11.10 we see that the assignment $p \longrightarrow pA^{**}p \cap A$ establishes bijective correspondences between the classes of open projections in A^{**} and the closed hereditary C^*-subalgebras of A, and this relation is very useful in sec. 3.

Before proceeding to the lemmas we fix some notations and discuss some important background materials. Given a C^*-algebra A, we denote $S(A)$ the state space, $P(A)$ the set of all pure states on A, and $\overline{P(A)}^{w^*}$ the pure state space (it is the weak*-closure of $P(A)$ in the dual of A). For each ϕ in $S(A)$, letting $(\pi_\phi, H_\phi, \xi_\phi)$ denote the cyclic representation associated with ϕ, we form the Hilbert space $H_S = \oplus_{\phi \in S} H_\phi$ and the representation $\pi_S = \oplus_{\phi \in S} \pi_\phi$ on H_S. Since S is separating for A, so π_S is faithful, and the strong closure of $\pi_S(A)$ in $B(H_S)$ is called the enveloping Von Neumann algebra of C^*-algebra A. In fact, $A^{**} = A''$ (see ref. [4] Prop.3.7.8), and we have

Proposition 1. Let A be a C^*-algebra, A^{**} be its enveloping Von Neumann algebra, for each non-trivial irreducible $*$ representation (π, H_π) on A , then $\tilde{\pi}(A^{**}) = B(H_\pi)$, where $\tilde{\pi}$ is the σ-weakly continuous extension representation of π to A^{**}.

Proof. Since π is an irreducible $*$ representation on A, so $\pi(A)' = \mathbb{C}$ (see ref. [3], Prop. 2.3.1). Note that $A^{**} = A''$ (see ref. [4], 3.7.8), and by ref. [4] Theorem 3.7.7 it follows that $\tilde{\pi}(A^{**}) = \tilde{\pi}(A'') = \pi(A)'' = B(H_\pi)$. Q.E.D.

We require more information of extension and restriction of representation as below: Given a C^*-algebra A and its C^*-subalgebra B, by Dixmier's dilation theorem (see ref. [3] proposition 2.10.2) we see that each non-trivial irreducible $*$ representation (π_B, H_B) on B has an extention irreducible representation (π_A, H_A) on A. In general, H_A is strictly larger than H_B, but if B is a closed two-sided ideal of A, then $H_B = H_A$ (see ref. [3] Proposition 2.10.4).

The next proposition which characterizes the pure state space of prime antiliminal C^*-algebra is needed also:

Proposition 2. Given a prime antiliminal C^*-algebra A, if A is unital then $\overline{P(A)}^{w^*} = S(A)$; if A is not unital then $\overline{P(A)}^{w^*} = S(A) \cup \{0\}$.

Proof. Since A is prime, that is, there is a faithful irreducible $*$ representation on A, so if A is unital, it follows from ref. [3] Lemma 11.2.4 that $\overline{P(A)}^{w^*} = S(A)$; if A is not unital, letting $A = A \oplus 1C$, it is easy to check that $\tilde{A} = A \oplus 1C$ is a prime and antiliminal C^*-algebra also, by ref. [3] Lemma 11.2.4 again that $\overline{P(\tilde{A})}^{w^*} = S(\tilde{A})$. Next we will show that $\overline{P(\tilde{A})}^{w^*} |A = \overline{P(A)}^{w^*} \cup \{0\}$. For this goal, let $f \in \overline{P(\tilde{A})}^{w^*}$, then f is a weak*-limit of pure states f_α of $\tilde{A}$. If $f|A$ is nonzero, we can suppose that f_α are all nonzero, so they are pure states of A, hence $f|A$ is in $\overline{P(A)}^{w^*}$ in this case, therefore $f|A$ is in $\overline{P(A)}^{w^*} \cup \{0\}$ in all cases. Note that if A is not unital, then 0 is in $\overline{P(A)}^{w^*}$ (see ref. [3] 2.12.13), so $\overline{P(\tilde{A})}^{w^*} |A = \overline{P(A)}^{w^*}$. On the other hand, we see that $S(\tilde{A})|A = S(A) \cup \{0\}$, so by the above discussion it implies that $\overline{P(A)}^{w^*} = S(A) \cup \{0\}$ if A is not unital. Q.E.D.

Based on the above materials we state the lemmas as below:

Lemma 1. Let A be a unital separable simple C^*-algebra, for each $a(\neq 0)$ in A, then there is a separable faithful irreducible representation (π, H_π) on A such that $\pi(a)$ has a non-trivial invariant subspace in H_π.

Lemma 1 was proved by Lin (see ref. [2] Lemma 1.1).

Lemma 2. Given a separable simple C^*-algebra $A, a(\neq 0)$ is in A, if A_α is a simple C^*-

subalgebra of A, where A_α is the C^*-subalgebra generated by a in A, then there is a separable faithful irreducible $*$ representation (π, H_π) on A such that $\pi(a)$ has a non-trivial invariant subspace in $B(H_\pi)$.

Proof. First, if A is liminal, for each non-trivial irreducible $*$ representation (π, H_π) on A, by the definition of liminal C^*-algebra we see that $\pi(A) = K(H_\pi)$, that is, each element of $\pi(A)$ is a compact operator, therefore $\pi(a)$ has a non-trivial invariant subspace in H_π. In the following we suppose that A is antiliminal, and discuss it in two cases.

Case 1. If $A_a = A$, we can regard A_a as acting irreducibly on a separable Hilbert space H, so $A_a \subset B(H)$, and $A^{**} = B(H)$. Let $\phi : B(H) \longrightarrow B(H)/K(H)$ be the canonical homomorphism, by the assumption that A is antiliminal, so $\phi(a) \neq 0$. Note that the Calkin algebra $B(H)/K(H)$ is simple , it follows from ref. [5] Proposition 7 that there exists a separable simple C^*-subalgebra B containing $\phi(1)$ and $\phi(a)$ in $B(H)/K(H)$, using Lemma 1 that there is a separable faithful irreducible $*$ representation (π_B, H_B) on B such that $\pi_B(\phi(a))$ has a non-trivial invariant subspace in H_B. Moreover, let $D = \phi^{-1}(B)$, so $D \subset B(H) = A^{**}$, and let $(\pi_D, H_D) = (\pi_B \circ \phi, H_B)$, so π_D is a non-trivial irreducible $*$ representation on D and $\pi_D(a)$ has a non-trivial invariant subspace in H_D. Since a is in D and A is generated by a, so $A \subset D$, and it follows that $A \subset D \subset A^{**}$. Put $(\pi, H_A) = (\pi_D|A, H_D)$, noting that $A^{**} = A''$, then from $\pi_D(A)' \supseteq \pi_D(D)' \supseteq \pi_D(A)'$ and $\pi_D(D)' = 1C$, it implies that $\pi_D(A)' = 1C$, that is $\pi(A)' = 1C$, showing that $\pi(= \pi_D|A)$ is an irreducible $*$ representation on A and $H_A = H_D = H_B$. By the above proof we show that $\pi(a) = \pi_D(a)$ has a non-trivial invariant subspace in H_π.

Case 2. Suppose that $A_a \neq A$. Since A_a is simple, for A_a we use the method of Case 1, so there is a non-trivial separable faithful irreducible $*$ representation (π_a, H_a) on A_a such that $\pi_a(a)$ has a non-trivial invariant subspace in H_a. By ref. [3] Proposition 2.10.2 we see that there exists a separable irreducible $*$ representation (π, H) on A such that $\pi|A_a = \pi_a$ and $H_a \subset H$, so $\pi(a) = \pi_a(a)$ has a non-trivial invariant subspace in $H_a(\subset H)$, and since A is simple, so (π, H) is faithful. Q.E.D.

2 The main theorems and proofs

Theorem 1. Let A be a separable simple C^*-algebra, for each $a(\neq 0)$ in A, then there is a separable and faithful irreducible $*$ representation (π, H_π) on A such that $\pi(a)$ has a non-trivial invariant subspace in H_π.

Proof. First, if A is unital, by Lemma 1 the conclusion is correct. In special if A is a finite dimensional C^*-algebra, then A has a unit (see ref. [6] I. Lemma 11.1), and by Lemma 1 again it holds also. If A is a liminal simple C^*-algebra, then for each non-trivial irreducible $*$ representation (π, H_π) , it follows that $\pi(A) = K(H_\pi)$, showing that $\pi(a)$ is a non-zero compact operator. In the following we suppose that A is a non-unital and infinite dimensional antiliminal simple C^*-algebra, and we discuss it by two cases. Here we fix a notation A_a which stands for the C^*-subalgebra generated by a in A.

Case 1. If $A_a \neq A$ and A_a contains at least one non-trivial closed two-sided ideal B_a, then there is a separable irreducible $*$ representation (π_a, H_a) on A_a such that $\pi_a(B_a) = \{0\}$. By ref. [3] Proposition 2.10.2 there is a separable entension irreducible $*$ representation (π, H_π) on A such that $\pi|A_a = \pi_a$. Since A is simple, so π is faithful and H_π is strictly larger than H_a(if not, then $H_\pi = H_a$, leading to that $\mathrm{Ker}\pi \supseteq B_a \neq \{0\}$, where $\mathrm{Ker}\pi$ is the kernal of π in A, and this is a controdiction with that A is simple), impling that $\pi(a)H_a = \pi_a(a)H_a \subseteq H_a \subset H_\pi$, showing that H_a is a non-trivial invariant subspace for $\pi(a)$ in H_π.

Case 2. If $A_a = A$, or $A_a(\neq A)$ is simple, this is Lemma 2, then we are done. Q.E.D.

It is well known that $B(H)$ is a prime C^*-algebra, that is, there is a faithful and irreducible

$*$ representation on $B(H)$. In the final of this section we discuss the invariant subspace problem in general prime C^*-algebra and obtain the following theorem:

Theorem 2. Given a separable prime C^*-algebra A, $a(\neq 0)$ in A, if there is a non-trivial hereditary C^*-subalgebra B of A containing a, then there exists a separable irreducible $*$ representation (π, H_π) on A such that $\pi(a)$ has a non-trivial invariant subspace in H_π.

Proof. We prove it by three cases.

Case 1. If B is liminal, it is similar to the proof of the above Theorem 1 and we are done.

Case 2. If B is antiliminal. Since A is prime and B is hereditary, it is easy to check that B is prime itself also, so by the above Proposition 2 that $S(B) \cup \{0\} = \overline{P(B)}^{w^*}$, and by ref. [4] Proposition 3.11.10, there is a non-trivial open projection p in A^{**} such that $B = pA^{**}p \cap A$ and $B^{**} = pA^{**}p$, showing that p is the unit of B^{**}, so $pa = ap$. For each irreducible $*$ representation (π, H_π) on B, it implies that $\pi(p) \neq 0$ (if not, this leads to that $\pi(B) = 0$, and this is a controdiction). Next we will show that there exists an irreducible $*$ representation (π_B, H_B) on B such that $\overline{\pi_B}(p) < 1$, where $\overline{\pi_B}$ is the σ-weakly continuous extension representation of π_B to B^{**}. For this goal, it is only to prove that there is a pure state ρ on B such that $\overline{\rho}(p) < 1$, where $\overline{\rho}$ is the extension pure state on B^{**} associated with ρ. Contrary to it, assume that for each pure state ρ on B it has that $\overline{\pi_B}(p) = 1$. Let f be an any normal state on B^{**}, note that each normal state of B^{**} is the extension of one corrsponding state on B, denoting g as the state on B associated with f, so $f|B = g$, and following the equation $S(B) \cup \{0\} = \overline{P(B)}^{w^*}$, it implies that there is a net of pure states h_α on B such that $g = \lim_\alpha h_\alpha$. Moreover, let h'_α be an extension of h_α to a pure state of B^{**}, therefore $f(p) = \lim_\alpha h'_\alpha(p)$ in the weak * topology. By the above assumption that $h'_\alpha(p) = 1$, so $f(p) = 1$, following it then $p = 1$, this is a controdiction with $p < 1$. Hence there is a pure state ρ on B such that $\overline{\rho}(p) < 1$, it is easy to prove that for the irreducible $*$ representation (π_B, H_B) on B associated with the ρ, then $\overline{\pi_B}(p) < 1$, where $\overline{\pi_B}$ is the σ-weakly continuous extension representation of π_B to B^{**} associated with $\overline{\rho}$, so $\overline{\pi}_B(a)\overline{\pi}_B(p) = \overline{\pi}_B(p)\overline{\pi}_B(a)$. This equation shows that $\pi_B(a) = \overline{\pi}_B(a)$ has a non-trivial invariant subspace in H_B. By ref. [3] Proposition 2.10.2 there is a separable irreducible extension $*$ representation (π, H_π) on A such that $\pi(a) = \pi_B(a)$ has a non-trivial invariant subspace in $H_B(\subset H_\pi)$.

Case 3. If B contains a non-trivial maximum liminal closed two-sided ideal J, so B/J is antiliminal, in this case, if a is in J, note that J is a hereditary C^*-subalgebra of A and J is liminal, by Case 1 we are done; if a is not in J, let $\Phi : B \to B/J$ be the canonical quotient homomorphism, so $\Phi(a)$ is not zero in B/J. Since B is prime and separable, so J is a primitive ideal of B, and B/J is a prime C^*-algebra also, similar to the proof of Case 2 we can see that there is an irreducible $*$ representation (π_1, H) on B/J such that $\pi_1(\Phi(a))$ has a non-trivial invariant subspace in H. Set $\pi_B = \pi_1 \circ \Phi$, so (π_B, H) is an irreducible $*$ representation on B. By ref. [3] Proposition 2.10.2 there is a separable irreducible extension $*$ representation (π, H_A) on A such that $\pi(a) = \pi_B(a) = \pi_1 \circ \Phi(a)$ has a non-trivial invariant subspace in $H(\subset H_A)$. Q.E.D.

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