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Identification of the gain system with quantized observations and bounded persistent excitations

GUO Jin & ZHAO YanLong*

Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190*, China*

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Abstract System identification with quantized observations and persistent excitations is a fundamental and difficult problem. As the first step, this paper takes the gain system for example to investigate the identification with quantized observations and bounded persistently exciting inputs. Firstly, the identification with single threshold quantization is considered. A projection recursive algorithm is proposed to estimate the unknown parameter. By use of the conditional expectation of quantized observations with respect to the estimates, the algorithm is shown to be both mean-square and almost surely convergent. The upper bound of the convergence rate is also obtained, which has the same order as the one of the optimal estimation in the case where the system output is exactly known. Secondly, for the multi-threshold quantization, the identification algorithm is similarly constructed and its asymptotic property is analyzed. Using a multi-linear transformation, the optimal scheme of quantization values and thresholds is given. A numerical example is simulated to demonstrate the effectiveness of the algorithms and the main results obtained.

Keywords system identification, quantized observation, bounded persistent excitation, binary-valued output, multi-threshold quantization, convergence, convergence rate

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1 Introduction

Since Norbert Wiener published his famous book "Cybernetics: or Control and Communication in the Animal and the Machine" [1] in 1948, the cybernetic thinking has penetrated into almost all the natural and social sciences. A number of classic methods and techniques have been developed gradually, such as system identification of least square [2] and maximum likelihood method [3], state estimation of Kalman filter [4] and Bayesian filter [5], control method of self-tuning regulator [6] and so on. The generation and application of these methods have greatly improved people's production and living standards.

It is noteworthy that all the above methods are based on the precondition that the system data (input, state or output) is known exactly or with a certain noise [7]. However, with the development of modernization and informationization, a class of new systems—quantized output systems [8]—have emerged in the practical fields. The output of such systems cannot be accurately measured, and what can

*Corresponding author (email: ylzhao@amss.ac.cn)

be measured is whether or not the output belongs to some known set. Taking the neuron system [9] as an example, though measuring its internal potential is very difficult, two states of excitation and inhibition, from the outside of neuron, can be detected, whose decisive factor is the potential threshold. When the potential is larger than the threshold, the neuron shows the excitation state, otherwise, the inhibition one.

The threshold is a key factor for quantized output systems, which can be fixed or adjustable with time. For example, the threshold of oxygen sensors [10] in industry is fixed, which depends on the physical characteristics of sensors and cannot been changed. An example of time-varying threshold is the coding process in communications, which is really a kind of protocol [11] that can be adjusted according to actual needs. On the other hand, the number of thresholds is also very important, which may be one or more. In fact, nothing supplies poorer information than the case of only one threshold. With the increasing numbers of thresholds, the available data will be enriched gradually and the most abundant one is the accurate data for the conventional systems.

On the practical side, quantized output systems are widely used in industrial production, technology of biological pharmacy, informational industry and many other fields. This poses a lot of problems of how to identify and control such types of systems. On the theoretical side, from the example of neuron system, it can be seen that quantized output systems cannot provide the exact observation except very limited information for identification and control, which spoils the basic precondition of classic methods. And, directly applying these classic methods to quantized systems, one may not reach the goal of identification and control, or even make some serious accidents. To study quantized output systems, we must start from their essential characters, and propose new methods to identify parameters and estimate states, as well as to design control laws to improve system performance.

Recently, the identification/state estimation of quantized output systems has attracted a lot of attention [8, 12–17]. Based on periodic inputs and empirical measures, Refs. [8, 17] gave a strongly consistent and asymptotically optimal parameter identification algorithm with the help of statistical properties of the system noises, and Ref. [12] studied the optimal identification error, complexity of space and time and impact of disturbances and unmodeled dynamics on the identification accuracy. Consequently, other models were also investigated, such as rational models, Wiener [18] and Hammerstein [19] systems and so on. Ref. [13] proposed a method for designing optimal periodic input to reduce the time complexity on parameter identification. Ref. [14] discussed the linear system identification with the colored noises and multi-sine input signal. Under the Gaussian assumption on the predicted density, Refs. [15, 16] investigated the minimum mean quare filtering using the quantized innovations.

All these papers have coped effectively with the parameter identification for quantized output systems, but they might not be applied to design adaptive control algorithm due to the restrictions of the periodicity or independent and identically distributed (i.i.d.) property on inputs. How to study the adaptive control for such systems? The following idea may be feasible. Firstly, construct the online recursive identification algorithm with quantized information and persistent excitations or even attenuating persistently exciting inputs. Then, prove the convergence and optimality with the correlation between quantized information and estimations. Finally, complete the control-oriented identification algorithm and the adaptive control.

The idea of control-oriented identification algorithms might have the potential to implement the adaptive control of the quantized output systems, but the design and analysis of them will bring new difficulties. On the one hand, the persistently exciting inputs usually have no periodic or i.i.d character. As a result, the existing methods, such as the empirical measure method under periodic inputs and the kernel function one under i.i.d. inputs and so on, do not work under persistent excitations. On the other hand, the correlation between quantized outputs and estimations is stronger under the control-oriented inputs than periodic or i.i.d. ones, and will become stronger and stronger with the increase in the number of system parameters, which makes it very difficult to find a general recursive equation of the estimation error, and many classic methods for the conventional systems seem to have little reference to the quantized output systems. As the first step towards the control-oriented identification of quantized systems, this paper only considers the gain system identification under bounded persistent excitations.

In order to show the basic idea, we begin with the most elementary quantization form—binary-valued observation, where only one threshold exists and the information is about whether or not the system output is larger than the threshold. A projection recursive algorithm is proposed to estimate the parameter that keeps the boundedness of the estimation. By use of the conditional expectation of the estimation errors and quantization values with respect to one-step forward estimations, it is shown that the algorithm is both mean-square and almost surely convergent. The upper bound of the convergence rate is also obtained.

For the more general case of multi-threshold quantization, the identification algorithm is constructed and its asymptotic property is analyzed. The thresholds and quantization values are optimized at the same time and an optimal scheme is given using the multi-linear transformation.

This paper is organized as follows. Section 2 formulates the identification problem. Section 3 studies the identification under single threshold quantization, including constructing the algorithm and analyzing its convergence and convergence rate. Section 4 develops the identification of multi-threshold quantization. The identification algorithm, its asymptotic property and the optimal scheme of quantization values and thresholds are given. A numerical example is simulated to demonstrate the effectiveness of the algorithms and the main results obtained in Section 5. Section 6 gives some concluding remarks. Some detailed mathematical results and proofs are addressed in Appendix.

2 Problem formulation

Consider the gain system:

$$y_k = \phi_k \theta + d_k, \quad q_k = \mathcal{Q}(y_k), \tag{1}$$

where $\phi_k \in \mathbb{R}$, $\theta \in \mathbb{R}$ and d_k are the system input, unknown but time-invariance parameter and noise, respectively; y_k is the system output, which cannot be exactly measured. What can be measured is only the quantized information q_k, which represents the comparison between y_k and one or more real numbers, namely, thresholds. Mathematically, q_k can be written as

$$q_k = \mathcal{Q}(y_k) = \sum_{i=1}^{N} \beta_i I_{[c_k^i \leqslant y_k < c_k^{i+1}]}, \tag{2}$$

where c_k^i are the time-varying thresholds, which are variables to be designed in this paper with $-\infty = c_k^1 < c_k^2 < \cdots < c_k^N < c_k^{N+1} = +\infty$; $I_{[c_k^i \leqslant y_k < c_k^{i+1}]}$ is the indicator function, which is 1 if $c_k^i \leqslant y_k < c_k^{i+1}$ and 0, otherwise; $\beta_i > 0$ is called quantization value with respect to the set $[c_k^i, c_k^{i+1})$, $i = 1, \dots, N$. It can be seen that q_k is one of $\beta_1, \dots, \beta_N$ and the key factors of deciding its value are the thresholds $c_k^2, \dots, c_k^N$.

The goal of this paper is to estimate the unknown parameter θ using the input ϕ_k and quantized observation q_k.

The case with quantized measurements q_k, which provides very limited information, is much more difficult than the conventional identification with accurate measurements of y_k, mainly because the relationship between the available measurement and the input is not one-to-one, but essentially nonlinear.

(1) and (2) show that the values of quantization, number of thresholds and system noise all have a great influence on the design and analysis of identification algorithms, whose detailed discussion will be given in the following sections.

Assumption 1. The prior information of θ is that $\theta \in [-\bar{\theta}, \bar{\theta}]$, where $\bar{\theta}$ is a known positive and finite constant.

Assumption 2. $\{d_k, k \geqslant 1\}$ is an independent and identically distributed (i.i.d.) stochastic sequence and d_1 is a normally distributed random variable with zero mean and known covariance σ^2.

Assumption 3. The system input follows the conditions that

$$|\phi_k| \leqslant M < \infty, \quad k = 1, 2, \dots \tag{3}$$

with M known constant and

$$\liminf_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\phi_i^2>0. \tag{4}$$

Remark 1. The noise in Assumption 2 can be generalized into the symmetrically distributed one with known covariance and continuous density function; the input given by Assumption 3 is called bounded persistently exciting input [20], which is a typical signal used in system identification and essentially different from the previous periodic or i.i.d. condition.

3 Identification with binary-valued observations

To investigate the identification with quantized observations, we start with the most elementary case—binary-valued observation, where only one threshold exists. Though the binary-valued observation is relatively simple, it possesses all the typical characters of quantized observations. In this section, q_k defined by (2) is binary-valued with the threshold c_k. And then, q_k can be rewritten as

$$q_k=\beta\left(I_{[y_k>c_k]}-I_{[y_k\leqslant c_k]}\right), \tag{5}$$

where $\beta>0$ is called quantization value.

3.1 Identification algorithm

To estimate θ, we propose the following recursive projection algorithm:

$$\hat\theta_k=\Pi_\Theta\left\{\hat\theta_{k-1}+\frac{P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2}q_k\right\}, \tag{6}$$

$$P_k=P_{k-1}-\gamma\frac{P_{k-1}^2\phi_k^2}{1+P_{k-1}\phi_k^2}, \tag{7}$$

$$q_k=\beta\left(I_{[y_k>\phi_k\hat\theta_{k-1}]}-I_{[y_k\leqslant\phi_k\hat\theta_{k-1}]}\right), \tag{8}$$

where $\Theta\triangleq[-\overline{\theta},\ \overline{\theta}]$; initial value $\hat\theta_0\in\Theta$ and $P_0>0$ can be arbitrarily chosen; γ is a positive real number; $\Pi_\Theta(\cdot)$ is a projection operator defined by $\Pi_\Theta(x)=\operatorname{argmin}_{z\in\Theta}|x-z|$ for any $x\in\mathbb{R}$.

Remark 2. (6) provides the recursion of estimations, and (7) is relevant to the value of γ. In fact, by selecting a suitable γ, P_k can represent the variance of the estimation error in a certain sense. Thus, (7) can be understood as the recursion of the variance of the estimation error. (8) implies the scheme of designing the time-varying threshold c_k, i.e., c_k is $\phi_k\hat\theta_{k-1}$, which can be seen as the prediction of the system output y_k based on the estimate $\hat\theta_{k-1}$.

Since P_k plays an important role in algorithm (6)–(8), we first give some of its properties whose proof is provided in Appendix. It is convenient to use "$a_n\sim b_n$" to express "$\lim_{n\to\infty}a_n/b_n=1$" for two real number sequences $\{a_n,n\geqslant1\}$ and $\{b_n,n\geqslant1\}$.

Proposition 1. If (3) and (4) hold, then P_k has the following properties:
i) P_k^{-1} satisfies the recursive equation:

$$P_k^{-1}=P_{k-1}^{-1}+\frac{\gamma\phi_k^2}{(1-\gamma)P_{k-1}\phi_k^2+1}, \tag{9}$$

and, for any initial value $P_0>0$, we have

$$0<P_{k+1}\leqslant P_k,\quad \lim_{k\to\infty}P_k=0. \tag{10}$$

ii) P_k can also be described as

$$P_k \sim \left(\gamma \cdot \sum_{i=1}^{k} \phi_i^2\right)^{-1} = O\left(\frac{1}{k}\right). \tag{11}$$

iii) The limit below holds:

$$\sum_{i=1}^{k} P_{i-1}\phi_i^2 \to \infty, \text{ as } k \to \infty. \tag{12}$$

Remark 3. (10) means that P_k decreases monotonously and tends zero. (11) clarifies the role of γ in the recursion of P_k and (12) is a direct result of persistent excitation condition (4).

3.2 Properties of the identification algorithm

Denote the estimate error by $\tilde{\theta}_k = \hat{\theta}_k - \theta$ $(k = 0, 1, \ldots)$. Noticing that Θ is a convex-compact set, by the property of the projection operator and (6), we have

$$\left|\tilde{\theta}_k\right| \leqslant \left|\tilde{\theta}_{k-1} + \frac{P_{k-1}\phi_k}{1 + P_{k-1}\phi_k^2} q_k\right|, \tag{13}$$

and q_k can be rewritten in the following form:

$$q_k = \beta\left(I_{[d_k > \phi_k\tilde{\theta}_{k-1}]} - I_{[d_k \leqslant \phi_k\tilde{\theta}_{k-1}]}\right), \tag{14}$$

since $y_k > \phi_k\hat{\theta}_{k-1}$ is equivalent to $d_k > \phi_k\tilde{\theta}_{k-1}$ by (1).

Theorem 1. For system (1) with binary-valued output (5), under the conditions of Assumptions 1–3, the parameter estimate given by algorithm (6)–(8) is both mean-square convergent and almost surely convergent, i.e.,

$$\lim_{k\to\infty} E\tilde{\theta}_k^2 = 0, \quad \lim_{k\to\infty} \tilde{\theta}_k = 0 \text{ a.s.}. \tag{15}$$

Proof. By (13) and (14) we have

$$\tilde{\theta}_k^2 \leqslant \tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1 + P_{k-1}\phi_k^2)^2} + 2\frac{P_{k-1}\phi_k}{1 + P_{k-1}\phi_k^2}\tilde{\theta}_{k-1}q_k,$$

such that

$$\begin{aligned} E[\tilde{\theta}_k^2|\tilde{\theta}_{k-1}] &\leqslant \tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1 + P_{k-1}\phi_k^2)^2} + 2\frac{P_{k-1}\phi_k}{1 + P_{k-1}\phi_k^2}\tilde{\theta}_{k-1}E[q_k|\tilde{\theta}_{k-1}] \\ &= \tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1 + P_{k-1}\phi_k^2)^2} + 2\frac{\beta P_{k-1}\phi_k}{1 + P_{k-1}\phi_k^2}\tilde{\theta}_{k-1}\big(1 - 2F(\phi_k\tilde{\theta}_{k-1})\big), \end{aligned}$$

where $F(\cdot)$ is the distribution function of d_1, i.e.,

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \mathrm{e}^{-u^2/(2\sigma^2)}\mathrm{d}u.$$

And then, we have

$$E\tilde{\theta}_k^2 \leqslant E\tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1 + P_{k-1}\phi_k^2)^2} - 2\frac{\beta P_{k-1}\phi_k}{1 + P_{k-1}\phi_k^2}E\tilde{\theta}_{k-1}\big(2F(\phi_k\tilde{\theta}_{k-1}) - 1\big). \tag{16}$$

From Assumption 1 and (6), we know $|\tilde{\theta}_k| \leqslant 2\bar{\theta}$. Let $\alpha = 2\bar{\theta}M$. Then, by [21, Lemma 1], there exists $\bar{B}_1 = \bar{B}_1(\alpha)$ such that $\phi_k\tilde{\theta}_{k-1}(1 - 2F(\phi_k\tilde{\theta}_{k-1})) \leqslant -\bar{B}_1\phi_k^2\tilde{\theta}_{k-1}^2$. Substituting it into (16) results in

$$E\tilde{\theta}_k^2 \leqslant \left(1 - \frac{2\bar{B}_1\beta P_{k-1}\phi_k}{1 + P_{k-1}\phi_k^2}\right)E\tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1 + P_{k-1}\phi_k^2)^2}.$$

Since $P_k \to 0$ by (10), we have

$$\frac{\beta^2 P_{k-1}^2 \phi_k^2}{(1+P_{k-1}\phi_k^2)^2} \Big/ \frac{2\bar{B}_1 \beta P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2} = \frac{\beta^2 P_{k-1}\phi_k^2}{2(1+P_{k-1}\phi_k^2)\bar{B}_1\beta\phi_k} \to 0, \quad \text{as } k \to \infty.$$

Hence, [22, Theorem 1.2.22] implies $\lim_{k\to\infty} E\tilde{\theta}_k^2 = 0$.

Noticing

$$x(1-2F(x)) = \frac{-2x}{\sqrt{2\pi}\sigma}\int_0^x \mathrm{e}^{-u^2/(2\sigma^2)}\mathrm{d}u \leqslant 0,$$

for any $x \in \mathbb{R}$, by (16) and (11) we have $E[\tilde{\theta}_k^2|\tilde{\theta}_{k-1}] \leqslant \tilde{\theta}_{k-1}^2 + \beta^2 P_{k-1}^2\phi_k^2/(1+P_{k-1}\phi_k^2)^2$ and

$$\sum_{k=1}^{\infty} \frac{\beta^2 P_{k-1}^2 \phi_k^2}{1+P_{k-1}\phi_k^2} \leqslant \beta^2 M^2 \sum_{k=1}^{\infty} P_{k-1}^2 < \infty.$$

Therefore, $|\tilde{\theta}_k|$ converges almost surely to a bounded limit [23, Lemma 1.2.2]. Notice that $\lim_{k\to\infty} E\tilde{\theta}_k^2 = 0$. Then, there is a subsequence of $\tilde{\theta}_k$ that converges almost surely to 0. Consequently, $\tilde{\theta}_k$ almost surely converges to 0. Thus, the theorem is proved.

Remark 4. Theorem 1 not only removes the requirement for periodic or i.i.d. inputs, but also makes the adaptive control for limited information systems possible [21].

Theorem 2. Under the condition of Theorem 1, if $2B\rho > 1$, then

$$\limsup_{k\to\infty} \left(E\tilde{\theta}_k^2 \cdot \sum_{i=1}^{k} \phi_i^2 \right) \leqslant \frac{\rho^2}{2B\rho - 1},$$

with $\rho = \beta/\gamma$ and $B = 2/(\sqrt{2\pi}\sigma)$.

Proof. Since $2B\rho > 1$, by Lemma A3, there exists m such that $E\tilde{\theta}_k^{2m} = o(1/k)$. Furthermore, by Lemma A2, we have $E\tilde{\theta}_k^{2r} = o(1/k)$ for $r = 2, \ldots, m$. Let $\alpha = 2\bar{\theta}M$. By (16) and [21, Lemma 1], one can get

$$\begin{aligned}
E\tilde{\theta}_k^2 &\leqslant E\tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1+P_{k-1}\phi_k^2)^2} + 2\frac{\beta P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2} E\tilde{\theta}_{k-1}\big(1-2F(\phi_k\tilde{\theta}_{k-1})\big) \\
&\leqslant E\tilde{\theta}_{k-1}^2 - E\left(\frac{2\beta B P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\tilde{\theta}_{k-1}^2\right) + \frac{P_{k-1}^2\phi_k^2\beta^2}{(1+P_{k-1}\phi_k^2)^2} \\
&\quad + \frac{2\beta P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2}\left(2\sum_{j=2}^{m-1}\frac{F^{(2j-1)}(0)}{(2j-1)!}E\tilde{\theta}_{k-1}^{2j} + \bar{B}_m E\tilde{\theta}_{k-1}^{2m}\right) \\
&= \left(1 - \frac{2\beta B P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\right)E\tilde{\theta}_{k-1}^2 + \frac{\beta^2 P_{k-1}^2\phi_k^2}{(1+P_{k-1}\phi_k^2)^2}\big(1+o(1)\big).
\end{aligned}$$

By $2B\rho > 1$ and $\rho = \beta/\gamma$, we have $2B\beta > \gamma$. Thus, Lemma A5 indicates $\limsup_{k\to\infty} E\tilde{\theta}_k^2/P_k \leqslant \beta^2/(2B\beta-\gamma)$, which together with $P_k \sim (\gamma\sum_{i=1}^k \phi_i^2)^{-1}$ in (11) implies this theorem.

Theorem 2 not only gives the upper bound of the convergence rate of estimation errors, but also tells us how the threshold and quantization value influence algorithm (6)–(8). Noticing

$$\frac{\rho^2}{2B\rho - 1} = \frac{1}{B^2 - (1/\rho - B)^2},$$

we know that $\rho = 1/B$ can minimize $\rho^2/(2B\rho - 1)$ and the minimum value is B^{-2} within the range of $B > 0$ and $\beta > (2B)^{-1}$. And then, the algorithm may have the fastest convergence rate.

Theorem 3. Under the condition of Theorem 2, $\rho = 1/B$ can minimize the upper bound of the convergence rate of $E\tilde{\theta}_k^2$ and then

$$\limsup_{k\to\infty} \left(E\tilde{\theta}_k^2 \cdot \sum_{i=1}^{k} \phi_i^2 \right) \leqslant \frac{1}{B^2}.$$

As we all know, the classic identification algorithm—least square (LS)—can be used to estimate the unknown parameter θ if the output is measured exactly, which means $q_k = y_k$ for system (1). Denote the estimation error by $\tilde{\theta}_k^{LS}$, $E(\tilde{\theta}_k^{LS})^2 \sim Ed_1^2/\sum_{i=1}^k \phi_i^2$ [7], which is $O(1/k)$ under the condition of Assumption 3. Theorems 2 and 3 show that, in the case of even binary-valued observations, algorithm (6)–(8) has the same order $O(1/k)$ as the LS one.

Then, what will happen for the multi-threshold quantization?

4 Identification with multi-threshold quantized observations

This section will discuss the identification with multi-threshold quantized observations. Multi-threshold quantized outputs bring us more information along with the increase of the number of thresholds than the binary-valued ones. However, subsequent challenges and difficulties are coming. A primary problem is how to select the thresholds and quantization values, because an irrational design of multiply thresholds may be worse than the binary case.

We will construct the identification algorithm, analyze its properties and give the optimal scheme of the thresholds and quantization values.

4.1 Identification algorithm

Similar to (6)–(8), we introduce the recursive identification algorithm as follows:

$$\hat{\theta}_k = \Pi_\Theta \left\{ \hat{\theta}_{k-1} + \frac{P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2} q_k \right\}, \tag{17}$$

$$P_k = P_{k-1} - \gamma \frac{P_{k-1}^2\phi_k^2}{1+P_{k-1}\phi_k^2}, \tag{18}$$

$$q_k = \sum_{i=1}^{N} \beta_i \left(I_{[c_i+\phi_k\hat{\theta}_{k-1} \leqslant y_k < c_{i+1}+\phi_k\hat{\theta}_{k-1}]} - I_{[-c_{i+1}+\phi_k\hat{\theta}_{k-1} < y_k \leqslant -c_i+\phi_k\hat{\theta}_{k-1}]} \right), \tag{19}$$

where $\gamma > 0$ and $0 = c_1 < c_2 < \cdots < c_N < c_{N+1} = +\infty$; $\beta_i > 0$, $i = 1, \ldots, N$, are called quantization values; initial value $\hat{\theta}_0 \in \Theta$ and $P_0 > 0$ can be arbitrarily selected; $\Pi_\Theta(\cdot)$ is the project operator in (6).

Remark 5. A significant difference between algorithm (17)–(19) and (6)–(8) is the quantized information of system outputs, i.e., quantization is different. q_k can only take two values in (6)–(8), but $2N$ values in (17)–(19). The thresholds in (19) are symmetric about $\phi_k\hat{\theta}_{k-1}$, which indicates that the difference between each threshold and $\phi_k\hat{\theta}_{k-1}$ are constants.

4.2 Properties of the identification algorithm

Firstly, the convergence and convergence rate of algorithm (17)–(19) can be obtained using the method for Theorems 1 and 2.

Theorem 4. For system (1), under the condition of Theorem 1, the parameter estimation error $\tilde{\theta}_k$ given by identification algorithm (17)–(19) has the convergence in the sense of

$$\lim_{k\to\infty} E\tilde{\theta}_k^2 = 0, \quad \lim_{k\to\infty} \tilde{\theta}_k = 0 \text{ a.s.} \tag{20}$$

As we might expect, $\tilde{\theta}_k$ is both mean-square and almost surely convergent, which is the same as the case of single threshold. From single threshold to multiple ones, what exact difference happens? The following theorem may be a good explanation.

Theorem 5. Under the condition of Theorem 4, if $\sum_{i=1}^N \rho_i(f(c_i)-f(c_{i+1}))>1/4$, then

$$\limsup_{k\to\infty}\left(E\tilde{\theta}_k^2\cdot\sum_{i=1}^k\phi_i^2\right)\leqslant R(N)\triangleq\frac{2\sum_{i=1}^N\rho_i^2\big(F(c_{i+1})-F(c_i)\big)}{4\sum_{i=1}^N\rho_i\big(f(c_i)-f(c_{i+1})\big)-1},\tag{21}$$

where $F(\cdot)$ and $f(\cdot)$ are the distribution and density functions of d_1, respectively; $\rho_i=\beta_i/\gamma$, $i=1,\dots,N$.

Proof. From (19), we have $q_k^2=\sum_{i=1}^N\beta_i^2 I_{[c_i\leqslant|d_k-\phi_k\tilde{\theta}_{k-1}|<c_{i+1}]}$. Then, denoting $\omega=\phi_k\tilde{\theta}_{k-1}$, by (17) one can get

$$\begin{aligned}
E[\tilde{\theta}_k^2|\tilde{\theta}_{k-1}]\leqslant{}&\tilde{\theta}_{k-1}^2+\frac{P_{k-1}^2\phi_k^2}{(1+P_{k-1}\phi_k^2)^2}E[q_k^2|\tilde{\theta}_{k-1}]+\frac{2P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2}\tilde{\theta}_{k-1}E[q_k|\tilde{\theta}_{k-1}]\\
={}&\tilde{\theta}_{k-1}^2+\frac{P_{k-1}^2\phi_k^2}{(1+P_{k-1}\phi_k^2)^2}\sum_{i=1}^N\beta_i^2\big(F(\omega+c_{i+1})-F(\omega+c_i)+F(\omega-c_i)-F(\omega-c_{i+1})\big)\\
&+\frac{2P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2}\tilde{\theta}_{k-1}\sum_{i=1}^N\beta_i\big(F(\omega+c_{i+1})-F(\omega+c_i)-F(\omega-c_i)+F(\omega-c_{i+1})\big)\\
={}&\tilde{\theta}_{k-1}^2+\frac{P_{k-1}^2\phi_k^2}{(1+P_{k-1}\phi_k^2)^2}\left(2\sum_{i=1}^N\beta_i^2(F(c_{i+1})-F(c_i))\right)\big(1+o(1)\big)\\
&-\frac{P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\left(4\sum_{i=1}^N\beta_i(f(c_i)-f(c_{i+1}))\right)\tilde{\theta}_{k-1}^2\big(1+o(1)\big)\\
={}&\left(1-\frac{P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\left(4\sum_{i=1}^N\beta_i(f(c_i)-f(c_{i+1}))\right)\big(1+o(1)\big)\right)\tilde{\theta}_{k-1}^2\\
&+\frac{P_{k-1}^2\phi_k^2}{(1+P_{k-1}\phi_k^2)^2}\left(2\sum_{i=1}^N\beta_i^2(F(c_{i+1})-F(c_i))\right)\big(1+o(1)\big).
\end{aligned}$$

From this, the theorem can be proved by use of the method in Theorem 2.

Theorem 5 gives an upper bound of the convergence rate of (17)–(19). Consequently, two questions need to be figured out: i) what is the limit of $R(N)$ with the increase of N; ii) how to choose c_i and β_i to minimize $R(N)$ for a fixed N. The subsequent discussion will concentrate on them.

The following theorem gives the limit characters of $R(N)$.

Theorem 6. If $\triangle_N=\sup_{2\leqslant i\leqslant N}(c_i-c_{i-1})$ and

$$\lim_{N\to\infty}\triangle_N=0,\quad \lim_{N\to\infty}c_N=+\infty,\quad c_i\leqslant\rho_i\leqslant c_{i+1},\ i=1,\dots,N,\tag{22}$$

then $\lim_{N\to\infty}R(N)=Ed_1^2=\sigma^2$.

Proof. From the definition of Lebesgue-Stieltjes integral [24, pp. 178], we have

$$\begin{aligned}
\lim_{N\to\infty}\sum_{i=1}^N\rho_i\big(f(c_i)-f(c_{i+1})\big)&=\lim_{c_N\to\infty}\lim_{\triangle_N\to0}\sum_{i=1}^N\rho_i\big(f(c_i)-f(c_{i+1})\big)=\lim_{c_N\to\infty}\int_0^{c_N}x\mathrm{d}\big(-f(x)\big)\\
&=-\lim_{c_N\to\infty}c_Nf(c_N)+\lim_{c_N\to\infty}\int_0^{c_N}f(x)\mathrm{d}x=\frac{1}{2}.
\end{aligned}\tag{23}$$

In the same way, one can get $\lim_{N\to\infty}\sum_{i=1}^N\rho_i^2(F(c_{i+1})-F(c_i))=\sigma^2/2$, which together with (23) implies the theorem.

Theorem 6 shows that algorithm (17)–(19) could have the same convergence rate as LS one when N is large enough. In fact, (22) provides a scheme of quantization values and thresholds such that $R(N)$ converges to Ed_1^2. Thus, (22) is surely a better scheme in the sense of limit. But for a fixed N, it may be not a good one. An optimal one will be given below.

4.3 Optimal scheme of the quantization values and thresholds

As was mentioned above, the quantized information may become richer with the increase in the number of thresholds, but the designing of thresholds and selecting of quantization values are also important.

To clarify this, let us look at an example. If $N=1$, then $\rho_1=(2f(0))^{-1}$ generates $R(1)=(2f(0))^{-2}=\pi/2=1.5708$; if $N=2$ and $c=1$, $\rho_1=20$ and $\rho_2=5$ make $R(2)=17.1377$; meanwhile, $\rho_1=0.5211$ and $\rho_2=1.7283$ do $R(2)=1.1332$.

This example shows that the convergence rate of (17)–(19) with two-threshold quantized observations may be slower than the one with single threshold, but faster if suitable quantization values are selected. Thus, the quantization values are important for the quantized information. Similar examples can also be given to illustrate the importance of thresholds.

Next, we will prove that $R(N+1)<R(N)$ if $R(N)$ is minimized for any fixed N. The theorem below implies the optimal scheme of quantization values and thresholds.

Theorem 7. Denoting $c^N=(c_1,\dots,c_N)$ and $\rho^N=(\rho_1,\dots,\rho_N)$, $R(N)$ given by (21) can reach its minimum value in the sense of

$$\min_{c^N,\,\rho^N} R(N)=\frac{1}{2}\left(\max_{c^N}\sum_{i=1}^{N}\frac{(f(c_i)-f(c_{i+1}))^2}{F(c_{i+1})-F(c_i)}\right)^{-1}.$$

Furthermore, if $(c_1^*,\dots,c_N^*)=\operatorname{argmax}_{c^N}\sum_{i=1}^N(f(c_i)-f(c_{i+1}))^2/(F(c_{i+1})-F(c_i))$, $\rho_i^*=\prod_{j=i}^{N-1}k_j\rho_N^*$, $i=1,\dots,N-1$, and $\rho_N^*=\big(\sum_{i=1}^{N-1}\prod_{j=i}^{N-1}k_j(f(c_i^*)-f(c_{i+1}^*))+f(c_N^*)\big)^{-1}/2$ with

$$k_j=\frac{F(c_{j+2}^*)-F(c_{j+1}^*)}{f(c_{j+1}^*)-f(c_{j+2}^*)}\cdot\frac{\sum_{i=1}^j k_i\big(f(c_i^*)-f(c_{i+1}^*)\big)}{\sum_{i=1}^j k_i^2\big(F(c_{i+1}^*)-F(c_i^*)\big)},\quad j=1,\dots,N-1,$$

then $(c_1^*,\dots,c_N^*,\,\rho_1^*,\dots,\rho_N^*)=\operatorname{argmin}_{c^N,\,\rho^N}R(N)$.

Proof. Since $\rho_i\neq 0$, $i=1,\dots,N$, we can define $k_1,\dots,k_{N-1}$ by

$$\rho_i=\prod_{j=i}^{N-1}k_j\rho_N,\quad i=1,\dots,N-1. \tag{24}$$

Then, by (21) one can get

$$R(N)=\frac{2\rho_N^2\,(1-F(c_N)+T_{N-1})}{4\rho_N\,(f(c_N)+S_{N-1})-1}=\frac{2\,(1-F(c_N)+T_{N-1})}{4(f(c_N)+S_{N-1})^2-(\rho_N^{-1}-2(f(c_N)+S_{N-1}))^2},$$

where $T_{N-1}=\sum_{i=1}^{N-1}\prod_{j=i}^{N-1}k_j^2(F(c_{i+1})-F(c_i))$ and $S_{N-1}=\sum_{i=1}^{N-1}\prod_{j=i}^{N-1}k_j(f(c_i)-f(c_{i+1}))$. Therefore, $R(N)$ is minimized by $\rho_N=(f(c_N)+S_{N-1})^{-1}/2$. Substituting it into $R(N)$ we have

$$R(N)=\frac{1-F(c_N)+T_{N-1}}{2(f(c_N)+S_{N-1})^2}=\frac{1}{2}\left(\frac{\big(f(c_N)+k_{N-1}\big(S_{N-2}+f(c_{N-1})-f(c_N)\big)\big)^2}{1-F(c_N)+k_{N-1}^2\big(T_{N-2}+F(c_N)-F(c_{N-1})\big)}\right)^{-1}. \tag{25}$$

Setting $x_1=f(c_N)$, $x_2=S_{N-2}+f(c_{N-1})-f(c_N)$, $x_3=1-F(c_N)$, $x_4=T_{N-2}+F(c_N)-F(c_{N-1})$, by Lemma A6 in Appendix and (25), we know that $R(N)$ is minimized by

$$k_{N-1}=\frac{F(c_{N+1})-F(c_N)}{f(c_N)-f(c_{N+1})}\cdot\frac{S_{N-2}+f(c_{N-1})-f(c_N)}{T_{N-2}+F(c_N)-F(c_{N-1})},$$

and the minimum value is

$$\frac{1}{2}\left(\frac{(f(c_N)-f(c_{N+1}))^2}{F(c_{N+1})-F(c_N)}+R_{N-2}\right)^{-1}, \tag{26}$$

where

$$R_{N-2}=\frac{(S_{N-2}+f(c_{N-1})-f(c_N))^2}{T_{N-2}+F(c_N)-F(c_{N-1})}=\frac{\big(k_{N-2}S_{N-3}+f(c_{N-1})-f(c_N)\big)^2}{k_{N-2}^2T_{N-3}+F(c_N)-F(c_{N-1})}.$$

Table 1 Optimal scheme of thresholds

N	$(c_1^*, c_2^*, \ldots, c_N^*)$	$\min(R(N))$
1	0.9816	1.1331
2	0.6579, 1.4463	1.0616
3	0.5008, 1.0496, 1.7475	1.0357
4	0.4052, 0.8343, 1.3247, 1.9657	1.0234
5	0.3400, 0.6957, 1.0855, 1.5395, 2.1399	1.0166

Similarly, by use of Lemma A6, R_{N-2} is maximized by

$$k_{N-2} = \frac{F(c_N) - F(c_{N-1})}{f(c_{N-1}) - f(c_N)} \cdot \frac{S_{N-3} + f(c_{N-2}) - f(c_{N-1})}{T_{N-3} + F(c_{N-1}) - F(c_{N-2})},$$

and the maximum value is

$$\frac{\left(f(c_{N-1}) - f(c_N)\right)^2}{F(c_N) - F(c_{N-1})} + R_{N-3},$$

where $R_{N-3} = \left(S_{N-3} + f(c_{N-2}) - f(c_{N-1})\right)^2 / (T_{N-3} + F(c_{N-1}) - F(c_{N-2}))$. Then, we can see that the minimum value of (26) is

$$\frac{1}{2}\left(\frac{(f(c_N) - f(c_{N+1}))^2}{F(c_{N+1}) - F(c_N)} + \frac{(f(c_{N-1}) - f(c_N))^2}{F(c_N) - F(c_{N-1})} + R_{N-3}\right)^{-1}.$$

Repeating this process, $R(N)$ is minimized by

$$k_j = \frac{F(c_{j+2}) - F(c_{j+1})}{f(c_{j+1}) - f(c_{j+2})} \cdot \frac{\sum_{i=1}^{j} k_i (f(c_i) - f(c_{i+1}))}{\sum_{i=1}^{j} k_i^2 (F(c_{i+1}) - F(c_i))}, \quad j = 1, \ldots, N-1,$$

and the minimum value is $(\sum_{i=1}^{N} (f(c_i) - f(c_{i+1}))^2 / (F(c_{i+1}) - F(c_i))^{-1}/2$, from which the theorem can be proved.

Corollary 1. Under the condition of Theorem 7, $R(N)$ has the strict monotonicity in the sense of $\min_{c^{N+1}, \rho^{N+1}} R(N+1) < \min_{c^N, \rho^N} R(N)$.

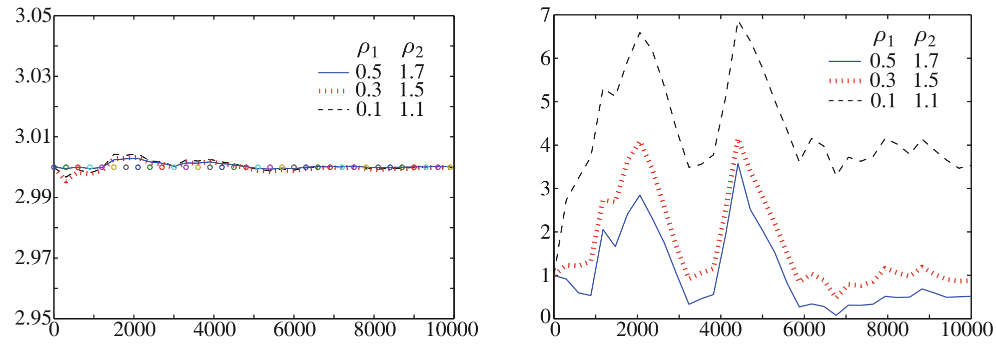
Remark 6. Corollary 1 shows that the upper bound of the convergence rate of algorithm (17)–(19) can gradually decrease along with the increase of the number of thresholds if we select the thresholds and quantization values according to the scheme in Theorem 7.

Remark 7. In the existing literature on the multi-threshold quantization, the quantization values are usually fixed in advance, and then the thresholds are optimized [16]. In this paper, the thresholds and quantization values are optimized at the same time, which brings more difficulties accordingly. Fortunately, the multi-linear transformation given by (24) can help us to get Theorem 7 and realizes the two terms optimization.

In the scheme of Theorem 7, quantization values can be expressed by the thresholds, which cannot be explicitly expressed but can be solved by

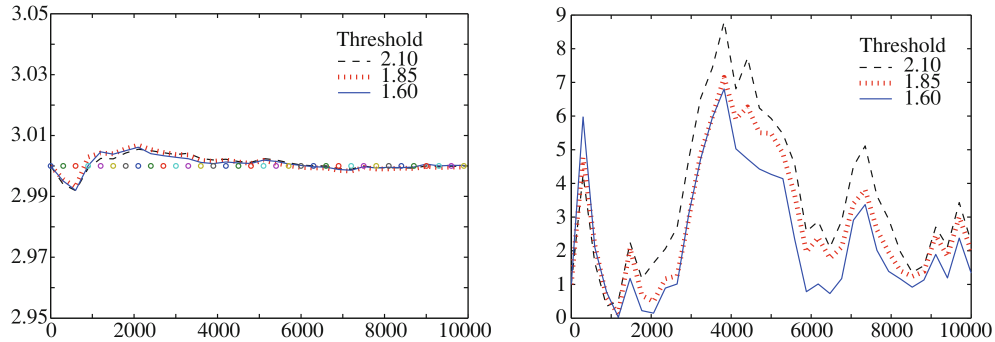
$$\underset{c^N}{\operatorname{argmax}} \sum_{i=1}^{N} \frac{\left(f(c_i) - f(c_{i+1})\right)^2}{F(c_{i+1}) - F(c_i)}.$$

For example, Table 1 gives the optimal c^N as $\sigma = 1$. It shows that $\min R(N)$ is decreasing with the increase of N, which is in keeping with Corollary 1.



(a) (b)

Figure 1 Convergence and convergence rate with identical thresholds and three sets of different quantization values. (a) Convergence of $\hat{\theta}_k$; (b) trajectories of $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2$.



(a) (b)

Figure 2 Convergence and convergence rate with identical quantization values and three sets of different thresholds. (a) Convergence of $\hat{\theta}_k$; (b) trajectories of $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2$.

5 Simulation

Consider a gain system: $y_k = \theta\phi_k + d_k$, whose quantized output is

$$q_k = \beta_1 \left(I_{[\hat{\theta}_{k-1}\phi_k \leqslant y_k < \hat{\theta}_{k-1}\phi_k + c]} - I_{[\hat{\theta}_{k-1}\phi_k - c < y_k < \hat{\theta}_{k-1}\phi_k]} \right) + \beta_2 \left(I_{[y_k \geqslant \hat{\theta}_{k-1}\phi_k + c]} - I_{[y_k \leqslant \hat{\theta}_{k-1}\phi_k - c]} \right),$$

where $\beta_i > 0$, $i = 1, 2$. The constant parameter $\theta = 3$ is unknown, but its upper bound $\bar{\theta} = 10$ is known; the system noise $\{d_k, k \geqslant 1\}$ satisfies Assumption 1 with $\sigma = 1$; the inputs follow $|\phi_k| \leqslant 8$ and $\liminf_{k\to\infty} \sum_{i=1}^k \phi_i^2/k > 0$.

The goal is to estimate the parameter θ based on the inputs and quantized output q_k. Identification algorithm (17)–(19) is adopted and simulated.

1) Identical thresholds and different quantization values. Let $c = 1$ and $\gamma = 1$, which implies that $\beta_1 = \rho_1$ and $\beta_2 = \rho_2$. Then, ρ_1 and ρ_2, respectively, take three different sets of data: 0.1 and 1.1, 0.3 and 1.5, 0.5 and 1.7. Figure 1 describes the trajectories of $\hat{\theta}_k$ and $\sum_{i=1}^k \phi_i^2 \tilde{\theta}_k^2$ for these three sets of data. Figure 1(a) shows the convergence of parameter estimates. Though there is a larger estimation error at the beginning, the estimates eventually converge to the true value, which is in keeping with Theorem 4.

The curves in Figure 1(b) have different heights and are bounded. From this, it follows that the convergence rate really is closely related to the quantization values and $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2 = O(1)$, which illustrates the results in Theorem 5. On the other hand, by $R(2; 0.1, 1.1) > R(2; 0.3, 1.5) > R(2; 0.5, 1.7)$, the height of $\sum_{i=1}^k \phi_i^2 \tilde{\theta}_k^2$ corresponds to the size of $R(2)$ and smaller of $R(2)$ would be lower of $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2$, which inspires us to minimize $R(N)$ for the fastest rate of algorithm (17)–(19) by selecting proper quantization values.

2) Identical quantization values and different thresholds. Let $\rho_1 = 1$ and $\rho_2 = 3$. The threshold c takes three different values: 2.10, 1.85 and 1.60. Figure 2 shows the trajectories of $\hat{\theta}_k$ and $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2$. Figure 2(a) displays the convergence of parameter estimates. The curves in Figure 2(b) have different

heights and are bounded, which demonstrates that the threshold has an important influence on the convergence rate and $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2 = O(1)$. As mentioned above, by $R(2; 2.10) > R(2; 1.85) > R(2; 1.60)$, the size of $R(2)$ decides the height of $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2$ and smaller of $R(2)$ would be lower of $\tilde{\theta}_k^2 \cdot \sum_{i=1}^k \phi_i^2$, which also inspires us to minimize $R(N)$ for the fastest rate by choosing suitable thresholds.

6 Conclusion

System identifications with quantized observations have already obtained a lot of excellent achievements, but these existing algorithms cannot be used to design adaptive control laws due to the restriction of the periodic or i.i.d. inputs. This paper tried to design the control-oriented identification algorithm and explore its difficulties. As a beginning, only single parameter system identification was studied with quantized observations and bounded persistent excitations. In the case of both single and multiple thresholds, the identification algorithms were constructed and proved to be both almost surely and mean square convergent, the convergence rate was studied, and the optimal scheme of quantization values and thresholds was given. A numerical example was simulated to support the results developed in this paper.

As we have declared, this is just a beginning work towards the control-oriented identification with quantized observations. There are many meaningful and interesting related topics, such as how to construct the most efficient identification algorithm and deal with the more general cases of system models, and how to design the adaptive control law, based on these algorithms, to regulate the system performance and so on.

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References

1 Wiener N. Cybernetics: or Control and Communication in the Animal and the Machine. John Wiley, 1948

2 Clarke D W. Generalized least squares estimation of the parameters of a dynamic model. In: Proceedings of IFAC Symposium on Identification in Automatic Control Systems, Prague, 1967. 1–11

3 Aström K J. Maximum likelihood and prediction error methods. Automatica, 1980, 16: 551–574

4 Kalman R E. A new approach to linear filtering and prediction problems. J Basic Eng, 1960, 82: 35–45

5 Ho Y C, Lee R C K. A Bayesian approach to problems in stochastic estimation and control. IEEE Trans Autom Control, 1964, 9: 333–339

6 Guo L. Convergence and logarithm laws of self-tuning regulators. Automatica, 1995, 31: 435–450

7 Ljung L. System Identification: Theory for the User. 2nd ed. Englewood Cliffs: Prentice-Hall, 1999

8 Wang L Y, Zhang J-F, Yin G. System identification using binary sensors. IEEE Trans Autom Control, 2003, 48: 1892–1907

9 Ghysen A. The origin and evolution of the nervous system. Int J Dev Biol, 2003, 47: 555–562

10 Wang L Y, Kim Y, Sun J. Prediction of oxygen storage capacity and stored NOx using HEGO sensor model for improved LNT control strategies. In: Proceeds of 2002 ASME International Mechanical Engineering Congress and Exposition, New Orleans, 2002. 17–22

11 Akyildiz I F, Su W, Sankarasubramaniam Y, et al. Wireless sensor networks: a survey. Comput Netw, 2002, 38: 393–422

12 Wang L Y, Yin G, Zhang J F, et al. System Identification with Quantized Observations. Boston: Birkäuser, 2010

13 Casini M, Garulli A, Vicino A. Time complexity and input design in worst-case identification using binary sensors. In: Proceedings of 46th IEEE Conference on Decision and Control, New Orleans, 2007. 5528–5533

14 Agüero J C, Goodwin G C, Yuz J I. System identification using quantized data. In: Proceedings of 46th IEEE Conference on Decision and Control, New Orleans, 2007. 4263–4268

15 Ribeiro A, Giannakis G B, Roumeliotis S I. SOI-KF: distributed Kalman filtering with low-cost communications using the sign of innovations. IEEE Trans Signal Process, 2006, 54: 4782–4795

16 You K, Xie L, Sun S, et al. Multiple-level quantized innovation Kalman filter. In: Proceedings of 17th International Federation of Automatic Control World Congress, Seoul, 2008. 1420–1425

17 Wang L Y, Yin G. Asymptotically efficient parameter estimation using quantized output observations. Automatica, 2007, 43: 1178–1191
18 Zhao Y L, Wang L Y, Yin G, et al. Identification of Wiener systems with binary-valued output observations. Automatica, 2007, 43: 1752–1765
19 Zhao Y L, Zhang J F, Wang L Y, et al. Identification of Hammerstein systems with quantized observations. SIAM J Control Optim, 2010, 48: 4352–4376
20 Green M, Moore J B. Persistence of excitation in linear systems. Syst Control Lett, 1986, 7: 351–360
21 Guo J, Zhang J F, Zhao Y L. Adaptive tracking control of a class of first-order systems with binary-valued observations and time-varying thresholds. IEEE Trans Autom Control, 2011, 56: 2991–2996
22 Guo L. Time-Varying Stochastic Systems—Stability, Estimation and Control. Changchun: Jilin Science and Technology Press, 1993
23 Chen H F. Stochastic Approximation and Its Application. Dordrecht: Kluwer Academic Publishers, 2002
24 Chow Y S, Teicher H. Probability Theory: Independence, Interchangeability, Martingales. 3rd ed. New York: Springer-Verlag, 1997

Appendix

Lemma A1. If the positive real number sequences $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ follow $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} b_n = \infty$ and $a_n \sim b_n$, then $\sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i$.

Proof. By Stolz theorem [1)], the lemma can be proved.

Proof of Proposition 1. i) From (7), it follows that

$$1 - \frac{\gamma P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2} = \frac{P_k}{P_{k-1}}, \quad \text{and} \quad \left(1 + \frac{\gamma P_{k-1}\phi_k^2}{1 + (1-\gamma)P_{k-1}\phi_k^2}\right)\left(1 - \frac{\gamma P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2}\right) = 1.$$

Thus, we have

$$1 + \frac{\gamma P_{k-1}\phi_k^2}{1 + (1-\gamma)P_{k-1}\phi_k^2} = \frac{P_{k-1}}{P_k},$$

which implies (9).

For any initial value $P_0 > 0$, by (9), one can get $0 < P_{k+1} \leqslant P_k$. Furthermore, by (3) we have

$$\frac{\gamma\phi_k^2}{(1-\gamma)P_{k-1}\phi_k^2 + 1} \geqslant \frac{\gamma\phi_k^2}{(1-\gamma)P_0 M^2 + 1},$$

which together with (9) and (4) implies

$$P_k^{-1} \geqslant P_0^{-1} + \frac{\gamma}{(1-\gamma)P_0 M^2 + 1}\sum_{i=1}^{k}\phi_i^2 \xrightarrow{k\to\infty} \infty.$$

Noticing $\lim_{k\to\infty} P_k = 0$, (10) is proved.

ii) By (9) and (10), we have

$$\frac{\gamma\phi_k^2}{1 + (1-\gamma)P_{k-1}\phi_k^2} \sim \gamma\phi_k^2, \quad \text{and} \quad P_k^{-1} = P_0^{-1} + \sum_{i=1}^{k}\frac{\gamma\phi_i^2}{1 + (1-\gamma)P_{i-1}\phi_i^2}.$$

Thus, by Lemma A1, $P_k^{-1} - P_0^{-1} \sim \gamma\sum_{i=1}^{k}\phi_k^2$, which together with (3) implies (11).

iii) If $\sum_{i=1}^{\infty}\phi_i^2/i < \infty$, then by Kronecker lemma [24, pp. 114], we know $\sum_{i=1}^{k}\phi_i^2/k \xrightarrow{k\to\infty} 0$, which is contradictory to (4). Thus, $\sum_{i=1}^{\infty}\phi_i^2/i = \infty$. By $P_k = O(1/k)$, we have

$$\sum_{i=1}^{k}\frac{1}{i}\phi_i^2 = O\left(\sum_{i=1}^{k}P_{i-1}\phi_i^2\right),$$

which together with $\sum_{i=1}^{\infty}\phi_i^2/i = \infty$ implies (12).

1) Chang G Z, Shi J H. Mathematical Analysis, Vol. 1. Nanjing: Jiangsu Education Press, 1998.

Lemma A2. Under the condition of Theorem 1, if $2B\rho > 1, m \geqslant 2$ and $E\tilde{\theta}_k^{2m} = o(1/k)$, then

$$E\tilde{\theta}_k^{2r} = o\left(1/k\right), \quad r = 2, 3, \ldots, m.$$

Proof. By (13), (11) and [21, Lemma 1], we have

$$\begin{aligned} E\tilde{\theta}_k^{2(m-1)} &\leqslant E\tilde{\theta}_{k-1}^{2(m-1)} + \frac{o(1)}{k^2} + \frac{2(m-1)\beta P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2} E\tilde{\theta}_{k-1}^{2(m-1)-1}\big(1-2F(\phi_k\tilde{\theta}_{k-1})\big) \\ &\leqslant \left(1 - \frac{2(m-1)\beta B P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\right) E\tilde{\theta}_{k-1}^{2(m-1)} - \frac{2(m-1)\beta\bar{B}_2 P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2} E\tilde{\theta}_{k-1}^{2m} + \frac{o(1)}{k^2} \\ &\leqslant \left(1 - \frac{\beta B P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\right)^2 E\tilde{\theta}_{k-1}^{2(m-1)} + \frac{o(1)}{k^2} \\ &= \left(1 - \frac{2\beta B P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\right) E\tilde{\theta}_{k-1}^{2(m-1)} + \frac{\beta^2 B^2 P_{k-1}^2\phi_k^4}{(1+P_{k-1}\phi_k^2)^2} E\tilde{\theta}_{k-1}^{2(m-1)} + \frac{o(1)}{k^2}, \end{aligned}$$

which together with $P_k = O(1/k)$ and $E\tilde{\theta}_{k-1}^2 = o(1)$ implies

$$E\tilde{\theta}_k^{2(m-1)} \leqslant \left(1 - \frac{2\beta B P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\right) E\tilde{\theta}_{k-1}^{2(m-1)} + \frac{o(1)}{k^2}.$$

By [21, Corollary 2], $E\tilde{\theta}_k^{2(m-1)} = o(1/k)$. Similarly, repeating this process, one can get the lemma.

Lemma A3. Under the condition of Theorem 1, if m is a positive integer following $m > (2\beta\bar{B}_1)^{-1}$, then we have $E\tilde{\theta}_k^{2m} = o\left(1/k\right)$.

Proof. Similar to the proof of Lemma A2, we have

$$\begin{aligned} E\tilde{\theta}_k^{2m} &\leqslant E\tilde{\theta}_{k-1}^{2m} + \frac{o(1)}{k^2} + \frac{2m\beta P_{k-1}\phi_k}{1+P_{k-1}\phi_k^2} E\tilde{\theta}_{k-1}^{2m-1}\big(1-2F(\phi_k\tilde{\theta}_{k-1})\big) \\ &\leqslant \left(1 - \frac{2m\beta\bar{B}_1 P_{k-1}\phi_k^2}{1+P_{k-1}\phi_k^2}\right) E\tilde{\theta}_{k-1}^{2m} + \frac{o(1)}{k^2}, \end{aligned}$$

which together with $P_k = O(1/k)$ and [21, Corollary 2] implies $E\tilde{\theta}_k^{2m} = o(1/k)$ for $m > (2\beta\bar{B}_1)^{-1}$.

Lemma A4. If $D > \gamma$ and (3)–(4) hold, then we have

$$P_k^{-1} \cdot \prod_{i=1}^{k} \left(1 - \frac{DP_{i-1}\phi_i^2}{1+P_{i-1}\phi_i^2}\right) \to 0, \quad k \to \infty.$$

Proof. By $P_k = O(1/k)$, there exists k_0 such that $1 - DP_{k-1}\phi_k^2/(1+P_{k-1}\phi_k^2) > 0$ for any $k \geqslant k_0 - 1$. By (7), we know

$$\frac{P_k}{P_{k_0-1}} = \prod_{i=k_0}^{k} \frac{P_i}{P_{i-1}} = \prod_{i=k_0}^{k} \left(1 - \frac{\gamma P_{i-1}\phi_i^2}{1+P_{i-1}\phi_i^2}\right),$$

which implies

$$P_k^{-1} \prod_{i=k_0}^{k} \left(1 - \frac{DP_{i-1}\phi_i^2}{1+P_{i-1}\phi_i^2}\right) = P_{k_0-1}^{-1} \prod_{i=k_0}^{k} \frac{1+(1-D)P_{i-1}\phi_i^2}{1+(1-\gamma)P_{i-1}\phi_i^2} = P_{k_0-1}^{-1} \prod_{i=k_0}^{k} \left(1 + \frac{(\gamma-D)P_{i-1}\phi_i^2}{1+(1-\gamma)P_{i-1}\phi_i^2}\right).$$

Let $\hat{D} = (\gamma - D)/(1+(1-\gamma)P_0M^2)$. By $D > \gamma$, we have $\hat{D} < 0$. Then,

$$P_k^{-1} \cdot \prod_{i=k_0}^{k} \left(1 - \frac{DP_{i-1}\phi_i^2}{1+P_{i-1}\phi_i^2}\right) \leqslant P_{k_0-1}^{-1} \prod_{i=k_0}^{k} \left(1 + \hat{D}P_{i-1}\phi_i^2\right). \tag{A1}$$

By Lemma A1 and (12), we have

$$\sum_{i=k_0}^{k} \log\left(1 + \hat{D}P_{i-1}\phi_i^2\right) \sim \hat{D} \sum_{i=k_0}^{k} P_{i-1}\phi_i^2 \xrightarrow{k\to\infty} -\infty,$$

which together with (A1) renders $P_k^{-1} \cdot \prod_{i=k_0}^{k}\left(1 - DP_{i-1}\phi_i^2/(1+P_{i-1}\phi_i^2)\right) \to 0$. From this, the lemma can be proved.

Lemma A5. Let $D_i > 0,\ i = 1, 2$. The sequence $\{Z_k, k \geqslant 1\}$ follows

$$Z_k \leqslant \left(1 - \frac{D_1 P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2}\right) Z_{k-1} + \frac{D_2 P_{k-1}^2 \phi_k^2}{(1 + P_{k-1}\phi_k^2)^2}. \tag{A2}$$

If $D_1 > \gamma$ and (3), (4) hold, then we have $\limsup_{k\to\infty} Z_k P_k^{-1} \leqslant D_2/(D_1 - \gamma)$.

Proof. By (7), we have

$$\begin{aligned}&\frac{D_2}{D_1 - \gamma} P_k - \frac{D_2}{D_1 - \gamma}\left(1 - \frac{D_1 P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2}\right) P_{k-1}\\ &\quad = \frac{D_2}{D_1 - \gamma}\left(P_{k-1} - \frac{\gamma P_{k-1}^2\phi_k^2}{1 + P_{k-1}\phi_k^2}\right) - \frac{D_2}{D_1 - \gamma}\left(1 - \frac{D_1 P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2}\right) P_{k-1} = \frac{D_2 P_{k-1}^2\phi_k^2}{1 + P_{k-1}\phi_k^2}.\end{aligned}$$

By (A2),

$$Z_k - \left(1 - \frac{D_1 P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2}\right) Z_{k-1} \leqslant \frac{D_2 P_{k-1}^2\phi_k^2}{(1 + P_{k-1}\phi_k^2)^2} \leqslant \frac{D_2 P_{k-1}^2\phi_k^2}{1 + P_{k-1}\phi_k^2}.$$

Furthermore, one can get

$$Z_k - \frac{D_2}{D_1 - \gamma} P_k \leqslant \left(1 - \frac{D_1 P_{k-1}\phi_k^2}{1 + P_{k-1}\phi_k^2}\right)\left(Z_{k-1} - \frac{D_2}{D_1 - \gamma} P_{k-1}\right) \leqslant \prod_{i=1}^{k}\left(1 - \frac{D_1 P_{i-1}\phi_i^2}{1 + P_{i-1}\phi_i^2}\right)\left(Z_0 - \frac{D_2}{D_1 - \gamma} P_0\right),$$

which together with Lemma A4 implies the lemma.

Lemma A6. If $g(x) = (x_1 + x_2 x)^2/(x_3 + x_4 x^2)$ with $x_i > 0,\ i = 1, \ldots, 4$, then $\max_{x>0} g(x) = x_1^2/x_3 + x_2^2/x_4$ and $\mathrm{argmax}_{x>0}\, g(x) = x_2 x_3/(x_1 x_4)$.

Proof. Noticing an equivalent form of $g(x)$ as follows

$$g(x) = \left[\frac{x_3 x_1^2 + x_4 x_2^2}{x_1^4}\left(\left(x + \frac{x_2}{x_1}\right)^{-1} - \frac{x_1 x_2 x_4}{x_3 x_1^2 + x_4 x_2^2}\right)^2 + \frac{x_3 x_4}{x_3 x_1^2 + x_4 x_2^2}\right]^{-1}, \tag{A3}$$

we know that $g(x)$ can reach its maximum value when $(x + x_2/x_1)^{-1} - x_1 x_2 x_4/(x_3 x_1^2 + x_4 x_2^2) = 0$, which implies that $x = x_2 x_3/(x_1 x_4)$. Thus, the maximum value of $g(x)$ is $x_1^2/x_3 + x_2^2/x_4$ by (A3) and the lemma is proved.