SCIENCE CHINA
Mathematics

• ARTICLES •

July 2014 Vol. 57 No. 7: 1435–1442
doi: 10.1007/s11425-014-4776-4

Induced orbit data and induced unitary representations for complex groups

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Received January 16, 2012; accepted April 19, 2013; published online February 12, 2014

Abstract We use induced orbit covers to define induced orbit data. By studying the space of regular functions on orbit cover, we know that the induced representation has close connection with the induced orbit datum under the meaning of Vogan's conjecture. Therefore, when verifying Vogan's conjecture, many cases can be reduced to the case of rigid orbit data.

Keywords Vogan's conjecture on quantization, induced orbit covers, induced orbit data, unitary representations

MSC(2010) 22E10, 22E46, 20G05

Citation: Zhang M J, Liang K. Induced orbit data and induced unitary representations for complex groups. Sci China Math, 2014, 57: 1435–1442, doi: 10.1007/s11425-014-4776-4

1 Introduction

One intriguing and difficult problem in the representation theory of reductive Lie groups is this: How to attach unitary representations to nilpotent coadjoint orbits? The orbit method (due to Kirillov and Kostant [11]) can sometimes predict how the representation is "attached" to an orbit should restrict to a maximal compact subgroup. About this, Vogan has given a conjecture.

Conjecture 1.1 (Vogan's conjecture on quantization, see also Conjecture 2.4)**.** Let G be a complex reductive Lie group, and $X = G \cdot \lambda \simeq G/G(\lambda)$ a nilpotent coadjoint orbit. Suppose that we are given an irreducible G-equivariant local system on X; equicalently, an irreducible representation (χ, V_χ) of the finite group $G(\lambda)/G(\lambda)_0$, or an indecomposable G-equivariant holomorphic vector bundle $\mathcal{V}_\chi \cong G \times_{G(\lambda)} V_\chi$ on X with a flat connection. Then there is attached to χ an irreducible unitary representation $\pi(\lambda, \chi)$ of G. The space of K-finite vectors of $\pi(\lambda, \chi)$ is isomorphic to the space of algebraic sections of the bundle $\mathcal{V}_\chi$.

Also we know, when a representation π_L is attached to a nilpotent orbit $\mathcal{O}_L$ of L, we can attach the representation $\pi_G := \mathrm{Ind}_P^G(\pi_L \otimes \mathbb{1})$ to the nilpotent orbit $\mathcal{O}_G := \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_L)$. Here, L is the Levi part of the parabolic subgroup P of G. Then a natural question is: When π_L is associated with an orbit datum $(\mathcal{O}_L, \chi_L)$ in the sense of Vogan's conjecture on quantization, which orbit datum will π_G be associated with?

In this paper, we define an induced orbit datum χ_G. We get a relationship between the space of K-finite vectors of π_G and the space of algebraic sections of the bundle $\mathcal{V}_{\chi_G}$. Here is the statement.

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Theorem 1.2 (See also Theorem 3.13). *Let G be a complex Lie group. Assume π_L to be a representation which satisfies Vogan's conjecture attached to the admissible orbit datum $(\mathcal{O}_u, \chi_L)$. Let $\mathcal{O}_v = \mathrm{Ind}_L^G(\mathcal{O}_u)$ be the induced orbit, and $(\mathcal{O}_v, \chi_G)$ the induced orbit datum of G. Write $\pi_G = \mathrm{Ind}_{LN}^G(\pi_L \otimes \mathbb{1})$, the parabolic induced representation. Then the space of K-finite vectors of π_G equals the space of algebraic sections of the bundle $G \times_{G^v} \chi_G$.*

Then we obtain the following corollary.

Corollary 1.3 (See also Corollary 3.14). *When χ_G is irreducible, if furthermore we assume π_G is irreducible, then π_G, which is attached to the orbit datum $(\mathcal{O}_G, \chi_G)$, will satisfies Vogan's conjecture on quantization.*

2 Admissible orbit data and Vogan's conjecture on quantization

In this section, we recall the definition of admissible orbit data, and Vogan's conjecture on quantization. We begin with a real reductive Lie group G in Harish-Chandra's class (see [6, Section 3]). We write $\mathfrak{g}$ for the Lie algebra of G; analogous notation will be used for other groups.

The "coadjoint orbits" of the orbit method consist of linear functionals on Lie algebras. First of all, we are interested in

$$i\mathfrak{g}^* = \text{imaginary-valued } \mathbb{R}\text{-linear functionals on } \mathfrak{g}.$$

A *coadjoint orbit* is by definition an orbit of G on $i\mathfrak{g}^*$. An orbit is called *nilpotent* if its closure is a cone.

The orbit method seeks to attach unitary representations not to arbitrary coadjoint orbits, but only to those satisfying a certain additional condition. In the original work of Kirillov and Kostant, this additional condition, called *integrality*, was formulated in a straightforward way. All nilpotent coadjoint orbits are integral. But Duflo [5] later found that a treatment of more general groups was possible only if integrality was replaced by the more subtle requirement of *admissibility*. Not every nilpotent coadjoint orbit is admissible, so we need to understand this condition.

Definition 2.1. Suppose that G is a real Lie group, and $\lambda \in i\mathfrak{g}^*$. Write $G(\lambda)$ for the isotropy group of the coadjoint action at λ, so that the coadjoint orbit $X = G \cdot \lambda$ may be identified with $G/G(\lambda)$. Recall that the tangent space $\mathfrak{g}/\mathfrak{g}(\lambda)$ to X at λ carries a natural $G(\lambda)$-invariant imaginary-valued non-degenerate symplectic form ω_λ, defined by

$$\omega_\lambda(A + \mathfrak{g}(\lambda), B + \mathfrak{g}(\lambda)) = \lambda([A, B]), \quad A, B \in \mathfrak{g}.$$

The isotropy action therefore gives a natural group homomorphism

$$j(\lambda) : G(\lambda) \to Sp(\omega_\lambda).$$

The symplectic group has a natural two-fold covering, *the metaplectic group*:

$$1 \to \{1, \epsilon\} \to Mp(\omega_\lambda) \xrightarrow{p} Sp(\omega_\lambda) \to 1.$$

This covering may be pulled back via the homomorphism $j(\lambda)$ to define the *metaplectic double cover*:

$$1 \to \{1, \epsilon\} \to \tilde{G}(\lambda) \xrightarrow{p(\lambda)} G(\lambda) \to 1.$$

Explicitly,

$$\tilde{G}(\lambda) = \{(g, m) \in G(\lambda) \times Mp(\omega_\lambda) \mid j(\lambda)(g) = p(m)\}.$$

A representation χ of $\tilde{G}(\lambda)$ is called *genuine* if $\chi(\epsilon) = -I$. It is called *admissible* if it is genuine, and in addition the differential of χ is a multiple of λ. Explicitly,

$$\chi(\exp A) = \exp(\lambda(A)) \cdot I, \quad A \in \mathfrak{g}(\lambda).$$

An *admissible orbit datum* is a pair (λ, χ) with $\lambda \in i\mathfrak{g}^*$ and χ an irreducible unitary admissible representation of $\tilde{G}(\lambda)$. The element λ (or the coadjoint orbit $G \cdot \lambda$) is called *admissible* if admissible orbit data (λ, χ) exist.

The notion of admissibility is a bit involved, but fortunately it simplifies for nilpotent orbits.

Proposition 2.2 (See [16, Theorem 5.7 and Observation 7.4])**.** *Suppose that G is a real reductive Lie group. Then an element $\lambda \in i\mathfrak{g}^*$ is nilpotent if and only if the restriction of λ to $\mathfrak{g}(\lambda)$ is zero.*

Suppose henceforth that λ is nilpotent. A representation χ of $\tilde{G}(\lambda)$ is admissible if and only if $\chi(\epsilon) = -I$, and χ is trivial on the identity component of $\tilde{G}(\lambda)$. Consequently λ is admissible if and only if the preimage under the metaplectic covering map $p(\lambda)$ of the identity component $G(\lambda)_0$ is disconnected.

In the case of complex groups, matters are even simpler.

Proposition 2.3 (See [1, Lemma 2.4])**.** *Suppose G is a complex Lie group and $\lambda \in i\mathfrak{g}^*$. Then the metaplectic double cover of $G(\lambda)$ is trivial*:

$$\tilde{G}(\lambda) \cong \{1, \epsilon\} \times G(\lambda).$$

Suppose in addition that G is reductive and λ is nilpotent. Then λ is admissible, and the admissible representations of $\tilde{G}(\lambda)$ are in one-in-one correspondence with irreducible representations of the group $G(\lambda)/G(\lambda)_0$ of connected components of $G(\lambda)$.

When G is a complex reductive Lie group, we have the Vogan's conjecture on quantization as follows.

Conjecture 2.4 (See [1])**.** Let G be a complex reductive Lie group, and $X = G \cdot \lambda \simeq G/G(\lambda)$ a nilpotent coadjoint orbit. Suppose that we are given an irreducible G-equivariant local system on X; equicalently, an irreducible representation (χ, V_χ) of the finite group $G(\lambda)/G(\lambda)_0$, or an indecomposable G-equivariant holomorphic vector bundle $\mathcal{V}_\chi \cong G \times_{G(\lambda)} V_\chi$ on X with a flat connection. Then there is attached to χ an irreducible unitary representation $\pi(\lambda, \chi)$ of G. The space of K-finite vectors of $\pi(\lambda, \chi)$ is isomorphic to the space of algebraic sections of the bundle $\mathcal{V}_\chi$.

More details about this section can be found in [1, 16].

3 Induced orbit data and unitary representations

Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be a parabolic subalgebra such that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ satisfies $\mathfrak{h} \subset \mathfrak{l}$ and $P = LN$ the corresponding parabolic subgroup. Then it is well known that, if π_M is a unitary representation of M,

$$\pi = \mathrm{Ind}\,_P^G(\pi_M \otimes \mathbb{1}) \quad \text{(unitary induction)}$$

is also unitary.

By [2, 7], we can attach to any representation π a set in the nilpotent cone in $\mathfrak{g}^*$. This set is denoted by $\mathrm{WF}\,(\pi)$ and called the *wavefront set*. In the case when π is irreducible and G is complex, $\mathrm{WF}\,(\pi)$ is the closure of one nilpotent orbit.

There is a simple relation between induction and the wavefront set. If $\pi = \mathrm{Ind}\,(\pi_M \otimes \mathbb{1})$, then

$$\mathrm{WF}\,(\pi) = \mathrm{Ind}\,_{\mathfrak{l}}^{\mathfrak{g}} \mathrm{WF}\,(\pi_M),$$

where induction of nilpotent orbits is as in [15]. More details about this can be found in [3].

So given a unitary representation π_L attached to a nilpotent orbit $\mathcal{O}_L$, the induced representation

$$\pi_G := \mathrm{Ind}\,_P^G(\pi_L \otimes \mathbb{1})$$

should be attached to the nilpotent orbit

$$\mathcal{O}_G := \mathrm{Ind}\,_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_L)$$

of $\mathfrak{g}$. Specially when π_G is irreducible, it is natural to wonder which orbit datum π_G will be associated with. We will consider this question in this section.

3.1 Orbit covers and induced orbit data

In this section, we will recall the definition of induced orbit covers and some properties about the space of regular functions on orbit covers. More details about this can be found in [12, 14]. Then we will use this to define the induced orbit data.

Definition 3.1. Let $\mathcal{O}$ be an adjoint orbit of G. An orbit cover of $\mathcal{O}$ for G is a G-space $\tilde{\mathcal{O}}$ with a G-invariant cover map $\pi : \tilde{\mathcal{O}} \to \mathcal{O}$.

For any $v \in \mathcal{O}$ and a subgroup Π of G^v/G^v_0, where G^v is the isotropic subgroup of v and G^v_0 the identity component of G^v, we have a natural G-invariant cover map

$$\pi : \tilde{\mathcal{O}} = G/G^v_\Pi \to G/G^v = \mathcal{O},$$

where G^v_Π is the open subgroup of G^v corresponding to Π. Then $\tilde{\mathcal{O}}$ is an orbit cover. (v, Π) is called a representation of $\tilde{\mathcal{O}}$. Obviously, every orbit cover has such a representation; and two representations correspond to the same orbit cover if and only if they are conjugated in G.

Suppose that G is a complex Lie group, and $P = LU_P$ a parabolic subgroup, with Levi factor L and unipotent component U_P. The Lie algebra $\mathfrak{p}$ of P has Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_P$. Here, $\mathfrak{u}_P$ is the nil-radical of $\mathfrak{p}$ and $\mathfrak{l}$ is a Levi-part.

Naturally, we have an inclusion $i \colon \mathfrak{p} \hookrightarrow \mathfrak{g}$ and the quotient by the nil radical map $j \colon \mathfrak{p} \twoheadrightarrow \mathfrak{l}$. The maps on the dual spaces i^* and j^* go in the other direction. Fix a nilpotent coadjoint orbit $\mathcal{O}_L$ of L. Then there is a unique nilpotent coadjoint orbit $\mathcal{O}_G$ such that $\mathcal{O}_G$ meets $(i^*)^{-1}(j^*(\mathcal{O}_L))$ in a dense open subset (see [15]). Such an orbit $\mathcal{O}_G$ is called the *nilpotent orbit induced from* $\mathcal{O}_L$ and we write $\mathcal{O}_G = \mathrm{Ind}\,^{\mathfrak{g}}_{\mathfrak{l}}(\mathcal{O}_L)$. Choose $v \in \mathcal{O}_G \cap ((i^*)^{-1}(j^*(\mathcal{O}_L)))$, we have $u \in \mathcal{O}_L$, $x \in \mathfrak{u}^*_P$ such that $v = u + x$.

Lemma 3.2 (See [15]). *With the notation as above, we have*

1. $P^v_0 = G^v_0$*, so* $\mathfrak{p}^v = \mathfrak{g}^v$*;*
2. $\mathcal{O}_G \cap (\mathcal{O}_L + \mathfrak{u}^*_P)$*, denoted by* $\mathcal{O}_P$*, is a single orbit of* P*;*
3. $P^v \subset L^u U_P$*, and* P^v *meets every component of* $L^u U_P$*, so we have a surjective homomorphism*

$$\theta : P^v/P^v_0 \to L^u/L^u_0.$$

Given an orbit cover $\tilde{\mathcal{O}}_L$ of L, find a representation (u, Π_L), and choose $v \in \mathcal{O}_P$ such that $v = u + x$ for some $x \in \mathfrak{u}^*_P$. Now, $\theta^{-1}(\Pi_L) \subset P^v/P^v_0 \subset G^v/G^v_0$. Hence, $(v, \theta^{-1}(\Pi_L))$ is a representation of the orbit cover $\tilde{\mathcal{O}}_G$ of $\mathcal{O}_G$.

Definition 3.3. $(v, \theta^{-1}(\Pi_L))$ defines an orbit cover of G, which is called *the parabolic induction* of $\tilde{\mathcal{O}}_L$ through P, denoted by $\mathrm{Ind}\,^G_L[P](\tilde{\mathcal{O}}_L)$.

Remark 3.4 (See [12, Lemma 2.2]). This definition is well-defined, and $\mathrm{Ind}\,^G_L[P](\tilde{\mathcal{O}}_L)$ is independent of the choice of $u \in \mathcal{O}_L$, of P with Levi factor L, and $v \in \mathcal{O}_P$. So we will denote it by $\mathrm{Ind}\,^G_L(\tilde{\mathcal{O}}_L)$ simply.

About the parabolic induction of orbits covers, there is an important property.

Proposition 3.5 (Induction-by-stage, see [12, Lemma 2.3]). *Let* $L_1 \supset L$ *be a Levi subgroup of* G*. So* $L_1 \cap P$ *and* $L_1 P$ *are parabolic subgroups of* L_1 *and* G*, respectively. Then*

$$\mathrm{Ind}\,^G_L[P](\tilde{\mathcal{O}}_L) = \mathrm{Ind}\,^G_{L_1}[L_1 P](\mathrm{Ind}\,^{L_1}_L[L_1 \cap P](\tilde{\mathcal{O}}_L)).$$

Let $\tilde{\mathcal{O}}$ be an orbit cover for G and $\mathcal{R}(\tilde{\mathcal{O}})$ the normal function ring on $\tilde{\mathcal{O}}$. Set $N(\tilde{\mathcal{O}}) = \mathrm{Spec}\mathcal{R}(\tilde{\mathcal{O}})$. Then $N(\tilde{\mathcal{O}})$ is the normalizer of the closure of $\tilde{\mathcal{O}}$ (Note here we choose $N(\tilde{\mathcal{O}}) = \mathrm{Max}\,\mathcal{R}(\tilde{\mathcal{O}})$). Obviously, $N(\tilde{\mathcal{O}})$ is a G-algebraic variety, $\tilde{\mathcal{O}} \subset N(\tilde{\mathcal{O}})$ and $\tilde{\mathcal{O}}$ is dense in $N(\tilde{\mathcal{O}})$. ($\tilde{\mathcal{O}}$ is homogeneous, hence every point in $\tilde{\mathcal{O}}$ is normal, so $\tilde{\mathcal{O}}$ can be regarded as a subset of $N(\tilde{\mathcal{O}})$.)

Let $\mathcal{O}_L$ be a nilpotent orbit and $\bar{\mathcal{O}}_L$ its closure. Then there exist a natural mapping $\phi : N(\tilde{\mathcal{O}}_L) \to \bar{\mathcal{O}}_L$. For any $(x, y) \in N(\tilde{\mathcal{O}}_L) \times \mathfrak{u}^*_P$, $p = ab \in P$, where $a \in L$, $b \in U_P$, we can define the action of P on $N(\tilde{\mathcal{O}}_L) \times \mathfrak{u}^*_P$ as the following:

$$p(x, y) := (ax, \mathrm{Ad}\, p(y) + \mathrm{Ad}\, b(\phi(x)) - \phi(x)). \tag{3.1}$$

It is easy to check that $N(\tilde{\mathcal{O}}_L) \times \mathfrak{u}_P^*$ is a normal P-algebraic variety. Similarly, we can define the action of P on $\tilde{\mathcal{O}}_L \times \mathfrak{u}_P^*$ and $\tilde{\mathcal{O}}_P$ can be defined similar to the definition of $\tilde{\mathcal{O}}_G$. More details about this part can be found in [12].

Proposition 3.6 (See [12, Proposition 2.1]). *With notation as before, then we have*

1. $\tilde{\mathcal{O}}_P$ *is an open dense subset of* $\tilde{\mathcal{O}}_L \times \mathfrak{u}_P^*$;
2. $N(\tilde{\mathcal{O}}_P) = N(\tilde{\mathcal{O}}_L) \times \mathfrak{u}_P^*$;
3. $\mathrm{Ind}_L^G(\tilde{\mathcal{O}}_L) = G \times_P \tilde{\mathcal{O}}_P$.

Since we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via an invariant bilinear form on $\mathfrak{g}$ (for example, the Killing form when $\mathfrak{g}$ is semisiple), we will consider the adjoint orbits on $\mathfrak{g}$ instead of the coadjoint orbits on $\mathfrak{g}^*$ from now. We have that $\mathcal{R}(\mathfrak{u}_P) = S(\mathfrak{u}_P^{\mathrm{opp}}) \cong S(\mathfrak{g})/S(\mathfrak{p})$. So

$$\mathcal{R}(\tilde{\mathcal{O}}_P) = \mathcal{R}(\tilde{\mathcal{O}}_L \times \mathfrak{u}_P) = \mathcal{R}(N(\tilde{\mathcal{O}}_L) \times \mathfrak{u}_P) \cong S(\mathfrak{g}) \otimes_{S(\mathfrak{p})} \mathcal{R}(\tilde{\mathcal{O}}_L). \tag{3.2}$$

Proposition 3.7 (See [12, Theorem 2.1]). *Since* $G \times_P \mathcal{R}(\tilde{\mathcal{O}}_P)$ *can be regarded as an induced sheaf on* G/P*, we have*

$$\mathcal{R}(\mathrm{Ind}_L^G(\tilde{\mathcal{O}}_L)) = \Gamma(G \times_P \mathcal{R}(\tilde{\mathcal{O}}_P)) = \Gamma(G \times_P (\mathcal{R}(\tilde{\mathcal{O}}_L)) \otimes_{S(\mathfrak{p})} S(\mathfrak{g})) = \mathrm{Ind}_L^G(\mathcal{R}(\tilde{\mathcal{O}}_L)).$$

With notation as above, let $\theta^{-1}(1) = P_1^v/P_0^v$. Given an admissible representation χ_L of $\mathcal{O}_L$, χ_L is corresponding to an irreducible representation of L^u/L_0^u. To be simple, we still sign this representation as χ_L. And also suppose the space χ_L acts on is V_{χ_L}. Then

$$P^v/P_0^v \xrightarrow{\theta} L^u/L_0^u \xrightarrow{\chi_L} \mathrm{End}\,(V_\chi).$$

So χ_L induces a unique irreducible representation χ_P of P^v/P_0^v, satisfying that it acts trivially on P_1^v/P_0^v. We can regard χ_P as an admissible representation of the orbit $\mathcal{O}_P$. We still sign it as χ_P to be simple.

By Lemma 3.2, $P^v/P_0^v \subset G^v/G_0^v$. Define $\chi_G = \mathrm{Ind}_{P^v}^{G^v}(\chi_P)$. We have the following definition.

Definition 3.8. With notation as before, we call $(\mathcal{O}_G, \chi_G)$ the induced orbit datum of G from $(\mathcal{O}_L, \chi_L)$.

Proposition 3.9. *By Remark* 3.4*, we know that this definition is independent of* P.

Theorem 3.10. *When* $P^v/P_0^v = G^v/G_0^v$, χ_G *is an irreducible representation of* G^v/G_0^v*. In this case,* $(\mathcal{O}_G, \chi_G)$ *is definitely an orbit datum of* G.

Table 1 Induction of orbit covers in classical cases

G	Type	G^ϕ/G_0^ϕ	$L^{\phi'}/L_0^{\phi'}$	Π_G	P^v/P_0^v
	I	$(\mathbb{Z}_2)^m$	$(\mathbb{Z}_2)^{m-1}$	$\psi_j^{-1}(\Pi_L)$	$(\mathbb{Z}_2)^m$
$\mathrm{Sp}(2n,\mathbb{C})$	II	$(\mathbb{Z}_2)^m$	$(\mathbb{Z}_2)^m$	Π_L	$(\mathbb{Z}_2)^m$
	III	$(\mathbb{Z}_2)^m$	$(\mathbb{Z}_2)^{m-1}$	$\varphi_i(\Pi_L)$	$(\mathbb{Z}_2)^{m-1}$
$\mathrm{SO}(N,\mathbb{C})$;	I	$S((\mathbb{Z}_2)^m)$	$S((\mathbb{Z}_2)^{m-1})$	$\psi_j^{-1}(\Pi_L)$	$S((\mathbb{Z}_2)^m)$
$\mathrm{Spin}(N,\mathbb{C})$ for d	II	$S((\mathbb{Z}_2)^m)$	$S((\mathbb{Z}_2)^m)$	Π_L	$S((\mathbb{Z}_2)^m)$
is not rather odd	III	$S((\mathbb{Z}_2)^m)$	$S((\mathbb{Z}_2)^{m-1})$	$\varphi_i(\Pi_L)$	$S((\mathbb{Z}_2)^{m-1})$
	I	$\overline{S((\mathbb{Z}_2)^m)}$	$S((\mathbb{Z}_2)^{m-1})$	$\overline{\psi_j^{-1}(\Pi_L)}$	$\overline{S((\mathbb{Z}_2)^m)}$
$\mathrm{Spin}(N,\mathbb{C})$ for d	II	$\overline{S((\mathbb{Z}_2)^m)}$	$S((\mathbb{Z}_2)^m)$	$\overline{\Pi_L}$	$\overline{S((\mathbb{Z}_2)^m)}$
is rather odd	IV	$\overline{S((\mathbb{Z}_2)^m)}$	$\overline{S((\mathbb{Z}_2)^m)}$	Π_L	$\overline{S((\mathbb{Z}_2)^m)}$
	V	$\overline{S((\mathbb{Z}_2)^m)}$	$S((\mathbb{Z}_2)^{m-2})$	Π_L	$S((\mathbb{Z}_2)^{m-2})$
	I	$(\mathbb{Z}_2)^m/\Delta'_d$	$(\mathbb{Z}_2)^{m-1}/\Delta'_p$	$\psi_j^{-1}(\Pi_L)$	$(\mathbb{Z}_2)^m/\Delta'_d$
$\mathrm{PSp}(2n,\mathbb{C})$	II	$(\mathbb{Z}_2)^m/\Delta'_p$	$(\mathbb{Z}_2)^m/\Delta'_d$	Π_L	$(\mathbb{Z}_2)^m/\Delta'_p$
	III	$(\mathbb{Z}_2)^m/\Delta'_d$	$(\mathbb{Z}_2)^{m-1}/\Delta'_p$	$\varphi_i(\Pi_L)$	$(\mathbb{Z}_2)^{m-1}/\Delta'_p$
	I	$S((\mathbb{Z}_2)^m)/S(\Delta'_d)$	$S((\mathbb{Z}_2)^{m-1})/S(\Delta'_p)$	$\psi_j^{-1}(\Pi_L)$	$S((\mathbb{Z}_2)^m)/S(\Delta'_d)$
$\mathrm{PSO}(2n,\mathbb{C})$	II	$S((\mathbb{Z}_2)^m)/S(\Delta'_d)$	$S((\mathbb{Z}_2)^m)/S(\Delta'_p)$	Π_L	$S((\mathbb{Z}_2)^m)/S(\Delta'_d)$
	III	$S((\mathbb{Z}_2)^m)/S(\Delta'_d)$	$S((\mathbb{Z}_2)^{m-1})/S(\Delta'_p)$	$\varphi_i(\Pi_L)$	$S((\mathbb{Z}_2)^{m-1})/S(\Delta'_p)$

This is easy to verify from the representation theory of finite groups. Moreover, when $P^v/P_0^v \neq G^v/G_0^v$, χ_G is almost never irreducible. In classical cases, the induction of orbit covers has been divided into three conditions except when $G = \mathrm{Spin}(N, \mathbb{C})$ and the partition associated with the nilpotent orbit is rather odd. In this case, it is a little complicated, and the induction of orbit covers is divided into four parts. In the last condition of each case, $P^v/P_0^v \neq G^v/G_0^v$. The induction of orbit covers is calculated clearly. We recall the result in the following table. Here, $\phi = \{h, v, v'\} \cong \mathfrak{sl}_2(\mathbb{C})$ is a standard three-dimensional simple Lie algebra attached to v by the Jacobson-Morozov theorem (see [4, Theorem 3.3.1]), while ϕ' is attached to u. And $G^\phi/G_0^\phi \cong G^v/G_0^v$, $L^{\phi'}/L_0^{\phi'} \cong L^u/L_0^u$. More details and notation can be found in [13].

From Table 1, we know that in classical cases, when $P^v/P_0^v \neq G^v/G_0^v$, χ_G is not irreducible. In exceptional cases, the induction of orbit covers was almost totally calculated. More details about this can be found in [14]. The tables in the exceptional cases are too long to list here. χ_G may be not irreducible only when $P^v/P_0^v \cong \mathbb{Z}_3$, and $G^v/G_0^v \cong S_3$.

3.2 Orbit data of induced representations and Vogan's conjecture

Let $\mathcal{O}_L = L\cdot u$ be a nilpotent orbit of L and $\tilde{\mathcal{O}}_L = L/L_0^u$ an orbit cover. Here L_0^u is the identity component of L^u. Then $\mathcal{R}(\tilde{\mathcal{O}}_L) = \mathcal{R}(L/L_0^u)$.

$$\mathcal{R}(\tilde{\mathcal{O}}_L) = \mathrm{Ind}\,_{L_0^u}^{L}(\mathbb{C}) = \mathrm{Ind}\,_{L^u}^{L}\mathrm{Ind}\,_{L_0^u}^{L^u}(\mathbb{C}) = \sum_{\chi_L \in \widehat{L^u/L_0^u}} \mathrm{Ind}\,_{L^u}^{L}(V_{\chi_L}). \tag{3.3}$$

Here, $\widehat{L^u/L_0^u}$ stands for the set of equivalent irreducible representations of L^u/L_0^u. So we can regard $\mathrm{Ind}\,_{L^u}^{L}(V_{\chi_L})$ as a subspace of $\mathcal{R}(\tilde{\mathcal{O}}_L)$. Choose $f \in \mathrm{Ind}\,_{L^u}^{L}(V_{\chi_L})$. That is, $f : L \to V_{\chi_L}$ satisfies $f(ll_1) = \chi_L(l_1)f(l)$, for all $l_1 \in L^u$. Write f as $\tilde{f} \in \mathcal{R}(\tilde{\mathcal{O}}_L)$ in the form: $\tilde{f}(l\tilde{u}) := f(l)$, where $\tilde{u} \in \tilde{\mathcal{O}}_L$ is a determined element.

Let $p = ln \in P$, where $l \in L$ and $n \in U_P$. According to the action (3.1) of P on $N(\tilde{\mathcal{O}}_L) \times \mathfrak{u}_P$,

$$p \cdot (\tilde{u}, x) = (l \cdot \tilde{u}, i_u(p \cdot (u + x))), \quad \text{for} \quad x \in \mathfrak{u}_P.$$

Also, $p_1 \cdot p_2 \cdot (\tilde{u}, x) = (l_1 \cdot l_2 \cdot \tilde{u}, i_u(p_1 \cdot p_2 \cdot (u + x)))$, for $p_1, p_2 \in P$.

According to (3.2), $\mathcal{R}(\tilde{\mathcal{O}}_P) \cong \mathcal{R}(\tilde{\mathcal{O}}_L) \otimes S(\mathfrak{u}^{\mathrm{opp}})$. Given $g \in S(\mathfrak{u}^{\mathrm{opp}})$, $\tilde{f} \otimes g \in \mathcal{R}(\tilde{\mathcal{O}}_P)$, we regard it as an element of $\mathcal{R}(P/P_1^v)$, named F.

$$F(p) = (\tilde{f} \otimes g)(p \cdot (\tilde{v})) = \tilde{f}(l \cdot \tilde{u}) \cdot g(i_u(p \cdot (u + x))).$$

Let $p \in P$ and $p_1 \in P^v$. Then we have

$$\begin{aligned} F(pp_1) &= \tilde{f}(ll_1\tilde{u}) \cdot g(i_u(pp_1(u + x))) \\ &= f(ll_1) \cdot g(i_u(p(u + x))) \\ &= \chi_L(l_1)f(l) \cdot g(i_u(p(u + x))) \\ &= \chi_L(l_1)\tilde{f}(l\tilde{u}) \cdot g(i_u(p(u + x))) \\ &= \chi_P(p_1)F(p). \end{aligned}$$

Here, l and l_1 are Levi parts of p, p_1, respectively. And also by the definition of χ_P, $\chi_P(p_1) = \chi_L(l_1)$.

Adding these, we get the following result.

Lemma 3.11. *With notation as above, we have* $\mathrm{Ind}\,_{L^u}^{L}(\chi_L) \otimes S(\mathfrak{u}^{\mathrm{opp}}) \subset \mathrm{Ind}\,_{P^v}^{P}(\chi_P)$.

Furthermore, we have the following theorem.

Theorem 3.12. *Let χ_G be defined as before. Then*

$$\mathrm{Ind}\,_{L}^{G}\mathrm{Ind}\,_{L^u}^{L}(\chi_L) \cong \mathrm{Ind}\,_{P}^{G}\mathrm{Ind}\,_{P^v}^{P}(\chi_P) = \mathrm{Ind}\,_{G^v}^{G}(\chi_G).$$

Proof. From the above theorem, we have

$$\begin{aligned}\mathrm{Ind}\,_L^G\mathrm{Ind}\,_{L^u}^L(\chi_L) &= \mathrm{Ind}\,_P^G\mathrm{Ind}\,_L^P\mathrm{Ind}\,_{L^u}^L(\chi_L)\\ &= \mathrm{Ind}\,_P^G(\mathrm{Ind}\,_{L^u}^L(\chi_L)\otimes S(\mathfrak{u}^{\mathrm{opp}}))\\ &\subset \mathrm{Ind}\,_P^G\mathrm{Ind}\,_{P^v}^P(\chi_P). \end{aligned}\tag{3.4}$$

By (3.3), we have

$$\mathcal{R}(\tilde{\mathcal{O}}_L) = \mathrm{Ind}\,_{L_0^u}^L(\mathbb{C}) = \sum_{\chi_L\in\widehat{L^u/L_0^u}} \mathrm{Ind}\,_{L^u}^L(V_{\chi_L}).$$

Similarly,

$$\mathcal{R}(\tilde{\mathcal{O}}_P) = \mathrm{Ind}\,_{P_1^v}^P(\mathbb{C}) = \sum_{\chi_P\in\widehat{P^v/P_1^v}} \mathrm{Ind}\,_{P^v}^P(V_{\chi_P}).$$

According to (3.2),

$$\begin{aligned}\mathrm{Ind}\,_L^G\mathcal{R}(\tilde{\mathcal{O}}_L) &= \mathrm{Ind}\,_P^G\mathrm{Ind}\,_L^P\mathcal{R}(\tilde{\mathcal{O}}_L) \cong \mathrm{Ind}\,_P^G(\mathcal{R}(\tilde{\mathcal{O}}_P))\\ &= \mathrm{Ind}\,_P^G\bigg(\sum_{\chi_P\in\widehat{P^v/P_1^v}} \mathrm{Ind}\,_{P^v}^P(V_{\chi_P})\bigg)\\ &= \sum_{\chi_P\in\widehat{P^v/P_1^v}} \mathrm{Ind}\,_P^G\mathrm{Ind}\,_{P^v}^P(V_{\chi_P}) \quad (\text{by the definition of } \chi_G)\\ &= \sum_{\chi_P\in\widehat{P^v/P_1^v}} \mathrm{Ind}\,_{G^v}^G(\chi_G).\end{aligned}$$

Also, because $P^v/P_1^v \cong L^u/L_0^u$ and the formula (3.4), we finish the proof. □

We will consider the connection between induction of admissible orbit data and the induction of unitary representations. Suppose that π_L is an irreducible unitary representation of L attached to $(\mathcal{O}_u,\chi_L)$. We choose a maximal compact subgroup K_L of L and then extend it to a maximal compact subgroup K of G. $\pi_L|_{K_L} = \mathrm{Ind}\,_{L^u}^L(\chi_L)$. We will consider the representation $\pi_G = \mathrm{Ind}\,_{LN}^G(\pi_L\otimes 1\!\!1)$. This is also unitary by unitary induction.

$$\pi_G|_K = \mathrm{Ind}\,_{LN}^G(\pi_L\otimes 1\!\!1)|_K = \mathrm{Ind}\,_{K\cap L}^K(\pi_L|_{K\cap L}).$$

When G is a complex reductive Lie group, $K_{\mathbb{C}}\cong G$. Instead of looking at the action of $K_{\mathbb{C}}$, we will look at the action of G. This is equivalent. Also by Weyl's unitary trick (see [9]), we can get

$$\text{the space of } K\text{-finite vectors of } \pi_G \cong \mathrm{Ind}\,_L^G(\mathrm{Ind}\,_{L^u}^L(\chi_L)), \quad \text{as a } G\text{-action}.$$

Then we get our main result.

Theorem 3.13. *Let G be a complex Lie group. With notation as before, assume π_L to be a representation which satisfies Vogan's conjecture attached to the admissible orbit datum $(\mathcal{O}_u,\chi_L)$. Let $\mathcal{O}_v = \mathrm{Ind}\,_L^G(\mathcal{O}_u)$ be the induced orbit, and $(\mathcal{O}_v,\chi_G)$ the induced orbit datum of G. Write $\pi_G = \mathrm{Ind}\,_{LN}^G(\pi_L\otimes 1\!\!1)$, the parabolic induced representation. Then the space of K-finite vectors of π_G equals the space of algebraic sections of the bundle $G\times_{G^v}\chi_G$.*

Proof. By Corollary 3.12, the space of K-finite vectors of π_G is

$$\begin{aligned}\pi_G|_K &= \mathrm{Ind}\,_L^G(\mathrm{Ind}\,_{L^u}^L(\chi_L))\\ &= \mathrm{Ind}\,_P^G\mathrm{Ind}\,_{P^v}^P(\chi_p) \quad (\text{by definition of } \chi_G)\\ &= \mathrm{Ind}\,_{G^v}^G(\chi_G).\end{aligned}$$

This is just the space of algebraic sections of the bundle $G\times_{G^v}\chi_G$. So we have finished the proof. □

Corollary 3.14. *When χ_G is irreducible, if furthermore we assume π_G is irreducible, then π_G, which is attached to the orbit datum $(\mathcal{O}_G, \chi_G)$, will satisfy Vogan's conjecture on quantization. Specially, when $P^v/P_0^v = G^v/G_0^v$, χ_G is irreducible.*

Example 3.15. Let $G = GL(n, \mathbb{C})$, and $\mathcal{O} = \mathcal{O}_p$ a nilpotent orbit of G, where $p := [p_1, p_2, \dots, p_r]$ is a partition of n. Assume $d := [d_1, d_2, \dots, d_s] = p^t$ to be the transpose partition of p. $L = GL(d_1, \mathbb{C}) \times GL(d_2, \mathbb{C}) \times \dots GL(d_s, \mathbb{C})$ is a Levi subgroup of G. The trivial representation is an irreducible unitary representation of L attached to the orbit datum $(\mathcal{O}_0, \mathbb{1})$. Here $\mathcal{O}_0$ is the 0-orbit of L, and $\mathbb{1}$ the trivial admissible representation. By [17], we know that $\pi_G := \mathrm{Ind}\,_{LN}^G(\mathbb{1})$ is irreducible and unitary. Also, by [4, Theorem 7.2.3], $\mathcal{O}_G = \mathrm{Ind}\,_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_0) = \mathcal{O}_p$. So by our result, π_G, attached to the orbit datum $(\mathcal{O}_p, \mathbb{1})$, satisfies Vogan's conjecture on quantization.

Example 3.16. We consider the representation $\mathrm{Ind}\,_P^G(\sigma \otimes e^\delta \otimes \mathbb{1}_N)$. Here, $P = MAN$ is a parabolic subgroup of G, σ a unitary admissible representation of M, and δ half-sum of all positive roots. From [8], we know that this representation is unitary. Specially, when P is a minimal parabolic subgroup and $\sigma = \mathbb{1}$, it is irreducible. By our results, this representation is attached to the principal nilpotent orbit. Actually, our result in the case amounts to Kostant's result in [9,10] that the restriction to K of $\mathrm{Ind}\,_{P_{\min}}^G(\mathbb{1} \otimes e^\delta \otimes \mathbb{1}_N)$ is isomorphic to the restriction to K of the adjoint action on the principal nilpotent orbit.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 10971103). The authors thank Professor Vogan for very helpful suggestions.

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