

WADGE DEGREES OF ω -LANGUAGES OF DETERMINISTIC TURING MACHINES *

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Abstract. We describe Wadge degrees of ω -languages recognizable by deterministic Turing machines. In particular, it is shown that the ordinal corresponding to these degrees is ξ^ω where $\xi = \omega_1^{\text{CK}}$ is the first non-recursive ordinal known as the Church–Kleene ordinal. This answers a question raised in [2].

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1. FORMULATION OF MAIN RESULT

Let $\{\Sigma_\alpha^0\}_{\alpha < \omega_1}$, where ω_1 is the first uncountable ordinal, denote the Borel hierarchy of subsets of the Cantor space 2^ω (all results below hold true also for the space $\{0, \dots, n+1\}^\omega$ for any $n < \omega$ but for notational simplicity we consider only the case $n = 0$) or the Baire space ω^ω . As usual, Π_α^0 denotes the dual class for Σ_α^0 while $\Delta_\alpha^0 = \Sigma_\alpha^0 \cap \Pi_\alpha^0$ – the corresponding ambiguous class. Let $\mathbf{B} = \cup_{\alpha < \omega_1} \Sigma_\alpha^0$ denote the class of all Borel sets.

In [18, 19] Wadge described the finest possible topological classification of Borel sets by means of a relation $\leq_W$ on subsets of a space $S \in \{2^\omega, \omega^\omega\}$ defined by

$$A \leq_W B \leftrightarrow A = f^{-1}(B),$$

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for some continuous function $f : S \rightarrow S$. He (and Martin) showed that the structure $(\mathbf{B}; \leq_W)$ is well-founded and proved that for all $A, B \in \mathbf{B}$ either $A \leq_W B$ or $\bar{B} \leq_W A$, where $\bar{B}$ stands for $S \setminus B$ (we call structures satisfying these two properties *almost well-ordered*). He also computed the corresponding (very large) ordinal ν . In [17, 21] it was shown that for any Borel set A which is non-self-dual (i.e., $A \not\leq_W \bar{A}$) exactly one of the principal ideals $\{X | X \leq_W A\}$, $\{X | X \leq_W \bar{A}\}$ has the separation property.

The results cited in the last paragraph give rise to the *Wadge hierarchy of Borel sets* which is, by definition, the sequence $\{\Sigma_\alpha\}_{\alpha < \nu}$ of all non-self-dual principal ideals of $(\mathbf{B}; \leq_W)$ not having the separation property [7] and satisfying for all $\alpha < \beta < \nu$ the strict inclusion $\Sigma_\alpha \subset \Delta_\beta$. As usual, we set $\Pi_\alpha = \{\bar{X} | X \in \Sigma_\alpha\}$ and $\Delta_\alpha = \Sigma_\alpha \cap \Pi_\alpha$. Note that the classes

$$\Sigma_\alpha \setminus \Pi_\alpha, \Pi_\alpha \setminus \Sigma_\alpha, \Delta_{\alpha+1} \setminus (\Sigma_\alpha \cup \Pi_\alpha) \quad (\alpha < \nu),$$

which we call *constituents* of the Wadge hierarchy, are exactly the equivalence classes induced by $\leq_W$ on Borel subsets of the Cantor space.

We warn the reader not to mistake Σ_α with Σ_α^0 since in general the equality $\Sigma_\alpha = \Sigma_\alpha^0$ fails, indeed we have e.g. $\Sigma_{\omega_1} = \Sigma_2^0$, $\Sigma_{\omega_1^{\omega_1}} = \Sigma_3^0$ and so on.

There is a well-known small difference between the Wadge hierachies in the Baire and in the Cantor space with respect to the question for which $\alpha < \nu$ the class Δ_α has a W -complete set (such sets correspond to the self-dual Wadge degrees). For the Cantor space, these are exactly the successor ordinals $\alpha < \nu$ while for the Baire space – the successor ordinals and the limit ordinals of countable cofinality [21]. This follows easily from the well-known fact that the Cantor space is compact while the Baire space is not.

The Wadge hierarchy on the Cantor space is of interest to the theory of ω -languages since in this theory people also try to classify “natural” classes of ω -languages according to their “complexity”. The order type of Wadge degrees of regular ω -languages is ω^ω [20]. In [11, 12, 14] the Wagner hierarchy of regular ω -languages was related to the Wadge hierarchy and to the author’s fine hierarchy. In [2] a description of the Wadge degrees containing regular ω -languages was obtained (this description is also implicitly contained in [11], if one takes into account the relationship of the fine hierarchy to the Wadge hierarchy [10]). In 2000 the author has proved that the Wadge degrees of regular star-free ω -languages coincide with the Wadge degrees of regular ω -languages (this result is still unpublished though it was reported at several seminars). In [2] the Wadge degrees of ω -languages recognizable by deterministic push-down automata were determined; the corresponding ordinal is $(\omega^\omega)^\omega$. In [2] a conjecture on the structure of Wadge degrees of ω -languages recognizable by deterministic Turing machines was formulated (for the Muller acceptance condition, see [16]) implying that the corresponding ordinal is ξ^ω , where $\xi = \omega_1^{\text{CK}}$ is the first non-recursive ordinal known also as the Church–Kleene ordinal.

In this paper we prove the conjecture from [2]. To formulate the corresponding result, define an encreasing function $e : \xi^\omega \rightarrow \omega_1^\omega$ by

$$e(\xi^n \alpha_n + \cdots + \xi^1 \alpha_1 + \alpha_0) = \omega_1^n \alpha_n + \cdots + \omega_1^1 \alpha_1 + \alpha_0,$$

where $n < \omega$ and $\alpha_i < \xi$. Note that we use some standard notation and facts from ordinal arithmetic (see, for example [5]).

As is well-known, any non-zero ordinal $\alpha < \xi^\omega$ ($\alpha < \omega_1^\omega$) is uniquely representable in the canonical form

$$\alpha = \xi^{n_0} \alpha_0 + \cdots + \xi^{n_k} \alpha_k \quad (\text{resp., } \alpha = \omega_1^{n_0} \alpha_0 + \cdots + \omega_1^{n_k} \alpha_k), \quad (1)$$

where $k < \omega$, $\omega > n_0 > \cdots > n_k$ and $0 < \alpha_i < \xi$ ($0 < \alpha_i < \omega_1$). The members of the sum (1) will be called *monomials* of the representation. The number n_k will be called *the height* of α .

If we have a similar canonical representation of another non-zero ordinal $\beta < \xi^\omega$ ($\beta < \omega_1^\omega$)

$$\beta = \xi^{m_0} \beta_0 + \cdots + \xi^{m_l} \beta_l \quad (\text{resp., } \beta = \omega_1^{m_0} \beta_0 + \cdots + \omega_1^{m_l} \beta_l),$$

then $\alpha < \beta$ iff the sequence $((n_0, \alpha_0), \dots, (n_k, \alpha_k))$ is lexicographically less than the sequence $((m_0, \beta_0), \dots, (m_l, \beta_l))$.

Let DTM_ω denote the class of subsets of the Cantor space recognized by deterministic Turing machines (using the Muller acceptance condition). Our main result is the following:

Theorem 1.1. (i) For every $\alpha < \xi^\omega$, any of the constituents

$$\Sigma_{e(\alpha)} \setminus \Pi_{e(\alpha)}, \quad \Pi_{e(\alpha)} \setminus \Sigma_{e(\alpha)}, \quad \Delta_{e(\alpha+1)} \setminus (\Sigma_{e(\alpha)} \cup \Pi_{e(\alpha)})$$

contains a set from DTM_ω .

(ii) All other constituents of the Wadge hierarchy do not contain sets from DTM_ω .

This result and the above-mentioned facts on the Wadge hierarchy imply the following:

Corollary 1.2. The structure $(DTM_\omega; \leq_W)$ is almost well-ordered with the corresponding ordinal ξ^ω .

2. SET-THEORETIC OPERATIONS

The first step toward the proof of the main result is to use a result from [16] stating, in our notation, that the class DTM_ω coincides with the boolean closure $bc(\Sigma_2^0)$ of the second level of the arithmetical hierarchy $\{\Sigma_n^0\}_{n < \omega}$ on the Cantor space. Please be careful in distinguishing the levels of the Borel hierarchy (denoted by boldface letters) and the corresponding levels of the arithmetical hierarchy (lightface letters). The result from [16] reduces the problem of this paper

to hierarchy theory since it becomes a question on the interplay of (a fragment of) the arithmetical hierarchy (being the effective version of the Borel hierarchy, see *e.g.* [7]) with the Wadge hierarchy. We will freely use some well-known terminology from computability theory, see *e.g.* [8].

It remains to describe Wadge degrees of sets in $bc(\Sigma_2^0)$. Note that this last problem makes sense not only for the Cantor space but also for the Baire space. We will get a solution for this case as a consequence of the proof for the Cantor space.

The second step toward the main theorem is to use a close relationship of the Wadge hierarchy to set-theoretic operations established in [19]; a version of this result appeared in [6]. These works describe all levels of the Wadge hierarchy in terms of some countable set-theoretic operations. Let us present a description of an initial segment of the Wadge hierarchy which is (with some notational changes) a particular case of the description in [6].

Let us first define the relevant set-theoretic operations. In definitions below, all sets are subsets of the Cantor or the Baire space. For classes $\mathcal{A}$ and $\mathcal{B}$ of sets, let $\mathcal{A} \cdot \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$, let $\bar{\mathcal{A}} = \{\bar{A} \mid A \in \mathcal{A}\}$ be the dual class for $\mathcal{A}$ (sometimes it is more convenient to denote the dual class by $co(\mathcal{A})$) and let $\bar{A} = A \cap \bar{A}$ be the corresponding ambiguous class.

Definition 2.1. For classes of sets $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} + \mathcal{B}$ denote the class of all symmetric differences $A \triangle B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

An ordinal α is called *odd* if $\alpha = 2\beta + 1$, for some ordinal β ; the non-odd ordinals are called *even*. For an ordinal α , let $r(\alpha) = 0$ if α is even and $r(\alpha) = 1$, otherwise.

Let us recall the well-known definition of the Hausdorff difference operation.

Definition 2.2. (i) For an ordinal α , define an operation D_α sending sequences of sets $\{A_\beta\}_{\beta < \alpha}$ to sets by

$$D_\alpha(\{A_\beta\}_{\beta < \alpha}) = \bigcup \{A_\beta \setminus \bigcup_{\gamma < \beta} A_\gamma \mid \beta < \alpha, r(\beta) \neq r(\alpha)\}.$$

For the sake of brevity, we denote in similar expressions below the set $A_\beta \setminus \bigcup_{\gamma < \beta} A_\gamma$ by A'_β .

(ii) For an ordinal α and a class of sets $\mathcal{A}$, let $D_\alpha(\mathcal{A})$ be the class of all sets $D_\alpha(\{A_\beta\}_{\beta < \alpha})$, where $A_\beta \in \mathcal{A}$ for all $\beta < \alpha$.

Now define another, more exotic, operation on sets playing a noticeable role in the theory of Wadge degrees.

Definition 2.3. For classes of sets $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{C}$, let $Bisep(\mathcal{A}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{C})$ be the class of all sets $A_0 B_0 \cup A_1 B_1 \cup \bar{A}_0 \bar{A}_1 C$, where XY denotes the intersection of X and Y , $A_0, A_1 \in \mathcal{A}$, $A_0 A_1 = \emptyset$, $B_i \in \mathcal{B}_i$ and $C \in \mathcal{C}$.

For the sake of brevity, we denote the set $Bisep(\Sigma_1^0, \mathcal{A}, co(\mathcal{A}), \mathcal{B})$ also by $\mathcal{A} * \mathcal{B}$.

Let us state some properties of the introduced operations.

Lemma 2.4. *Let classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and their duals be closed under intersections with $\Sigma_1^0 \cup \Pi_1^0$ -sets. Then it holds:*

- (i) $X \in \mathcal{A} * \mathcal{B}$ iff there are disjoint $U_0, U_1 \in \Sigma_1^0$ such that $XU_0 \in \mathcal{A}$, $XU_1 \in \check{\mathcal{A}}$ and $X\bar{U}_0\bar{U}_1 \in \mathcal{B}$;
- (ii) $co(\mathcal{A} * \mathcal{B}) = \mathcal{A} * co(\mathcal{B})$;
- (iii) $\mathcal{A} * (\mathcal{B} * \mathcal{C}) \subseteq (\mathcal{A} * \mathcal{B}) * \mathcal{C}$.

Proof. (i) and (ii) are easy, we check as an example only (ii). Let $X \in co(\mathcal{A} * \mathcal{B})$, then, by (i),

$$\bar{X}A_0 \in \mathcal{A}, \bar{X}A_1 \in \check{\mathcal{A}} \text{ and } \bar{X}\bar{A}_0\bar{A}_1 \in \mathcal{B}$$

for some disjoint sets $A_0, A_1 \in \Sigma_1^0$. Taking the complements and intersecting them respectively with A_0, A_1 , and $\bar{A}_0\bar{A}_1$ we get

$$XA_0 \in \check{\mathcal{A}}, XA_1 \in \mathcal{A} \text{ and } X\bar{A}_0\bar{A}_1 \in \check{\mathcal{B}},$$

hence $X \in \mathcal{A} * co(\mathcal{B})$. The converse inclusion is checked by a similar computation.

(iii) Let $X \in \mathcal{A} * (\mathcal{B} * \mathcal{C})$, then

$$XA_0 \in \mathcal{A}, XA_1 \in \check{\mathcal{A}} \text{ and } X\bar{A}_0\bar{A}_1 \in \mathcal{B} * \mathcal{C} \quad (2)$$

for some disjoint $A_0, A_1 \in \Sigma_1^0$. Let B_0, B_1 be disjoint recursively enumerable (r.e.) sets such that

$$X\bar{A}_0\bar{A}_1B_0 \in \mathcal{B}, X\bar{A}_0\bar{A}_1B_1 \in \check{\mathcal{B}} \text{ and } X\bar{A}_0\bar{A}_1\bar{B}_0\bar{B}_1 \in \mathcal{C}.$$

Let (C_0, C_1) be a pair of r.e. sets reducing the pair $(A_0 \cup A_1 \cup B_0, A_0 \cup A_1 \cup B_1)$. Then

$$XC_0 \in \mathcal{A} * \mathcal{B}, XC_1 \in \mathcal{A} * \check{\mathcal{B}} \text{ and } X\bar{C}_0\bar{C}_1 \in \mathcal{C}$$

(e.g., for the first assertion) we get from (2) that

$$XC_0A_0 \in \mathcal{A}, XC_0A_1 \in \check{\mathcal{A}} \text{ and } XC_0\bar{A}_0\bar{A}_1 = XC_0\bar{A}_0\bar{A}_1B_0 \in \mathcal{B}.$$

This completes the proof of the lemma.

Next we formulate a result describing the initial segment $\{\Sigma_\alpha\}_{\alpha < \omega_1^\omega}$ of the Wadge hierarchy in terms of the introduced operations. The result is a (reformulation of a) particular case of a result from [6, 19] providing a similar (quite complicated) description for all levels of the Wadge hierarchy. Our description uses an induction on ordinals and the canonical representation (1) described at the end of the previous section.

Theorem 2.5. (i) For $\alpha < \omega_1$, $\Sigma_\alpha = D_\alpha(\Sigma_1^0)$.

(ii) For a monomial $\alpha = \omega_1^n(\gamma + 1)$, $0 < n < \omega$, $\gamma < \omega_1$, $\Sigma_\alpha = \Sigma_\gamma + D_n(\Sigma_2^0)$.

(iii) For a monomial $\alpha = \omega_1^n \lambda$, $0 < n < \omega$, $\lambda < \omega_1$, λ a limit ordinal, Σ_α coincides with the class of all sets of the form

$$(\{A'_\beta Y | \beta < \alpha, r(\beta) = 1\}) \cup (\{A'_\beta \bar{Y} | \beta < \alpha, r(\beta) = 0\}),$$

where $\{A_\beta\}_{\beta < \alpha}$ is a sequence of Σ_1^0 -sets and $Y \in \Sigma_{\omega_1^n}$.

(iv) If $\alpha = \beta + \omega_1^n \gamma$, where $0 < n < \omega$, $0 < \gamma < \omega_1$ and β is a non-zero ordinal of height $> n$, then $\Sigma_\alpha = \text{Bisep}(\Sigma_1^0, \Sigma_\beta^0, \Pi_\beta^0, \Sigma_{\omega_1^n \gamma}^0)$.

(v) If $\alpha = \beta + 1 + \gamma$, where $\gamma < \omega_1$ and β is a non-zero ordinal of height > 0 , then $\Sigma_\alpha = \text{Bisep}(\Sigma_1^0, \Sigma_\beta^0, \Pi_\beta^0, \Sigma_\gamma^0)$.

Notice that $\Sigma_0 = \{\emptyset\}$, $\Sigma_{\omega_1^n} = D_n(\Sigma_2^0)$ for $0 < n < \omega$, and $\bigcup_{\alpha < \omega_1^n} \Sigma_\alpha = bc(\Sigma_2^0)$.

3. EFFECTIVE WADGE HIERARCHY

The third step toward the proof of the main theorem consists in defining an effective analog $\{\mathcal{S}_\alpha\}_{\alpha < \xi^\omega}$ of the sequence $\{\Sigma_\alpha\}_{\alpha < \omega_1^\omega}$. To do this, we turn Theorem 2.5 into a definition by taking the lightface classes Σ_1^0, Σ_2^0 in place of the boldface ones Σ_1^0, Σ_2^0 and considering recursive well-orderings in place of the countable ordinals.

Recall [8] that a *recursive well-ordering* is a well-ordering of the form $(R; \prec)$ where R is a recursive subset of ω and $\prec$ is a recursive relation on R . Let $r : R \rightarrow \{0, 1\}$ be the function induced by the corresponding function on ordinals defined in the last section. As in [3], we will consider only the recursive well-orderings such that r is a partial recursive (p.r.) function, and the set of limit elements of $(R; \prec)$ is recursive. Alternatively, one could use the Kleene notation system for recursive ordinals [8].

For a recursive well-ordering $(R; \prec)$ of order type α and a sequence of sets $\{A_x\}_{x \in R}$, let

$$D_\alpha(\{A_x\}_{x \in R}) = \bigcup \{A'_x | x \in R, r(x) \neq r(\alpha)\}, \quad A'_x = A_x \setminus \bigcup_{y \prec x} A_y.$$

The next definition of classes $\mathcal{S}_\alpha$ closely mimicks Theorem 2.5.

Definition 3.1. (i) For $\alpha < \xi$, let $\mathcal{S}_\alpha$ be the class of all sets $D_\alpha(\{A_x\}_{x \in R})$, where $(R; \prec)$ is a recursive well-ordering of order type α and $\{A_x\}_{x \in R}$ is a uniform r.e. sequence.

(ii) For a monomial $\alpha = \xi^n(\gamma + 1)$, $0 < n < \omega$, $\gamma < \xi$, set $\mathcal{S}_\alpha = \mathcal{S}_\gamma + D_n(\Sigma_2^0)$.

(iii) For a monomial $\alpha = \xi^n \lambda$, $0 < n < \omega$, $\lambda < \xi$, λ a limit ordinal, let $\mathcal{S}_\alpha$ consist of all sets of the form

$$(\{A'_x Y | x \in R, r(x) = 1\}) \cup (\{A'_x \bar{Y} | x \in R, r(x) = 0\}), \quad (3)$$

where again $(R; \prec)$ is a recursive well ordering of type λ , $\{A_x\}_{x \in R}$ is a r.e. sequence, and $Y \in \mathcal{S}_{\xi^n}$.

- (iv) If $\alpha = \beta + \xi^n \gamma$ where $0 < n < \omega$, $0 < \gamma < \xi$ and β is a non-zero ordinal of height $> n$ then set $\mathcal{S}_\alpha = \mathcal{S}_\beta * \mathcal{S}_{\xi^n \gamma}$.
- (v) If $\alpha = \beta + 1 + \gamma$ where $\gamma < \xi$ and β is a non-zero ordinal of height > 0 then set $\mathcal{S}_\alpha = \mathcal{S}_\beta * \mathcal{S}_\gamma$.

Let us state an immediate corollary of the last definition and of Theorem 2.5.

Corollary 3.2. *For any $\alpha < \xi^\omega$, $\mathcal{S}_\alpha \subseteq \Sigma_{e(\alpha)}$.*

Note that Definition 3.1 resembles the definition of the so called fine hierarchy studied in [13] which was first defined (for the case of subsets of ω) in [9] in terms of some jump operations independently of the work on Wadge degrees. Quite similar to [13] one can check some natural properties of the sequence $\{\mathcal{S}_\alpha\}_{\alpha < \xi^\omega}$, e.g.

Lemma 3.3. (i) *For all $\alpha < \beta < \xi^\omega$, $\mathcal{S}_\alpha \subseteq \tilde{\mathcal{S}}_\beta$.*

(ii) *If $X \in \mathcal{S}_\alpha$ and $F : 2^\omega \rightarrow 2^\omega$ is recursive then $F^{-1}(X) \in \mathcal{S}_\alpha$.*

(iii) *For $n > 1$, $\mathcal{S}_{\xi^n} = \tilde{\mathcal{S}}_{\xi^{n-1}} \cdot \mathcal{S}_\xi = \mathcal{S}_{\xi^{n-1}} + \mathcal{S}_\xi$.*

(iv) *For $n > 1$, $\tilde{\mathcal{S}}_{\xi^n} = \mathcal{S}_{\xi^{n-1}} + \tilde{\mathcal{S}}_\xi$.*

(v) *If $0 < \alpha < \xi^\omega$ and $e(\alpha)$ is an ordinal of uncountable cofinality then the classes $\mathcal{S}_\alpha$, $\tilde{\mathcal{S}}_\alpha$ and $\tilde{\mathcal{S}}_\alpha$ are closed under intersections with Δ_2^0 -sets.*

(vi) *If $X_0, X_1 \in \Sigma_1^0$ and $X_0 Y, X_1 Y \in \mathcal{S}_\alpha$ then $(X_0 \cup X_1) Y \in \mathcal{S}_\alpha$. The same holds true for the class $\tilde{\mathcal{S}}_\alpha$ provided that α is a non-zero ordinal of height > 0 .*

(vii) *For $0 < n < \omega$, $0 < \gamma < \xi$, it holds $\tilde{\mathcal{S}}_{\xi^n(\gamma+1)} \subseteq \mathcal{S}_{\xi^n \gamma} * \tilde{\mathcal{S}}_{\xi^n}$.*

(viii) *If $\alpha = \beta + \xi^n \gamma$, where $0 < n < \omega$, $0 < \gamma < \xi$, and β is a non-zero ordinal of height $> n$ then $\tilde{\mathcal{S}}_\alpha \subseteq \mathcal{S}_\beta * \tilde{\mathcal{S}}_{\xi^n \gamma}$.*

Proof. (sketch). The assertions (i, ii) are similar to corresponding assertions in [12]. The assertion (iii) is a well-known fact on the finite difference hierarchy (see e.g. [3, 4, 13]).

(iv) The inclusion from right to left follows from (iii), hence it remains to check the inclusion $\tilde{\mathcal{S}}_{\xi^n} \subseteq \mathcal{S}_{\xi^{n-1}} + \tilde{\mathcal{S}}_\xi$. Let $X \in \tilde{\mathcal{S}}_{\xi^n}$, then $X, \bar{X} \in \mathcal{S}_{\xi^n}$. By (iii),

$$X = Y A \text{ and } \bar{X} = Z B, \text{ for some } Y, Z \in \tilde{\mathcal{S}}_{\xi^{n-1}} \text{ and } A, B \in \mathcal{S}_\xi. \quad (4)$$

Then $A \cup B = 2^\omega$ and $A, B \in \Sigma_2^0$. By Σ_2^0 -reduction, there is an $R \in \tilde{\mathcal{S}}_\xi = \Delta_2^0$ with $R \subseteq A$ and $\bar{R} \subseteq B$. From (4) we get $X R = Y R$ and $X \bar{R} = \bar{Z} \bar{R}$, hence $X \in Y R \cup \bar{Z} \bar{R}$. Then $X = T \triangle R$, where $T = \bar{Z} \bar{R} \cup \bar{Y} R \in \mathcal{S}_{\xi^{n-1}}$. Hence, $X \in \mathcal{S}_{\xi^{n-1}} + \tilde{\mathcal{S}}_\xi$, as desired.

The assertions (v) and (vi) are proved as similar statements in [13].

(vii) Let $X \in \tilde{\mathcal{S}}_{\xi^n(\gamma+1)}$. Then

$$X = Y \triangle A \text{ and } \bar{X} = Z \triangle B, \text{ for some } Y, Z \in \mathcal{S}_{\xi^n} \text{ and } A, B \in \mathcal{S}_\gamma.$$

Let $(R; \prec)$ be a recursive well ordering of type γ and $\{A_x\}, \{B_x\}_{x \in R}$ be r.e. sequences satisfying $A = D_\gamma(\{A_x\})$ and $B = D_\gamma(\{B_x\})$. Let $A^* = \bigcup_{x \in R} A_x$ and $B^* = \bigcup_{x \in R} B_x$. We have:

$$X = (\cup\{A'_x Y | x \in R, r(x) = r(\gamma)\}) \cup (\cup\{A'_x \bar{Y} | x \in R, r(x) \neq r(\gamma)\}) \cup \bar{A}^* Y,$$

$$\bar{X} = (\cup\{B'_x Z | x \in R, r(x) = r(\gamma)\}) \cup (\cup\{B'_x \bar{Z} | x \in R, r(x) \neq r(\gamma)\}) \cup \bar{B}^* Z.$$

From the second equality it follows that

$$X = (\cup\{B'_x \bar{Z} | x \in R, r(x) = r(\gamma)\}) \cup (\cup\{B'_x Z | x \in R, r(x) \neq r(\gamma)\}) \cup \bar{B}^* \bar{Z}.$$

By Σ_1^0 -reduction, there are disjoint r.e. sets $C \subseteq A^*, D \subseteq B^*$ with $C \cup D = A^* \cup B^*$. Then

$$XC = (\cup\{A'_x Y C | x \in R, r(x) = r(\gamma)\}) \cup (\cup\{A'_x \bar{Y} C | x \in R, r(x) \neq r(\gamma)\}),$$

$$XD = (\cup\{B'_x \bar{Z} D | x \in R, r(x) = r(\gamma)\}) \cup (\cup\{B'_x Z D | x \in R, r(x) \neq r(\gamma)\}),$$

and $X\bar{C}\bar{D} = Y\bar{C}\bar{D} = \bar{Z}\bar{C}\bar{D}$. By the last equation, $X\bar{C}\bar{D} \in \tilde{\mathcal{S}}_{\xi^n}$. The other two equations show that $XC \in \check{\mathcal{S}}_{\xi^n \gamma}$ (consider the r.e. sequence $\{C_x\}, C_x = A_x C$) and $XD \in \mathcal{S}_{\xi^n \gamma}$ (consider the r.e. sequence $\{D_x\}, D_x = B_x D$). By Lemma 2.4(i), $X \in \mathcal{S}_{\xi^n \gamma} * \tilde{\mathcal{S}}_{\xi^n}$.

(viii) $X \in \tilde{\mathcal{S}}_\alpha$. Then

$$XU_0 \in \mathcal{S}_\beta, XU_1 \in \check{\mathcal{S}}_\beta, X\bar{U}_0\bar{U}_1 \in \mathcal{S}_{\xi^n \gamma},$$

for some disjoint r.e. sets U_0, U_1 and

$$\bar{X}V_0 \in \mathcal{S}_\beta, \bar{X}V_1 \in \check{\mathcal{S}}_\beta, \bar{X}\bar{V}_0\bar{V}_1 \in \mathcal{S}_{\xi^n \gamma},$$

for some disjoint r.e. sets V_0, V_1 . From the last line we get

$$XV_0 \in \check{\mathcal{S}}_\beta, XV_1 \in \mathcal{S}_\beta, X\bar{V}_0\bar{V}_1 \in \check{\mathcal{S}}_{\xi^n \gamma}.$$

From (v) and (vi) we get

$$XW_0 \in \mathcal{S}_\beta, XW_1 \in \check{\mathcal{S}}_\beta, X\bar{W}_0\bar{W}_1 \in \tilde{\mathcal{S}}_{\xi^n \gamma},$$

where $W_0 = U_0 \cup V_1$ and $W_1 = V_0 \cup U_1$. Hence, $X \in \mathcal{S}_\beta * \tilde{\mathcal{S}}_{\xi^n \gamma}$, completing the proof of the lemma.

4. COMPLETE SETS

Now we are in a position to prove the assertion (i) of Theorem 1.1. We do this by constructing for any $\alpha < \xi^\omega$ a set $C_\alpha \subseteq 2^\omega$ such that $C_\alpha \in \mathcal{S}_\alpha$ and any set

from $\Sigma_{e(\alpha)}$ is W -reducible to C_α . This really proves the assertion (i) since, by Corollary 3.2, $C_\alpha \in \Sigma_{e(\alpha)}$ and *a fortiori* $C_\alpha \in \Sigma_{e(\alpha)} \setminus \Pi_{e(\alpha)}$. For the dual class, the set $\bar{C}_\alpha$ makes the job while for the Δ -level we have

$$C_\alpha \oplus \bar{C}_\alpha \in \Delta_{e(\alpha+1)} \setminus (\Sigma_{e(\alpha)} \cup \Pi_{e(\alpha)}),$$

where $\oplus$ is the join operator on subsets of the Cantor space defined by

$$A \oplus B = \{0 \smallfrown f \mid f \in A\} \cup \{1 \smallfrown f \mid f \in B\}.$$

Here $i \smallfrown f$ is the concatenation of a number i and a function f considered as a sequence. For the construction of the specified sets we need a pair (U_0, U_1) of disjoint Σ_1^0 -sets and a set $V \in \Sigma_2^0$ such that:

- any pair (X_0, X_1) of disjoint Σ_1^0 -sets is W -reducible to (U_0, U_1) ;
- any Σ_2^0 -set is W -reducible to V .

These conditions of course imply that U_0 and V are W -complete in Σ_1^0 and Σ_2^0 , respectively. For existence of such sets (which can be chosen even as regular ω -languages) see *e.g.* [14].

We will also need canonical computable bijections between sets $2^\omega \times 2^\omega$, $(2^\omega)^\omega$ and the set 2^ω defined by

$$\langle f, g \rangle(2n) = f(n), \langle f, g \rangle(2n+1) = g(n), \langle f_0, f_1, \dots \rangle \langle m, n \rangle = f_m(n),$$

where $\langle m, n \rangle$ is a computable bijection between $\omega \times \omega$ and ω . As usual, one can also define a computable bijection between $2^\omega \times \dots \times 2^\omega$ ($n+1$ terms, $n < \omega$) and 2^ω , which is denoted also by $\langle f_0, \dots, f_n \rangle$.

The following definition of the sets C_α uses the same induction scheme (and the same conditions on ordinals and their heights) as Definition 3.1.

Definition 4.1. (i) Let $\alpha < \xi$. For $\alpha = 0$, set $C_0 = \emptyset$. For $0 < \alpha < \omega$, set

$$C_\alpha = D_\alpha(\{Z_i\}_{i < \alpha}), \text{ where } Z_i = \{\langle f_0, \dots, f_{\alpha-1} \rangle \mid f_i \in U_0\}.$$

For $\omega \leq \alpha < \xi$, choose a recursive well ordering $(R; <)$ of type α and a recursive bijection $p: \omega \rightarrow R$ and set

$$C_\alpha = D_\alpha(\{Z_x\}_{x \in R}), \text{ where } Z_{p(i)} = \{\langle f_0, f_1, \dots \rangle \mid f_i \in U_0\}.$$

(ii) Let $\alpha = \xi^n(\gamma + 1)$. For $\gamma = 0$, set

$$C_\alpha = D_n(\{Z_i\}_{i < n}), \text{ where } Z_i = \{\langle f_0, \dots, f_{n-1} \rangle \mid f_i \in V\}.$$

For $\gamma > 0$, set

$$C_\alpha = X \triangle Y, \text{ where } X = \{\langle f, g \rangle \mid f \in C_{\xi^n}\}, Y = \{\langle f, g \rangle \mid g \in C_\gamma\}.$$

(iii) For $\alpha = \xi^n \lambda$, let C_α be of the form (3), where $(R; \prec)$ is a recursive well ordering of type λ , $\{Z_x\}_{x \in R}$ is the sequence defined as in (i) above, and

$$A_x = \{\langle f, g \rangle \mid f \in Z_x\}, Y = \{\langle f, g \rangle \mid g \in C_{\xi^n}\}.$$

(iv) For $\alpha = \beta + \xi^n \gamma$, set $C_\alpha = W_0 X_0 \cup W_1 X_1 \cup \bar{W}_0 \bar{W}_1 Y$, where

$$W_i = \{\langle f, g_0, g_1, h \rangle \mid f \in U_i\}, X_i = \{\langle f, g_0, g_1, h \rangle \mid g_i \in C_\beta\},$$

and $Y = \{\langle f, g_0, g_1, h \rangle \mid h \in C_{\xi^n \gamma}\}$.

(v) For $\alpha = \beta + 1 + \gamma$, C_α is defined as in (iv), with γ in place of $\xi^n \gamma$.

It remains to prove the following:

Proposition 4.2. *For every $\alpha < \xi$, $C_\alpha \in \mathcal{S}_\alpha$, and any $\Sigma_{e(\alpha)}$ -set is W -reducible to C_α .*

Proof. (sketch). The first assertion is immediate by induction, using Definitions 4.1 and 3.1 and Lemma 3.3 (take into account that the projections $\langle f_0, \dots, f_n \rangle \mapsto f_i$ and $\langle f_0, f_1, \dots \rangle \mapsto f_i$ to any coordinate are computable).

The second assertion is also by a straightforward induction (see also a similar proof in [14]). As an example, consider the case (i) of Definition 4.1 for $\omega \leq \alpha < \xi$. Let $T \in \Sigma_\alpha$ (in this case $e(\alpha) = \alpha$). By Theorem 2.5(i), $T = D_\alpha(\{T_\beta\}_{\beta < \alpha})$ for some Σ_α -sequence $\{T_\beta\}_{\beta < \alpha}$. The sequence $\{T_\beta\}_{\beta < \alpha}$ may be written as $\{T_x\}_{x \in R}$, since $(R; \prec)$ is of type α . For any $x = p(i) \in R$, $T_{p(i)} \leq_W U_0$ by means of a continuous function $F_i : 2^\omega \rightarrow 2^\omega$, i.e. $T_{p(i)} = F_i^{-1}(U_0)$. Then for the continuous function $F(f) = \langle F_0(f), F_1(f), \dots \rangle$ we have $T_{p(i)} = F^{-1}(Z_{p(i)})$, hence

$$T = D_\alpha(\{T_x\}_{x \in R}) = D_\alpha(\{F^{-1}(Z_x)\}_{x \in R}) = F^{-1}(D_\alpha(\{Z_x\}_{x \in R})) = F^{-1}(C_\alpha),$$

and $T \leq_W C_\alpha$. This completes the proof.

5. EFFECTIVE HAUSDORFF THEOREM

Here we make the fourth step to proving the main theorem by establishing an effective version of the following classical result of Hausdorff: a set A is Δ_2^0 iff $A = D_\alpha(\{T_\beta\}_{\beta < \alpha})$, for some $\alpha < \omega_1$ and some sequence $\{T_\beta\}_{\beta < \alpha}$ of open sets. In notation of Section 1 it looks as follows: $\Delta_{\omega_1} = \bigcup_{\alpha < \omega_1} \Sigma_\alpha$.

The effective version of the Hausdorff theorem looks like $\Delta_2^0 = \bigcup \{\mathcal{S}_\alpha \mid \alpha < \xi\}$. For the case of subsets of ω , the effective version was established in [3]; in this case, the equality may be even sharpened to $\Delta_2^0 = \bigcup \{\mathcal{S}_\alpha \mid \alpha \leq \omega\}$. To the best of my knowledge, for the Cantor (or Baire) space the proof of the corresponding assertion was never published (though it appeared in handwritten notes [10] accessible only to a small group of recursion theorists). For this reason, let us reproduce here (with minor changes) the proof from [10]. Note that for the case of Cantor and

Baire space the inclusion $\cup\{\mathcal{S}_\alpha \mid \alpha < \gamma\} \subset \Delta_2^0$ is, according to the last section, strict for any $\gamma < \xi$. This is in contrast with the cited result from [3].

We need the following connection of Δ_2^0 -sets to limiting computations.

Proposition 5.1. *Let A be a subset of the Cantor (or Baire) space. Then A is Δ_2^0 iff there is a recursive function $G : 2^\omega \times \omega \rightarrow \{0, 1\}$ such that $A(f) = \lim_{n \rightarrow \infty} G(f, n)$.*

Proof. First note that we identify A with its characteristic function, i.e. $A(f) = 1$ for $f \in A$ and $A(f) = 0$, otherwise. From right to left, the assertion follows from the Tarski–Kuratowski algorithm.

Conversely, let $A \in \Delta_2^0$, then $A = \cap_n B_n$ and $\bar{A} = \cap_n C_n$ for some Σ_1^0 -sequences $\{B_n\}, \{C_n\}_{n < \omega}$. For any $n < \omega$, it holds $B_n \cup C_n = 2^\omega$. By Σ_1^0 -reduction, there are Δ_1^0 -sequences $\{B_n^*\}$ and $\{C_n^*\}$ such that

$$B_n^* \subseteq B_n, C_n^* \subseteq C_n, B_n^* \cap C_n^* = \emptyset, B_n^* \cup C_n^* = 2^\omega.$$

Set $G(f, n) = 1$ for $f \in B_n^*$ and $G(f, n) = 0$ for $f \in C_n^*$. Then the function G has the desired property, completing the proof.

There is a deep and useful connection of the effective difference hierarchy with limiting computations of a special kind which we would like to describe now. Let Φ be a partial function from $S \times \omega$ (where again S is one of $2^\omega, \omega^\omega$) to ω . Relate to Φ and to any recursive well ordering $(R; <)$ a partial function $m = m_{a, \Phi}$ from S to R as follows: $m(f)$ is the least element (if any) of $(\{x \in R \mid \Phi(f, x) \downarrow\}; <)$. Note that $m(f) \downarrow$ implies $\Phi(f, m(f)) \downarrow$.

Definition 5.2. (i) A function $F : S \rightarrow \omega$ is called k - R -computable if there is a p.r. function $\Phi : S \times \omega \rightarrow \omega$ (called a k - R -computation of F) such that $F(f) = k$ for $m(f) \uparrow$ and $F(f) = \Phi(f, m(f))$ otherwise.

(ii) A function $F : S \rightarrow \omega$ is called R -computable if $F(f) = \Phi(f, m(f))$ for some p.r. function $\Phi : S \times \omega \rightarrow \omega$.

(iii) A set $A \subseteq S$ is called k - R -computable (R -computable) if its characteristic function is k - R -computable (R -computable). Here $k \leq 1$.

Let C_R^k (C_R) denote the class of all k - R -computable (R -computable) functions. Note that any set $A \in C_R^k$ ($k \leq 1$) is k - R -computable by a p.r. function Φ with $\text{rng}(\Phi) \subseteq \{0, 1\}$, and similarly for C_R (if Ψ is a p.r. k - R -computation of A then the function Φ defined by

$$\Phi(f, x) = 0 \text{ for } \Psi(f, x) \text{ even and } \Phi(f, x) = 1 \text{ for } \Psi(f, x) \text{ odd,}$$

is also a k - R -computation of A).

If a p.r. function Φ is a k - R -computation of F then any effective stepwise enumerations $\{\Phi^s\}, \{R^s\}$ of Φ and of the r.e. set R induce the limiting computations $\{m^s\}, \{F^s\}$ of m and F as follows:

$$\begin{aligned} m^s(f) &\text{ is the least element of } (\{x \in R^s \mid \Phi^s(f, x) \downarrow\}; <), \\ F^s(f) &= k \text{ for } m^s(f) \uparrow \text{ and } F^s(f) = \Phi(f, m^s(f)) \text{ otherwise.} \end{aligned}$$

From the well-foundedness of $(R; \prec)$ follows that $m(f) = \lim_s m^s(f)$ and $F(f) = \lim_s F^s(f)$.

Let us relate the introduced classes of functions to the effective difference hierarchy. Let $\mathcal{S}_R$ denote the class of all sets $D_\alpha(\{A_x\})$, where $\{A_x\}_{x \in R}$ is an r.e. sequence and α is the order type of $(R; \prec)$. Of course, $\mathcal{S}_\alpha$ coincides with the union of classes $\mathcal{S}_R$, for all recursive well orderings of type α .

Proposition 5.3. *A set $A \subseteq S$ belongs to $\mathcal{S}_R$ ($\tilde{\mathcal{S}}_R$, $\tilde{\mathcal{S}}_R$) iff $A \in C_R^0$ (C_R^1 , C_R).*

Proof. It suffices to prove the assertion for $\mathcal{S}_R$. Let $A = D_\alpha(\{A_x\}) \in \mathcal{S}_R$, and let $\{A_x^s\}$, $\{R^s\}$ be effective enumerations of the r.e. sequence $\{A_x\}_{x \in R}$ and r.e. set R . Define a partial function Φ as follows:

$$\begin{aligned} \Phi(f, x) &= 1 \leftrightarrow r(x) \neq r(\alpha) \wedge \exists s(f \in A_x^s \wedge x \in R^s \wedge \forall y \in R^s(f \in A_y^s \rightarrow x \preceq y)), \\ \Phi(f, x) &= 0 \leftrightarrow r(x) = r(\alpha) \wedge \exists s(f \in A_x^s \wedge x \in R^s \wedge \forall y \in R^s(f \in A_y^s \rightarrow x \preceq y)), \\ \Phi(f, x) &\uparrow \text{ in all other cases.} \end{aligned}$$

Then Φ is a p.r. 0- R -computation of A , hence $A \in C_R^0$.

Conversely, let $A \in C_R^0$ and let Φ be a p.r. 0- R -computation of A with $\text{rng}(\Phi) \subseteq \{0, 1\}$. Fix effective enumerations $\{\Phi^s\}$, $\{R^s\}$ and define sets A_x ($x \in R$) as follows:

$$\begin{aligned} f \in A_x &\leftrightarrow \exists s \exists y \in R^s (\Phi^s(f, y) \downarrow \wedge \forall z \in R^s (\Phi^s(f, z) \downarrow \rightarrow y \preceq z) \wedge \\ &\quad (x = y \text{ or } x \text{ is a successor of } y) \wedge (\Phi(f, y) = 1 \leftrightarrow r(x) \neq r(\alpha))). \end{aligned}$$

We claim that $A = D_\alpha(\{A_x\})$. If $m(f) \uparrow$ then $f \notin A$ and $f \notin \bigcup_{x \in R} A_x \supseteq D_\alpha(\{A_x\})$.

Now let $y = m(f) \downarrow$. If $\Phi(f, y) = 0$ and the successor of y is the greatest element of $(R; \prec)$ then $f \notin A$ and $f \notin \bigcup_{x \in R} A_x \supseteq D_\alpha(\{A_x\})$. Otherwise, for the unique number $x \in \{y, y'\}$ (y' is the successor of y) satisfying

$$\Phi(f, y) = 1 \leftrightarrow r(x) \neq r(\alpha),$$

we have

$$x \in R, f \in A_y \setminus \bigcup_{z \prec y} A_z, \text{ and } f \in A \leftrightarrow f \in D_\alpha(\{A_x\}),$$

completing the proof of the proposition.

Now we prove the effective version of the Hausdorff theorem.

Theorem 5.4. *The effective difference hierarchy is an exhaustive refinement of the arithmetical hierarchy in the second level, i.e. $\bigcup_{\alpha < \xi} \mathcal{S}_\alpha = \Delta_2^0$.*

Proof. By Proposition 5.3, it suffices to check that A is Δ_2^0 iff $A \in C_R$ for some recursive well ordering $(R; \prec)$. From right to left, the assertion follows from the Tarski-Kuratowski algorithm.

Conversely, let $A \in \Delta_2^0$. By Proposition 5.1, there is a recursive function $G: S \times \omega \rightarrow \{0, 1\}$ with $A(f) = \lim_s G(f, s)$. We shall construct a recursive well-ordering $(R; \prec)$ and a p.r. function Ψ from $S \times R$ to $\{0, 1\}$ which $(R; \prec)$ -computes A .

Let $G(f, s) = \varphi_a^f(s)$, where φ is the standard numbering of p.r. functions with oracles. Let $2^*(\omega^*)$ be the set of all finite strings of 0's and 1's (of natural numbers). For a string $\sigma \in \omega^*$, let $\varphi_a^\sigma(s) \downarrow$ denote that the algorithm which computes $\varphi_a^f(s)$ terminates after $lh(\sigma)$ steps ($lh(\sigma)$ is the length of σ), all questions q to the oracle during this computation are less than $lh(\sigma)$ and the answers of the oracle are $\sigma(q)$. The output of this computation is denoted by $\varphi_a^\sigma(s)$. Let $\varphi_a^\sigma(s) \uparrow$ mean the negation of $\varphi_a^\sigma(s) \downarrow$.

Define sets $R_n \subseteq 2^* \times \omega (\omega^* \times \omega)$ by induction on $n < \omega$ as follows. Let

$$R_0 = \{(\sigma, 0) \mid \varphi_a^\sigma(0) \downarrow \wedge \neg \exists \rho \subset \sigma (\varphi_a^\rho(0) \downarrow)\},$$

where $\subset$ is the prefix relation on strings. Let R_{n+1} consist of all (τ, t) such that there is $(\sigma, s) \in R_n$ with $\sigma \subseteq \tau$, $s \leq t$, $\forall p \leq t (\varphi_a^\tau(p) \downarrow, \varphi_a^\tau(t) \neq \varphi_a^\tau(s), \varphi_a^\tau(p) = \varphi_a^\sigma(s) \text{ for } s \leq p < t \text{ and } \neg \exists \rho \subset \tau \forall p \leq t (\varphi_a^\rho(p) \downarrow))$. It is clear that:

- $\varphi_a^\sigma(s) \downarrow$ for $(\sigma, s) \in R_n$;
- if $(\sigma, s), (\sigma, s_1) \in R_n$ then $s = s_1$;
- for all $(\sigma, s), (\sigma_1, s) \in R_n$, the strings σ, σ_1 are $\subseteq$ -incomparable;
- for any $(\tau, t) \in R_{n+1}$ there is a unique $(\sigma, s) \in R_n$ with $\sigma \subseteq \tau$;
- the set $R = \bigcup_n R_n$ is recursive.

The partial ordering $(E; \supseteq)$, where $E = \{\sigma \mid \exists s ((\sigma, s) \in R)\}$, is well-founded (suppose not: $\sigma_i \in S$ and $\sigma_i \subset \sigma_{i+1}$ for all i ; let s_i and n_i satisfy $(\sigma_i, s_i) \in R_{n_i}$, then, by the properties of R_n , the sequence $\{\varphi_a^{h_i}(s)\}_s$, $h = \bigcup_i \sigma_i$, changes infinitely often contradicting the equality $A(h) = \lim_s G(h, s)$. As is well known [8], there is a recursive well ordering $(E; \sqsubset)$ such that $\sigma \supset \tau$ implies $\sigma \sqsubset \tau$.

Now define a recursive linear ordering $(R; <)$ by

$$(\sigma, s) < (\sigma_1, s_1) \leftrightarrow \sigma \sqsubset \sigma_1 \vee (\sigma = \sigma_1 \wedge s_1 < s).$$

This ordering is also well-founded. Suppose not: $(\sigma_i, s_i) \succ (\sigma_{i+1}, s_{i+1})$ for all i . Then $\sigma_0 \supseteq \sigma_1 \supseteq \dots$, hence, by the preceding paragraph, there is k such that $\sigma_j = \sigma_k$ for $j \geq k$. Then $s_k < s_{k+1} < \dots$ and $(\sigma_i, s_i) \in R_{n_i}$ for some n_i . By the properties of R_n , the sequence $\{\varphi_a^{\sigma_k}(s)\}_s$ changes infinitely often, again contradicting to $A(h) = \lim_s G(h, s)$ for $h \supseteq \bigcup_i \sigma_i$.

Define a p.r. function Ψ from $S \times R$ to ω by

$$\Psi(h, (\sigma, s)) \downarrow \leftrightarrow (\sigma, s) \in R \wedge \sigma \subseteq h, \text{ and}$$

$$\Psi(h, (\sigma, s)) = \varphi_a^\sigma(s) = G(h, s) \text{ for } \Psi(h, (\sigma, s)) \downarrow.$$

Then Ψ is an $(R; <)$ -computation of F . This completes the proof of the theorem.

For any oracle $h \in 2^\omega$, let $\xi(h)$ be the first ordinal non-recursive in h , and let $\mathcal{S}_\alpha^h (\alpha < \xi(h))$ be the effective difference hierarchy relative to h . The class $\mathcal{S}_\alpha^h$ is defined just as $\mathcal{S}_\alpha$ except this time well orderings $(R; <)$ have to be recursive in h , and sequences $\{A_x\}_{x \in R}$ r.e. in h . As usual, $\Sigma_n^{0,h}$ denotes the relativization

of Σ_n^0 to h , and analogously for other classes of the arithmetical hierarchy. A straightforward relativization of the proof above yields the following:

Corollary 5.5. *For any $h \in 2^\omega$, $\Delta_2^{0,h} = \cup\{\mathcal{S}_\alpha^h \mid \alpha < \xi(h)\}$.*

6. EXTENDING THE EFFECTIVE HAUSDORFF THEOREM

Here we extend the result of the previous section to some levels of the effective Wadge hierarchy from Section 3. The effective Hausdorff theorem is a particular case of the following result for $\alpha = \xi$.

Theorem 6.1. *If $\alpha < \xi^\omega$ and $e(\alpha)$ is a limit ordinal of uncountable cofinality then $\tilde{\mathcal{S}}_\alpha = \cup\{\mathcal{S}_\beta \mid \beta < \alpha\}$.*

Proof. The inclusion from right to left was established in Lemma 3.3(i), hence it remains to prove the inclusion $\tilde{\mathcal{S}}_\alpha \subseteq \cup\{\mathcal{S}_\beta \mid \beta < \alpha\}$. The condition on $e(\alpha)$ is equivalent to saying that the height of α is non-zero and the last coefficient α_k in the canonical representation (1) of α is a successor ordinal. In other words, we have to consider only the cases $\alpha = \xi^n(\gamma + 1)$ and $\alpha = \beta + \xi^n(\gamma + 1)$ where n and β satisfy conditions of items (ii) and (iv) of Definition 3.1.

Assume that $\alpha = \xi^n(\gamma + 1)$. Let first $\gamma = 0$, hence $\alpha = \xi^n$, $n > 0$. For $n = 1$, $\tilde{\mathcal{S}}_\alpha = \tilde{\mathcal{S}}_\xi = \Delta_2^0$, hence Theorem 5.4 applies. So let $n > 1$ and $X \in \tilde{\mathcal{S}}_\alpha$. By Lemma 3.3(iv), $X = T \triangle R$ for some $T \in \mathcal{S}_{\xi^{n-1}}$ and $R \in \tilde{\mathcal{S}}_\xi$. By Theorem 5.4, $R \in \mathcal{S}_\delta$ for some $\delta < \xi$. Hence, $X \in \mathcal{S}_{\xi^{n-1}} + \mathcal{S}_\delta = \mathcal{S}_{\xi^{n-1}(\delta+1)}$ and $\xi^{n-1}(\delta+1) < \alpha$, as desired.

Now let $\gamma > 0$ and $X \in \tilde{\mathcal{S}}_\alpha$. By Lemma 3.3(vii), $X \in \mathcal{S}_{\xi^n\gamma} * \tilde{\mathcal{S}}_{\xi^n}$, i.e. $X = U_0 X_0 \cup U_1 \bar{X}_1 \cup \bar{U}_0 \bar{U}_1 Y$, for some disjoint sets $U_i \in \Sigma_1^0$, and some $X_i \in \mathcal{S}_{\xi^n\gamma}$, $Y \in \tilde{\mathcal{S}}_{\xi^n}$. By the last paragraph, $Y \in \mathcal{S}_\delta$ for some $\delta < \xi^n$. By Definition 3.1(ii), $X \in \mathcal{S}_\varepsilon$ for $\varepsilon = \xi^n\gamma + 1 + \delta < \alpha$.

It remains to consider the case $\alpha = \beta + \xi^n(\gamma + 1)$. Let $X \in \tilde{\mathcal{S}}_\alpha$. By Lemma 3.3(viii), $X \in \mathcal{S}_\beta * \tilde{\mathcal{S}}_{\xi^n(\gamma+1)}$, hence $XU_0 \in \mathcal{S}_\beta$, $XU_1 \in \tilde{\mathcal{S}}_\beta$ and $X\bar{U}_0\bar{U}_1 \in \tilde{\mathcal{S}}_\delta$ for some disjoint sets $U_i \in \Sigma_1^0$, where $\delta = \xi^n(\gamma + 1)$. Assume first that $\gamma = 0$ and $n = 1$. Then, by Theorem 5.4, $X\bar{U}_0\bar{U}_1 \in \tilde{\mathcal{S}}_\rho$ for some $\rho < \xi$. Hence, $X \in \mathcal{S}_\beta * \mathcal{S}_\rho = \mathcal{S}_{\beta+1+\rho}$ and $\beta + 1 + \rho < \alpha$.

In case $\gamma = 0$ and $n > 1$ we have, by a case considered above, $X\bar{U}_0\bar{U}_1 \in \mathcal{S}_{\xi^{n-1}\rho}$ for some $\rho < \xi$. Hence, $X \in \mathcal{S}_\beta * \mathcal{S}_{\xi^{n-1}\rho} = \mathcal{S}_{\beta+\xi^{n-1}\rho}$ and $\beta + \xi^{n-1}\rho < \alpha$.

Finally, let $\gamma > 0$. Then $X\bar{U}_0\bar{U}_1 \in \mathcal{S}_{\xi^n\gamma+\rho} = \mathcal{S}_{\xi^n\gamma} * \mathcal{S}_\rho$ for some $\rho < \xi^n$. Hence, $X \in \mathcal{S}_\beta * (\mathcal{S}_{\xi^{n-1}\gamma} * \mathcal{S}_\rho)$. By Lemma 2.4(iii), $X \in (\mathcal{S}_\beta * \mathcal{S}_{\xi^{n-1}\gamma}) * \mathcal{S}_\rho = \mathcal{S}_{\beta+\xi^{n-1}\gamma} * \mathcal{S}_\rho = \mathcal{S}_\delta$, where $\delta = \beta + \xi^{n-1}\gamma + 1 + \rho < \alpha$. This completes the proof of the theorem.

We will need the following straightforward relativization of the preceding theorem.

Corollary 6.2. *For all $h \in 2^\omega$ and $\alpha < \xi(h)$, if $e(\alpha)$ is a limit ordinal of uncountable cofinality then $\tilde{\mathcal{S}}_\alpha^h = \cup\{\mathcal{S}_\beta^h \mid \beta < \alpha\}$.*

7. PROOF OF MAIN THEOREM

The fifth step to the proof of the main theorem are two important facts on the hyperarithmetical sets. The first one states that any ordinal recursive in a hyperarithmetical set is (absolutely) recursive (see *e.g.* [7, 8]). In other words, for any hyperarithmetical oracle $h \in \Delta_1^1$ it holds $\xi(h) = \xi$.

The second fact is related to an effective “hyperarithmetical” version of a theorem in [6] mentioned in Section 2. From definitions in [6] and in Sections 2 and 3 it is not hard to see that the following assertion is a reformulation of a particular case of Theorem 2.4 in [6].

Proposition 7.1. *For all $\alpha < \xi^\omega$ and $A \in \Delta_1^1 \cap \Sigma_{e(\alpha)}$ there exists a hyperarithmetical oracle $h \in \Delta_1^1$ with $A \in \mathcal{S}_\alpha^h$. The same holds true with $\Delta_{e(\alpha)}$ in place of $\Sigma_{e(\alpha)}$ and $\tilde{\mathcal{S}}_\alpha^h$ in place of $\mathcal{S}_\alpha^h$.*

Proof of the main theorem. We have to verify the assertion (ii) of the main theorem. Let a set $A \in DTM_\omega$ belong to one of the constituents

$$\Sigma_\alpha \setminus \Pi_\alpha, \Pi_\alpha \setminus \Sigma_\alpha, \Delta_{\alpha+1} \setminus (\Sigma_\alpha \cup \Pi_\alpha)$$

of the Wadge hierarchy. We consider only the case $A \in \Sigma_\alpha \setminus \Pi_\alpha$ the other two cases being similar.

Since $DTM_\omega \subseteq \bigcup_{\alpha < \omega_1^\omega} \Sigma_\alpha$, it holds $\alpha < \omega_1^\omega$. We have to show that indeed $\alpha = e(\alpha^*)$, for some $\alpha^* < \xi^\omega$. We may of course assume α to be non-zero, hence there is a canonical representation

$$\alpha = \omega_1^{n_0} \alpha_0 + \cdots + \omega_1^{n_k} \alpha_k.$$

It suffices to show that all coefficients $\alpha_0, \dots, \alpha_k$ are recursive ordinals since then it holds $\alpha = e(\alpha^*)$, where

$$\alpha^* = \xi^{n_0} \alpha_0 + \cdots + \xi^{n_k} \alpha_k.$$

We have $A \in \Delta_{\beta, \beta} = \omega_1^{n_0+1}$. By Proposition 7.1, $A \in \tilde{\mathcal{S}}_{\xi^{n_0+1}}^h$, for some $h \in \Delta_1^1$. By Corollary 6.2, $A \in \mathcal{S}_{\xi^{n_0}\gamma}^h$ for some $\gamma < \xi(h) = \xi$. By a relativization of Corollary 3.2, $A \in \Sigma_{\omega_1^{n_0}\gamma}$ for some $\gamma < \xi$. But $A \in \Sigma_\alpha \setminus \Pi_\alpha$, hence $\alpha_0 < \xi$.

If $k = 0$, the proof is over. Otherwise, $A \in \Delta_{\beta, \beta} = \omega_1^{n_0} \alpha_0 + \omega_1^{n_1+1}$. Arguing as in the last paragraph, we deduce that $A \in \Sigma_\delta$, $\delta = \omega_1^{n_0} \alpha_0 + \omega_1^{n_1} \gamma$ for some $\gamma < \xi$. But $A \in \Sigma_\alpha \setminus \Pi_\alpha$, hence $\alpha_1 < \xi$. Continuing in this manner, we deduce that really all the ordinals $\alpha_0, \dots, \alpha_k$ are recursive, completing the proof of the theorem.

Notice that the proof works for any hyperarithmetical set $A \in \bigcup_{\alpha < \omega_1^\omega} \Sigma_\alpha$. In other words, the Wadge degrees of the hyperarithmetical $bc(\Sigma_2^0)$ -sets are the same as the degrees of DTM_ω -sets.

8. CONCLUSION

For the Baire space, the formulation of the main result looks as follows:

Theorem 8.1. (i) For every $\alpha < \xi^\omega$, any of the constituents

$$\Sigma_{e(\alpha)} \setminus \Pi_{e(\alpha)}, \Pi_{e(\alpha)} \setminus \Sigma_{e(\alpha)}, \Delta_{e(\alpha+1)} \setminus (\Sigma_{e(\alpha)} \cup \Pi_{e(\alpha)}), \Delta_{e(\lambda)} \setminus \bigcup_{\beta < \lambda} \Sigma_{e(\beta)},$$

where $\lambda < \xi^\omega$ is a limit ordinal such that $e(\lambda)$ is a limit ordinal of countable cofinality, contains a set from $bc(\Sigma_2^0)$.

(ii) All other constituents of the Wadge hierarchy do not contain sets from $bc(\Sigma_2^0)$.

Proof. (sketch). (i) We have to consider only the limit ordinals λ , and for any such an ordinal to find a complete set C_λ with the same properties as the complete sets from Section 4. It suffices to consider the following cases: $\lambda < \xi$; $\lambda = \xi^n \mu$, where $n > 0$ and μ is a limit ordinal $< \xi$; $\lambda = \beta + \mu$, where β is a non-zero ordinal of height > 0 and μ is a limit ordinal $< \xi$; $\lambda = \beta + \xi^n \mu$, where β is a non-zero ordinal of height $> n$, $n > 0$ and μ is a limit ordinal $< \xi$. In all these cases, the construction of C_α uses the operator of join of an infinite sequence $B_0, B_1, \dots$ defined by

$$\bigoplus_{n < \omega} B_n = \{0 \smallfrown f \mid f \in B_0\} \cup \{1 \smallfrown f \mid f \in B_1\} \cup \dots$$

Consider as an example the simplest case $\lambda < \xi$ (other cases use the same idea and the idea behind Def. 4.1). Let $(R; \prec)$ be a recursive well ordering of type λ and g be a recursive function such that $g(0) \prec g(1) \prec \dots$ and for any $x \in R$ there is n with $x \prec g(n)$. Let C_n be the set constructed as in the Definition 4.1 from the recursive well ordering $(\{x \in R \mid x \preceq g(n)\}; \prec)$. Then the set $C_\lambda = \bigoplus_{n < \omega} C_n$ has the desired properties.

Note that for the case of Cantor space the operator of infinite join cannot be defined.

The assertion (ii) follows from the main theorem and the well known properties of the Wadge degrees on the Baire space described in detail *e.g.* in [1, 21]. This completes the proof.

Now we say a couple of words about Lipschitz degrees of $bc(\Sigma_2^0)$ -sets. As is well known (see *e.g.* [1, 21]), any non-self-dual Wadge degree forms a single Lipschitz degree. Any self-dual Wadge degree splits into an encreasing ω -chain of self-dual Lipschitz degrees over the Cantor space, and into an encreasing ω_1 -chain of self-dual Lipschitz degrees over the Baire space. Accordingly, Lipschitz degrees of DTM_ω -sets are obtained from Wadge degrees of such sets by splitting every self-dual Wadge degrees into an encreasing ω -chain of self-dual Lipschitz degrees. For the Baire space, Lipschitz degrees of $bc(\Sigma_2^0)$ -sets are obtained from Wadge degrees of such sets by splitting every self-dual Wadge degree into an encreasing ξ -chain of self-dual Lipschitz degrees.

It seems that the method of this paper applies also to characterizing Wadge degrees of the arithmetical sets and of the hyperarithmetical sets. At the same

time, we do not know any answer to some natural questions similar to those answered above (*e.g.* whether the effective version of the Hausdorff theorem in the formulation similar to that from Sect. 5 may be lifted to the higher levels of the hyperarithmetical hierarchy or not).

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