Chen's theorem in arithmetical progressions*

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Abstract Let N be a sufficiently large even integer and

$$q \geqslant 1,\ (l_i, q) = 1\ (i = 1,2),$$
$$l_1 + l_2 \equiv N(\bmod q).$$

It is proved that the equation

$$N = p + P_2,\ p \equiv l_1(\bmod q),\ P_2 \equiv l_2(\bmod q)$$

has infinitely many solutions for almost all $q \leqslant N^{\frac{1}{37}}$, where p is a prime and P_2 is an almost prime with at most two prime factors.

Keywords: **Chen's Theorem, sieve, mean value theorem.**

In 1937 Vinogradov[1] proved the well-known Goldbach-Vinogradov theorem. Afterwards, some mathematicians generalized Goldbach-Vinogradov theorem to arithmetical progressions[2—5]. In particular, recently Liu and Zhan[6] proved that for sufficiently large odd integer N and

$$q \geqslant 1,\ (l_i, q) = 1\ (i = 1,2,3),$$
$$l_1 + l_2 + l_3 \equiv N(\bmod q),$$

the equation

$$N = p_1 + p_2 + p_3,\ p_i \equiv l_i(\bmod q)\ (i = 1,2,3), \tag{0.1}$$

is solvable in primes p_1, p_2, p_3 for $q \leqslant N^{\delta}$, where δ denotes an effective positive constant. In ref. [7] it is shown that for almost all $q \leqslant N^{\frac{1}{8} - \varepsilon}$, where $\varepsilon > 0$, eq. (1) is solvable.

In 1966 Chen Jingrun[8] made considerable progress in the research of the binary Goldbach conjecture. He proved[9] the well-known Chen's Theorem: let N be a sufficiently large even integer, then the equation

$$N = p + P_2$$

is solvable, where p is a prime and P_2 is an almost prime with at most two prime factors. In fact, Chen's theorem can be stated in a more exact quantitative form.

Inspired by refs. [6,7], it is interesting to generalize Chen's theorem to arithmetical progressions with large moduli. In this paper, we obtain the following result.

Theorem 1. *Let N be a sufficiently large even integer, and let*

$$q \geqslant 1,\ (l_i, q) = 1\quad (i = 1,2),$$
$$l_1 + l_2 \equiv N(\bmod q).$$

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Then the equation

$$N = p + P_2,\ p \equiv l_1(\bmod q), \qquad P_2 \equiv l_2(\bmod q) \tag{0.2}$$

has infinitely many solutions for almost all $q \leqslant N^{\frac{1}{37}}$.

Theorem 1 is a simple corollary of the following result.

Theorem 2. *Under the conditions in Theorem* 1, *let* $S(N, q)$ *be the number of solutions of eq.* (0.2). *Then for* $q \leqslant N^{\frac{1}{37}}$, *except for* $O(N^{\frac{1}{37}}\log^{-5}N)$ *exceptional values*,

$$S(N, q) \geqslant \frac{0.001\, C(N, q) N}{\varphi(q)\log^2 N}, \tag{0.3}$$

where $\varphi(q)$ *is the Euler's function*, *and*

$$C(N, q) = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p \mid N, q, p>2}\frac{p-1}{p-2}. \tag{0.4}$$

By similar arguments we have

Theorem 3. *Let* x *be a sufficiently large real number*, *and*

$$q \geqslant 1,\ (l_i, q) = 1 \quad (i = 1, 2),$$
$$l_1 + 2 \equiv l_2(\bmod q).$$

Then the equation

$$p + 2 = P_2,\ p \equiv l_1(\bmod q),\ P_2 \equiv l_2(\bmod q),\ p \leqslant x$$

has infinitely many solutions for almost all $q \leqslant x^{\frac{1}{37}}$.

Remark. By an improved sieve procedure used in refs. [9, 10], $\frac{1}{37}$ may be increased to 0.028.

1 Some preliminary lemmas

Let $\mathscr{A}$ denote a finite set of integers, $\mathscr{P}$ denote an infinite set of primes, $\overline{\mathscr{P}}$ denote the set of primes not belonging to $\mathscr{P}$. Let $z \geqslant 2$, and put

$$P(z) = \prod_{p<z,\ p\in\mathscr{P}} p,\quad S(\mathscr{A}, \mathscr{P}, z) = \sum_{a\in\mathscr{A},\ (a, P(z))=1} 1,$$
$$\mathscr{A}_d = \{a \mid a \in \mathscr{A}, a \equiv 0 \ (\bmod\ d)\}.$$

Lemma 1[11]. *If*

$$A_1)\ |\mathscr{A}_d| = \frac{\omega(d)}{d}X + r_d,\ \mu(d) \neq 0,\ (d, \overline{\mathscr{P}}) = 1;$$

$$A_2)\ \sum_{z_1 \leqslant p < z_2}\frac{\omega(p)}{p} = \log\frac{\log z_2}{\log z_1} + O\left(\frac{1}{\log z_1}\right),\quad z_2 > z_1 \geqslant 2,$$

where $\omega(d)$ *is a multiplicative function*, $0 \leqslant \omega(p) < p$, $X > 1$ *is independent of* d, *then*

$$S(\mathscr{A}, \mathscr{P}, z) \geqslant XV(z)\left\{f(s) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} - R_D,$$

$$S(\mathscr{A}, \mathscr{P}, z) \leqslant XV(z)\left\{F(s) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} + R_D,$$

where

$$s = \frac{\log D}{\log z},\quad R_D = \sum_{d<D,\ d\mid P(z)} |r_d|,$$

$$V(z) = C(\omega)\frac{e^{-\gamma}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right),$$

$$C(\omega) = \prod_p\left(1 - \frac{\omega(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1},$$

where γ denotes the Euler's constant, and $f(s)$ and $F(s)$ are determined by the following differential-difference equation:

$$\begin{cases} F(s) = \dfrac{2e^{\gamma}}{s},\ f(s) = 0, & 0 < s \leqslant 2, \\ (sF(s))' = f(s-1),\ (sf(s))' = F(s-1), & s \geqslant 2. \end{cases}$$

Lemma 2[12].

$$F(s) = \frac{2e^{\gamma}}{s}, \quad 0 < s \leqslant 3;$$

$$F(s) = \frac{2e^{\gamma}}{s}\left(1 + \int_2^{s-1}\frac{\log(t-1)}{t}\mathrm{d}t\right), \quad 3 \leqslant s \leqslant 5;$$

$$f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, \quad 2 \leqslant s \leqslant 4;$$

$$f(s) = \frac{2e^{\gamma}}{s}\left(\log(s-1) + \int_3^{s-1}\frac{\mathrm{d}t}{t}\int_2^{t-1}\frac{\log(u-1)}{u}\mathrm{d}u\right), \quad 4 \leqslant s \leqslant 6.$$

Lemma 3[12]. *Let* $0 \leqslant g(x) \ll 1$, $0 \leqslant \beta < 1$,

$$\pi(y;\ a, d, l) = \sum_{ap \leqslant y,\ ap \equiv l (\mathrm{mod}\, d)} 1, \quad (l, d) = 1.$$

Then for any given constant $A > 0$, there exists a constant $B = B(A) > 0$ such that

$$\sum_{d \leqslant D}\max_{(l,d)=1}\max_{y \leqslant x}\tau(d)\left|\sum_{a \leqslant x^{\beta},\ (a,d)=1} g(a)\left(\pi(y; a, d, l) - \frac{\mathrm{Li}\left(\frac{y}{a}\right)}{\varphi(d)}\right)\right| \ll \frac{x}{\log^A x},$$

where

$$\mathrm{Li}y = \int_2^{y}\frac{\mathrm{d}t}{\log t},\ D = \frac{x^{\frac{1}{2}}}{\log^B x},\ \tau(n) = \sum_{d \mid n} 1.$$

Lemma 4. *In the notations in Lemma 3, let*

$$R(D, q) = \sum_{d \leqslant \frac{D}{q}}\max_{(l,dq)=1}\max_{y \leqslant N}\left|\sum_{a \leqslant N^{\beta},\ (a,d)=1} g(a)\left(\pi(y; a, dq, l) - \frac{\mathrm{Li}\left(\frac{y}{a}\right)}{\varphi(dq)}\right)\right|.$$

Then for any $A > 0$, there exists a constant $B = B(A) > 0$ such that for $q \leqslant N^{\frac{1}{37}}$, except for $O(N^{\frac{1}{37}}\log^{-A}N)$ exceptional values,

$$R(D, q) \ll \frac{N^{\frac{36}{37}}}{\log^A N},$$

where $D = \dfrac{N^{\frac{1}{2}}}{\log^B N}$.

Proof. By Lemma 3 we have

$$\sum_{q \leqslant N^{\frac{1}{37}}} R(D, q) = \sum_{q \leqslant N^{\frac{1}{37}}}\sum_{d \leqslant \frac{D}{q}}\max_{y \leqslant N}\max_{(l,dq)=1}\left|\sum_{a \leqslant N^{\beta},\ (a,d)=1} g(a)\left(\pi(y; a, dq, l) - \frac{\mathrm{Li}\left(\frac{y}{a}\right)}{\varphi(dq)}\right)\right|$$

$$\leqslant \sum_{d \leqslant D} \tau(d) \max_{y \leqslant N} \max_{(l,dq)=1} \left| \sum_{a \leqslant N^{\beta},\ (a,d)=1} g(a) \left(\pi(y;a,d,l) - \frac{\mathrm{Li}\left(\frac{y}{a}\right)}{\varphi(d)} \right) \right|$$

$$\leqslant \frac{N}{\log^{2A} N},$$

$$\sum_{\substack{q \leqslant N^{\frac{1}{37}} \\ R(D,q) > \frac{N^{36/37}}{\log^A N}}} 1 \ll \frac{\log^A N}{N^{\frac{36}{37}}} \sum_{q \leqslant N^{\frac{1}{37}}} R(D,q) \ll \frac{N^{\frac{1}{37}}}{\log^A N}.$$

Lemma 4 is proved.

2 Weighted sieve method

Let N be a sufficiently large even integer,

$$1 \leqslant q \leqslant N^{\frac{1}{37}}, \quad (l_i, q) = 1 \quad (i = 1,2),$$

$$l_1 + l_2 \equiv N(\mathrm{mod}\, q),$$

$$\mathscr{A} = \{N - p \mid p < N, p \equiv l_1(\mathrm{mod}\, q)\}, \tag{2.1}$$

$$\mathscr{P} = \{p \mid (p, Nq) = 1\}. \tag{2.2}$$

If $a \in \mathscr{A}$, then

$$a = N - p \equiv l_1 + l_2 - p \equiv l_2(\mathrm{mod}\, q).$$

Lemma 5.

$$S(N,q) \geqslant \sum_{\substack{a \in \mathscr{A} \\ (a, P(N^{\frac{1}{10.93}}))=1}} \left(1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a)\right) + O(N^{\frac{9.93}{10.93}}),$$

where

$$\rho_1(a) = \sum_{\substack{p \mid a,\ (p,\ Nq)=1 \\ N^{\frac{1}{10.93}} \leqslant p < N^{\frac{1}{3.3}}}} 1,$$

$$\rho_2(a) = \begin{cases} 1, & a = p_1p_2p_3,\ N^{\frac{1}{10.93}} \leqslant p_1 < N^{\frac{1}{3.3}} \leqslant p_2 < p_3,\ (a, Nq) = 1; \\ 0, & \textit{otherwise}. \end{cases}$$

$$\rho_3(a) = \begin{cases} 1, & a = p_1p_2p_3,\ N^{\frac{1}{3.3}} \leqslant p_1 < p_2 < p_3,\ (a, Nq) = 1; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. Let

$$v_1(a) = \sum_{p \mid a} 1, \quad v_2(a) = \sum_{p^m \mid a} 1, \quad \lambda(a) = \begin{cases} 1, & v_2(a) \leqslant 2, \\ 0, & v_2(a) > 2. \end{cases}$$

Then

$$S(N,q) \geqslant \sum_{\substack{a \in \mathscr{A} \\ (a, P(N^{\frac{1}{10.93}}))=1}} \lambda(a)$$

$$= \sum_{\substack{a \in \mathscr{A},\ (a,\ Nq)=1 \\ (a, P(N^{\frac{1}{10.93}}))=1}} \lambda(a) + O(v_1^{10}(Nq))$$

$$= \sum_{\substack{a \in \mathscr{A},\ (a,\ Nq) = 1 \\ (a,\ P(N^{\frac{1}{10.93}})) = 1}} \mu^2(a)\lambda(a) + O(N^{\frac{9.93}{10.93}}).$$

On the other hand,

$$\sum_{\substack{a \in \mathscr{A} \\ (a,\ P(N^{\frac{1}{10.93}})) = 1}} \left(1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a)\right)$$

$$= \sum_{\substack{a \in \mathscr{A},\ (a,\ Nq) = 1 \\ (a,\ P(N^{\frac{1}{10.93}})) = 1}} \mu^2(a)\left(1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a)\right) + O(N^{\frac{9.93}{10.93}}).$$

For

$$\mu^2(a) = 1,\ (a,\ P(N^{\frac{1}{10.93}})) = 1, \quad (a, Nq) = 1$$

we have

1) $v_2(a) \leqslant 2$.

$$\lambda(a) = 1 \geqslant 1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a).$$

2) $v_2(a) \geqslant 3$. If $\rho_1(a) \geqslant 2$, then

$$\lambda(a) = 0 \geqslant 1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a).$$

If $\rho_1(a) = 1$, then $v_1(a) = v_2(a) = 3$, and $\rho_2(a) = 1$; hence

$$\lambda(a) = 0 = 1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a).$$

If $\rho_1(a) = 0$, then $\rho_3(a) = 1$, and

$$\lambda(a) = 0 = 1 - \frac{1}{2}\rho_1(a) - \frac{1}{2}\rho_2(a) - \rho_3(a).$$

Combining the above arguments we complete the proof of Lemma 5.

3 Proof of Theorem 2

In this section, sets $\mathscr{A}$ and $\mathscr{P}$ are defined by (2.1) and (2.2), respectively, $\delta = \frac{1}{37}$ and

$$X = \frac{\mathrm{Li}N}{\varphi(q)} \sim \frac{N}{\varphi(q)\log N}.$$

For $(d, Nq) = 1$, by Chinese remainder theorem we have

$$\begin{aligned}\mathscr{A}_d &= \{N - p \mid p < N,\ p \equiv l_1(\bmod q),\ p \equiv N(\bmod d)\} \\ &= \{N - p \mid p < N,\ p \equiv l(\bmod qd)\},\end{aligned}$$

where $(l, dq) = 1$; hence

$$r_d = \pi(N;\ dq, l) - \frac{\mathrm{Li}N}{\varphi(dq)},$$

$$\omega(d) = \frac{d}{\varphi(d)}, \quad \mu(d) \neq 0, \quad (d,\ Nq) = 1.$$

By Lemma 5 we get

$$S(N, q) \geqslant S - \frac{1}{2}S_1 - \frac{1}{2}S_2 - S_3 + O(N^{\frac{9.93}{10.93}}), \tag{3.1}$$

$$S=\sum_{\substack{a\in\mathscr{A},\ (a,\ Nq)=1\\ (a,\ P(N^{\frac{1}{10.93}}))=1}}1,\quad S_1=\sum_{\substack{N^{\frac{1}{10.93}}\leqslant p<N^{\frac{1}{3.3}}\\ (p,\ Nq)=1}}S(\mathscr{A}_p,\mathscr{P},N^{\frac{1}{10.93}}),$$

$$S_2=\sum_{\substack{a\in\mathscr{A},\ (a,\ Nq)=1\\ (a,\ P(N^{\frac{1}{10.93}}))=1}}\rho_2(a),\quad S_3=\sum_{\substack{a\in\mathscr{A},\ (a,\ Nq)=1\\ (a,\ P(N^{\frac{1}{10.93}}))=1}}\rho_3(a).$$

All the following arguments are valid for $q\leqslant N^{\frac{1}{37}}$ with $O(N^{\frac{1}{37}}\log^{-5}N)$ exceptional values excluded.

1) The lower bound for S.

Let $D=\frac{N^{\frac{1}{2}}}{\log^B N}$ with $B=B(5)>0$ (see Lemma 4). By Lemma 4 we get

$$R(D,q)=\sum_{d\leqslant\frac{D}{q}}\left|\pi(N;\ dq,l)-\frac{\mathrm{Li}N}{\varphi(dq)}\right|$$

$$\leqslant\sum_{d\leqslant\frac{D}{q}}\max_{y\leqslant N}\max_{(l,dq)=1}\left|\pi(y;\ dq,l)-\frac{\mathrm{Li}y}{\varphi(dq)}\right|\ll\frac{N^{\frac{36}{37}}}{\log^5 N},\tag{3.2}$$

and

$$\begin{aligned}C(\omega)&=\prod_p\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}\\&=\prod_{p\mid Nq}\left(1-\frac{1}{p}\right)^{-1}\prod_{(p,Nq)=1}\left(1-\frac{1}{p-1}\right)\left(1-\frac{1}{p}\right)^{-1}\\&=\prod_{p\mid Nq}\frac{p}{p-1}\prod_{(p,Nq)=1}\frac{p(p-2)}{(p-1)^2}\\&=2\prod_{p\mid Nq,\ p>2}\left(\frac{p}{p-1}\cdot\frac{(p-1)^2}{p(p-2)}\right)\prod_{p>2}\frac{p(p-2)}{(p-1)^2}\\&=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\mid Nq,p>2}\frac{p-1}{p-2}\\&=2C(N,q).\end{aligned}\tag{3.3}$$

By Lemma 1, (3.2) and (3.3) we have

$$\begin{aligned}S&\geqslant(1+o(1))\frac{8C(N,q)}{1-2\delta}\frac{N}{\varphi(q)\log^2N}\Big(\log(4.465-10.93\delta)\\&\quad+\int_2^{3.465-10.93\delta}\frac{\log(s-1)}{s}\log\frac{4.465-10.93\delta}{s+1}\mathrm{d}s\Big)\\&\geqslant12.2598\,C(N,q)\frac{N}{\varphi(q)\log^2N}.\end{aligned}\tag{3.4}$$

2) The upper bound for S_1.

Let $D=\frac{N^{\frac{1}{2}}}{\log^B N}$ with $B=B(5)>0$ (see Lemma 4). For $N^{\frac{1}{10.93}}\leqslant p<N^{\frac{1}{3.3}}$, by Lemma 1 we get

$$S(\mathscr{A}_p,\mathscr{P},N^{\frac{1}{10.93}})\leqslant21.86(1+o(1))C(N,q)e^{-\gamma}$$

$$\times \frac{N}{p\varphi(q)\log^2 N} F\left(5.465 - 10.93\delta - 10.93\frac{\log p}{\log N}\right) + R_D(p), \tag{3.5}$$

where
$$R_D(p) = \sum_{d<\frac{D}{pq},\, d\mid P(N^{\frac{1}{10.93}})} |r_{dp}|.$$

By Lemma 4 we have

$$\sum_{\substack{N^{\frac{1}{10.93}} \leqslant p < N^{\frac{1}{3.3}} \\ (p,Nq)=1}} R_D(p)$$

$$= \sum_{\substack{N^{\frac{1}{10.93}} \leqslant p < N^{\frac{1}{3.3}} \\ (p,Nq)=1}} \sum_{d<\frac{D}{pq},\, d\mid P(N^{\frac{1}{10.93}})} \left| \pi(N; dpq, l) - \frac{\mathrm{Li}N}{\varphi(dpq)} \right|$$

$$\leqslant \sum_{d \leqslant \frac{D}{q}} \max_{(l,dq)=1} \max_{y \leqslant N} \tau(d) \left| \pi(y; dq, l) - \frac{\mathrm{Li}y}{\varphi(dq)} \right| \ll \frac{N^{\frac{36}{37}}}{\log^5 N}. \tag{3.6}$$

By (3.5), (3.6), the prime number theorem and through summation by parts we get

$$S_1 \leqslant 21.86(1+o(1))C(N,q)e^{-\gamma}\frac{N}{\varphi(q)\log^2 N}$$

$$\times \sum_{\substack{N^{\frac{1}{10.93}} \leqslant p < N^{\frac{1}{3.3}} \\ (p,Nq)=1}} \frac{1}{p} F\left(5.465 - 10.93\delta - 10.93\frac{\log p}{\log N}\right)$$

$$= 21.86(1+o(1))C(N,q)e^{-\gamma}\frac{N}{\varphi(g)\log^2 N}$$

$$\times \int_{N^{\frac{1}{10.93}}}^{N^{\frac{1}{3.3}}} \frac{1}{u\log u} F\left(5.465 - 10.93\delta - 10.93\frac{\log u}{\log N}\right) \mathrm{d}u$$

$$= (1+o(1))\frac{8C(N,q)}{1-2\delta}\frac{N}{\varphi(q)\log^2 N}\left(\log\frac{8.93 - 21.86\delta}{1.3 - 6.6\delta}\right.$$

$$\left. + \int_2^{3.465-10.93\delta} \frac{\log(s-1)}{s}\log\frac{(4.465-10.93\delta)(4.465-10.93\delta - s)}{s+1}\mathrm{d}s\right)$$

$$\leqslant 17.70495\frac{C(N,q)N}{\varphi(q)\log^2 N}. \tag{3.7}$$

3) The upper bounds for S_2, S_3.

By the definition of $\rho_2(a)$ we have

$$S_2 = \sum_{\substack{N^{\frac{1}{10.93}} \leqslant p_1 < N^{\frac{1}{3.3}} \leqslant p_2 < \left(\frac{N}{p_1}\right)^{\frac{1}{2}} \\ (p_1p_2,Nq)=1}} \sum_{\substack{a\in\mathscr{A},\ a=p_1p_2p_3 \\ p_2<p_3,\ (p_3,Nq)=1}} 1 = \sum_{\substack{N^{\frac{1}{10.93}} \leqslant p_1 < N^{\frac{1}{3.3}} \leqslant p_2 < \left(\frac{N}{p_1}\right)^{\frac{1}{2}} \\ (p_1p_2,Nq)=1}} \sum_{\substack{p=N-p_1p_2p_3,\ p\equiv l_1(\mathrm{mod}\, q) \\ p_2<p_3<\frac{N}{p_1p_2},\ (p_3,Nq)=1}} 1.$$

Consider the sets

$$\mathscr{E} = \left\{ e \,\middle|\, e = p_1p_2,\ N^{\frac{1}{10.93}} \leqslant p_1 < N^{\frac{1}{3.3}} \leqslant p_2 < \left(\frac{N}{p_1}\right)^{\frac{1}{2}},\ (p_1p_2, Nq) = 1 \right\},$$

$$\mathscr{L} = \{ l \mid l = N - ep,\, e\in\mathscr{E},\ ep < N,\ (p, Nq) = 1,\ l \equiv l_1(\mathrm{mod}\ q) \}.$$

We have

$$|\mathscr{E}| \leqslant \sum_{N^{\frac{1}{10.93}} \leqslant p_1 < N^{\frac{1}{3.3}}} \left(\frac{N}{p_1}\right)^{\frac{1}{2}} < N^{\frac{43}{66}},$$

$$e \geqslant N^{\frac{13}{33}},\ e \in \mathscr{E}.$$

The number of elements in the set $\mathscr{L}$ which are less than $N^{\frac{13}{33}}$ does not exceed $N^{\frac{43}{66}}$. S_2 does not exceed the number of primes in $\mathscr{L}$. Thereforee

$$S_2 \leqslant S(\mathscr{L},\mathscr{P},z) + O\left(N^{\frac{43}{66}}\right) z \leqslant N^{\frac{13}{33}}. \tag{3.8}$$

Now we employ Lemma 1 to estimate the upper bound of $S(\mathscr{L},\mathscr{P},z)$. For the set $\mathscr{L}$,

$$X = \sum_{e\in\mathscr{E}} \frac{1}{\varphi(q)} \cdot \mathrm{Li}\left(\frac{N}{e}\right),$$

$$\omega(d) = \frac{d}{\varphi(d)}, \quad \mu(d) \neq 0, \quad (d,\ Nq) = 1,$$

$$D = \frac{N^{\frac{1}{2}}}{\log^B N}, \quad z = \left(\frac{D}{q}\right)^{\frac{1}{2}},$$

where $B = B(5) > 0$ (see Lemma 4). Then by Lemma 1 we get

$$S(\mathscr{L},\mathscr{P},z) \leqslant \frac{8}{1-2\delta}(1+o(1))C(N,q)\frac{X}{\log N} + R_1 + R_2, \tag{3.9}$$

where

$$R_1 = \sum_{d\leqslant\frac{D}{q},\ (d,\ Nq)=1} \left| \sum_{e\in\mathscr{E},\ (e,d)=1} \left(\pi(N;\ e,\ dq,\ l) - \frac{\mathrm{Li}\left(\frac{N}{e}\right)}{\varphi(dq)}\right) \right|,$$

$$R_2 = \sum_{d\leqslant\frac{D}{q},\ (d,\ Nq)=1} \frac{1}{\varphi(dq)} \sum_{e\in\mathscr{E},\ (e,d)>1} \mathrm{Li}\left(\frac{N}{e}\right).$$

In view of $N^{\frac{13}{33}} \leqslant e < N^{\frac{2}{3.3}}$ for $e\in\mathscr{E}$, we have

$$R_1 \leqslant \sum_{d\leqslant\frac{D}{q}} \max_{y\leqslant N} \max_{(l,dq)=1} \left| \sum_{N^{\frac{13}{33}} \leqslant a < N^{\frac{2}{3.3}},\ (a,d)=1} g(a)\left(\pi(y;a,\ dq,l) - \frac{\mathrm{Li}\left(\frac{y}{a}\right)}{\varphi(dq)}\right) \right|,$$

where

$$g(a) = \sum_{a=e,\,e\in\mathscr{E}} 1 \ll 1.$$

By Lemma 4 we get

$$R_1 \ll \frac{N^{\frac{36}{37}}}{\log^5 N}. \tag{3.10}$$

Now

$$R_2 \ll \frac{N}{\varphi(q)\log N} \sum_{d\leqslant\frac{D}{q}} \frac{1}{\varphi(d)} \sum_{a<N^{\frac{2}{3.3}},\ (a,d)\geqslant N^{\frac{1}{10.93}}} \frac{1}{a}$$

$$\ll \frac{N}{\varphi(q)\log N} \sum_{d\leqslant D} \frac{1}{\varphi(d)} \sum_{m\mid d,\ m\geqslant N^{\frac{1}{10.93}}} \sum_{a<N^{\frac{2}{3.3}},\ (a,d)=m} \frac{1}{a}$$

$$\ll \frac{N}{\varphi(q)} \sum_{d \leqslant D} \frac{1}{\varphi(d)} \sum_{m \mid d,\ m \geqslant N^{\frac{1}{10.93}}} \frac{1}{m}$$

$$\ll \frac{N}{\varphi(q)} \sum_{N^{\frac{1}{10.93}} \leqslant m \leqslant D} \frac{1}{m} \sum_{d \leqslant D,\ m \mid d} \frac{1}{\varphi(d)} \ll \frac{N^{\frac{21}{22}}}{\varphi(q)}. \tag{3.11}$$

By the prime number theorem and integration by parts we have

$$\begin{aligned} X &= (1 + o(1)) \frac{1}{\varphi(q)} \sum_{e \in \mathscr{E}} \frac{N}{e \log \frac{N}{\varepsilon}} \\ &= (1 + o(1)) \frac{N}{\varphi(q)} \int_{N^{\frac{1}{10.93}}}^{N^{\frac{1}{3.3}}} \frac{\mathrm{d}t}{t \log t} \int_{N^{\frac{1}{3.3}}}^{\left(\frac{N}{t}\right)^{\frac{1}{2}}} \frac{\mathrm{d}u}{u \log u \log \frac{N}{ut}} \\ &= (1 + o(1)) \frac{N}{\varphi(q) \log N} \int_{2.3}^{9.93} \frac{\log\left(2.3 - \frac{3.3}{s+1}\right)}{s} \mathrm{d}s. \end{aligned} \tag{3.12}$$

By (3.8)—(3.12) we get

$$S_2 \leqslant 6.48743 C(N, q) \frac{N}{\varphi(q) \log^2 N}. \tag{3.13}$$

Similarly, we have

$$\begin{aligned} S_3 &\leqslant (1 + o(1)) \frac{8 C(N, q) N}{(1 - 2\delta) \varphi(q) \log^2 N} \int_2^{2.3} \frac{\log(s - 1)}{s} \mathrm{d}s \\ &\leqslant 0.15823 C(N, q) \frac{N}{\varphi(q) \log^2 N}. \end{aligned} \tag{3.14}$$

By (3.4), (3.7), (3.13), (3.14) and (3.1), we have

$$\begin{aligned} S(N, q) &\geqslant \left(12.2598 - \frac{17.70495}{2} - \frac{6.48743}{2} - 0.15823\right) C(N, q) \frac{N}{\varphi(q) \log^2 N} \\ &\geqslant 0.001 \frac{C(N, q) N}{\varphi(q) \log^2 N}. \end{aligned}$$

The proof of Theorem 2 is completed.

References

1 Vinogradov, I. M., Representation of odd numbers as the sum of three primes, *Dokl. SSSR*, 1937, 15: 291.

2 Tames, R. D., Weyl, H., Elementary note on prime-number problems of Vinogradov's type, *Amer. J. Math.*, 1942, 64: 539.

3 Richert, H. E., Aus der additiven Primzahltheorie, *J. für Math.*, 1953, 191: 179.

4 Van der Corput, J. G., Proprietes addtives, *Acta Arith.*, 1939, 33: 181.

5 Zulauf, A., Beweis einer Erweiterung des Satzes von Goldbach-Vinogradov, *J. Reine Angew. Math.*, 1952, 190: 169.

6 Liu Ming-chit, Zhan Tao, The Goldbach problem with primes in arithmetical progressions, in *Lecture Notes Math.* 247, Cambridge: Cambridge University Press, 227—252.

7 Liu Jianya, Zhan Tao, The ternary Goldbach problem in arithmetical progressions, *Acta Arith.*, 1997, 82: 197.

8 Chen Jingrun, On the representation of a large even integer as the sum of a prime and the product of at most two primes, *Kexue Tongbao*, 1966, 17: 385.

9 Chen Jingrun, On the representation of a large even integer as the sum of a prime and the product of at most two primes, *Sci. Sin.*, 1973, 16: 157.

10 Chen Jingrun, On the representation of a large even integer as the sum of a prime and the preduct of at most two primes, *Scin. Sin.*, 1978, 21: 421.

11 Iwaniec, H., *Rosser's Sieve*, *Recent Progress in Analytic Number Theory* II, New York: Academic Press, 1981, 203—230.

12 Pan Chengdong, Pan Chengbiao, *Goldbach Conjecture* (in Chinese), Beijing: Science Press, 1981.