多层渗流方程组合系统的迎风分数步长差分方法和应用*

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摘要　对多层渗流方程组合系统提出适合并行计算的二阶和一阶两类迎风分数步长差分格式，利用变分形式、能量方法、二维和三维格式的配套、隐显格式的相互结合、差分算子乘积交换性、高阶差分算子的分解、先验估计的理论和技巧，对二阶格式得到收敛性的最佳阶的 L^2 误差估计．对一阶格式亦得到收敛性的 L^2 误差估计．该方法已成功地应用到多层油资源运移聚集的评估生产实际中，得到了很好的数值模拟结果．

关键词　组合系统　渗流力学　二类迎风分数步长差分法　收敛性　能源数值模拟

在多层地下渗流驱动问题的非稳定流计算中，当第 1、第 3 层近似地认为水平流速，而置于它们中间的层（弱渗透层）仅有垂直流速时，需要求解下述一类多层对流扩散耦合系统的初边值问题[1~5]：

$$\Phi_1(x,y)\frac{\partial u}{\partial t}+\boldsymbol{a}(x,y,t)\cdot\nabla u-\nabla\cdot(K_1(x,y,t)\nabla u)+K_2(x,y,z,t)\left.\frac{\partial w}{\partial z}\right|_{z=H}$$

$$=Q_1(x,y,t,u),\quad (x,y)^{\mathrm{T}}\in\Omega_1,\ t\in J=(0,\ T],\tag{1a}$$

$$\Phi_2(x,y,z)\frac{\partial w}{\partial t}=\frac{\partial}{\partial z}\left(K_2(x,y,z,t)\frac{\partial w}{\partial z}\right),\quad (x,y,z)^{\mathrm{T}}\in\Omega,\ t\in J,\tag{1b}$$

$$\Phi_3(x,y)\frac{\partial v}{\partial t}+\boldsymbol{b}(x,y,t)\cdot\nabla v-\nabla\cdot(K_3(x,y,t)\nabla v)-K_2(x,y,z,t)\left.\frac{\partial w}{\partial z}\right|_{z=0}$$

$$=Q_3(x,y,t,v),\quad (x,y)^{\mathrm{T}}\in\Omega_1,\ t\in J,\tag{1c}$$

此处 $\Omega=\{(x,y,z)\mid(x,y)\in\Omega_1,\ 0<z<H\}$，$\Omega_1$ 为平面有界区域，$\partial\Omega$，$\partial\Omega_1$ 分别为 Ω 和 Ω_1 的边界，如图 1 所示．

初始条件

$$\begin{aligned}u(x,y,o)&=\Psi_1(x,y),\quad (x,y)^{\mathrm{T}}\in\Omega_1,\\ w(x,y,z,o)&=\Psi_2(x,y,z),\quad (x,y,z)^{\mathrm{T}}\in\Omega,\\ v(x,y,o)&=\Psi_3(x,y),\quad (x,y)^{\mathrm{T}}\in\Omega_1.\end{aligned}\tag{2}$$

边界条件是第一型的：

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$$u(x,y,t)|_{\partial\Omega_1}=0,\quad w(x,y,z,t)|_{z=0,H,\partial\Omega_1}=0,\quad v(x,y,t)|_{\partial\Omega_1}=0, \tag{3a}$$

$$w(x,y,z,t)|_{z=H}=u(x,y,t),$$
$$w(x,y,z,t)|_{z=0}=v(x,y,t),\quad (x,y)^{\mathrm{T}}\in\Omega_1(\text{内边界条件}). \tag{3b}$$

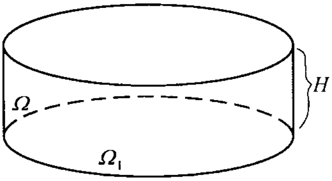

图 1 区域 Ω, Ω_1 示意图

在渗流力学中,待求函数 u, w, v 为位势函数,∇u, ∇v, $\frac{\partial w}{\partial z}$ 为 Darcy 速度, Φ_α 为孔隙度函数, $K_1(x,y,t)$, $K_2(x,y,z,t)$ 和 $K_3(x,y,t)$ 为渗透率函数, $\boldsymbol{a}(x,y,t)=(a_1(x,y,t), a_2(x,y,t))^{\mathrm{T}}$, $\boldsymbol{b}(x,y,t)=(b_1(x,y,t), b_2(x,y,t))^{\mathrm{T}}$ 为相应的对流系数, $Q_1(x,y,t,u)$, $Q_3(x,y,t,v)$ 为产量项.

对于对流扩散问题已有 Douglas 和 Russell 的著名特征差分方法[6,7]克服经典方法可能出现数值解的振荡和失真[1,8,9],解决了用差分方法处理以对流为主的问题. 但特征差分方法有着处理边界条件带来的计算复杂性[1,7],Ewing, Lazarov 等提出用迎风差分格式来解决这类问题[10,11]. 为解决大规模科学与工程计算(节点个数可多达数万乃至数十万个)需要采用分数步技术,将高维问题化为连续解几个一维问题的计算[8,9,12]. 本文从油气资源勘探、开发和地下水渗流计算的实际问题出发,研究多层地下渗流耦合系统驱动问题的非稳定渗流计算,提出适合并行计算的二阶和一阶两类组合迎风分数步差分格式,利用变分形式、能量方法、二维和三维格式的配套、隐显格式的相互结合,差分算子乘积交换性、高阶差分算子的分解、先验估计的理论和技巧,对二阶格式得到收敛性的最佳阶 L^2 误差估计, 对一阶格式亦得到收敛性的 L^2 误差估计. 该方法已成功地应用到多层油资源运移聚集数值模拟计算和工程实践中1). 到目前为止还未见这方面成果发表[11],2). 我们成功地解决了这一著名的计算数学和石油地质问题[1,4,5].

通常问题是正定的,即满足

$$0<\Phi_*\leqslant\Phi_\alpha\leqslant\Phi^*,\quad 0<K_*\leqslant K_\alpha\leqslant K^*,\quad \alpha=1,2,3, \tag{4}$$

此处 Φ_*, Φ^*, K_*, K^* 均为正常数.

假定(1)~(4)式的精确解是正则的,

$$\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 v}{\partial t^2}\in L^\infty(L^\infty(\Omega_1)),\quad u,\ v\in L^\infty(W^{4,\infty}(\Omega_1))\cap W^{1,\infty}(W^{1,\infty}(\Omega_1)),$$

$$\frac{\partial^2 w}{\partial t^2}\in L^\infty(L^\infty(\Omega)),\quad w\in L^\infty(W^{4,\infty}(\Omega)),$$

且 $Q_1(x,y,t,u)$, $Q_3(x,y,t,v)$ 在解的 ε_0-邻域满足 Lipschitz 连续条件,即存在常数 M, 当 $|\varepsilon_i|\leqslant\varepsilon_0(1\leqslant i\leqslant 4)$ 时,有

$$|Q_1(u(x,y,t)+\varepsilon_1)-Q_1(u(x,y,t)+\varepsilon_2)|\leqslant M|\varepsilon_1-\varepsilon_2|,$$

$$|Q_3(v(x,y,t)+\varepsilon_3)-Q_3(v(x,y,t)+\varepsilon_4)|\leqslant M|\varepsilon_3-\varepsilon_4|,\quad (x,y,t)\in\Omega\times J.$$

记号 M 和 ε 分别表示正常数和充分小的正数,在不同地方可以不同.

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2) Ewing R E. Mathematical Modeling and Simulation for Multiphase Flow in Porous Media, An International Workshop on Computation Physics: Fluid Flow and Transport in Porous Media, August 2~6, 1999, Beijing

1　二阶迎风分数步长差分格式

为了用差分方程求解，我们用网格区域 $\Omega_{1,h}$ 代替 Ω_1. 在平面 (x,y) 上步长为 h_1，$x_i = ih_1$，$y_j = jh_1$，

$$\Omega_{1,h} = \{(x_i, y_j) \mid i_1(j) < i < i_2(j), \quad j_1(i) < j < j_2(i)\}.$$

在 z 方向步长为 h_2，$z_k = kh_2$，$h_2 = H/N$，$t^n = n\Delta t$，用 Ω_h 代替 Ω，

$$\Omega_h = \{(x_i, y_j, z_k) \mid i_1(j) < i < i_2(j), \quad j_1(i) < j < j_2(i), 0 < k < N\}.$$

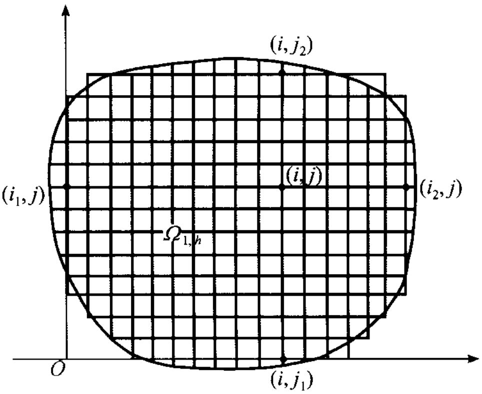

图 2　网域 $\Omega_{1,h}$ 示意图

用 $\partial\Omega_{1,h}$，$\partial\Omega_h$ 分别表示 Ω_h 和 $\Omega_{1,h}$ 的边界. 设 $U(x_i, y_j, t^n) = U_{ij}^n$，$V(x_i, y_j, t^n) = V_{ij}^n$，$W(x_i, y_j, z_k, t^n) = W_{ijk}^n$，$\delta_x$，$\delta_y$，$\delta_z$，$\delta_{\bar{x}}$，$\delta_{\bar{y}}$，$\delta_{\bar{z}}$ 分别为 x，y 和 z 方向向前、向后差商算子，$d_t U^n$ 为网格函数 U_{ij}^n 在 $t = t^n$ 的向前差商.

为了得到高精度计算格式，对方程(1a)在 $(x, y, z_{N-1/2}, t)$ 点展开，得

$$\left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_{N-1/2} = \left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_N - \frac{h_2}{2}\left[\frac{\partial}{\partial z}(K_2(x,y,z,t))\frac{\partial w}{\partial z}\right]_N + O(h_2^2),$$

于是得到

$$\left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_N = \left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_{N-1/2} + \frac{h_2}{2}\left[\Phi_2(x,y,z)\frac{\partial w}{\partial t}\right]_N + O(h_2^2).$$

类似地对方程(1c)可得

$$\left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_0 = \left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_{1/2} - \frac{h_2}{2}\left[\Phi_2(x,y,z)\frac{\partial w}{\partial t}\right]_0 + O(h_2^2).$$

在点 (x, y, H, t)，有

$$\Phi_1(x,y)\frac{\partial u}{\partial t} + \boldsymbol{a}(x,y,t)\cdot\nabla u - \nabla\cdot(K_1(x,y,t)\nabla u) + \frac{h_2}{2}\left[\Phi_2(x,y,H)\frac{\partial u}{\partial t}\right] + \left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_{N-1/2} + O(h_2^2) = Q(u),$$

即

$$\hat{\Phi}_1(x,y,h_2)\frac{\partial u}{\partial t} + \boldsymbol{a}(x,y,t)\cdot\nabla u - \nabla\cdot(K_1(x,y,t)\nabla u) = -\left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_{N-1/2} + Q_1(u) + O(h_2^2), \tag{5a}$$

此处 $\hat{\Phi}_1(x,y,h_2) = \Phi_1(x,y) + \frac{h_2}{2}\Phi_2(x,y,H)$.

类似地在点 $(x, y, 0, t)$，有

$$\hat{\Phi}_3(x,y,h_2)\frac{\partial v}{\partial t} + \boldsymbol{b}(x,y,t)\cdot\nabla v - \nabla\cdot(K_3(x,y,t)\nabla v) = \left[K_2(x,y,z,t)\frac{\partial w}{\partial z}\right]_{1/2} + Q_3(v) + O(h_2^2), \tag{5b}$$

此处 $\hat{\Phi}_3(x,y,h_2)=\Phi_3(x,y)+\frac{h_2}{2}\Phi_2(x,y,0)$. 设

$K_{i+1/2,j}^n=[K(x_i,y_j,t^n)+K(x_{i+1},y_j,t^n)]/2,\ K_{i,j+1/2}^n=[K(x_i,y_j,t^n)+K(x_i,y_{j+1},t^n)]/2,$

定义

$$\delta_x(K^n\delta_{\bar{x}}u^{n+1})_{ij}=h_1^{-2}[K_{i+1/2,j}^n(u_{i+1,j}^{n+1}-u_{ij}^{n+1})-K_{i-1/2,j}^n(u_{ij}^{n+1}-u_{i-1,j}^{n+1})],$$

$$\delta_y(K^n\delta_{\bar{y}}u^{n+1})_{ij}=h_1^{-2}[K_{i,j+1/2}^n(u_{i,j+1}^{n+1}-u_{ij}^{n})-K_{i,j-1/2}^n(u_{ij}^{n}-u_{i,j-1}^{n+1})],$$

$$\nabla_h(K^n\nabla_h u^{n+1})_{ij}=\delta_x(K^n\delta_{\bar{x}}u^{n+1})_{ij}+\delta_y(K^n\delta_{\bar{y}}u^{n+1})_{ij},$$

类似地定义 $\delta_z(K^n\delta_{\bar{z}}w^n)_{ijk}=h_2^{-2}[K_{ij,k+1/2}^n(W_{ij,k+1}^n-W_{ijk}^n)-K_{ij,k-1/2}^n(W_{ijk}^n-W_{ij,k-1}^n)]$.

1.1 迎风分数步长格式 I

方程(5a)可近似分裂为

$$\left(1-\frac{\Delta t}{\hat{\Phi}_1}\frac{\partial}{\partial x}\left(K_1\frac{\partial}{\partial x}\right)+\frac{\Delta t}{\hat{\Phi}_1}a_1\frac{\partial}{\partial x}\right)\left(1-\frac{\Delta t}{\hat{\Phi}_1}\frac{\partial}{\partial y}\left(K_1\frac{\partial}{\partial y}\right)+\frac{\Delta t}{\hat{\Phi}_1}a_2\frac{\partial}{\partial y}\right)u^{n+1}$$

$$=u^n-\frac{\Delta t}{\hat{\Phi}_1}\left\{\left(K_2\frac{\partial w^{n+1}}{\partial z}\right)_{N-1/2}-Q_1(x,y,t^{n+1},u^{n+1})\right\},\tag{6}$$

其对应的二阶迎风分数步长差分格式为

$$\left(\hat{\Phi}_1-\Delta t\left(1+\frac{h_1}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}})+\Delta t\ \delta_{a_1^n,x}\right)U_{ij}^{n+1/2}$$

$$=\hat{\Phi}_{1,ij}U_{ij}^n+\Delta t\{-K_{2,ij,N-1/2}^n\delta_{\bar{z}}W_{ij,N}^n+Q(x_i,y_j,t^n,U_{ij}^n)\},\quad i_1(j)<i<i_2(j),\tag{7a}$$

$$U_{ij}^{n+1/2}=0,\quad (x_i,y_j)\in\partial\Omega_{1,h},\tag{7b}$$

$$\left(\hat{\Phi}_1-\Delta t\left(1+\frac{h_1}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}})+\Delta t\delta_{a_2^n,y}\right)U_{ij}^{n+1}=\hat{\Phi}_{1,ij}U_{ij}^{n+1/2},\quad j_1(i)<j<j_2(i),\tag{7c}$$

$$U_{ij}^{n+1}=0,\quad (x_i,y_j)\in\partial\Omega_{1,h},\tag{7d}$$

此处 $\delta_{a_1^n,x}u_{ij}=a_{1,ij}^n[H(a_{1,ij}^n)K_{1,ij}^{n,-1}K_{1,i-1/2,j}^n\delta_{\bar{x}}+(1-H(a_{1,ij}^n))K_{1,ij}^{n,-1}K_{1,i+1/2,j}^n\delta_x]u_{ij}$, $\delta_{a_2^n,y}u_{ij}=a_{2,ij}^n[H(a_{2,ij}^n)K_{1,ij}^{n,-1}K_{1,i,j-1/2}^n\delta_{\bar{y}}+(1-H(a_{2,ij}^n))K_{1,ij}^{n,-1}K_{1,i,j+1/2}^n\delta_y]u_{ij}$, $K_{1,ij}^{n,-1}=(K_{1,ij}^n)^{-1}$,

$$H(z)=\begin{cases}1, & z\geqslant 0,\\ 0, & z<0.\end{cases}$$

方程(1b)的差分格式是

$$\Phi_{2,ijk}\frac{W_{ijk}^{n+1}-W_{ijk}^n}{\Delta t}=\delta_z(K_2^n\delta_{\bar{z}}W^n)_{ijk},\quad 0<k<N,\ (i,j)\in\Omega_{1,h}.\tag{8}$$

方程(1c)可近似分裂为

$$\left(1-\frac{\Delta t}{\hat{\Phi}_3}\frac{\partial}{\partial x}\left(K_3\frac{\partial}{\partial x}\right)+\frac{\Delta t}{\hat{\Phi}_3}\delta_{b_1^n,x}\right)\left(1-\frac{\Delta t}{\hat{\Phi}_3}\frac{\partial}{\partial y}\left(K_3\frac{\partial}{\partial y}\right)+\frac{\Delta t}{\hat{\Phi}_3}\delta_{b_2^n,y}\right)v_{ij}^{n+1}$$

$$=v_{ij}^n+\frac{\Delta t}{\hat{\Phi}_3}\left\{\left(K_2\frac{\partial w^{n+1}}{\partial z}\right)_{1/2}+Q(x_i,y_j,t^{n+1},v^{n+1})\right\},\tag{9}$$

对应的迎风分数步长差分格式为

$$\left(\hat{\Phi}_3-\Delta t\left(1+\frac{h_1}{2}\frac{|b_1^n|}{K_3^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}})+\Delta t\ \delta_{b_1^n,x}\right)V_{ij}^{n+1/2}$$

$$=\hat{\Phi}_{3,ij}V_{ij}^n+\Delta t\{K_{2,ij,1/2}^n\delta_zW_{ij,0}^n+Q(x_i,y_j,t^n,V_{ij}^n)\},\quad i_1(j)<i<i_2(j),\tag{10a}$$

$$V_{ij}^{n+1/2} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{10b}$$

$$\left(\hat{\Phi}_3 - \Delta t\left(1 + \frac{h_1}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar{y}}) + \Delta t\delta_{b_2^n,y}\right)V_{ij}^{n+1} = \hat{\Phi}_{3,ij}V_{ij}^{n+1/2}, \quad j_1(i) < j < j_2(i), \tag{10c}$$

$$V_{ij}^{n+1} = 0, \quad (x_i, y_j) \in \Omega_{1,h}, \tag{10d}$$

此处 $\delta_{b_1^n,x}v_{ij} = b_{1,ij}^n[H(b_{1,ij}^n)K_{3,ij}^{n,-1}K_{3,i-1/2,j}^n\delta_{\bar{x}} + (1 - H(b_{1,ij}^n))K_{3,ij}^{n,-1}K_{3,i+1/2,j}^n\delta_x]v_{ij}$，$\delta_{b_2^n,y}v_{ij} = b_{2,ij}^n[H(b_{2,ij}^n)K_{3,ij}^{n,-1}K_{3,i,j-1/2}^n\delta_{\bar{y}} + (1 - H(b_{2,ij}^n))K_{3,ij}^{n,-1}K_{3,i,j+1/2}^n\delta_y]v_{ij}$. 差分格式(7)、(8)和(10)式的计算程序是：若已知 $t = t^n$ 的差分解 $\{U_{ij}^n, W_{ijk}^n, V_{ij}^n\}$ 时，寻求下一时刻 t^{n+1} 的 $\{U_{ij}^{n+1}, W_{ijk}^{n+1}, V_{ij}^{n+1}\}$. 首先由(7a)和(7b)式用追赶法求出过渡层的解 $\{U_{ij}^{n+1/2}\}$，再由(7c)和(7d)式求出 $\{U_{ij}^{n+1}\}$. 与此同时可并行的由(10a)和(10b)式用追赶法求出过渡层的解 $\{V_{ij}^{n+1/2}\}$，再由(10c)和(10d)式求出 $\{V_{ij}^{n+1}\}$. 最后由(8)式利用内边界条件(3b)式求出 $\{W_{ijk}^{n+1}\}$. 由正定性条件(4)，此差分解存在且惟一.

1.2　迎风分数步长差分格式 Ⅱ

对应于方程(1a)的迎风分数步长差分格式为

$$\left(\hat{\Phi}_1 - \Delta t\left(1 + \frac{h_1}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}}) + \Delta t\delta_{a_1^n,x}\right)U_{ij}^{n+1/2}$$

$$= \hat{\Phi}_{1,ij}U_{ij}^n + \Delta t\{-K_{2,ij,N-1/2}^n\delta_{\bar{z}}W_{ij,N}^{n+1} + Q(x_i, y_j, t^{n+1}, U_{ij}^{n+1})\}, \quad i_1(j) < i < i_2(j), \tag{7a'}$$

$$U_{ij}^{n+1/2} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{7b'}$$

$$\left(\hat{\Phi}_1 - \Delta t\left(1 + \frac{h_1}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}}) + \Delta t\delta_{a_2^n,y}\right)U_{ij}^{n+1} = \hat{\Phi}_{1,ij}U_{ij}^{n+1/2}, \quad j_1(i) < j < j_2(i), \tag{7c'}$$

$$U_{ij}^{n+1} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}. \tag{7d'}$$

在实际计算时(7a′)式中 $\delta_{\bar{z}}W_{ij,N}^{n+1}$ 近似地取为 $\delta_{\bar{z}}W_{ij,N}^n$，$U_{ij}^{n+1}$ 近似取为 U_{ij}^n.

对应于方程(1b)的差分格式为

$$\Phi_{2,ijk}\frac{W_{ijk}^{n+1} - W_{ijk}^n}{\Delta t} = \delta_z(K_2^n\delta_{\bar{z}}W^{n+1})_{ijk}, \quad 1 \leqslant k \leqslant N-1, (i, j) \in \Omega_{1,h}. \tag{8'}$$

方程(1c)的迎风分数步长差分格式为

$$\left(\hat{\Phi}_3 - \Delta t\left(1 + \frac{h_1}{2}\frac{|b_1^n|}{K_3^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}}) + \Delta t\delta_{b_1^n,x}\right)V_{ij}^{n+1/2}$$

$$= \hat{\Phi}_{3,ij}V_{ij}^n + \Delta t\{K_{2,ij,1/2}\delta_z W_{ij,0}^{n+1} + Q(x_i, y_j, t^{n+1}, V_{ij}^{n+1})\}, \quad i_1(j) < i < i_2(j), \tag{10a'}$$

$$V_{ij}^{n+1/2} = 0, \quad (x_i, y_i) \in \partial\Omega_{1,h}, \tag{10b'}$$

$$\left(\hat{\Phi}_3 - \Delta t\left(1 + \frac{h_1}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar{y}}) + \Delta t\delta_{b_2^n,y}\right)V_{ij}^{n+1} = \hat{\Phi}_{3,ij}V_{ij}^{n+1/2}, \quad j_1(i) < j < j_2(i), \tag{10c'}$$

$$V_{ij}^{n+1} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{10d'}$$

在实际计算时(10a′)式中 $\delta_z W_{ij,0}^{n+1}$ 近似地取为 $\delta_z W_{ij,0}^{n}$, V_{ij}^{n+1} 近似地取为 V_{ij}^{n}.

格式Ⅱ的计算过程和格式Ⅰ是类似的.

2 二阶格式的收敛性分析

为理论分析简便,设 $\Omega=\{(x,y,z)|0<x<1,0<y<1,\ 0<z<1\}$, $\Omega_1=\{(x,y)|0<x<1,0<y<1\}$, $h=1/N$, $t^n=n\Delta t$. 定义网格函数空间 $\hat{H}_h$, H_h 的内积[13~15], 对三维网格区域 Ω_h, 有

$$(\omega,\chi)=\sum_{i,j,k=1}^{N-1}\omega_{ijk}\chi_{ijk}h^3,\quad (\omega,\ \chi]=\sum_{i,j=1}^{N-1}\sum_{k=1}^{N}\omega_{ijk}\chi_{ijk}h^3,\quad \forall\,\omega,\chi\in\hat{H}_h,$$

对二维网格区域 $\Omega_{1,h}$,则有

$$\langle u,v\rangle=\sum_{i,j=1}^{N-1}u_{ij}v_{ij}h^2,\quad \langle u,v^{(1)}\rangle=\sum_{i=1}^{N}\sum_{j=1}^{N-1}u_{ij}v_{ij}h^2,\quad \langle u,v^{(2)}\rangle=\sum_{i=1}^{N-1}\sum_{j=1}^{N}u_{ij}v_{ij}h^2,\quad \forall\,u,v\in H_h,$$

其相应的范数为

$$\|\omega^n\|=\left(\sum_{i,j,k=1}^{N-1}(\omega_{ijk}^n)^2h^2\right)^{1/2},\qquad \|\delta_{\bar z}\omega^n\|=\left(\sum_{i,j=1}^{N-1}\sum_{k=1}^{N}(\delta_{\bar z}\,\omega_{ijk}^n)^2h^3\right)^{1/2},$$

$$\|u^n\|=\left(\sum_{i,j=1}^{N-1}(u_{ij}^n)^2h^2\right)^{1/2},\qquad \|\delta_{\bar x}u^n\|=\left(\sum_{i=1}^{N}\sum_{j=1}^{N-1}(\delta_{\bar x}\,u_{ijk}^n)^2h^2\right)^{1/2},$$

$$\|\delta_{\bar y}u^n\|=\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N}(\delta_{\bar y}u_{ij}^n)^2h^2\right)^{1/2}.$$

首先对格式Ⅰ进行收敛性分析. 设 u, v, w 为问题(1)~(4)的精确解, U, V, W 为格式Ⅰ的差分解,记误差函数为 $\xi=u-U$, $\zeta=v-V$, $\omega=w-W$. 方程(7a)~(7d)消去 $U^{n+1/2}$可得下述等价的差分方程:

$$\begin{aligned}
&\hat{\Phi}_{1,ij}\frac{U_{ij}^{n+1}-U_{ij}^n}{\Delta t}-\left\{\left(1+\frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar x})+\left(1+\frac{h}{2}\frac{|a_{2,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar y})\right\}U_{ij}^{n+1}\\
&+\delta_{a_1^n,x}U_{ij}^{n+1}+\delta_{a_2^n,y}U_{ij}^{n+1}+\Delta t\left(1+\frac{h}{2}\frac{|a_{1,jj}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x\Bigg(K_1^n\delta_{\bar x}\Bigg(\hat{\Phi}_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\\
&\times\delta_y(K_1^n\delta_{\bar y}U^{n+1})\Bigg)\Bigg)_{ij}-\Delta t\left\{\left(1+\frac{h}{2}\frac{|a_{1,jj}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar x}(\Phi_1^{-1}(\delta_{a_2^n,y}U^{n+1})))_{ij}\right.\\
&\left.+\delta_{a_1^n,x}\left(\hat{\Phi}_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar y}U^{n+1})\right)_{ij}-\delta_{a_1^n,x}(\hat{\Phi}_1^{-1}\delta_{a_2^n,y}U^{n+1})_{ij}\right\}\\
&=-K_{2,ij,N-1/2}^n\delta_{\bar z}W_{ij,N}^n+Q(U_{ij}^n),\quad 1\leqslant i,\ j\leqslant N-1,
\end{aligned}\tag{11a}$$

$$U_{ij}^{n+1}=0,\quad (x_i,\ y_j)\in\partial\Omega_1,\tag{11b}$$

方程(10a)~(10d)消去 $V^{n+1/2}$可得下述等价的差分方程:

$$\hat{\Phi}_{3,ij}\frac{V_{ij}^{n+1}-V_{ij}^n}{\Delta t}-\left\{\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar x})+\left(1+\frac{h}{2}\frac{|b_{2,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar y})\right\}V_{ij}^{n+1}$$

$$+\delta_{b_1^n,x}V_{ij}^{n+1}+\delta_{b_2^n,y}V_{ij}^{n+1}+\Delta t\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x\left(K_3^n\delta_{\bar{x}}\left(\Phi_3^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_2^n}\right)^{-1}\right.\right.$$

$$\left.\left.\times\,\delta_y(K_3^n\delta_{\bar{y}}V^{n+1})\right)\right)_{ij}-\Delta t\left\{\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}}(\Phi_3^{-1}(\delta_{b_2^n,y}V^{n+1})))_{ij}\right.$$

$$\left.+\,\delta_{b_1^n,x}\left(\hat{\Phi}_3^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_2^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar{y}}V^{n+1})\right)_{ij}-\delta_{b_1^n,x}(\hat{\Phi}_3^{-1}\delta_{b_2^n,y}V^{n+1})_{ij}\right\}$$

$$=K_{2,ij,1/2}^n\delta_z W_{ij,0}^n+Q(V_{ij}^n),\quad 1\leqslant i,\ j\leqslant N-1, \tag{12a}$$

$$V_{ij}^{n+1}=0,\quad (x_i,\ y_j)\in\partial\Omega_1, \tag{12b}$$

若问题(1)~(4)的精确解 u, v, w 是正则的,则有下述误差方程:

$$\hat{\Phi}_{1,ij}\frac{\xi_{ij}^{n+1}-\xi_{ij}^n}{\Delta t}-\left\{\left(1+\frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}}\xi^{n+1})_{ij}+\left(1+\frac{h}{2}\frac{|a_{2,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}}\xi^{n+1})_{ij}\right\}$$

$$+\delta_{a_1^n,x}\ \xi_{ij}^{n+1}+\delta_{a_2^n,y}\ \xi_{ij}^{n+1}+\Delta t\left(1+\frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x\left(K_1\delta_{\bar{x}}\left(\Phi_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_2^n}\right)^{-1}\right.\right.$$

$$\left.\left.\times\,\delta_y(K_1^n\delta_{\bar{y}}\xi^{n+1})\right)\right)_{ij}-\Delta t\left\{\left(1+\frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}}(\Phi_1^{-1}(\delta_{a_2^n,y}\xi^{n+1})))_{ij}\right.$$

$$\left.+\,\delta_{a_1^n,x}\left(\hat{\Phi}_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_2^n}\right)^{-1}\delta_y(K_1^n\xi_{\bar{y}}\xi^{n+1})\right)_{ij}-\delta_{a_1^n,x}(\hat{\Phi}_1^{-1}\delta_{a_2^n,y}^n\xi^{n+1})_{ij}\right\}$$

$$=-K_{2,ij,N-1/2}^n\delta_{\bar{z}}\omega_{ij,N}^n+Q(u_{ij}^{n+1})-Q(U_{ij}^n)+\varepsilon_{1,ij}^{n+1},\quad 1\leqslant i,\ j\leqslant N-1, \tag{13a}$$

$$\xi_{ij}^{n+1}=0,\quad (x_i,\ y_j)\in\partial\Omega_1, \tag{13b}$$

此处 $|\varepsilon_{1,ij}^{n+1}|\leqslant M\left\{\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^\infty(L^\infty)}\|u\|_{L^\infty(W^{4,\infty})}\right\}(\Delta t+h^2)$.

$$\hat{\Phi}_{3,ij}\frac{\zeta_{ij}^{n+1}-\zeta_{ij}^n}{\Delta t}-\left\{\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}}\zeta^{n+1})+\left(1+\frac{h}{2}\frac{|b_{2,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}}\xi^{n+1})\right\}$$

$$+\delta_{b_1^n,x}\zeta^{n+1}+\delta_{b_2^n,y}\zeta^{n+1}+\Delta t\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x\left(K_3\delta_{\bar{x}}\left(\Phi_3^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_2^n}\right)^{-1}\right.\right.$$

$$\left.\left.\times\,\delta_y(K_3^n\delta_{\bar{y}}\zeta^{n+1})\right)\right)_{ij}-\Delta t\left\{\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}}(\Phi_3^{-1}(\delta_{b_2^n,y}\zeta^{n+1})))_{ij}\right.$$

$$\left.+\,\delta_{b_1^n,x}\left(\hat{\Phi}_3^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar{y}}\zeta^{n+1})\right)_{ij}-\delta_{b_1^n,x}(\hat{\Phi}_3^{-1}\delta_{b_2^n,y}^n\zeta^{n+1})_{ij}\right\}$$

$$=K_{2,ij,1/2}^n\delta_z\omega_{ij,0}^n+Q_3(v_{ij}^{n+1})-Q_3(V_{ij}^n)+\varepsilon_{3,ij}^{n+1},\quad 1\leqslant i,\ j\leqslant N-1, \tag{14a}$$

$$\zeta_{ij}^{n+1}=0,\quad (x_i,\ y_j)\in\partial\Omega_1, \tag{14b}$$

此处 $|\varepsilon_{3,ij}^{n+1}|\leqslant M\left\{\left\|\frac{\partial^2 v}{\partial t^2}\right\|_{L^\infty(L^\infty)},\ \|v\|_{L^\infty(W^{4,\infty})}\right\}(\Delta t+h^2)$.

$$\Phi_{2,ijk}\frac{\omega_{ijk}^{n+1}-\omega_{ijk}^{n}}{\Delta t}=\delta_z(K_2^n\delta_{\bar z}\omega^n)_{ijk}+\varepsilon_{2,ijk}^{n+1},\qquad 1\le i,j,k\le N-1,\tag{15}$$

其中 $|\varepsilon_{2,ijk}^{n+1}|\le M\left\{\left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^\infty(L^\infty)},\ \|w\|_{L^\infty(W^{4,\infty})}\right\}(\Delta t+h^2)$.

对方程(13a),(14a),(15)分别乘以 $2\Delta t\,\xi_{ij}^{n+1}$, $2\Delta t\,\zeta_{ij}^{n+1}$, $2\Delta t\,\omega_{ijk}^{n+1}$ 作内积、分步求和并利用(13b)、(13c)和(3b)式可得

$$\begin{aligned}
&\{\|\hat\Phi_1^{1/2}\xi^{n+1}\|^2-\|\hat\Phi_1^{1/2}\xi^n\|^2\}+(\Delta t)^2\|\hat\Phi_1^{1/2}\xi_t^n\|^2\\
&+2\Delta t\left\{\left\langle K_1^n\delta_{\bar x}\xi^{n+1},\ \delta_{\bar x}\left(\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\xi^{n+1}\right)\right\rangle+\left\langle K_1^n\delta_{\bar y}\xi^{n+1},\ \delta_{\bar y}\left(\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\xi^{n+1}\right)\right\rangle\right\}\\
=&-2\Delta t\langle\delta_{a_1^n,x}\xi^{n+1}+\delta_{a_2^n,y}\xi^{n+1},\ \xi^{n+1}\rangle-2(\Delta t)^2\left\langle\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_x\left(K_1^n\delta_{\bar x}\left(\hat\Phi_1^{-1}\left(1+\right.\right.\right.\right.\\
&\left.\left.\left.\left.\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar y}\xi^{n+1})\right)\right),\ \xi^{n+1}\right\rangle+2(\Delta t)^2\left\langle\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar x}(\hat\Phi_1^{-1}\delta_{a_2^n,y}\xi^{n+1}))\right.\\
&\left.+\delta_{a_1^n,x}\left(\hat\Phi_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar y}\xi^{n+1})\right)-\delta_{a_1^n,x}(\hat\Phi_1^{-1}\delta_{a_2^n,y}\xi^{n+1}),\ \xi^{n+1}\right\rangle\\
&-2\Delta t\sum_{i,j=1}^{N-1}K_{2,ij,N-1/2}^n\delta_z\ \omega_{ij,N}^n\ \xi_{ij}^{n+1}h^2+2\Delta t\langle Q_1(u^{n+1})-Q_1(U^n),\ \xi^{n+1}\rangle+2\Delta t\langle\xi_1^{n+1},\ \xi^{n+1}\rangle,
\end{aligned}\tag{16}$$

$$\begin{aligned}
&\{\|\hat\Phi_3^{1/2}\zeta^{n+1}\|^2-\|\hat\Phi_3^{1/2}\zeta^n\|^2\}+(\Delta t)^2\|\hat\Phi_3^{1/2}\zeta_t^n\|^2\\
&+2\Delta t\left\{\left\langle K_3^n\delta_{\bar x}\zeta^{n+1},\ \delta_{\bar x}\left(\left(1+\frac{h}{2}\frac{|b_1^n|}{K_3^n}\right)^{-1}\zeta^{n+1}\right)\right\rangle+\left\langle K_3^n\delta_{\bar y}\zeta^{n+1},\ \delta_{\bar y}\left(\left(1+\frac{h}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\zeta^{n+1}\right)\right\rangle\right\}\\
=&-2\Delta t\langle\delta_{b_1^n,x}\zeta^{n+1}+\delta_{b_2^n,y}\zeta^{n+1},\ \zeta^{n+1}\rangle-2(\Delta t)^2\left\langle\left(1+\frac{h}{2}\frac{|b_1^n|}{K_2^n}\right)^{-1}\delta_x\left(K_3^n\delta_{\bar x}\left(\hat\Phi_3^{-1}\left(1+\right.\right.\right.\right.\\
&\left.\left.\left.\left.\frac{h}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar y}\zeta^{n+1})\right)\right),\ \zeta^{n+1}\right\rangle+2(\Delta t)^2\left\{\left\langle\left(1+\frac{h}{2}\frac{|b_1^n|}{K_2^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar x}(\hat\Phi_3^{-1}\delta_{b_2^n,y}\zeta^{n+1}))\right.\right.\\
&\left.\left.+\delta_{b_1^n,x}\left(\hat\Phi_1^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar y}\zeta^{n+1})\right)-\delta_{b_1^n,x}(\hat\Phi_3^{-1}\delta_{b_2^n,y}\zeta^{n+1}),\ \zeta^{n+1}\right\rangle\right\}\\
&+2\Delta t\sum_{i,j=1}^{N-1}K_{2,ij,1/2}^n\delta_z\omega_{ij,0}^n\zeta_{ij}^{n+1}h^2+2\Delta t\langle Q_3(v^{n+1})-Q_3(V^n),\ \zeta^{n+1}\rangle+2\Delta t\langle\varepsilon_3^{n+1},\ \zeta^{n+1}\rangle,
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\{\|\hat\Phi_2^{1/2}\omega^{n+1}\|^2-\|\hat\Phi_2^{1/2}\omega^n\|^2\}+(\Delta t)^2\|\hat\Phi_2^{1/2}\omega_t^n\|^2\\
&=2\Delta t(\delta_z(K_3^n\delta_{\bar z}\omega^n),\ \omega^{n+1})+2\Delta t(\varepsilon_2^{n+1},\ \zeta^{n+1})\\
&=2\Delta t\sum_{i,j,k=1}^{N-1}\delta_z(K_2^n\delta_{\bar z}\omega_{ijk}^n)\omega_{ijk}^{n+1}h^3+2\Delta t(\varepsilon_2^{n+1},\ \zeta^{n+1})\\
&=2\Delta t\sum_{i,j=1}^{N-1}\sum_{k=1}^{N-1}\omega_{ijk}^n\delta_z(K_2^n\delta_{\bar z}\omega_{jik}^n)h^3+2\Delta t(\varepsilon_2^{n+1},\ \zeta^{n+1})\\
&=-2\Delta t\sum_{i,j=1}^{N-1}h^2\left\{\sum_{k=1}^{N}K_{2,ijk}^n\ \delta_{\bar z}\omega_{ijk}^n\ \delta_{\bar z}\ w_{ijk}^{n+1}\ h-\omega_{ij,0}^{n+1}K_{2,ij,1/2}^n\delta_z\ \omega_{ij,0}^n+\omega_{ij,N}^{n+1}K_{2,ij,N-1/2}^n\delta_{\bar z}\,\omega_{ij,N}^n\right\}
\end{aligned}$$

$$+2\Delta t(\varepsilon_2^{n+1},\ \zeta^{n+1}) = -2\Delta t(K_2^n\delta_{\bar z}\omega^n,\ \delta_{\bar z}\omega^{n+1}] + 2\Delta t\sum_{i,j=1}^{N-1}\{K_{2,ij,N-1/2}^n\delta_{\bar z}\omega_{ij,N}^n\cdot\xi_{ij}^{n+1}$$

$$-K_{2,ij,1/2}^n\delta_z\omega_{ij,0}^n\zeta_{ij}^{n+1}\}h^2 + 2\Delta t(\varepsilon_2^{n+1},\ \zeta^{n+1}),$$

注意到

$$2\Delta t(K_2^n\delta_{\bar z}\omega^n,\ \delta_{\bar z}\omega^{n+1}] = 2\Delta t(K_2^n\delta_{\bar z}\omega^{n+1},\ \delta_{\bar z}\omega^n]$$

$$=2\Delta t\{(K_2^n\delta_{\bar z}(\omega^{n+1}-\omega^n),\ \delta_{\bar z}\omega^n] + (K_2^n\delta_{\bar z}\omega^n,\ \delta_{\bar z}\omega^n]\}$$

$$\geqslant\Delta t\{(K_2^n\delta_{\bar z}\omega^{n+1},\ \delta_{\bar z}\omega^{n+1}] - (K^n\delta_{\bar z}\omega^n,\ \delta_{\bar z}\omega^n]\} + 2\Delta t(K_2^n\delta_{\bar z}\omega^n,\ \delta_{\bar z}\omega^n]$$

$$=\Delta t\{(K_2^n\delta_{\bar z}\omega^{n+1},\delta_{\bar z}\omega^{n+1}] + (K_2^n\delta_{\bar z}\omega^n,\ \delta_{\bar z}\omega^n]\} = \Delta t\{\|K_2^{n,1/2}\delta_{\bar z}\omega^{n+1}\|^2 + \|K_2^{n,1/2}\delta_{\bar z}\omega^n\|^2\},$$

此处 $K_2^{n,1/2}=(K_2^n)^{1/2}$，于是有

$$\{\|\hat\Phi_2^{1/2}\omega^{n+1}\|^2 - \|\hat\Phi_2^{1/2}\omega^n\|^2\} + (\Delta t)^2\|\hat\Phi_2^{1/2}\omega_t^n\|^2 + \Delta t\{\|K_2^{n,1/2}\delta_{\bar z}\omega^{n+1}\|^2 + \|K^{n,1/2}\delta_{\bar z}\omega^n\|^2\}$$

$$\leqslant 2\Delta t\sum_{i,j=1}^{N-1}\{K_{2,ij,N-1/2}^n\delta_{\bar z}\omega_{ij,N}^n\cdot\xi_{ij}^{n+1} - K_{2,ij,1/2}^n\delta_z\omega_{ij,0}^n\cdot\zeta_{ij}^{n+1}\}h^2 + 2\Delta t(\varepsilon_2^{n+1},\ \omega^{n+1}).\tag{18}$$

引入归纳法假定：

$$\sup_{1\leqslant n\leqslant L}\max\{\|\xi^n\|_{0,\infty},\ \|\zeta^n\|_{0,\infty}\}\to 0,\quad (h,\ \Delta t)\to 0,\tag{19}$$

现在估计(16)式左端第 3 项，因为 K_1 是正定的，当 h 适当小，有

$$2\Delta t\left\{\left\langle K_1^n\delta_{\bar x}\xi^{n+1},\ \delta_{\bar x}\left(\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\xi^{n+1}\right)\right\rangle + \left\langle K_1^n\delta_{\bar y}\xi^{n+1},\ \delta_{\bar y}\left(\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\xi^{n+1}\right)\right\rangle\right\}$$

$$=2\Delta t\left\{\left\langle K_1^n\delta_{\bar x}\xi^{n+1},\ \left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_{\bar x}\xi^{n+1}\right\rangle + \left\langle K_1^n\delta_{\bar y}\xi^{n+1},\ \left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_{\bar y}\xi^{n+1}\right\rangle\right\}$$

$$+2\Delta t\left\{\left\langle K_1^n\delta_{\bar x}\xi^{n+1},\ \delta_{\bar x}\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\cdot\xi^{n+1}\right\rangle + \left\langle K_1^n\delta_{\bar y}\xi^{n+1},\ \delta_{\bar y}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\cdot\xi^{n+1}\right\rangle\right\}$$

$$\geqslant\Delta t\{\|K_1^{n,1/2}\delta_{\bar x}\xi^{n+1}\|^2 + \|K_1^{n,1/2}\delta_{\bar y}\xi^{n+1}\|^2\} - M\|\xi^{n+1}\|^2\Delta t.\tag{20}$$

类似地估计(17)式左端第 3 项，我们有

$$2\Delta t\left\{\left\langle K_3^n\delta_{\bar x}\zeta^{n+1},\ \delta_{\bar x}\left(\left(1+\frac{h}{2}\frac{|b_1^n|}{K_3^n}\right)^{-1}\zeta^{n+1}\right)\right\rangle + \left\langle K_3^n\delta_{\bar y}\zeta^{n+1}\cdot\ \delta_{\bar y}\left(\left(1+\frac{h}{2}\frac{|b_2^n|}{K_1^n}\right)^{-1}\cdot\xi^{n+1}\right)\right\rangle\right\}$$

$$\geqslant\Delta t\{\|K_2^{n,1/2}\delta_{\bar x}\zeta^{n+1}\|^2 + \|K_3^{n,1/2}\delta_{\bar y}\zeta^{n+1}\|^2\} - M\|\xi^{n+1}\|^2\Delta t,\tag{21}$$

将估计(16)～(18)式相加，并利用(20)和(21)式可得

$$\{\|\hat\Phi_1^{1/2}\xi^{n+1}\|^2 + \|\hat\Phi_3^{1/2}\zeta^{n+1}\|^2 + \|\hat\Phi_2^{1/2}\omega^{n+1}\|^2\} - \{\|\hat\Phi_1^{1/2}\xi^n\|^2 + \|\hat\Phi_3^{1/2}\zeta^n\|^2 + \|\hat\Phi_2^{1/2}\omega^n\|^2\}$$

$$+(\Delta t)^2\{\|\hat\Phi_1^{1/2}\xi_t^n\|^2 + \|\hat\Phi_3^{1/2}\zeta_t^n\|^2 + \|\Phi_2^{1/2}\omega_t^n\|^2\}$$

$$+\Delta t\Big\{[\|K_1^{n,1/2}\delta_{\bar x}\xi^{n+1}\|^2 + \|K_1^{n,1/2}\delta_{\bar y}\xi^{n+1}\|^2] + [\|K_3^{n,1/2}\delta_{\bar x}\zeta^{n+1}\|^2 + \|K_3^{n,1/2}\delta_{\bar y}\zeta^{n+1}\|^2]$$

$$+\frac{1}{2}[\|\Phi_2^{1/2}\delta_{\bar z}\omega^{n+1}\|^2 + \|\Phi_2^{1/2}\delta_{\bar z}\omega^n\|^2]\Big\}$$

$$\geqslant -2\Delta t\{\langle\delta_{a_1^n,x}\xi^{n+1} + \delta_{a_2^n,y}\xi^{n+1},\ \xi^{n+1}\rangle + \langle\delta_{b_1^n,x}\zeta^{n+1} + \delta_{b_2^n,y}\zeta^{n+1},\ \zeta^{n+1}\rangle\}$$

$$+ [K^n_{1,i,j-1/2}\delta_{\bar{y}}(K^n_{1,i-1/2,j}\hat{\Phi}^{-1}_{1,ij}R^n_{a_2,ij})\delta_{\bar{x}}(R^n_{a_1,ij}\xi^{n+1}_{ij}) + K^n_{1,i-1/2,j}\hat{\Phi}^{-1}_{1,ij}R^n_{a_2,ij}\delta_{\bar{x}}K^n_{1,i,j-1/2}\delta_{\bar{y}}\xi^{n+1}_{ij}$$

$$+ K^n_{1,i,j-1/2}K^n_{1,j-1/2,j}\delta_{\bar{x}}(\hat{\Phi}^{-1}_{ij}R^n_{a_2,ij})\delta_{\bar{y}}\xi^{n+1}_{ij}]\delta_{\bar{x}}\delta_{\bar{y}}\xi^{n+1}_{ij}\}h_2$$

$$\leqslant -(\Phi_*)^2(\Phi^*)^{-1}(\Delta t)^2\sum_{i,j=1}^{N}[\delta_{\bar{x}}\delta_{\bar{y}}\xi^{n+1}_{ij}]^2h^2 + M(\Delta t)^2\{\|\delta_{\bar{x}}\xi^{n+1}\|^2 + \|\delta_{\bar{y}}\xi^{n+1}\|^2 + \|\xi^{n+1}\|^2\},$$

$$-2(\Delta t)^2\sum_{i,j=1}^{N}\{[\delta_{\bar{x}}K^n_{1,i,j-1/2}\delta_{\bar{y}}(\hat{\Phi}^{-1}_{1,ij}R^n_{a_2,ij}K^n_{1,i-1/2,j})R^n_{a_1,ij} + K^n_{1,i,j-1/2}\delta_{\bar{y}}(\delta_{\bar{x}}(\hat{\Phi}^{-1}_{1,ij}R^n_{a_2,ij})K^n_{1,i-1/2,j})$$

$$\times R^n_{a_1,ij}]\delta_{\bar{x}}\xi^{n+1}_{ij}\delta_{\bar{y}}\xi^{n+1}_{ij} + [K^n_{1,i-1/2,j}K^n_{1,i,j-1/2}\delta_{\bar{x}}\delta_{\bar{y}}R^n_{a_1,ij}\xi^{n+1}_{ij}\delta_{\bar{y}}\xi^{n+1}_{ij}$$

$$+ K^n_{1,i,j-1/2}\delta_{\bar{y}}(\delta_{\bar{x}}(\hat{\Phi}^{-1}_{1,ij}\cdot R^n_{a_2,ij})K^n_{1,i-1/2,j})\delta_{\bar{x}}R^n_{a_1,ij}\xi^{n+1}_{ij}\delta_{\bar{y}}\xi^{n+1}_{ij}]\}h^2$$

$$\leqslant M(\Delta t)^2\{\|\delta_{\bar{x}}\xi^{n+1}\|^2 + \|\delta_{\bar{y}}\xi^{n+1}\|^2 + \|\xi^{n+1}\|^2\}.$$

对于第 2 项中的另一项估计是类似的，于是有

$$-2(\Delta t)\Big\{\langle R^n_{a_1}\delta_x(K^n_1\delta_{\bar{x}}(\hat{\Phi}^{-1}_1R^n_{a_2}\delta_y(K^n_1\delta_{\bar{y}}\xi^{n+1}))),\ \xi^{n+1}\rangle$$

$$+\langle R^n_{b_1}\delta_x(K^n_3\delta_{\bar{x}}(\hat{\Phi}^{-1}_3R^n_{b_2}\delta_y(K^n_3\delta_{\bar{y}}\zeta^{n+1}))),\ \zeta^{n+1}\rangle\Big\}$$

$$\leqslant -2(\Phi_*)^2(\Phi^*)^{-1}(\Delta t)^2\sum_{i,j=1}^{N}[\delta_{\bar{x}}\delta_{\bar{y}}\xi^{n+1}_{ij}]^2h^2 + M(\Delta t)^2$$

$$\times\{\|\delta_{\bar{x}}\xi^{n+1}\|^2 + \|\delta_{\bar{y}}\xi^{n+1}\|^2 + \|\xi^{n+1}\|^2 + \|\delta_{\bar{x}}\zeta^{n+1}\|^2 + \|\delta_{\bar{y}}\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|^2\}. \tag{24}$$

现估计(22)式右端第 3 项，有

$$2(\Delta t)^2\Big\{\langle R^n_{a_1}\delta_x(K^n_1\delta_{\bar{x}}(\hat{\Phi}^{-1}_1\delta_{a^n_2,y}\xi^{n+1})) + \cdots,\xi^{n+1}\rangle + \langle R^n_{b_1}\delta_x(K^n_3\delta_{\bar{x}}(\hat{\Phi}^{-1}_3\delta_{b^n_2,y}\zeta^{n+1})) + \cdots,\ \zeta^{n+1}\rangle\Big\}$$

$$\leqslant \varepsilon(\Delta t)^2\sum_{i,j=1}^{N}\{|\delta_{\bar{x}}\delta_{\bar{y}}\xi^{n+1}_{ij}|^2 + |\delta_{\bar{x}}\delta_{\bar{y}}\zeta^{n+1}_{ij}|^2\}h^2 + M(\Delta t)^2\Big\{\|\delta_{\bar{x}}\xi^{n+1}\|^2$$

$$+\|\delta_{\bar{y}}\xi^{n+1}\|^2 + \|\xi^{n+1}\|^2 + \|\delta_{\bar{x}}\zeta^{n+1}\|^2 + \|\delta_{\bar{y}}\zeta^{n+1}\|^2 + \|\zeta^{n+1}\|^2\Big\}. \tag{25}$$

对第 4 项，由 ε_0-Lipschitz 条件和归纳法假定(19)式有

$$2\Delta t\{\langle Q_1(u^{n+1}) - Q_1(U^n),\ \xi^{n+1}\rangle + \langle Q_3(v^{n+1}) - Q_3(V^n),\ \zeta^{n+1}\rangle\}$$

$$\leqslant 2\Delta t\{(\Delta t)^2 + \|\xi^n\|^2 + \|\zeta^n\|^2\}. \tag{26}$$

对第 5 项有

$$2\Delta t\{\langle\varepsilon^{n+1}_1,\ \xi^{n+1}\rangle + \langle\varepsilon^{n+1}_3,\ \zeta^{n+1}\rangle + (\varepsilon^{n+1}_2,\ \omega^{n+1})\}$$

$$\leqslant M\Delta t\{(\Delta t)^2 + h^4 + \|\xi^{n+1}\|^2 + \|\zeta^{n+1}\|^2 + \|\omega^{n+1}\|^2\}, \tag{27}$$

对误差方程(22)，应用估计(23) ~ (27)式，当 ε, Δt 足够小时，整理可得

$$\{\|\hat{\Phi}^{1/2}_1\xi^{n+1}\|^2 + \|\hat{\Phi}^{1/2}_3\zeta^{n+1}\|^2 + \|\Phi^{1/2}_2\omega^{n+1}\|^2\}$$

$$-\{\|\hat{\Phi}^{1/2}_1\xi^n\|^2 + \|\hat{\Phi}^{1/2}_3\zeta^n\|^2 + \|\Phi^{1/2}_2\omega^n\|^2\}$$

$$+(\Delta t)^2\{\|\hat{\Phi}^{1/2}_1\xi^n_t\|^2 + \|\hat{\Phi}^{1/2}_3\zeta^n_t\|^2 + \|\hat{\Phi}^{1/2}_2\omega^n_t\|^2\}$$

$$+(\Delta t)^2\sum_{i,j=1}^{N}\{[\delta_{\bar{x}}\delta_{\bar{y}}\xi^{n+1}_{ij}]^2 + [\delta_{\bar{x}}\delta_{\bar{y}}\zeta^{n+1}_{ij}]^2\}h^2$$

$$+\Delta t\{[\|K^{n,1/2}_1\delta_{\bar{x}}\xi^{n+1}\|^2 + \|K^{n,1/2}_1\delta_{\bar{y}}\xi^{n+1}\|^2]$$

$$+ [\| K_3^{n,1/2}\delta_{\bar{x}}\zeta^{n+1}\|^2 + \| K_3^{n,1/2}\delta_{\bar{y}}\zeta^{n+1}\|^2] + \|\Phi_2^{1/2}\delta_{\bar{z}}\omega^{n+1}\|^2\}$$

$$\leqslant M\{(\Delta t)^2 + h^4 + \|\xi^{n+1}\|^2 + \|\zeta^{n+1}\|^2 + \|\omega^{n+1}\|^2\}\Delta t, \tag{28}$$

上式对时间 t 求和($0 \leqslant n \leqslant L$), 并注意到 $\xi^0 = \zeta^0 = \omega^0 = 0$, 故有

$$\{\|\hat{\Phi}_1^{1/2}\xi^{L+1}\|^2 + \|\hat{\Phi}_3^{1/2}\zeta^{L+1}\|^2 + \|\Phi_2^{1/2}\omega^{L+1}\|^2\} + \Delta t\sum_{n=0}^{L}\Big\{[\|\hat{\Phi}_1^{1/2}\xi_t^n\|^2$$

$$+ \|\hat{\Phi}_3^{1/2}\zeta_t^n\|^2 + \|\Phi_2^{1/2}\omega_t^n\|^2] + \sum_{i,j=1}^{N}[(\delta_{\bar{x}}\delta_{\bar{y}}\xi_{ij}^{n+1})^2 + (\delta_{\bar{x}}\delta_{\bar{y}}\zeta_{ij}^{n+1})^2 h^2]\Big\}\Delta t$$

$$+ \sum_{n=0}^{L}\{\|K_1^{n,1/2}\delta_{\bar{x}}\xi^{n+1}\|^2 + \|K_1^{n,1/2}\delta_{\bar{y}}\xi^{n+1}\|^2 + \|K_3^{n,1/2}\delta_{\bar{x}}\zeta^{n+1}\|^2$$

$$+ \|K_3^{n,1/2}\delta_{\bar{y}}\zeta^{n+1}\|^2 + \|\Phi_2^{1/2}\delta_{\bar{z}}\omega^{n+1}\|^2\}\Delta t$$

$$\leqslant M\Big\{\sum_{n=0}^{L}[\|\xi^{n+1}\|^2 + \|\zeta^{n+1}\|^2 + \|\omega^{n+1}\|^2]\Delta t + (\Delta t)^2 + h^4\Big\}, \tag{29}$$

应用 Gronwall 引理可得

$$\{\|\hat{\Phi}_1^{1/2}\xi^{L+1}\|^2 + \|\hat{\Phi}_3^{1/2}\zeta^{L+1}\|^2 + \|\Phi_2^{1/2}\omega^{L+1}\|^2\} + \Delta t\sum_{n=0}^{L}\{[\|\hat{\Phi}_1^{1/2}\xi_t^n\|^2$$

$$+ \|\hat{\Phi}_3^{1/2}\zeta_t^n\|^2 + \|\Phi_2^{1/2}\omega_t^n\|^2] + \sum_{i,j=1}^{N}[(\delta_{\bar{x}}\delta_{\bar{y}}\xi_{ij}^{n+1})^2 + (\delta_{\bar{x}}\delta_{\bar{y}}\zeta_{ij}^{n+1})^2 h^2]\}\Delta t$$

$$+ \sum_{n=0}^{L}\{[\|K_1^{n,1/2}\delta_{\bar{x}}\xi^{n+1}\|^2 + \|K_1^{n,1/2}\delta_{\bar{y}}\xi^{n+1}\|^2 + \|K_3^{n,1/2}\delta_{\bar{x}}\zeta^{n+1}\|^2$$

$$+ \|K_3^{n,1/2}\delta_{\bar{y}}\zeta^{n+1}\|^2 + \|\Phi_2^{1/2}\delta_{\bar{z}}\omega^{n+1}\|^2]\}\Delta t \leqslant M\{(\Delta t)^2 + h^4\}, \tag{30}$$

定理1 假定问题(1)~(4)的精确解满足光滑性条件: $\dfrac{\partial^2 u}{\partial t^2}, \dfrac{\partial^2 v}{\partial t^2} \in L^\infty(L^\infty(\Omega_1))$, $u, v \in L^\infty(W^{4,\infty}(\Omega_1)) \cap W^{1,\infty}(W^{1,\infty}(\Omega_1))$, $\dfrac{\partial^2 w}{\partial t^2} \in L^\infty(L^\infty(\Omega))$, $w \in L^\infty(W^{4,\infty}(\Omega_1))$. 采用迎风分数步长差分格式Ⅰ的(7)、(8)和(10)式逐层计算,则下述误差估计式成立:

$$\|u - U\|_{\bar{L}^\infty(J;\, l^2)} + \|v - V\|_{\bar{L}^\infty(J;\, l^2)} + \|w - W\|_{\bar{L}^\infty(J;\, l^2)} + \|u - U\|_{\bar{L}^2(J;\, h^1)}$$

$$+ \|v - V\|_{\bar{L}^2(J;\, h^1)} + \|w - W\|_{\bar{L}^2(J;\, h^1)} \leqslant M\{\Delta t + h^2\}, \tag{31}$$

此处 $\|g\|_{\bar{L}^\infty(J;\,X)} = \sup\limits_{n\Delta t \leqslant T}\|f^n\|_X$, $\|g\|_{\bar{L}^2(J;\,X)} = \sup\limits_{L\Delta t \leqslant T}\Big\{\sum\limits_{n=0}^{L}\|g^n\|_X^2\Delta t\Big\}^{1/2}$, M 依赖于函数 u, v, w 及其导函数.

下面讨论格式Ⅱ的收敛性分析. 类似于格式Ⅰ可以建立等价于(7a′)~(7d′)的差分方程:

$$\hat{\Phi}_{1,ij}\frac{U_{ij}^{n+1} - U_{ij}^n}{\Delta t} - \left\{\left(1 + \frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}}) + \left(1 + \frac{h}{2}\frac{|a_{2,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}})\right\}U_{ij}^{n+1}$$

$$+ \delta_{a_1^n,x}U_{ij}^{n+1} + \delta_{a_2^n,y}U_{ij}^{n+1} + \Delta t\left(1 + \frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x\left(K_1^n\delta_{\bar{x}}\left(\hat{\Phi}_1^{-1}\left(1 + \frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\right.\right.$$

$$\left.\left.\times\,\delta_y(K_1^n\delta_{\bar{y}}U^{n+1})\right)\right)_{ij} - \Delta t\left\{\left(1 + \frac{h}{2}\frac{|a_{1,ij}^n|}{K_{1,ij}^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}}(\hat{\Phi}_1^{-1}(\delta_{a_2^n,y}U^{n+1})))_{ij}\right.$$

$$+\delta_{a_1^n,x}\left(\hat{\Phi}_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}}U^{n+1})\right)_{ij}-\delta_{a_1^n,x}(\hat{\Phi}_1^{-1}\delta_{a_2^n,y}U^{n+1})_{ij}\Big\}$$

$$=-K_{2,ij,N-1/2}^n\delta_{\bar{z}}W_{ij,N}^{n+1}+Q(U_{ij}^{n+1}),\quad 1\leqslant i,j\leqslant N-1, \tag{32a}$$

$$U_{ij}^{n+1}=0,\quad (x_i,\ y_j)\in\partial\Omega_1, \tag{32b}$$

等价于(10a′)～(10d′)式的差分方程：

$$\hat{\Phi}_{3,ij}\frac{V_{ij}^{n+1}-V_{ij}^n}{\Delta t}-\left\{\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}})+\left(1+\frac{h}{2}\frac{|b_{2,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar{y}})\right\}V_{ij}^{n+1}$$

$$+\delta_{b_1^n,x}V_{ij}^{n+1}+\delta_{b_2^n,y}V_{ij}^{n+1}+\Delta t\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x\left(K_3^n\delta_{\bar{x}}\left(\hat{\Phi}_3^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\right.\right.$$

$$\left.\left.\times\,\delta_y(K_3^n\delta_{\bar{y}}V^{n+1})\right)\right)_{ij}-\Delta t\left\{\left(1+\frac{h}{2}\frac{|b_{1,ij}^n|}{K_{3,ij}^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}}(\hat{\Phi}_1^{-1}(\delta_{b_2^n,y}V^{n+1})))_{ij}\right.$$

$$\left.+\delta_{b_1^n,x}\left(\hat{\Phi}_1^{-1}\left(1+\frac{h}{2}\frac{|b_2^n|}{K_3^n}\right)^{-1}\delta_y(K_3^n\delta_{\bar{y}}V^{n+1})\right)_{ij}-\delta_{b_1^n,x}(\hat{\Phi}_3^{-1}\delta_{b_2^n,y}V^{n+1})_{ij}\right\}$$

$$=K_{2,ij,1/2}^n\delta_zW_{ij,0}^{n+1}+Q(V_{ij}^{n+1}),\quad 1\leqslant i,j\leqslant N-1, \tag{33a}$$

$$V_{ij}^{n+1}=0,\ (x_i,\ y_j)\in\partial\Omega_1, \tag{33b}$$

方程(8′)此时写为

$$\Phi_{2,ijk}\frac{W_{ijk}^{n+1}-W_{ijk}^n}{\Delta t}=\delta_z(K_2^n\delta_{\bar{z}}W^{n+1})_{ijk},\ 1\leqslant k\leqslant N-1, \tag{34a}$$

内边界条件：

$$W_{ij,N}^{n+1}=U_{ij}^{n+1},\quad W_{ij,0}^{n+1}=V_{ij}^{n+1},\quad (x_i,\ y_j)\in\Omega_1, \tag{34b}$$

从(32)，(33)和(34)式能够得到误差方程和下述误差估计：

$$\{\|\hat{\Phi}_1^{1/2}\xi^{n+1}\|^2+\|\hat{\Phi}_3^{1/2}\zeta^{n+1}\|^2+\|\Phi_2^{1/2}\omega^{n+1}\|^2\}$$

$$-\{\|\hat{\Phi}_1^{1/2}\xi^n\|^2+\|\hat{\Phi}_3^{1/2}\zeta^n\|^2+\|\Phi_2^{1/2}\omega^n\|^2\}$$

$$+(\Delta t)^2\{\|\hat{\Phi}_1^{1/2}\xi_t^n\|^2+\|\hat{\Phi}_3^{1/2}\zeta_t^n\|^2+\|\Phi_2^{1/2}\omega_t^n\|^2\}+2\Delta t\Big\{\big[\|K_1^{n,1/2}\delta_{\bar{x}}\xi^{n+1}\|^2$$

$$+\|K_1^{n,1/2}\delta_{\bar{y}}\xi^{n+1}\|^2\big]+\big[\|K_3^{n,1/2}\delta_{\bar{x}}\zeta^{n+1}\|^2+\|K_3^{n,1/2}\delta_{\bar{y}}\zeta^{n+1}\|^2\big]+\|\Phi_2^{1/2}\delta_{\bar{z}}\omega^{n+1}\|^2\Big\}$$

$$=-2\Delta t\left\{\langle\delta_{a_1^n,x}\xi^{n+1}+\delta_{a_2^n,y}\xi^{n+1},\ \xi^{n+1}\rangle+\cdots\right\}$$

$$-2(\Delta t)^2\left\{\left\langle\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_x\left(K_1^n\delta_{\bar{x}}\left(\hat{\Phi}_1^{-1}\left(1+\frac{h}{2}\frac{|a_2^n|}{K_1^n}\right)^{-1}\delta_y(K_1^n\delta_{\bar{y}}\xi^{n+1})\right)\right),\ \xi^{n+1}\right\rangle+\cdots\right\}$$

$$+2(\Delta t)^2\left\{\left\langle\left(1+\frac{h}{2}\frac{|a_1^n|}{K_1^n}\right)^{-1}\delta_x(K_1^n\delta_{\bar{x}}(\hat{\Phi}_1^{-1}\delta_{a_2^n,y}\xi^{n+1}))+\cdots,\ \xi^{n+1}\right\rangle\right\}$$

$$+\left\langle\left(1+\frac{h}{2}\frac{|b_1^n|}{K_3^n}\right)^{-1}\delta_x(K_3^n\delta_{\bar{x}}(\hat{\Phi}_3^{-1}\delta_{b_2^n,y}\zeta^{n+1}))+\cdots,\zeta^{n+1}\right\rangle$$

$$+2\Delta t\{\langle Q(u^{n+1})-Q(U^{n+1}),\ \xi^{n+1}\rangle+\langle Q_3(v^{n+1})-Q_3(V^{n+1}),\ \zeta^{n+1}\rangle\}$$

$$+2\Delta t\{\langle\varepsilon_1^{n+1},\xi^{n+1}\rangle+\langle\varepsilon_3^{n+1},\xi^{n+1}\rangle+(\varepsilon_2^{n+1},\ \omega^{n+1})\}, \tag{35}$$

最后，同样可以得到相应的误差估计并建立下述收敛性定理：

定理 2 假定问题(1)~(4)的精确解满足光滑性条件：$\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 v}{\partial t^2} \in L^\infty(L^\infty(\Omega_1))$，$u, v \in L^\infty(W^{4,\infty}(\Omega_1)) \cap w^{1,\infty}(W^{1,\infty}(\Omega_1))$，$\frac{\partial^2 w}{\partial t^2} \in L^\infty(L^\infty(\Omega))$，$w \in L^\infty(W^{4,\infty}(\Omega))$. 采用迎风分数步差分格式Ⅱ的(7′)，(8′)，(10′)式逐层计算，则下述误差估计式成立：

$$\|u-U\|_{\bar{L}^\infty(J;\,l^2)} + \|v-V\|_{\bar{L}^\infty(J;\,l^2)} + \|w-W\|_{\bar{L}^\infty(J;\,l^2)} + \|u-U\|_{\bar{L}^2(J;\,h^1)}$$
$$+ \|v-V\|_{\bar{L}^2(J;\,h^1)} + \|w-W\|_{\bar{L}^2(J;\,h^1)} \leqslant M\{\Delta t + h^2\}, \tag{36}$$

3 一阶迎风分数步长差分格式及其收敛性分析

对于一般问题的非高精度计算，通常可采用简便的一阶分数步长差分格式.

3.1 迎风分数步长格式Ⅲ

一阶迎风分数步长差分格式为：

$$(\hat{\Phi}_1 - \Delta t\delta_x(K_1^n\delta_{\bar{x}}) + \Delta t\delta_{a_1^n,x})U_{ij}^{n+1/2}$$
$$= \hat{\Phi}_{1,ij}U_{ij}^n + \Delta t\{-K_{2,ij,N-1/2}^n\delta_{\bar{z}}W_{ij,N}^n + Q(x_i, y_j, t^n, U_{ij}^n)\},\quad i_1(j) < i < i_2(j), \tag{37a}$$

$$U_{ij}^{n+1/2} = 0,\quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{37b}$$

$$(\hat{\Phi}_1 - \Delta t\delta_y(K_1^n\delta_{\bar{y}}) + \Delta t\delta_{a_2^n,y})U_{ij}^{n+1/2} = \hat{\Phi}_{1,ij}U_{ij}^{n+1/2},\quad j_1(i) < j < j_2(i), \tag{37c}$$

$$U_{ij}^{n+1} = 0,\quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{37d}$$

此处，$\delta_{a_1^n,x}u_{ij} = a_{1,ij}^n[H(a_{1,ij}^n)\delta_{\bar{x}} + (1 - H(a_{1,ij}^n))\delta_x]u_{ij}$，$\delta_{a_2^n,y}u_{ij} = a_{2,ij}^n[H(a_{2,ij}^n)\delta_{\bar{y}} + (1 - H(a_{2,ij}^n))\delta_y]u_{ij}$，$H(z) = \begin{cases}1, & z \geqslant 0,\\ 0, & z < 0.\end{cases}$

$$\Phi_{2,ijk}\frac{W_{ijk}^{n+1} - W_{ijk}^n}{\Delta t} = \delta_z(K_2^n\delta_{\bar{z}}W^n)_{ijk},\quad 0 < k < N,\ (i, j) \in \Omega_{1,h}, \tag{38}$$

$$(\hat{\Phi}_3 - \Delta t\delta_x(K_3^n\delta_{\bar{x}}) + \Delta t\delta_{b_1^n,x})V_{ij}^{n+1/2}$$
$$= \hat{\Phi}_{3,ij}U_{ij}^n + \Delta t\{K_{2,ij,1/2}^n\delta_zW_{ij,0}^n + Q(x_i, y_j, t^n, V_{ij}^n)\},\quad i_1(j) < i < i_2(j), \tag{39a}$$

$$V_{ij}^{n+1/2} = 0,\quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{39b}$$

$$(\hat{\Phi}_3 - \Delta t\delta_y(K_3^n\delta_{\bar{y}}) + \Delta t\delta_{b_2^n,y})V_{ij}^{n+1} = \hat{\Phi}_{3,ij}V_{ij}^{n+1/2},\quad j_1(i) < j < j_2(i), \tag{39c}$$

$$V_{ij}^{n+1} = 0,\quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{39d}$$

此处 $\delta_{b_1^n,y}v_{ij} = b_{1,ij}^n[H(b_{1,ij}^n)\delta_{\bar{x}} + (1 - H(b_{1,ij}^n))\delta_x]v_{ij}$，$\delta_{b_2^n,y}v_{ij} = b_{2,ij}^n[H(b_{2,ij}^n)\delta_{\bar{y}} + (1 - H(b_{2,ij}^n))\delta_y]v_{ij}$，计算过程和格式Ⅰ是类似的.

3.2 迎风分数步长差分格式Ⅳ

一阶迎风分数步长差分格式为：

$$(\hat{\Phi}_1 - \Delta t\delta_x(K_1^n\delta_{\bar{x}}) + \Delta t\delta_{a_1^n,x})U_{ij}^{n+1/2}$$
$$= \hat{\Phi}_{1,ij}U_{ij}^n + \Delta t\{-K_{2,ij,N-1/2}^n\delta_{\bar{z}}W_{ij,N}^{n+1} + Q(x_i, y_j, t^{n+1}, U_{ij}^{n+1})\}, i_1(j) < i < i_2(j), \tag{37a′}$$

$$U_{ij}^{n+1/2} = 0,\quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{37b′}$$

$$(\hat{\Phi}_1 - \Delta t\delta_y(K_1^n\delta_{\bar{y}}) + \Delta t\delta_{a_2^n, y})U_{ij}^{n+1/2} = \hat{\Phi}_{1,ij}U_{ij}^{n+1/2}, \quad j_1(i) < j < j_2(i), \tag{37c'}$$

$$U_{ij}^{n+1} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{37d'}$$

$$\Phi_{2,ijk}\frac{W_{ijk}^{n+1} - W_{ijk}^n}{\Delta t} = \delta_z(K_2^n\delta_{\bar{z}}W^{n+1})_{ijk}, \quad 0 < k < N, (i, j) \in \Omega_{1,h}, \tag{38'}$$

$$(\hat{\Phi}_3 - \Delta t\delta_x(K_3^n\delta_{\bar{x}}) + \Delta t\delta_{b_1^n, x})V_{ij}^{n+1/2} = \hat{\Phi}_{3,ij}V_{ij}^n + \Delta t\{K_{2,ij,1/2}^n\delta_z W_{ij,0}^{n+1} + Q(x_i, y_j, t^n, V_{ij}^n)\}, i_1(j) < i < i_2(j), \tag{39a'}$$

$$V_{ij}^{n+1/2} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{39b'}$$

$$(\hat{\Phi}_3 - \Delta t\delta_y(K_3^n\delta_{\bar{y}}) + \Delta t\delta_{b_2^n, y})V_{ij}^{n+1} = \hat{\Phi}_{3,ij}V_{ij}^{n+1/2}, \quad j_1(i) < j < j_2(i), \tag{39c'}$$

$$V_{ij}^{n+1} = 0, \quad (x_i, y_j) \in \partial\Omega_{1,h}, \tag{39d'}$$

计算过程和格式Ⅱ是类似的.

3.3 收敛性定理

定理 3 假定问题(1)～(4)的精确解满足光滑性条件：$\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 v}{\partial t^2} \in L^\infty(L^\infty(\Omega_1))$, $u, v \in L^\infty(W^{4,\infty}(\Omega_1))$, $\frac{\partial^2 w}{\partial t^2} \in L^\infty(L^\infty(\Omega))$, $w \in L^\infty(W^{4,\infty}(\Omega))$. 采用迎风分数步差分格式Ⅲ，Ⅳ逐层计算，则下述误差估计式成立：

$$\| u - U \|_{\bar{L}^\infty(J; l^2)} + \| v - V \|_{\bar{L}^\infty(J; l^2)} + \| w - W \|_{\bar{L}^\infty(J; l^2)} + \| u - U \|_{\bar{L}^2(J; h^1)} + \| v - V \|_{\bar{L}^2(J; h^1)} + \| w - W \|_{\bar{L}^2(J; h^1)} \leqslant M\{\Delta t + h\}. \tag{40}$$

4 应用

迎风分数步长差分格式除在多层地下渗流的非稳定流计算中得到应用外，最近也应用到多层油资源运移聚集的软件系统和胜利油田油资源评估中[1)]. 问题的数学模型为

$$\nabla\cdot\left(K_1\frac{k_{ro}}{\mu_o}\nabla\psi_o\right) + B_0 q - \left(K_2\frac{k_{ro}}{\mu_o}\frac{\partial\psi_o}{\partial z}\right)_{z=H} = -\Phi_1\dot{s}\left(\frac{\partial\psi_o}{\partial t} - \frac{\partial\psi_w}{\partial t}\right), \quad X = (x, y)^{\mathrm{T}} \in \Omega_1, \quad t \in J = (0, T], \tag{41a}$$

$$\nabla\cdot\left(K_1\frac{k_{rw}}{\mu_w}\nabla\psi_w\right) + B_w q - \left(K_2\frac{k_{rw}}{\mu_w}\frac{\partial\psi_w}{\partial z}\right)_{z=H} = \Phi_1\dot{s}\left(\frac{\partial\psi_o}{\partial t} - \frac{\partial\psi_w}{\partial t}\right), \quad X \in \Omega_1, t \in J, \tag{41b}$$

$$\frac{\partial}{\partial z}\left(K_2\frac{k_{ro}}{\mu_o}\frac{\partial\psi_o}{\partial z}\right) = -\Phi_2\dot{s}\left(\frac{\partial\psi_o}{\partial t} - \frac{\partial\psi_w}{\partial t}\right), \quad X = (x, y, z)^{\mathrm{T}} \in \Omega, t \in J, \tag{42a}$$

$$\frac{\partial}{\partial z}\left(K_2\frac{k_{rw}}{\mu_w}\frac{\partial\psi_w}{\partial z}\right) = \Phi_2\dot{s}\left(\frac{\partial\psi_o}{\partial t} - \frac{\partial\psi_w}{\partial t}\right), \quad X \in \Omega, t \in J, \tag{42b}$$

$$\nabla\cdot\left(K_3\frac{k_{ro}}{\mu_o}\nabla\psi_o\right) + B_0 q + \left(K_2\frac{k_{ro}}{\mu_o}\frac{\partial\psi_o}{\partial z}\right)_{z=0} = -\Phi_3\dot{s}\left(\frac{\partial\psi_o}{\partial t} - \frac{\partial\psi_w}{\partial t}\right), \quad X = (x, y)^{\mathrm{T}} \in \Omega_1, t \in J. \tag{43a}$$

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$$\nabla \cdot \left(K_3 \frac{k_{rw}}{\mu_w} \nabla \psi_w \right) + B_w q + \left(K_2 \frac{k_{rw}}{\mu_w} \frac{\partial \psi_w}{\partial z} \right)_{z=0} = \Phi_3 \dot{s} \left(\frac{\partial \psi_o}{\partial t} - \frac{\partial \psi_w}{\partial t} \right), X \in \Omega_1,\ t \in J. \tag{43b}$$

应用本文的计算方法,对胜利油田运移聚集的实际问题进行了数值模拟，结果符合油水运移聚集规律,可清晰地看到油在下层运移聚集的情况,并由中间层进一步运移到上层,最后形成油藏的全过程,其成藏位置基本上和实际油田的位置一致.

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$$- 2(\Delta t)^2\Big\{\Big\langle\Big(1 + \frac{h}{2}\frac{|a_1^n|}{K_1^n}\Big)^{-1}\delta_x\Big(K_1^n\delta_{\bar x}\Big(\hat\Phi_1^{-1}\Big(1 + \frac{h}{2}\frac{|a_2^n|}{K_1^n}\Big)^{-1}\delta_y(K_1^n\delta_{\bar y}\xi^{n+1})\Big)\Big),\ \xi^{n+1}\Big\rangle$$

$$+ \Big\langle\Big(1 + \frac{h}{2}\frac{|b_1^n|}{K_3^n}\Big)^{-1}\delta_x\Big(K_3^n\delta_{\bar x}\Big(\hat\Phi_3^{-1}\Big(1 + \frac{h}{2}\frac{|b_2^n|}{K_3^n}\Big)^{-1}\delta_y(K_3^n\delta_{\bar y}\xi^{n+1})\Big)\Big),\ \zeta^{n+1}\Big\rangle\Big\}$$

$$+ 2(\Delta t)^2\Big\{\Big\langle\Big(1 + \frac{h}{2}\frac{|a_1^n|}{K_1^n}\Big)^{-1}\delta_x(K_1^n\delta_{\bar x}(\hat\Phi_1^{-1}\delta_{a_2^n,y}\xi^{n+1}))$$

$$+ \delta_{a_1^n,x}\Big(\hat\Phi_1^{-1}\Big(1 + \frac{h}{2}\frac{|a_2^n|}{K_1^n}\Big)^{-1}\delta_y(K_1^n\delta_{\bar y}\xi^{n+1})\Big) - \delta_{a_1^n,x}(\hat\Phi_1^{-1}\delta_{a_2^n,y}\xi^{n+1}),\ \xi^{n+1}\Big\rangle$$

$$+ \Big\langle\Big(1 + \frac{h}{2}\frac{|b_1^n|}{K_3^n}\Big)^{-1}\delta_x(K_3^n\delta_{\bar x}(\hat\Phi_3^{-1}\delta_{b_2^n,y}\zeta^{n+1}))$$

$$+ \delta_{b_1^n,x}\Big(\hat\Phi_3^{-1}\Big(1 + \frac{h}{2}\frac{|b_2^n|}{K_3^n}\Big)^{-1}\delta_y(K_1^n\delta_{\bar y}\zeta^{n+1})\Big) - \delta_{b_1^n,x}(\hat\Phi_3^{-1}\delta_{b_2^n,y}\zeta^{n+1}),\ \zeta^{n+1}\Big\rangle\Big\}$$

$$+ 2\Delta t\{\langle Q_1(u^{n+1}) - Q_1(U^n),\ \xi^{n+1}\rangle + \langle Q_3(v^{n+1}) - Q_3(V^n),\ \zeta^{n+1}\rangle\}$$

$$+ 2\Delta t\{\langle\varepsilon_1^{n+1},\xi^{n+1}\rangle + \langle\varepsilon_3^{n+1},\zeta^{n+1}\rangle + (\varepsilon_2^{n+1},\omega^{n+1})\} - M\{\|\xi^{n+1}\|^2 + \|\zeta^{n+1}\|^2\}\Delta t, \quad (22)$$

依次分析(22)式右端诸项,对第 1 项,

$$- 2\Delta t\langle\delta_{a_1^n,x}\xi^{n+1} + \delta_{a_2^n,y}\xi^{n+1},\ \xi^{n+1}\rangle \leqslant \varepsilon\{\|\delta_{\bar x}\xi^{n+1}\|^2 + \|\delta_{\bar y}\xi^{n+1}\|^2\}\Delta t + M\|\xi^{n+1}\|^2\Delta t, \quad (23)$$

对于第 2 项尽管 $-\delta_x(K_1^n\delta_{\bar x})$, $-\delta_y(K_1^n\delta_{\bar y})$, $\cdots$ 是自共轭、正定算子,空间区域为正方形,但它们的乘积一般是不可交换的,记 $R_{a_1}^n = \Big(1 + \frac{h}{2}\frac{|a_1^n|}{K_1^n}\Big)^{-1}$, $R_{a_2}^n = \Big(1 + \frac{h}{2}\frac{|a_2^n|}{K_2^n}\Big)^{-1}$, 利用 $\delta_x\delta_y = \delta_y\delta_x$, $\delta_x\delta_{\bar y} = \delta_{\bar y}\delta_x$, $\delta_{\bar x}\delta_y = \delta_y\delta_{\bar x}$, $\cdots$ 有

$$- 2(\Delta t)^2\langle R_{a_1}^n\delta_x(K_1^n\delta_{\bar x}(\hat\Phi_1^{-1}R_{a_2}^n\delta_y(K_1^n\delta_{\bar y}\xi^{n+1}))),\ \xi^{n+1}\rangle$$

$$= - 2(\Delta t)^2\{\langle K_1^n\delta_{\bar x}\delta_y\xi^{n+1} + \delta_{\bar x}K_1\delta_{\bar y}\xi^{n+1},\ \hat\Phi_1^{-1}R_{a_2}^nK_1^n\delta_{\bar x}\delta_{\bar y}\xi^{n+1} + \delta_{\bar y}(\hat\Phi_1^{-1}R_{a_2}^nK_1^n)\delta_{\bar x}(R_{a_1}^n\xi^{n+1})\rangle$$

$$+ \langle K_1^n\delta_{\bar y}\xi^{n+1},\ \delta_{\bar x}(\hat\Phi_1^{-1}R_{a_2}^n)K_1\delta_x\delta_y(R_{a_1}^n\xi^{n+1}) + \delta_{\bar y}(\delta_{\bar x}(\Phi_1^{-1}R_{a_2}^n)K_1^n)\delta_{\bar x}(R_{a_1}^n\xi^{n+1})\rangle\}$$

$$= - 2(\Delta t)^2\sum_{i,j=1}^{N}\{K_{1,i,j-1/2}^nK_{1,i-1/2,j}^n\hat\Phi_{ij}^{-1}R_{a,ij}^n[\delta_{\bar x}\delta_{\bar y}\xi_{ij}^{n+1}]^2$$

$$+ [K_{1,i,j-1/2}^n\delta_{\bar y}(K_{1,i-1/2,j}\hat\Phi_{1,ij}^{-1}R_{a_2,ij}^n)\cdot\delta_{\bar x}(R_{a_1,ij}^n\xi_{ij}^{n+1}) + K_{1,i-1/2,j}^n\hat\Phi_{1,ij}^{-1}R_{a_2,ij}^n\,\delta_{\bar x}K_{1,i,j-1/2}^n\delta_{\bar y}\xi_{ij}^{n+1}$$

$$+ K_{1,i,j-1/2}^n\cdot K_{1,i-1/2,j}^n\,\delta_{\bar x}(\hat\Phi_{1,ij}^{-1}R_{a_2,ij}^n)\delta_{\bar y}\xi_{ij}^{n+1}]\delta_{\bar x}\delta_{\bar y}\xi_{ij}^{n+1} + [\delta_{\bar x}K_{1,i,j-1/2}^n\delta_{\bar y}(\hat\Phi_{1,ij}^{-1}$$

$$\times R_{a_2,ij}^nK_{1,i-1/2,j})R_{a_1,ij}^n + K_{1,i,j-1/2}^n\delta_{\bar y}(\delta_{\bar x}(\hat\Phi_{1,ij}^{-1}R_{a_2,ij}^n)K_{1,i-1/2,j}^n)R_{a_1,ij}^n]\delta_{\bar x}\xi_{ij}^{n+1}\delta_{\bar y}\xi_{ij}^{n+1}$$

$$+ K_{1,i,j-1/2,j}^nK_{1,i-1/2,j}^nR_{a_1,ij}^n\cdot\xi_{ij}^{n+1}\cdot\delta_{\bar y}\xi_{ij}^{n+1} + K_{1,i,j-1/2}^n\delta_{\bar y}$$

$$\times(\delta_{\bar x}(\hat\Phi_{1,ij}^{-1}\cdot R_{a_2,ij}^n)K_{1,i-1/2,j}^n)\delta_{\bar x}R_{a_1,ij}^n\cdot\xi_{ij}^{n+1}\cdot\delta_{\bar y}\xi_{ij}^{n+1}\}h^2.$$

对上述表达式的前两项,应用 K_1, $\hat\Phi_1^{-1}$, $R_{a_2}^n$ 的正定性,可分离出高阶差商项 $\delta_{\bar x}\delta_{\bar y}\xi_{ij}^{n+1}$. 现利用 Cauchy 不等式可得

$$- 2(\Delta t)^2\sum_{i,j=1}^{N}\{K_{1,i,j-1/2}^nK_{1,i-1/2,j}^n\,\hat\Phi_{1,ij}^{-1}R_{a_2,ij}^n[\delta_{\bar x}\delta_{\bar y}\xi_{ij}^{n+1}]^2$$