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The growth of H-harmonic functions on the Heisenberg group

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Abstract We discuss the relationship between the frequency and the growth of H-harmonic functions on the Heisenberg group. Precisely, we prove that an H-harmonic function must be a polynomial if its frequency is globally bounded. Moreover, we show that a class of H-harmonic functions are homogeneous polynomials provided that the frequency of such a function is equal to some constant.

Keywords Heisenberg group, H-harmonic function, frequency

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1 Introduction

It is well known that a harmonic function on $\mathbb{R}^n$ is a homogeneous polynomial if and only if its frequency is equal to a constant. For a harmonic function u in the unit ball $B_1 \subset \mathbb{R}^n$, the frequency of u on B_r is defined by

$$N(r) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}, \quad \text{for any } r < 1.$$

In this paper, we study the growth of H-harmonic functions on the Heisenberg group.

The Heisenberg group $\mathbb{H}_n$ is a nilpotent Lie group of step two whose underlying manifold is $\mathbb{R}^{2n} \times \mathbb{R}$ with coordinates

$$(z,t) = (x,y,t) = (x_1,\dots,x_n,y_1,\dots,y_n,t)$$

and whose group action $\circ$ is given by

$$(x_0,y_0,t_0) \circ (x,y,t) = \left(x+x_0, y+y_0, t+t_0+2\sum_{i=1}^{n}(x_i y_{0_i} - y_i x_{0_i})\right). \tag{1.1}$$

It is easy to check that (1.1) does indeed make $\mathbb{R}^{2n} \times \mathbb{R}$ into a group whose identity is the origin $(0,0)$, and where the inverse is given by $(z,t)^{-1} = (-z,-t)$. Let us denote by δ_λ the Heisenberg group dilation

$$\delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0. \tag{1.2}$$

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Then $\mathbb{H}_n = (\mathbb{R}^{2n+1}, \circ, \delta_\lambda)$ is a homogeneous group. We denote $Q = 2n + 2$ and call it the homogeneous dimension of $\mathbb{H}_n$. A basis of the Lie algebra of $\mathbb{H}_n$ is given by

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad X_{n+i} = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \ldots, n, \quad T = \frac{\partial}{\partial t}. \tag{1.3}$$

From (1.3), it is easy to check that X_i and X_{n+j} satisfy

$$[X_i, X_{n+j}] = -4T\delta_{ij}, \quad [X_i, X_j] = [X_{n+i}, X_{n+j}] = 0, \quad i, j = 1, \ldots, n.$$

Therefore, the vector fields X_i, X_{n+i} $(i = 1, \ldots, n)$ and their first order commutators span the whole Lie algebra. The horizontal gradient of a function u is defined as

$$\nabla_H u = (X_1 u, \ldots, X_n u, X_{n+1} u, \ldots, X_{2n} u).$$

Given a function $u : \mathbb{H}_n \to \mathbb{R}$, we say that u is homogeneous of H-degree $k \in \mathbb{Z}$ if for every $\lambda > 0$,

$$u \circ \delta_\lambda = \lambda^k u. \tag{1.4}$$

For $(z, t) \in \mathbb{H}_n$, we recall the Heisenberg norm

$$\rho(z, t) = \left(\left(\sum_{i=1}^{n} (x_i^2 + y_i^2) \right)^2 + t^2 \right)^{\frac{1}{4}} \equiv (|z|^4 + t^2)^{\frac{1}{4}}. \tag{1.5}$$

Obviously, we have

$$\rho((z, t)^{-1}) = \rho(z, t) \quad \text{and} \quad \rho(\delta_\lambda(z, t)) = \lambda \rho(z, t).$$

In addition, ρ satisfies the triangle inequality [6]

$$\rho((z_1, t_1) \circ (z_2, t_2)) \leqslant \rho(z_1, t_1) + \rho(z_2, t_2).$$

The associated distance between (z, t) and (z_0, t_0) is defined by

$$d(z, t; z_0, t_0) = \rho((z_0, t_0)^{-1} \circ (z, t)).$$

It is clear that $d(z, t; z_0, t_0)$ satisfies the symmetric property: $d(z, t; z_0, t_0) = d(z_0, t_0; z, t)$.

In the sequel we let

$$B_r = \{(z, t) \in \mathbb{H}_n \mid \rho(z, t) < r\}, \quad \partial B_r = \{(z, t) \in \mathbb{H}_n \mid \rho(z, t) = r\}, \tag{1.6}$$

and call these sets a Heisenberg-ball and a Heisenberg-sphere centred at the origin with radius r, respectively. Since $\rho \in C^\infty(\mathbb{H}_n \backslash \{0\})$, the outward unit normal on ∂B_r is $\overrightarrow{n} = |\nabla \rho|^{-1} \nabla \rho$, where $\nabla \rho$ means the Euclidean gradient of ρ.

Introducing the function

$$\psi(z, t) = |\nabla_H \rho(z, t)|^2 = \frac{|z|^2}{\rho(z, t)^2}, \tag{1.7}$$

we define

$$|B_r| = \int_{B_r} \psi(z, t) dH^{2n+1}, \tag{1.8}$$

where dH^{2n+1} denotes the $(2n + 1)$-dimensional Hausdorff measure in $\mathbb{R}^{2n+1}$. It is easy to check that there exists a constant $C_Q > 0$ depending only on Q such that

$$|B_r| = C_Q r^Q. \tag{1.9}$$

The sub-Laplacian on $\mathbb{H}_n$ is defined by

$$\Delta_H = -\sum_{i=1}^{2n} X_i^2, \tag{1.10}$$

which is an analog to the Laplacian $\Delta = -\sum_{i=1}^{N} \partial^2/\partial x_i^2$ on $\mathbb{R}^N$. Since the appearance of Hörmander's fundamental work [22], the study of properties of solutions to $\Delta_H u = 0$ has received increasing attention, see [4, 5, 9, 23, 27–29]. The sub-Laplacian arises in a wide range of applications, from several complex variables and CR geometry (see, for example, [9] and [31]) to control theory and financial mathematics (see, for example, [2]).

A solution u to $\Delta_H u = 0$ is called an H-harmonic function. Garofalo and Lanconelli [11] introduced a definition of the frequency $N(r)$ for H-harmonic functions (the special definition will be given in Section 2), and showed that if u is a homogeneous H-harmonic polynomial (see Definition 2.1) of H-degree k, then $N(r) = k$. On the other hand, a natural reverse question arises: If $N(r) =$ constant, does u reduce to be a homogeneous polynomial? To our knowledge, nothing is known for this question. One of our main theorems is to give a positive answer for a special class of H-harmonic functions. In the following, the constant C in distinct inequalities may be different. Our main results are as follows:

Theorem 1.1. *Let u be a nonconstant entire H-harmonic function on $\mathbb{H}_n$ satisfying*

$$\frac{r\int_{B_r} |\nabla_H u|^2}{\int_{\partial B_r} u^2 \frac{\psi}{|\nabla \rho|}} \leqslant N_0$$

for some positive constant N_0 and any $r > 0$. Then for any $R_0 > 1$,

$$\|u\|_{L^\infty(B_{R_0})} \leqslant C b^{2N_0+Q/2} \mathrm{e}^{1+Q/(2N_0)} N_0^{Q/2} (R_0)^{N_0} \|u\|_{L^\infty(B_1)},$$

where C is a positive constant depending only on the homogeneous dimension Q.

Using the asymptotic Liouville-type theorem for H-harmonic functions on $\mathbb{H}_n$ [3, Theorem 5.8.8 and Remark 5.8.10], Theorem 1.1 implies the following corollary:

Corollary 1.2. *Under the assumptions in Theorem* 1.1*, u is a polynomial with H-degree not exceeding* $[N_0]$ (*the integer part of* N_0).

Theorem 1.3. *Let u be an H-harmonic function in B_1 satisfying $\widetilde{T}u = \sum_{j=1}^{n}(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j})u = 0$. If there exists a positive constant $r_0 < 1$ such that for any $r \in (0, r_0)$, $N(r) \equiv N_0$ for some constant N_0, then N_0 is an integer, and u is a homogeneous polynomial of H-degree N_0 in B_{r_0}.*

The structure of the paper is as follows. In Section 2, we recall some facts about the sub-Laplacian and introduce the frequency of H-harmonic functions on $\mathbb{H}_n$. In Section 3, we prove Theorem 1.1 with the help of a Caccioppoli type inequality and a comparison result between integrals on different Heisenberg-balls by translations and scaling. In Section 4, we introduce the relationship between sub-Laplacian and Grushin operator, and prove Theorem 1.3 by using the orthogonality of Grushin-harmonic polynomials.

2 Frequency of H-harmonic functions

In this section, we collect some known facts about the sub-Laplacian (1.10) which will be useful later on, and introduce the frequency of H-harmonic functions on $\mathbb{H}_n$.

It is useful to represent sub-Laplacian Δ_H as a divergence form operator

$$\Delta_H = -\mathrm{div}(A(z)\nabla), \tag{2.1}$$

where

$$A(z) = \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 2y & -2x & 4|z|^2 \end{pmatrix}. \tag{2.2}$$

It is easy to check that the horizontal gradient ∇_H and the standard gradient ∇ in $\mathbb{R}^{2n+1}$ satisfy

$$\nabla \cdot A(z) \cdot \nabla = \nabla_H \cdot \nabla_H. \tag{2.3}$$

Let $\widetilde{T}$ and $\widetilde{X}$ be the vector fields,

$$\widetilde{T} = \sum_{j=1}^{n}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right), \quad \widetilde{X} = \sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right) + 2t\frac{\partial}{\partial t}. \tag{2.4}$$

A direct computation gives

$$\widetilde{X}\rho = \rho, \tag{2.5}$$

$$-\Delta_H\rho = \frac{Q-1}{\rho}\psi \quad \text{in } \mathbb{H}_n\backslash\{0\}, \tag{2.6}$$

$$A(z)\nabla\rho(z,t) = \frac{1}{\rho(z,t)}\left(\frac{|z|^2}{\rho(z,t)^2}\begin{pmatrix} x \\ y \\ 2t \end{pmatrix} + \frac{t}{\rho(z,t)^2}\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}\right). \tag{2.7}$$

Definition 2.1. A polynomial on $\mathbb{H}_n$ is a function which can be expressed as

$$P(x,y,t) = \sum_I a_I x^{\alpha_1}y^{\alpha_2}t^{\alpha_3},$$

where $I = (\alpha_1,\alpha_2,\alpha_3) = (\alpha_1^1,\ldots,\alpha_1^n;\alpha_2^1,\ldots,\alpha_2^n;\alpha_3)$ is a multi-index, a_I are real numbers, and

$$x^{\alpha_1}y^{\alpha_2}t^{\alpha_3} = x_1^{\alpha_1^1}\cdots x_n^{\alpha_1^n}y_1^{\alpha_2^1}\cdots y_n^{\alpha_2^n}t^{\alpha_3}.$$

The H-degree of a monomial $x^{\alpha_1}y^{\alpha_2}t^{\alpha_3}$ is given by the sum

$$|I|_{\mathbb{H}} = |\alpha_1| + |\alpha_2| + 2\alpha_3 = \alpha_1^1 + \cdots + \alpha_1^n + \alpha_2^1 + \cdots + \alpha_2^n + 2\alpha_3.$$

It is easy to check that, if u is a homogeneous polynomial of H-degree k, then $\widetilde{X}u = ku$.

Now, we introduce the frequency of H-harmonic functions on $\mathbb{H}_n$.

Definition 2.2. Let u be a solution to $\Delta_H u = 0$ in B_1, and $r < 1$. We define

$$H(r) = \int_{\partial B_r} u^2\frac{\psi}{|\nabla\rho|}dH^{2n}, \tag{2.8}$$

$$D(r) = \int_{B_r}|\nabla_H u|^2 dH^{2n+1}. \tag{2.9}$$

The generalized Almgren's frequency of u on B_r is defined by

$$N(r) = \frac{rD(r)}{H(r)}. \tag{2.10}$$

For harmonic functions on $\mathbb{R}^n$, the frequency was first introduced by Almgren [1], who proved its monotonicity with respect to r. There are many important applications of the frequency including doubling conditions and the control of growth of harmonic functions [21]. Garofalo and Lin [12, 13] proved unique continuation for elliptic operators by using the properties of frequency. Lin [25] gave estimates on Hausdorff measure of nodal sets of harmonic functions in terms of the frequency. Han et al. [20] and Han [19] obtained estimates on Hausdorff measure of singular sets of solutions to elliptic differential equations.

The corresponding frequency for H-harmonic functions on $\mathbb{H}_n$ as in (2.10) was introduced by Garofalo and Lanconelli in [11]. They showed that if u is a homogeneous H-harmonic polynomial of H-degree k, then $N(r) = k$.

By a result of Hörmander [22], Δ_H is C^∞ hypoelliptic. Folland [8] produced an explicit fundamental solution of the operator Δ_H with singularity at the origin

$$\Gamma(z,t) = \frac{C_Q}{\rho(z,t)^{Q-2}}.$$

Making use of the real analyticity (out of the origin) of the fundamental solution Γ, we obtain that Δ_H on $\mathbb{H}_n$ is real analytic hypoelliptic. For more information on the analytic hypoellipticity of sub-Laplacians, one can see [24] and monograph [3, pp. 280–287].

We derive a relationship between the value of frequency and the vanishing order. We begin with a definition of vanishing order of a function $u : \mathbb{H}_n \to \mathbb{R}$.

Definition 2.3. Suppose $u : \mathbb{H}_n \to \mathbb{R}$ is a smooth function,

$$I = (\alpha_1, \alpha_2, \alpha_3) = (\alpha_1^1, \ldots, \alpha_1^n; \alpha_2^1, \ldots, \alpha_2^n; \alpha_3)$$

is a multi-index, and k is a positive integer. The vanishing order of u at p is said to be k, if for any $|I|_{\mathbb{H}} < k$, there holds $X^I u(p) = 0$, and there exists $|I|_{\mathbb{H}} = k$, such that $X^I u(p) \neq 0$, where

$$X^I = X_1^{\alpha_1^1} \cdots X_n^{\alpha_1^n} X_{n+1}^{\alpha_2^1} \cdots X_{2n}^{\alpha_2^n} T^{\alpha_3}.$$

Proposition 2.4. *Let u be a non-constant H-harmonic function. Then $\lim_{r\to 0^+} N(r)$ equals to the vanishing order of u at* 0.

Proof. Let u be a non-constant H-harmonic function. Because of its analyticity, we write

$$u(z,t) = \sum_{j=0}^{\infty} P_j(z,t)$$

for (z,t) near the origin, where each P_j is a homogeneous polynomial of H-degree j, and the series converges absolutely near the origin.

We claim that each P_j is H-harmonic. In fact, P_0 and P_1 are H-harmonic obviously. For any fixed λ, let $(z,t) = \delta_\lambda(\widetilde{z}, \widetilde{t})$, we have

$$P_j(z,t) = P_j(\delta_\lambda(\widetilde{z}, \widetilde{t})) = \lambda^j P_j(\widetilde{z}, \widetilde{t}).$$

Therefore

$$0 = \Delta_H u = \sum_{j=0}^{\infty} \Delta_H P_j(z,t) = \sum_{j=2}^{\infty} \lambda^{j-2} \Delta_H P_j(\widetilde{z}, \widetilde{t}). \tag{2.11}$$

Let $\lambda \to 0$ in (2.11). We get $\Delta_H P_2(\widetilde{z}, \widetilde{t}) = 0$, i.e., P_2 is H-harmonic. Then

$$0 = \sum_{j=3}^{\infty} \lambda^{j-2} \Delta_H P_j(\widetilde{z}, \widetilde{t}). \tag{2.12}$$

Dividing λ in (2.12), and taking $\lambda \to 0$, we have $\Delta_H P_3(\widetilde{z}, \widetilde{t}) = 0$, i.e., P_3 is H-harmonic. Repeating the above argument, we have that each P_j $(j \geqslant 2)$ is H-harmonic.

Assume that the vanishing order of u at the origin is $k \in \mathbb{N}$. Then we have

$$u(z,t) = P_k(z,t) + \sum_{j=k+1}^{\infty} P_j(z,t).$$

Denote by $R(z,t) = \sum_{j=k+1}^{\infty} P_j(z,t)$, i.e., $u(z,t) = P_k(z,t) + R(z,t)$, where P_k is a non-zero homogeneous polynomial of H-degree k. Then we have (see [11, Proposition 4.1])

$$\frac{r\int_{B_r} |\nabla_H P_k|^2}{\int_{\partial B_r} P_k^2 \frac{\psi}{|\nabla\rho|}} = k.$$

Therefore,

$$\lim_{r\to 0^+} N(r) = \lim_{r\to 0^+} \frac{r\int_{B_r} |\nabla_H P_k + \nabla_H R|^2}{\int_{\partial B_r} (P_k + R)^2 \frac{\psi}{|\nabla\rho|}} = \lim_{r\to 0^+} \frac{r\int_{B_r} |\nabla_H P_k|^2}{\int_{\partial B_r} P_k^2 \frac{\psi}{|\nabla\rho|}} = k.$$

All terms involving R have a higher order of r and hence vanish as $r \to 0$. The proof is completed. □

In [11], Garofalo and Lanconelli proved a monotonicity property of the generalized frequency (2.10) and obtained the strong unique continuation property. Here we collect a lemma from [11] that will be useful later on.

Lemma 2.5. *Let u be an H-harmonic function in B_1. Then for a.e. $r \in (0,1)$,*

$$H'(r) = \frac{Q-1}{r}H(r) + 2D(r). \tag{2.13}$$

3 Proof of Theorem 1.1

In this section, our goal is to prove Theorem 1.1. We begin with a technical theorem, which will allow us to compare integrals of functions on different Heisenberg-balls by translations and scaling. The corresponding version on $\mathbb{R}^n$ is obvious. To be precise, we recall the following Lagrange mean value theorem on $\mathbb{H}_n$ [3, Theorem 20.3.1]: There exist absolute constants $C_0 > 0$ and $b > 1$ depending only on Q, such that

$$|f(q\circ h) - f(q)| \leqslant C_0\rho(h) \sup_{\xi;\rho(\xi)\leqslant b\rho(h)} |\nabla_H f(q\circ\xi)|. \tag{3.1}$$

Theorem 3.1. *Let u be an H-harmonic function in $B_{2R}(0)$. Then for any $p \in B_{R/(4b)}(0)$ and any $r > 0$ such that $B_r(p) \subset B_{R/(3b)}(0)$,*

$$\int_{B_r(p)} u^2(z,t)\psi(p^{-1}\circ(z,t)) \leqslant C\int_{B_{2R}(0)} u^2(z,t)\psi(z,t), \tag{3.2}$$

where $b > 1$ is the constant in (3.1) *and $C > 0$ is a constant depending on Q, C_0 and b.*

Proof. If we let $p^{-1}\circ(z',t') = (z,t)$, we can then write

$$\int_{B_r(p)} u^2(z',t')\psi(p^{-1}\circ(z',t')) = \int_{B_r(0)} u^2((z,t)\circ(z,t)^{-1}\circ p\circ(z,t))\psi(z,t).$$

Using (3.1) with $q = (z,t)$ and $h = (z,t)^{-1}\circ p\circ(z,t)$, we have

$$\begin{aligned}&\int_{B_r(p)} u^2(z,t)\psi(p^{-1}\circ(z,t))\\ &\quad\leqslant 2\int_{B_r(0)} u^2(z,t)\psi(z,t) + 2C_0^2\int_{B_r(0)}\rho^2(h)\Big(\max_{\xi;\rho(\xi)\leqslant b\rho(h)}|\nabla_H u((z,t)\circ\xi)|\Big)^2\psi(z,t).\end{aligned}$$

For any $p \in B_{R/(4b)}(0)$, $(z,t)\in B_r(0)$ and $B_r(p)\subset B_{R/(3b)}(0)$, we have

$$\rho(h) \leqslant \rho((z,t)^{-1}) + \rho(p) + \rho(z,t) \leqslant R/b, \quad \rho((z,t)\circ\xi) \leqslant \rho(z,t) + \rho(\xi) \leqslant 5R/4.$$

Then

$$\begin{aligned}&\int_{B_r(p)} u^2(z,t)\psi(p^{-1}\circ(z,t))\\ &\quad\leqslant 2\int_{B_r(0)} u^2(z,t)\psi(z,t) + 2C_0^2 C_Q r^Q\Big(\frac{R}{b}\Big)^2\Big(\max_{B_{5R/4}(0)}|\nabla_H u|\Big)^2,\end{aligned}$$

where C_Q is the constant in (1.9) depending only on Q. By using the sub-elliptic estimates of H-harmonic functions [22]

$$|\nabla_H u|_{L^\infty(B_{5R/4}(0))} \leqslant C\Big(\fint_{B_{6R/4}(0)} |\nabla_H u|^2\Big)^{\frac12},$$

where $\fint_E u = \frac{1}{|E|}\int_E u$ denotes the average of u over the set E, we have

$$\int_{B_r(p)} u^2(z,t)\psi(p^{-1}\circ(z,t))$$

$$\leqslant 2\int_{B_r(0)} u^2(z,t)\psi(z,t) + 2C_0^2 C_Q C^2 r^Q \left(\frac{R}{b}\right)^2 C_Q^{-1}\left(\frac{6R}{4}\right)^{-Q}\int_{B_{6R/4}(0)} |\nabla_H u|^2.$$

By using the following Caccioppoli inequality for H-harmonic functions [32]

$$\int_{B_{6R/4}(0)} |\nabla_H u|^2 \leqslant \frac{C}{R^2}\int_{B_{2R}(0)} u^2(z,t)\psi(z,t),$$

we get

$$\int_{B_r(p)} u^2(z,t)\psi(p^{-1}\circ(z,t)) \leqslant C\int_{B_{2R}(0)} u^2(z,t)\psi(z,t),$$

where C is a constant depending only on C_0, Q and b. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We claim that for any $r > 0$ and $\eta \geqslant 1$,

$$\fint_{B_{\eta r}(0)} u^2(z,t)\psi(z,t) \leqslant \eta^{2N_0}\fint_{B_r(0)} u^2(z,t)\psi(z,t). \tag{3.3}$$

In fact, (2.13) implies

$$\frac{d}{dr}\log\frac{H(r)}{r^{Q-1}} = 2\frac{D(r)}{H(r)} = 2\frac{N(r)}{r}. \tag{3.4}$$

Integrating (3.4) from r to ηr and using the boundedness of $N(r)$, we have

$$\frac{H(\eta r)}{(\eta r)^{Q-1}} \leqslant \eta^{2N_0}\frac{H(r)}{r^{Q-1}}. \tag{3.5}$$

Integrating (3.5) from 0 to r, the claimed (3.3) is proved.

Fixing an $R_0 > 1$ and considering any $p \in \partial B_{R_0}$ and $s > bR_0$, we note the mean value formula of H-harmonic functions [15]

$$u(p) = \fint_{B_{s-bR_0}(p)} u(z,t)\psi(p^{-1}\circ(z,t)).$$

By using the Hölder's inequality, we have

$$|u(p)| \leqslant \left(\fint_{B_{s-bR_0}(p)} u^2(z,t)\psi(p^{-1}\circ(z,t))\right)^{1/2}.$$

Let us choose $R = 4bs$, so that $R_0 < s/b \leqslant s = R/(4b)$, i.e., $\partial B_{R_0} \subset B_{R/(4b)}(0)$ and in particular $p \in B_{R/(4b)}(0)$. Let $r = s - bR_0$. If $q \in B_r(p)$, then

$$d(q,0) \leqslant d(q,p) + d(p,0) \leqslant r + R_0 = s - bR_0 + R_0 \leqslant s = R/(4b) < R/(3b),$$

i.e., $B_r(p) \subset B_{R/(3b)}(0)$. Applying Theorem 3.1 with the data u, $R = 4bs$ and $r = s - bR_0$ as above and (3.3), we have

$$\begin{aligned}|u(p)| &\leqslant C\frac{1}{|B_{s-bR_0}(p)|^{1/2}}\left(\int_{B_{8bs}(0)} u^2(z,t)\psi(z,t)\right)^{1/2}\\ &\leqslant C(8b)^{N_0+\frac{Q}{2}}\left(\frac{s^{Q+2N_0}}{(s-bR_0)^Q}\right)^{1/2}\left(\fint_{B_1(0)} u^2(z,t)\psi(z,t)\right)^{1/2}.\end{aligned}$$

Now, we choose s appropriately to minimize the function

$$f(s) = \frac{s^{Q+2N_0}}{(s-bR_0)^Q}, \quad s > bR_0.$$

A direct calculation shows that $f(s)$ attains its minimum in $(bR_0, +\infty)$ at

$$s_{\min} = \left(1 + \frac{Q}{2N_0}\right) bR_0.$$

Then the corresponding minimum value of f is

$$f_{\min} = f\left(s_{\min}\right) = \left(\frac{2}{Q} + \frac{1}{N_0}\right)^{Q} \left(1 + \frac{Q}{2N_0}\right)^{2N_0} N_0^{Q} b^{2N_0} R_0^{2N_0}.$$

Noting that

$$\left(1 + \frac{Q}{2N_0}\right)^{2N_0} \leqslant \mathrm{e}^{Q}, \quad \left(\frac{2}{Q} + \frac{1}{N_0}\right)^{Q} \leqslant \mathrm{e}^{2+Q/N_0}\mathrm{e}^{-Q},$$

we have

$$f_{\min} \leqslant \mathrm{e}^{2+Q/N_0} N_0^{Q} b^{2N_0} R_0^{2N_0}.$$

Therefore,

$$|u(p)| \leqslant C(8b)^{N_0+Q/2}\mathrm{e}^{1+Q/(2N_0)} N_0^{Q/2} b^{N_0} R_0^{N_0} \left(\fint_{B_1(0)} u^2(z,t)\psi(z,t)\right)^{1/2}. \tag{3.6}$$

Applying the maximum principle for H-harmonic functions, i.e., [3, Theorem 8.2.19, p. 409], and using the following estimate for H-harmonic functions,

$$\fint_{B_1} u^2\psi = \frac{1}{|B_1|}\int_{B_1} u^2\psi \leqslant \|u\|^2_{L^\infty(B_1)},$$

we obtain for any $R_0 > 1$,

$$\|u\|_{L^\infty(B_{R_0})} \leqslant Cb^{2N_0+Q/2}\mathrm{e}^{1+Q/(2N_0)} N_0^{Q/2} R_0^{N_0} \|u\|_{L^\infty(B_1)}.$$

The proof is completed. □

4 Grushin-harmonic polynomials and the proof of Theorem 1.3

In this section, we introduce the relationship between sub-Laplacian and Grushin operator, and prove Theorem 1.3 by using the orthogonality of Grushin-harmonic polynomials.

Let u be an H-harmonic function satisfying $\widetilde{T}u = 0$. Then u solves the equation

$$0 = \mathcal{L}u = \Delta_z u + 4|z|^2\partial_t^2 u = \sum_{i=1}^{2n+1} Y_i^2, \tag{4.1}$$

where $z \in \mathbb{R}^{2n}, t \in \mathbb{R}$ and the vector fields are given by

$$Y_k = \frac{\partial}{\partial x_k}, \quad k = 1, \ldots, 2n, \quad Y_{2n+1} = 2|z|\frac{\partial}{\partial t}. \tag{4.2}$$

The operator (4.1) was studied by Grushin [17, 18], who established its hypoellipticity. The Grushin operator $\mathcal{L}$ is elliptic for $z \neq 0$ and degenerates on the characteristic sub-manifold $\{0\} \times \mathbb{R}$ of $\mathbb{R}^{2n+1}$. We rewrite the Grushin operator (4.1) as the divergence form

$$\mathcal{L} = \mathrm{div}(B(z)\nabla), \tag{4.3}$$

where

$$B(z) = \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 4|z|^2 \end{pmatrix}. \tag{4.4}$$

The corresponding ρ, ψ, $\widetilde{X}$ and Q are the same as for $\mathbb{H}_n$. A direct computation gives [10]

$$\mathcal{L}\rho = \frac{Q-1}{\rho}\psi, \tag{4.5}$$

$$\nabla u \cdot B(z) \cdot \nabla \rho = \frac{1}{\rho}\psi \widetilde{X} u. \tag{4.6}$$

Let $\nabla_{\mathcal{L}} u = (Y_1 u, \ldots, Y_{2n} u, Y_{2n+1} u)$. We recall the definition of frequency for a solution u to $\mathcal{L}u = 0$ as follows [10],

$$N(r) = \frac{r D_{\mathcal{L}}(r)}{H(r)}, \tag{4.7}$$

where

$$H(r) = \int_{\partial B_r} u^2 \frac{\psi}{|\nabla \rho|}, \quad D_{\mathcal{L}}(r) = \int_{B_r} |\nabla_{\mathcal{L}} u|^2 = \int_{\partial B_r} u \frac{\langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} \rho \rangle}{|\nabla \rho|}. \tag{4.8}$$

Definition 4.1. A spherical $\mathcal{L}$-harmonic of H-degree k $(k = 0, 1, 2, \ldots)$ is a polynomial in x, y and t, which is a solution to (4.1) and is homogeneous of H-degree k.

Lemma 4.2. *Let u_{k_1} and u_{k_2} be two spherical $\mathcal{L}$-harmonics of H-degrees k_1 and k_2 in B_1, respectively. Then for any $0 < r < 1$, we have*

$$\int_{\partial B_r} u_{k_1} u_{k_2} \frac{\psi}{|\nabla \rho|} = 0, \quad \textit{for} \quad k_1 \neq k_2.$$

Proof. Let u_{k_1} and u_{k_2} be two spherical $\mathcal{L}$-harmonics of H-degrees k_1 and k_2, respectively. Then by divergence theorem and (4.3) we have

$$0 = \int_{B_r} (u_{k_1} \mathcal{L} u_{k_2} - u_{k_2} \mathcal{L} u_{k_1}) = \int_{\partial B_r} \left(u_{k_1} B \cdot \nabla u_{k_2} \cdot \frac{\nabla \rho}{|\nabla \rho|} - u_{k_2} B \cdot \nabla u_{k_1} \cdot \frac{\nabla \rho}{|\nabla \rho|} \right).$$

By (4.6) and $\widetilde{X} u_{k_i} = k_i u_{k_i} (i = 1, 2)$, we get

$$0 = \frac{1}{r} \int_{\partial B_r} (u_{k_1} \widetilde{X} u_{k_2} - u_{k_2} \widetilde{X} u_{k_1}) \frac{\psi}{|\nabla \rho|} = \frac{1}{r} \int_{\partial B_r} (k_2 - k_1) u_{k_1} u_{k_2} \frac{\psi}{|\nabla \rho|}.$$

Hence

$$\int_{\partial B_r} u_{k_1} u_{k_2} \frac{\psi}{|\nabla \rho|} = 0, \quad \text{for} \quad k_1 \neq k_2.$$

□

Remark 4.3. The above orthogonality fails to general H-harmonic polynomials on $\mathbb{H}_n$. For example, $x^2 - y^2$ and $x^4 - y^4 + 3xyt$ are H-harmonic polynomials of H-degrees 2 and 4, respectively, a direct calculation yields

$$\int_{\partial B_r} (x^2 - y^2) \cdot (x^4 - y^4 + 3xyt) \frac{\psi}{|\nabla \rho|} \neq 0.$$

Let us introduce polar coordinates on $\mathbb{H}_n$, which were first introduced by Greiner [16] for $\mathbb{H}_1$ and then extended by Dunkl [7] to $\mathbb{H}_n$. Let

$$\begin{cases} x_1 = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-2} \sin \theta_{2n-1}, \\ y_1 = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-2} \cos \theta_{2n-1}, \\ x_2 = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-3} \cos \theta_{2n-2}, \\ y_2 = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-4} \cos \theta_{2n-3}, \\ \quad \vdots \\ x_n = \rho \sin^{1/2} \phi \sin \theta_1 \cos \theta_2, \\ y_n = \rho \sin^{1/2} \phi \cos \theta_1, \\ t = \rho^2 \cos \phi. \end{cases} \tag{4.9}$$

Here $0 \leqslant \phi < \pi$, $0 \leqslant \theta_i < \pi$, $i = 1, \ldots, 2n-2$ and $0 \leqslant \theta_{2n-1} < 2\pi$. Then, a straightforward computation gives

$$\mathcal{L} = \sin\phi \left\{ \frac{\partial^2}{\partial \rho^2} + \frac{Q-1}{\rho}\frac{\partial}{\partial \rho} + \frac{4}{\rho^2}\mathcal{L}^* \right\}, \tag{4.10}$$

and

$$\mathcal{L}^* = \frac{\partial^2}{\partial \phi^2} + n\cot\phi \frac{\partial}{\partial \phi} + \frac{1}{(2\sin\phi)^2}\Delta_{S^{2n-1}}, \tag{4.11}$$

where $\Delta_{S^{2n-1}}$ denotes the Laplace-Beltrami operator on S^{2n-1}.

Garofalo and Shen [14] obtained a polynomial solution to $\mathcal{L}u = 0$ by separation of variables as follows.

Lemma 4.4. *Let u_k be spherical $\mathcal{L}$-harmonics of H-degree k $(k = 0, 1, 2, \ldots)$. Then u_k are sums of*

$$\rho^k \sin^{l/2}\phi\, \mathbb{C}^{\frac{l+n}{2}}_{\frac{k-l}{2}}(\cos\phi) Y_l(\omega),$$

where $0 \leqslant l \leqslant k$, $l \equiv k \pmod 2$, $Y_l(\omega)$ are $(2n-1)$-dimension spherical harmonics of degree l, and the ultraspherical polynomials (Gegenbauer polynomials) $\mathbb{C}^\lambda_k(x)$, $-1 \leqslant x \leqslant 1$, are defined by the generating function

$$\sum_{k=0}^{\infty} \omega^k \mathbb{C}^\lambda_k(x) = (1 - 2x\omega + \omega^2)^{-\lambda}.$$

Fix an integer $l \geqslant 0$ and denote by $\{Y_{l,j}\}$ $(j = 1, 2, \ldots, d_l)$ an orthonormal basis for the space of spherical harmonics of degree l on S^{2n-1}. In the present situation, we have the following theorem.

Theorem 4.5. *Let u be a solution to $\mathcal{L}u = 0$ in B_1. If there exists a positive constant $0 < r_0 < 1$ such that $N(r) = N_0$ for some constant N_0 and any $r \in (0, r_0)$, then N_0 is an integer, and u is a homogeneous polynomial of H-degree N_0 in B_{r_0}.*

Proof. Because of the analyticity of u [26], we may assume u is given by

$$u = \sum_{k=0}^{\infty} a_k u_k = \sum_{k=0}^{\infty} a_k \rho^k \sum_{l=0}^{k} b_k^l \sin^{l/2}\phi\, \mathbb{C}^{\frac{l+n}{2}}_{\frac{k-l}{2}}(\cos\phi) \sum_{j=0}^{d_l} c_l^j Y_{l,j}(\omega),$$

for $\rho \leqslant r_0$, where u_k is a spherical $\mathcal{L}$-harmonics of H-degree k, i.e., $\widetilde{X}u_k = ku_k$, and a_k, b_k^l, c_l^j are constants. Then, by (4.8) and (4.6), we have

$$\begin{aligned} D_{\mathcal{L}}(r) &= \int_{\partial B_r} u \frac{\langle \nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} \rho \rangle}{|\nabla \rho|} = \frac{1}{r}\int_{\partial B_r} u\widetilde{X}u \frac{\psi}{|\nabla \rho|} \\ &= \frac{1}{r}\int_{\partial B_r} \left(\sum_{k=0}^{\infty} a_k u_k\right)\left(\sum_{k=0}^{\infty} k a_k u_k\right)\frac{\psi}{|\nabla \rho|}. \end{aligned}$$

Similarly,

$$H(r) = \int_{\partial B_r} u^2 \frac{\psi}{|\nabla \rho|} = \int_{\partial B_r} \left(\sum_{k=0}^{\infty} a_k u_k\right)\left(\sum_{k=0}^{\infty} a_k u_k\right)\frac{\psi}{|\nabla \rho|}.$$

By using the orthogonality of $Y_{l,j}(\omega)$, $\mathbb{C}^\lambda_k(x)$ (see [30, p. 81]) and Lemma 4.2, we get

$$\begin{aligned} N(r) = \frac{rD_{\mathcal{L}}(r)}{H(r)} &= \frac{\int_{\partial B_r} (\sum_{k=0}^{\infty} a_k u_k)(\sum_{k=0}^{\infty} k a_k u_k)\frac{\psi}{|\nabla \rho|}}{\int_{\partial B_r} (\sum_{k=0}^{\infty} a_k u_k)(\sum_{k=0}^{\infty} a_k u_k)\frac{\psi}{|\nabla \rho|}} \\ &= \frac{\sum_{k=0}^{\infty} k a_k^2 \sum_{l=0}^{k} (b_k^l)^2 \sum_{j=0}^{d_l} (c_l^j)^2 \rho^{2k}}{\sum_{k=0}^{\infty} a_k^2 \sum_{l=0}^{k} (b_k^l)^2 \sum_{j=0}^{d_l} (c_l^j)^2 \rho^{2k}} \equiv \frac{\sum_{k=0}^{\infty} k\widetilde{a_k}^2 \rho^{2k}}{\sum_{k=0}^{\infty} \widetilde{a_k}^2 \rho^{2k}} = N_0, \end{aligned}$$

where

$$\widetilde{a_k}^2 = a_k^2 \sum_{l=0}^{k} (b_k^l)^2 \sum_{j=0}^{d_l} (c_l^j)^2.$$

Therefore,

$$\sum_{k=0}^{\infty}(k-N_0)\,\widetilde{a_k}^2\rho^{2k}=0,\quad \text{for any small } \rho,$$

i.e.,

$$\sum_{k>N_0}(k-N_0)\,\widetilde{a_k}^2\rho^{2k}=\sum_{k\leqslant N_0}(N_0-k)\,\widetilde{a_k}^2\rho^{2k}.$$

Letting $\rho\to 0$, we get $\widetilde{a}_0=0$. Going on the same computation, we get $\widetilde{a_k}=0$, when $k<N_0$. Therefore, we get

$$\sum_{k>N_0}(k-N_0)\,\widetilde{a_k}^2\rho^{2N_0}\rho^{2k-2N_0}=0.$$

Hence $\widetilde{a_k}=0$, when $k>N_0$. This concludes the proof of the theorem. □

Proof of Theorem 1.3. Noting that an H-harmonic function with $\widetilde{T}u=0$ is automatically a solution to $\mathcal{L}u=0$, we can immediately obtain the desired result from Theorem 4.5. □

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